AVERAGE NORMS OF POLYNOMIALS

Toufik Mansour [1]

Department of Mathematics, Chalmers University of Technology, S-412 96 Göteborg, Sweden

toufik@math.chalmers.se

Abstract

In this paper we study the average $\mathcal{L}_{2\alpha}$-norm over T-polynomials, where α is a positive integer. More precisely, we present an explicit formula for the average $\mathcal{L}_{2\alpha}$-norm over all the polynomials of degree exactly n with coefficients in T, where T is a finite set of complex numbers, α is a positive integer, and $n \geq 0$. In particular, we give a complete answer for the cases of Littlewood polynomials and polynomials of a given height. As a consequence, we derive all the previously known results for this kind of problems, as well as many new results.

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1. Introduction

The set of polynomials with special coefficients has given much attention and there are many old research questions concerning it. Erdös and Szekere; Hilbert; Littlewood; Prouhet, Tarry, and Escott, are suggested number of these questions (see [B3, Pages 5-7]). For example, Prouhet, Tarry, and Escott asked to find a polynomial with integer coefficients that is divisible by $(z-1)^n$ and has smallest sum of the absolute values of the coefficients, and Erdös and Szekere asked to find the minimum of $||\prod_{j=1}^{n}(1-z^{a_j})||_\infty$ where the a_j are positive integers, for given n. The problem to find the maximum and the minimum norms of polynomials with special coefficients is a specific old and difficult problem. In the $\mathcal{L}_4$-norm this problem is often called Golay's "Merit Factor" problem. In the supremum norm this problem is due to Littlewood. These problems are at least fifty years old and never had solved. In this paper we interested in the following problem (for particular cases see [B3, page 35]): Find the average $\mathcal{L}_{2\alpha}$-norm over polynomials of degree exactly n with coefficients in a given finite set T, where α is a positive integer. In this paper, we give a complete answer for this problem.

Let $T = \{x_1, x_2, \ldots, x_d\}$ be any finite set of complex numbers. A polynomial $p(z) = a_n z^n + \cdots + a_1 z + a_0$ is said to be *T-polynomial* if $a_i \in T$ for all i, $0 \leq i \leq n$. We denote by $\mathfrak{T}_T(n)$ the set of all T-polynomials of degree exactly n. For example, if $T = \{0, 1, 2\}$ then the set of T-polynomials of degree exactly 1 is given by $\mathfrak{T}_T(1) = \{z, 2z, z+1, 2z+1, z+2, 2z+2\}$. The cardinality of this set, $\mathfrak{T}_T(n)$, is denoted by $N_T(n)$. Clearly, for all $n \geq 0$,

$$N_T(n) = \begin{cases} d^{n+1}, & 0 \notin T \text{ and } n \geq 1, \\ (d-1)d^n, & 0 \in T \text{ and } n \geq 1, \\ d, & n = 0. \end{cases}$$

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A polynomial $p(z)$ is said to be *Littlewood polynomial* if it $\{-1,1\}$-polynomial. A polynomial $p(z)$ is said to be *polynomial of height* h if it $\{-h,-h+1,\ldots,h-1,h\}$-polynomial. For example, $p(z)=z^2-z+1$ is a Littlewood polynomial of degree 2, and $p(z)=z^3-2z^2-1$ is a polynomial of height 2 and degree 3.

Let $p(z)$ be any T-polynomial of degree exactly n. For any positive integer α, the $\mathcal{L}_\alpha$*-norm* on the boundary of the unit disk is defined by

$$||p||_\alpha=\left(\frac{1}{2\pi}\int_0^{2\pi}|p(e^{i\theta})|^\alpha\,d\theta\right)^{\frac{1}{\alpha}}.$$

Let f,g,h be any three $\mathbb{R}$-polynomials, the (f,g,h)*-average* over T-polynomials of degree exactly n is defined by

$$E_T(n;f,g,h)=\frac{1}{2\pi N_T(n)}\sum_{p\in\mathfrak{T}_T(n)}\int_0^{2\pi}h(e^{i\theta})f(p(e^{i\theta}))g(p(e^{-i\theta}))\,d\theta, \tag{1}$$

for any $n\geq 1$. We denote by $e_T(n,s,t,m)$ the (z^s,z^t,z^m)-average over T-polynomials of degree exactly n, where $m\in\mathbb{Z}$ and $s,t,n\geq 0$.

We define the *average* $\mathcal{L}_\alpha$*-norm* over T-polynomials of degree exactly n by

$$\mu_T^\alpha(n)=e_T(n;\alpha/2,\alpha/2,0)=\frac{1}{N_T(n)}\sum_{p\in\mathfrak{T}_T(n)}||p||_\alpha^\alpha=\frac{1}{2\pi N_T(n)}\sum_{p\in\mathfrak{T}_n}\int_0^{2\pi}|p(e^{i\theta})|^\alpha\,d\theta, \tag{2}$$

for any positive integer α. For $\alpha=0$ we define $\mu_T^0(n)=1$ for all $n\geq 1$.

Some one could ask to find an explicit formula for $\mu_T^{2\alpha}(n)$, where T is a finite set of complex numbers and α is a positive integer (for particular cases, see the research problem in [B3, Page 35]). While the cases of Littlewood polynomials and polynomials of height h have attracted much attention (for example, see [B3, BC, NB]), the case of other sets T. The case of $T=\{-1,1\}$, Littlewood polynomials, considered by several authors as follows. In 1990, Newman and Byrnes [NB] found $\mu_{\{-1,1\}}^4(n)=2n^2+3n+1$. In 2002, Borwein and Choi [BC] proved

$$\begin{aligned}\mu_{\{-1,1\}}^6(n)&=6n^3+9n^2+4n+1,\\ \mu_{\{-1,1\}}^8(n)&=24n^4+30n^3+4n^2+5n+4-3(-1)^n.\end{aligned}$$

In the case $T=\{-1,0,1\}$, polynomials of height 1, Borwein [B2] proved $\mu_{\{-1,0,1\}}^2(n)=\frac{2}{3}(n+1)$, $\mu_{\{-1,0,1\}}^4(n)=\frac{2}{9}(4n^2+7n+3)$, and $\mu_{\{-1,0,1\}}^6(n)=\frac{2}{9}(8n^3+18n^2+13n+3)$. More generally, Borwein [B2] found for any $h\geq 0$,

$$\begin{aligned}\mu_{\{-h,-h+1,\ldots,h-1,h\}}^2(n)&=\tfrac{h(h+1)}{3}(n+1),\\ \mu_{\{-h,-h+1,\ldots,h-1,h\}}^4(n)&=\tfrac{h(h+1)}{45}(10h(h+1)n^2+(19h^2+19h-3)n+3(3h^2+3h-1)).\end{aligned}$$

In this paper we suggest a general approach to study the average $\mathcal{L}_{2\alpha}$-norm over T-polynomials of degree exactly n, for any positive integer α and any finite set T of complex numbers, which allows one to get an explicit expression for $\mu_T^{2\alpha}(n)$. More precisely, we find an explicit expression for the generating function $e_T(x,u,v,w)$. Using this generating function we get an explicit expression for $e_T(n;s,t,m)$ in general, and $\mu_T^{2\alpha}(n)=e_T(n,\alpha,\alpha,0)$ in particular, where $m\in\mathbb{Z}$, $s,t,n\geq 0$, α is a positive integer, and T is a finite set of complex numbers. As a consequence, we derive all the previously known results for this kind of problems, as well as many new results.

The main result of this paper can be formulated as follows. We denote by $e_T(x,u,v,w)$ the generating function for the sequence $\{e_T(n,s,t,m)\}_{n,s,t,m}$, that is,

$$e_T(x,u,v,w)=\sum_{m\in\mathbb{Z}}\sum_{p,q,n\geq 0} e_T(n,s,t,m)x^nu^sv^tw^m.$$

Theorem 1.1. *Let $T=\{x_1,x_2,\ldots,x_d\}$ be a finite set of complex numbers. Then the generating function $e_T(x,u,v,w)$ is given by*

$$\sum_{n\geq1}\left[\frac{1}{d^n}\sum_{j_1,\ldots,j_n=1}^{d}\frac{x^{n-1}}{(1-x_{j_1}u-x_{j_2}uw^{-1}-\cdots-x_{j_n}uw^{-n+1})\,(1-\overline{x_{j_1}}v-\overline{x_{j_2}}vw-\cdots-\overline{x_{j_n}}vw^{n-1})}\right].$$

Moreover, $e_T(n,s,t,m)$ is given by

$$\frac{1}{d^{n+1}}\sum_{j_1,\ldots,j_{n+1}=1}^{d}\sum_{s,t\geq0}\sum_{\substack{k_1+\cdots+k_{n+1}=s\\ \ell_1+\cdots+\ell_{n+1}=t\\ \sum_{a=1}^{n+1}(a-1)(\ell_a-k_a)=m}}\prod_{a=1}^{n+1}(x_{j_a}^{t_a}\overline{x_{j_a}}^{r_a})\binom{s}{k_1,\ldots,k_{n+1}}\binom{t}{\ell_1,\ldots,\ell_{n+1}}.$$

Using Theorem 1.1 we get an explicit expression for the average $\mathcal{L}_{2\alpha}$-norm over T-polynomials of degree exactly n, namely $\mu_T^{2\alpha}(n)$.

Corollary 1.2. *Let $T=\{x_1,x_2,\ldots,x_d\}$ be a finite set of complex numbers. Then the average $\mathcal{L}_{2\alpha}$-norm over T-polynomials of degree exactly n, namely $\mu_T^{2\alpha}(n)$, is given by*

$$\frac{1}{d^{n+1}}\sum_{j_1,\ldots,j_{n+1}=1}^{d}\sum_{\substack{k_1+\cdots+k_{n+1}=\alpha\\ \ell_1+\cdots+\ell_{n+1}=\alpha\\ \sum_{a=1}^{n+1}(a-1)(\ell_a-k_a)=0}}\prod_{a=1}^{n+1}(x_{j_a}^{k_a}\overline{x_{j_a}}^{\ell_a})\binom{\alpha}{k_1,\ldots,k_{n+1}}\binom{\alpha}{\ell_1,\ldots,\ell_{n+1}}.$$

Proof. If applying Theorem 1.1 and the identity $e_T(n,\alpha,\alpha,0)=\mu_T^{2\alpha}(n)$, then we get the desired result. □

The paper is organized as follows. The proof of our main result, Theorem 1.1, is presented in Section 2. In section 3 we present a general application for our results. In particular, we give an explicit formulas up to $\alpha=4$ where $\sum_{t\in T}t=0$ (as in the case of Littlewood polynomials and polynomials of height h). In Section 4 we apply our main result for ceratin sets T, which allows us to get more details in the cases of Littlewood polynomials and polynomials of height 1. Finally, in Section 5 we suggest several directions to generalize the results of the previous sections.

2. Proofs

Let us start by introduce the quantity that plays the crucial role in the proof of Theorem 1.1.

Theorem 2.1. *Let $T=\{x_1,x_2,\ldots,x_d\}$ be a set of complex numbers, $n\geq1$, m any integer, and $s,t\geq0$. Then*

$$e_T(n,s,t,m)=\frac{1}{d}\sum_{j=1}^{d}\sum_{k=0}^{a}\sum_{\ell=0}^{b}x_j^{s-k}\,\overline{x_j}^{t-\ell}\binom{s}{k}\binom{t}{\ell}e_T(n-1,k,\ell,m+k-\ell). \tag{3}$$

Proof. Let $z = e^{i\theta}$ and $f_{s,t,m}(p(z)) = z^m p^s(z)\overline{p}^t(z)$, where $\overline{p}(z)$ is the conjugate polynomial of the polynomial $p(z) \in \mathfrak{T}_T(n)$. Since for any T-polynomial $p(z) \in \mathfrak{T}_T(n)$ there exists an unique polynomial $q(z) \in \mathfrak{T}_T(n-1)$ and exists j, $1 \leq j \leq d$, such that $p(z) = zq(z) + x_j$, we have that

$$\sum_{p(z)\in\mathfrak{T}_T(n)} f_{s,t,m}(p(z)) = z^m \sum_{j=1}^{d} \sum_{q(z)\in\mathfrak{T}_T(n-1)} (zq(z)+x_j)^s(\overline{z}\,\overline{q}(z)+\overline{x_j})^t.$$

Using $(x+y)^k = \sum_{j=0}^k \binom{k}{j} x^j y^{k-j}$ and $\overline{z} = z^{-1}$ we get

$$\sum_{p(z)\in\mathfrak{T}_T(n)} f_{s,t,m}(p(z)) = \sum_{q(z)\in\mathfrak{T}_T(n-1)} \sum_{j=1}^{d}\sum_{k=0}^{s}\sum_{\ell=0}^{t} x_j^{s-k}\,\overline{x_j}^{\,t-\ell}\binom{s}{k}\binom{t}{\ell} f_{k,\ell,m+k-\ell}(q(z)).$$

Therefore, using Definition 1 we get the desired result. □

To present Recurrence 3 in terms of generating functions we need the following lemma.

Lemma 2.2. *Let $F(x,y) = \sum_{s,t\geq 0} d_{s,t}x^s y^t$ be a generating function with two variables. Then*

$$\sum_{s,t\geq 0} x^s y^t \left(\sum_{k=0}^{s}\sum_{\ell=0}^{t} a^{s-k}b^{t-\ell}d_{k,\ell}\binom{s}{k}\binom{t}{\ell}\right) = \frac{1}{(1-ax)(1-by)}F\left(\frac{x}{1-ax},\frac{y}{1-by}\right).$$

Proof. By definitions we have

$$\frac{1}{(1-ax)(1-by)}F\left(\frac{x}{1-ax},\frac{y}{1-by}\right) = \sum_{s,t\geq 0} a^{-s}b^{-t}d_{s,t}\frac{(ax)^s(by)^t}{(1-ax)^{s+1}(1-by)^{t+1}}.$$

Using $\frac{x^r}{(1-x)^r} = \sum_{n\geq 0}\binom{n}{r}x^r$ we get

$$\frac{1}{(1-ax)(1-by)}F\left(\frac{x}{1-ax},\frac{y}{1-by}\right) = \sum_{s,t\geq 0}\left(\sum_{k,\ell\geq 0} a^{k-s}b^{\ell-t}d_{s,t}\binom{k}{s}\binom{\ell}{t}x^k y^\ell\right),$$

equivalently,

$$\frac{1}{(1-ax)(1-by)}F\left(\frac{x}{1-ax},\frac{y}{1-by}\right) = \sum_{s,t\geq 0} x^s y^t \left(\sum_{k=0}^{s}\sum_{\ell=0}^{t} a^{s-k}b^{t-\ell}d_{k,\ell}\binom{s}{k}\binom{t}{\ell}\right),$$

as claimed. □

Remark 2.3. *Lemma 2.2 can be generalized as follows. Let*

$$F(v_1,\ldots,v_k) = \sum_{s_1,s_2,\ldots,s_k\geq 0} e_{s_1,\ldots,s_k}\prod_{d=1}^{k} v_d^{s_d}$$

be a generating function with k variables. Then

$$\sum_{s_1,s_2,\ldots,s_k\geq 0}\prod_{d=1}^{k} v_d^{s_d}\left(\sum_{j_1=0}^{s_1}\sum_{j_2=0}^{s_2}\cdots\sum_{j_k=0}^{s_k} e_{j_1,j_2,\ldots,j_k}\prod_{d=1}^{k}\binom{s_d}{j_d}w_d^{s_d-j_d}\right)$$

is given by

$$\frac{1}{\prod_{d=1}^{k}(1-v_d w_d)}F\left(\frac{v_1}{1-w_1v_1},\frac{v_2}{1-w_2v_2},\ldots,\frac{v_k}{1-w_kv_k}\right).$$

Now we are ready to prove our main result, namely Theorem 1.1.

Theorem 2.4. *Let $T = \{x_1, x_2, \ldots, x_d\}$ be a finite set of complex numbers. Then the generating function $e_T(x, v, u, w)$ is given by*

$$\sum_{n\geq 1}\left[\frac{1}{d^n}\sum_{j_1,\ldots,j_n=1}^{d}\frac{x^{n-1}}{(1-x_{j_1}u-x_{j_2}uw^{-1}-\cdots-x_{j_n}uw^{-n+1})(1-\overline{x_{j_1}}v-\overline{x_{j_2}}vw-\cdots-\overline{x_{j_n}}vw^{n-1})}\right].$$

Proof. If multiplying Equation 3 by $x^n u^s v^t w^m$, and summing over all $m \in \mathbb{Z}$, $s, t \geq 0$, and $n \geq 1$, together with using Lemma 2.2, then we arrive to

$$e_T(x,u,v,w)-\sum_{m\in\mathbb{Z}}\sum_{s,t\geq 0}e(0,s,t,m)u^sv^tw^m=\frac{x}{d}\sum_{j=1}^{d}\left[\frac{1}{(1-x_ju)(1-\overline{x_j}v)}e_T\left(x,\frac{u}{w(1-x_ju)},\frac{wv}{1-\overline{x_j}v},w\right)\right].$$

On the other hand, by definitions we have that $e_T(0, s, t, m) = \frac{1}{d}\sum_{j=1}^{d}\delta_m x_j^s\overline{x_j}^t$ for any $m \in \mathbb{Z}$ and $s, t \geq 0$, where $\delta_m = 1$ if $m = 0$, otherwise $\delta_m = 0$. So

$$\sum_{m\in\mathbb{Z}}\sum_{s,t\geq 0}e_T(0,s,t,m)u^sv^tw^m=\frac{1}{d}\sum_{j=1}^{d}\frac{1}{(1-x_ju)(1-\overline{x_j}v)}.$$

Therefore, by combining the above two equations we get that

$$e_T(x,u,v,w)=\frac{1}{d}\sum_{j=1}^{d}\frac{1}{(1-x_ju)(1-\overline{x_j}v)}\left[1+xe_T\left(x,\frac{u}{w(1-x_ju)},\frac{wv}{1-x_jv},w\right)\right]. \tag{4}$$

An infinite number of applications of this identity concludes the proof. □

Remark 2.5. *Theorem 2.4 yields the generating function $e_T(x, u, v, w)$ is symmetric under the translation $(u, v, w) \to (v, u, w^{-1})$, that is, $e_T(x, u, v, w) = e_T(x, v, u, w^{-1})$.*

Theorem 2.4 can be presented as follows.

Corollary 2.6. *Let $T = \{x_1, x_2, \ldots, x_d\}$ be a finite set of complex numbers. Then the generating function $e_T(x, v, u, w)$ is given by*

$$\sum_{n\geq 1}\sum_{j_1,\ldots,j_n=1}^{d}\sum_{s,t\geq 0}\sum_{\substack{k_1+\cdots+k_n=s\\ \ell_1+\cdots+\ell_n=t}}\prod_{a=1}^{n}(x_{j_a}^{t_a}\overline{x_{j_a}}^{r_a})\binom{s}{k_1,\ldots,k_n}\binom{t}{\ell_1,\ldots,\ell_n}\frac{x^{n-1}u^sv^tw^{\left(\sum_{a=1}^{n}(a-1)(r_a-t_a)\right)}}{d^n}.$$

Proof. Using Theorem 2.4 we get that the generating function $e_T(x, v, u, w)$ is given by

$$\sum_{n\geq 1}\left[\sum_{j_1,j_2,\ldots,j_n=1}^{d}\sum_{s,t\geq 0}\left(x_{j_1}+x_{j_2}w^{-1}+\cdots+x_{j_n}w^{-n+1}\right)^s\left(\overline{x_{j_1}}+\overline{x_{j_2}}w+\cdots+\overline{x_{j_n}}w^{n-1}\right)^t\frac{x^{n-1}u^sv^t}{d^n}\right].$$

the rest is easy to check by the identity $(a_1 + \ldots + a_n)^s = \sum_{k_1+\cdots+k_n=s}\binom{s}{k_1,\ldots,k_n}\prod_{j=1}^{n}a_j^{k_j}$. □

Let us denote by $\mu_T^\alpha(x)$ the generating function for the sequence $\{\mu_T^\alpha(n)\}_{n\geq 0}$, that is, $\mu_T^\alpha(x) = \sum_{n\geq 0}\mu_T^\alpha(n)x^n$. Corollary 1.2 gives a complete answer to find the generating function $\mu_T^{2\alpha}(x)$ for any given finite set T and positive integer α.

Example 2.7. *Using Corollary 1.2 for $\alpha = 1$ we get that*

$$\mu_T^2(x) = \sum_{n\geq 1} \sum_{j_1,\ldots,j_n=1}^{d} \left(\sum_{k=1}^{n} |x_{j_k}|^2 \right) \frac{x^{n-1}}{d^n} = \sum_{n\geq 1} \left(nd^{n-1} \sum_{j=1}^{d} |x_j|^2 \right) \frac{x^{n-1}}{d^n},$$

so it is easy to see that $\mu_T^2(x) = \frac{1}{d(1-x)} \sum_{k=1}^{n} |x_{j_k}|^2 + x\mu_T^2(x)$, hence

$$\mu_T^2(x) = \sum_{n\geq 0} \left(\frac{n+1}{d} \sum_{j=1}^{d} |x_j|^2 \right) x^n.$$

In particular, we have that $\mu_{\{-1,1\}}(n;2) = n+1$, and $\mu_{\{-1,0,1\}}(n;2) = \frac{2}{3}(n+1)$.

Corollary 1.2 provides a finite algorithm for finding the average $\mu_T^{2\alpha}(n)$ where n, T, and α are given. This algorithm has been implemented in MAPLE, and yields explicit results for given n, T, and α (see the tables below).

Remark 2.8. *To apply Corollary 1.2 we have to consider d^n possibilities for j_i and $\binom{n+\alpha-1}{\alpha}^2$ possibilities for k_i and ℓ_i, namely we have to consider $d^n\binom{n+\alpha-1}{\alpha}^2$ possibilities. Therefore, Corollary 1.2, for n large, works very slowly.*

3. Exact formulas

Corollary 1.2 provides a close formula for finding the average $\mathcal{L}_{2\alpha}$-norm over T-polynomials of degree exactly n for any given $n \geq 0$ and $\alpha \geq 1$. Remark 2.8 yields that the problem to find exact formula for $\mu_T^{2\alpha}(n)$ with given only $\alpha \geq 1$ it is a hard problem by using Corollary 1.2. Thus, we suggest here another approach to find an explicit formula for $\mu_T^{2\alpha}(n)$.

First of all, let us denote by $e_T^{s,t}(x,w)$ the $(s+t)$-derivative of the generating function $e_T(x,u,v,w)$ with respect u^s and then with respect v^t at $u = v = 0$, that is,

$$e_T^{s,t} = e_T^{s,t}(x,w) = \left. \frac{\partial^{s+t}}{\partial u^s \partial v^t} e_T(x,u,v,w) \right|_{u=v=0}.$$

For any $s,t \geq 0$, we define

$$A_T^{s,t} = \sum_{j=1}^{d} x_j^s \overline{x_j}^t.$$

Now let us consider Equation 4. This equation provide a finite algorithm, *μ-algorithm*, for finding $e_T(n,s,t,m)$ in general, and $\mu_T^{2\alpha}(n)$ in particular, since $s!\,t!\,e_T(n,s,t,m)$ is the coefficient of $w^m x^n$ in the $(s+t)$-derivative of the generating function $e_T(x,u,v,w)$ with respect to u^s and then with respect v^t at $u = v = 0$, namely $e_T^{s,t}(x,w)$, and $\mu_T^{2\alpha}(n) = e_T(n;\alpha,\alpha,0)$. Therefore, the μ-algorithm with input α and output $\mu_T^{2\alpha}(x)$ can be constructed as follows:

(1) Apply the derivative operator with respect u^s and then with respect v^t on Equation 4 for all s,t, where $0 \leq s,t \leq \alpha$.
(2) Find explicitly $e_T^{s,t}$ for all s,t, where $0 \leq s,t \leq \alpha$. This by solving the system equations which is obtained from step 1.
(3) Find $\mu_T^{2\alpha}(x)$, which is the free coefficient of w in $e_T^{\alpha,\alpha}(x,w)$.

This algorithm has been implemented in MAPLE , and yields explicit results for given α. Below we present several explicit calculations.

3.1. **Formula for $\mu_T^2(n)$.** Let us start by apply the μ-algorithm for $\alpha = 1$. The first step of the μ-algorithm gives

$$\begin{cases} e_T^{0,0} = 1 + xe_T^{0,0}, \\ e_T^{0,1} = \frac{1}{d}A_T^{0,1} + \frac{x}{d}A_T^{0,1}e_T^{0,0} + \frac{x}{w}e_T^{0,1}, \\ e_T^{1,0} = \frac{1}{d}A_T^{1,0} + \frac{x}{d}A_T^{1,0}e_T^{0,0} + \frac{x}{w}e_T^{1,0}, \\ e_T^{1,1} = \frac{1}{d}A_T^{1,1} + \frac{x}{d}A_T^{1,1}e_T^{0,0} + \frac{xw}{d}A_T^{1,0}e_T^{0,1} + \frac{x}{dw}A_T^{0,1}e_T^{1,0} + xe_T^{1,1}. \end{cases}$$

Equivalently (the second step of the μ-algorithm),

$$e_T^{0,0} = \frac{1}{1-x}, \quad e_T^{1,0} = \frac{A_T^{1,0}}{d(1-x)(1-xw^{-1})}, \quad e_T^{0,1} = \frac{A_t^{0,1}}{d(1-x)(1-xw)},$$

and

$$e_T^{1,1} = \tfrac{1}{d(1-x)^2}A_T^{1,1} + \left(\tfrac{xw}{d^2(1-x)^2(1-xw)} + \tfrac{xw^{-1}}{d^2(1-x)^2(1-xw^{-1})}\right)A_T^{1,0}A_T^{0,1}.$$

Therefore, the third step of the μ-algorithm gives $\mu_T^2(x)$, which is the free coefficient of w in $e_T^{1,1}(x,w)$. Hence, we get the following result.

Corollary 3.1. *We have*

$$\mu_T^2(x) = \frac{1}{d(1-x)^2}A_T^{1,1} = \frac{1}{d(1-x)^2}\sum_{j=1}^{d}|x_j|^2.$$

Moreover, for all $n \geq 0$,

$$\mu_T^2(n) = \frac{n+1}{d}A_T^{1,1} = \frac{n+1}{d}\sum_{j=1}^{d}|x_j|^2.$$

For example, in the case of Littlewood polynomials, namely $T = \{-1, 1\}$, we have that $\mu_T^2(n) = n+1$, and in the case of polynomials of height h, namely $T = \{-h, -h+1, \ldots, h-1, h\}$, we have that $\mu_T^2(n) = \frac{h(h+1)}{3}(n+1)$.

3.2. **Formula for $\mu_T^4(n)$.** Again, using the μ-algorithm for $\alpha = 2$ we get that

$$\begin{aligned} e_T^{0,2} &= \tfrac{2}{d(1-x)(1-xw^2)}A_T^{0,2} + \tfrac{2xw}{d(1-xw^2)}e_T^{0,1}A_T^{0,1}, \\ e_T^{2,0} &= \tfrac{2}{d(1-x)(1-xw^{-2})}A_T^{2,0} + \tfrac{2x}{dw(1-xw^{-2})}e_T^{1,0}A_T^{1,0}, \\ e_T^{1,2} &= \tfrac{2}{d(1-x)(1-xw)}A_T^{1,2} + \tfrac{4xw}{d(1-xw)}e_T^{0,1}A_T^{1,1} + \tfrac{xw^2}{d(1-xw)}e_T^{0,2}A_T^{1,0} + \tfrac{2x}{dw(1-xw)}e_T^{1,0}A_T^{0,2} \\ &\quad + \tfrac{4x}{d(1-xw)}e_T^{1,1}A_T^{0,1}, \\ e_T^{2,1} &= \tfrac{2}{d(1-x)(1-xw^{-1})}A_T^{2,0} + \tfrac{4x}{dw(1-xw^{-1})}e_T^{1,0}(x,w)A_T^{1,1} + \tfrac{x}{dw^2(1-xw^{-1})}e_T^{0,2}A_T^{0,1} + \tfrac{2xw}{d(1-xw^{-1})}e_T^{1,0}A_T^{2,0} \\ &\quad + \tfrac{4x}{d(1-xw^{-1})}e_T^{1,1}A_T^{1,0}, \\ e_T^{2,2} &= 2\tfrac{4}{d(1-x)^2}A_T^{2,2} + \tfrac{8xw}{d(1-x)}e_T^{0,1}A_T^{2,1} + \tfrac{2xw^2}{d(1-x)}e_T^{0,2}A_T^{2,0} + \tfrac{2x}{dw^2(1-x)}e_T^{2,0}A_T^{0,2} + \tfrac{8x}{dw(1-x)}e_T^{1,0}A_T^{1,2} \\ &\quad + \tfrac{16x}{d(1-x)}e_T^{1,1}A_T^{1,1} + \tfrac{4xw}{d(1-x)}e_T^{1,2}A_T^{1,0} + \tfrac{4x}{dw(1-x)}e_T^{2,1}A_T^{0,1} \end{aligned}$$

Solving this linear system in $e_T^{s,t}$ where $0 \le s, t \le 2$, we get an explicit expression for $e_T^{2,2}(x,w)$ (it is long to present here). Using this expression we find the free coefficient w of $e_T^{2,2}(x,w)$, hence we get the following result.

Corollary 3.2. *We have*

$$\begin{aligned}\mu_T^4(x) &= \tfrac{1}{d(1-x)^2}A_T^{2,2} + \tfrac{4x}{d^2(1-x)^3}\left(A_T^{1,1}\right)^2 + \tfrac{2x^2(1+x)^2}{d^3(1-x^2)^3}\left[\left(A_T^{1,0}\right)^2 A_T^{0,1} + \left(A_T^{0,1}\right)^2 A_T^{1,0}\right]\\ &\quad + \tfrac{8x^3}{d^4(1-x)^4(1+x)}\left(A_T^{1,0}\right)^2\left(A_T^{0,1}\right)^2.\end{aligned}$$

Moreover, for all $n \ge 0$,

$$\begin{aligned}\mu_T^4(n) &= \tfrac{2\left(A_T^{1,0}\right)^2\left(A_T^{0,1}\right)^2}{3d^4}n^3 + \left\{\tfrac{2\left(A_T^{1,1}\right)^2}{d^2} + \tfrac{\left[\left(A_T^{1,0}\right)^2 A_T^{0,1} + \left(A_T^{0,1}\right)^2 A_T^{1,0}\right]}{2d^3} - \tfrac{\left(A_T^{1,0}\right)^2\left(A_T^{0,1}\right)^2}{d^4}\right\}n^2\\ &\quad + \left\{\tfrac{2\left(A_T^{1,1}\right)^2}{d^2} + \tfrac{A_T^{2,2}}{d} - \tfrac{2\left(A_T^{1,0}\right)^2\left(A_T^{0,1}\right)^2}{3d^4}\right\}n + \tfrac{A_T^{2,2}}{d} + \tfrac{(1-(-1)^n)\left(A_T^{1,0}\right)^2\left(A_T^{0,1}\right)^2}{2d^4}\\ &\quad - \tfrac{(1-(-1)^n)\left[\left(A_T^{1,0}\right)^2 A_T^{0,1} + \left(A_T^{0,1}\right)^2 A_T^{1,0}\right]}{3d^3}.\end{aligned}$$

For example, in the case of Littlewood polynomials we have that $\mu_{\{-1,1\}}^4(n) = 2n^2 + 3n + 1$, and in the case of polynomials of height h we get that

$$\mu_{\{-h,-h+1,\ldots,h-1,h\}}^4(n;4) = \frac{h(h+1)}{45}(10h(h+1)n^2 + (19h^2 + 19h - 3)n + 3(3h^2 + 3h - 1)).$$

3.3. **Formula for $\mu_T^{2\alpha}(n)$ where $A_T^{1,0} = 0$.** Similarly to the previous subsection, our results can be extended to the case of $\mu_T^6(n)$. Since the answers become very cumbersome, we present here only the simplest case when $A_T^{1,0} = \sum_{j=1}^d x_j = 0$. Therefore, if we apply our approach for finding $\mu_T^{2\alpha}(n)$ where $A_T^{1,0} = 0$ (so $A_T^{0,1} = 0$), then we get the following result.

Corollary 3.3. *Let $T = \{x_1, x_2, \ldots, x_d\}$ such that $\sum_{j=1}^d x_j = 0$. Then*

(i)

$$\mu_T^2(x) = \frac{1}{d(1-x)^2}A_T^{1,1}.$$

(ii)

$$\mu_T^4(x) = \frac{1}{d(1-x)^2}A_T^{2,2} + \frac{4x}{d^2(1-x)^3}\left(A_T^{1,1}\right)^2.$$

(iii)

$$\mu_T^6(x) = \frac{1}{d(1-x)^2}A_T^{3,3} + \frac{18x}{d^2(1-x)^3}A_T^{1,1}A_T^{2,2} + \frac{36x^2}{d^3(1-x)^4}(A_T^{1,1})^3.$$

(iv)

$$\begin{aligned}\mu_t^8(x) &= \tfrac{1}{d(1-x)^2}A_T^{4,4} + \tfrac{32x}{d^2(1-x)^3}A_T^{1,1}A_T^{3,3} + \tfrac{36x^2}{d^2(1-x)^4}(A_T^{2,2})^2 + \tfrac{432x^2}{d^3(1-x)^4}A_T^{2,2}(A_T^{1,1})^2\\ &\quad + \tfrac{72x^4(3-2x-2x^2+3x^3-x^4)}{d^4(1-x)^4(1+x)}(A_T^{0,2})^2(A_T^{2,0})^2 + \tfrac{576x^3}{d^4(1-x)^5}(A_T^{1,1})^4\\ &\quad + \tfrac{48x^3}{d^3(1-x)^2(1-x^3)}(A_T^{0,3}A_T^{2,0}A_T^{2,1} + A_T^{3,0}A_T^{0,2}A_T^{1,2})\\ &\quad + \tfrac{72x^2}{d^3(1-x)^2(1-x^2)}((A_T^{1,2})^2A_T^{2,0} + (A_T^{2,1})^2A_T^{0,2})\\ &\quad + \tfrac{6x^2}{d^3(1-x)^2(1-x^2)}((A_T^{2,0})^2A_T^{0,4} + (A_T^{0,2})^2A_T^{4,0}).\end{aligned}$$

4. Applications

In this section we discus particular cases of Theorem 1.1 and Corollary 1.2. More precisely, we apply Corollary 1.2 for ceratin finite sets of complex numbers (special real numbers), and then we present explicit formulas for $\mu_T^{2\alpha}$ for certain positive integers α.

4.1. **Littlewood polynomials.** Here we suggest formulas for $\mu_T^{2\alpha}(n)$, where $T = \{-1, 1\}$ (Littlewood polynomials). Corollary 1.2 for $T = \{-1, 1\}$ gives the following result.

Corollary 4.1. *The average $\mathcal{L}_{2\alpha}$-norm over Littlewood polynomials of degree exactly n, namely $\mu_{\{-1,1\}}^{2\alpha}(n)$, is given by*

$$\frac{1}{2^{n+1}} \sum_{j_1,\dots,j_{n+1}=1}^{2} \sum_{\substack{k_1+\cdots+k_{n+1}=\alpha \\ \ell_1+\cdots+\ell_{n+1}=\alpha \\ \sum_{s=1}^{n+1}(s-1)(\ell_s-k_s)=0}} (-1)^{\sum_{s=1}^{n+1} j_s(k_s+\ell_s)} \binom{\alpha}{k_1,\dots,k_{n+1}} \binom{\alpha}{\ell_1,\dots,\ell_{n+1}}.$$

Using Lemma 4.1 we quickly generate the numbers $\mu_{\{-1,1\}}^{2\alpha}(n)$; the first few of these numbers are given in Table 1.

$n\backslash\alpha$	0	1	2	3	4	5
0	1	1	1	1	1	1
1	1	2	6	20	70	252
2	1	3	15	93	651	4913
3	1	4	28	256	2812	35024
4	1	5	45	545	8149	143945
5	1	6	66	996	18882	433116
6	1	7	81	1645	37759	1062697
7	1	8	120	2528	68152	2272128
8	1	9	153	3681	113961	4385969
9	1	10	190	5140	179710	7839260
10	1	11	231	6941	270451	13178561

Table 1. Values of $\mu_{\{-1,1\}}^{2\alpha}(n)$.

On the other hand, if applying Equation 4 for $T = \{-1, 1\}$ then we get that

$$\begin{aligned} e_{\{-1,1\}}(x,u,v,w) &= \tfrac{1}{2(1-u)(1-v)}\left[1 + xe_{\{-1,1\}}\left(x, \tfrac{u}{w(1-u)}, \tfrac{wv}{1-v}, w\right)\right] \\ &\quad + \tfrac{1}{2(1+u)(1+v)}\left[1 + xe_{\{-1,1\}}\left(x, \tfrac{u}{w(1+u)}, \tfrac{wv}{1+v}, w\right)\right]. \end{aligned} \tag{5}$$

Now, if applying μ-algorithm on Equation 5 for $T = \{-1, 1\}$ and $\alpha = 0, 1, \dots, 5$, then we get the following result.

Corollary 4.2. *For all $n \geq 0$,*

(i) $\mu_{\{-1,1\}}^{0}(n) = 1$,

(ii) $\mu_{\{-1,1\}}^{2}(n) = n + 1$,

(iii) $\mu^4_{\{-1,1\}}(n) = 2n^2 + 3n + 1$,

(iv) $\mu^6_{\{-1,1\}}(n) = 6n^3 + 9n^2 + 4n + 1$,

(v) $\mu^8_{\{-1,1\}}(n) = 24n^4 + 30n^3 + 4n^2 + 5n + 4 - 3(-1)^n$,

(vi) $\mu^{10}_{\{-1,1\}}(n) = 120n^5 + 150n^4 - 350n^3 + 265n^2 + 281n - 144 - 5(-1)^n(15n - 29)$.

4.2. **Polynomials of height 1.** Here we suggest, in the case of polynomials of height 1, some formulas for $\mu^{2\alpha}_{\{-1,0,1\}}(n)$. Using Lemma 4.1 we quickly generate the numbers $\mu^{2\alpha}_{\{-1,0,1\}}(n)$; the first few of these numbers are given in Table 2.

$n\backslash\alpha$	0	1	2	3	4	5
0	1	$\frac{2}{3}$	$\frac{6}{9}$	$\frac{6}{9}$	$\frac{6}{9}$	$\frac{18}{27}$
1	1	$\frac{4}{3}$	$\frac{28}{9}$	$\frac{84}{9}$	$\frac{284}{9}$	$\frac{3036}{27}$
2	1	$\frac{6}{3}$	$\frac{66}{9}$	$\frac{330}{9}$	$\frac{2018}{9}$	$\frac{42334}{27}$
3	1	$\frac{8}{3}$	$\frac{120}{9}$	$\frac{840}{9}$	$\frac{7480}{9}$	$\frac{239832}{27}$
4	1	$\frac{10}{3}$	$\frac{190}{9}$	$\frac{1710}{9}$	$\frac{19902}{9}$	$\frac{856010}{27}$
5	1	$\frac{12}{3}$	$\frac{276}{9}$	$\frac{3036}{9}$	$\frac{43604}{9}$	$\frac{2348788}{27}$
6	1	$\frac{14}{3}$	$\frac{378}{9}$	$\frac{4914}{9}$	$\frac{83866}{9}$	$\frac{5410646}{27}$
7	1	$\frac{16}{3}$	$\frac{496}{9}$	$\frac{7440}{9}$	$\frac{147056}{9}$	$\frac{11040304}{27}$
8	1	$\frac{18}{3}$	$\frac{630}{9}$	$\frac{10710}{9}$	$\frac{240502}{9}$	$\frac{20567042}{27}$
9	1	$\frac{20}{3}$	$\frac{780}{9}$	$\frac{14820}{9}$	$\frac{372620}{9}$	$\frac{35735180}{27}$
10	1	$\frac{22}{3}$	$\frac{946}{9}$	$\frac{19866}{9}$	$\frac{552786}{9}$	$\frac{58715598}{27}$

TABLE 2. Values of $\mu^{2\alpha}_{\{-1,0,1\}}(n)$.

Applying Equation 4 for $T = \{-1, 0, 1\}$ then we get that

$$\begin{aligned} e_{\{-1,0,1\}}(x,u,v,w) &= \tfrac{1}{3}\left[1 + xe_{\{-1,0,1\}}\left(x, \tfrac{u}{w}, wv, w\right)\right] \\ &+ \tfrac{1}{3(1-u)(1-v)}\left[1 + xe_{\{-1,0,1\}}\left(x, \tfrac{u}{w(1-u)}, \tfrac{wv}{1-v}, w\right)\right] \\ &+ \tfrac{1}{3(1+u)(1+v)}\left[1 + xe_{\{-1,0,1\}}\left(x, \tfrac{u}{w(1+u)}, \tfrac{wv}{1+v}, w\right)\right]. \end{aligned} \tag{6}$$

Now, by applying μ-algorithm on Equation 6 for $T = \{-1, 0, 1\}$ and $\alpha = 0, 1, \ldots, 5$ we get as follows.

Corollary 4.3. *For all $n \geq 0$,*

(i) $\mu^0_{\{-1,0,1\}}(n) = 1$,

(ii) $\mu^2_{\{-1,0,1\}}(n) = \frac{2}{3}(n + 1)$,

(iii) $\mu^4_{n,\{-1,0,1\}}(n) = \frac{2}{9}(4n^2 + 7n + 3)$,

(iv) $\mu^6_{\{-1,0,1\}}(n) = \frac{2}{9}(8n^3 + 18n^2 + 13n + 3)$,

(v) $\mu^8_{\{-1,0,1\}}(n) = \frac{2}{27}(64n^4 + 176n^3 + 128n^2 + 37n + 15 - 6(-1)^n)$,

(vi) $\mu^{10}_{\{-1,0,1\}}(n) = \frac{2}{81}(640n^5 + 2400n^4 + 630n^2 + 1337n - 363 - 30(-1)^n(10n - 13)$.

4.3. **Polynomials with coefficients** $0,1$. Here we suggest another case, the case of polynomials with coefficients $0,1$, which allows us to find explicit formulas for $\mu^{2\alpha}_{\{0,1\}}(n)$. Using Lemma 4.1 we quickly generate the numbers $\mu^{2\alpha}_{\{0,1\}}(n)$; the first few of these numbers are given in Table 3.

$n\backslash\alpha$	0	1	2	3
0	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{4}{8}$
1	1	$\frac{2}{2}$	$\frac{4}{2}$	$\frac{44}{8}$
2	1	$\frac{3}{2}$	$\frac{10}{2}$	$\frac{204}{8}$
3	1	$\frac{4}{2}$	$\frac{19}{2}$	$\frac{592}{8}$
4	1	$\frac{5}{2}$	$\frac{32}{2}$	$\frac{1397}{8}$
5	1	$\frac{6}{2}$	$\frac{49}{2}$	$\frac{2826}{8}$
6	1	$\frac{7}{2}$	$\frac{71}{2}$	$\frac{5206}{8}$
7	1	$\frac{8}{2}$	$\frac{98}{2}$	$\frac{8876}{8}$
8	1	$\frac{9}{2}$	$\frac{131}{2}$	$\frac{14334}{8}$
9	1	$\frac{10}{2}$	$\frac{170}{2}$	$\frac{22084}{8}$
10	1	$\frac{11}{2}$	$\frac{216}{2}$	$\frac{32828}{8}$

TABLE 3. Values of $\mu^{2\alpha}_{\{0,1\}}(n)$.

Applying Equation 4 for $T=\{0,1\}$ then we get that
(7)
$$e_{\{0,1\}}(x,u,v,w) \quad = \tfrac{1}{2}\left[1+xe_{\{0,1\}}\left(x,\tfrac{u}{w},wv,w\right)\right]+\tfrac{1}{2(1-u)(1-v)}\left[1+xe_{\{0,1\}}\left(x,\tfrac{u}{w(1-u)},\tfrac{wv}{1-v},w\right)\right].$$
Now, if applying μ-algorithm on Equation 7 for $T=\{0,1\}$ and $\alpha=0,1,\ldots,5$, then we get the following result.

Corollary 4.4. *For all* $n\geq 0$,

(i) $\mu^{0}_{\{0,1\}}(n)=1$,

(ii) $\mu^{2}_{\{0,1\}}(n)=\frac{1}{2}(n+1)$,

(iii) $\mu^{4}_{\{0,1\}}(n)=\frac{1}{96}(4n^3+54n^2+92n+45+3(-1)^n)$,

(iv) $\mu^{6}_{\{0,1\}}(n)=\frac{1}{2560}(22n^5+460n^4+3100n^3+5600n^2+4143n+1130+75(-1)^n(3n+2))$.

4.4. **Example of polynomials with coefficients** $0,i$ **where** $i^2=-1$. Here we extend our examples (see the Section 1) to the case of sets of complex numbers. For example, if we apply the main results of the pervious sections for $T=\{0,i\}$ then we get the following result.

Corollary 4.5. *For all* $n\geq 0$,

(i) $\mu^{0}_{\{0,i\}}(n)=1$,

(ii) $\mu^{2}_{\{0,i\}}(n)=\frac{1}{2}n+\frac{1}{2}$,

(iii) $\mu^{4}_{\{0,i\}}(n)=\frac{1}{96}(4n^3+54n^2+92n+45+3(-1)^n)$,

(iv) $$\mu^6_{\{0,i\}}(n) = \frac{1}{5120}(46n^5+890n^4+6320n^3+11200n^2+7789n+2455 \\ +15(-1)^n(23+29n)-60i^n(2+3i-in)+60(-i)^n(-2+3i+in)),$$

where $i^2=-1$.

5. Further results

In this section we suggest several directions to generalize the results of the previous sections.

5.1. **Average norms of T-polynomials with weights.** The first of these directions is to obtain an exact formula for the average $\mathcal{L}_{2\alpha}$-norm over T-polynomials of degree exactly n with weight z^m for given α, n, and m. Let us define,

$$\mu^\alpha_T(n;m)=e_T(n;\alpha/2,\alpha/2,m)=\frac{1}{2\pi N_T(n)}\sum_{p\in\mathfrak{T}_n}\int_0^{2\pi}e^{im\theta}|p(e^{i\theta})|^\alpha\,d\theta, \tag{8}$$

Clearly, $\mu^\alpha_T(n)=\mu^\alpha_T(n;0)$ for all n, and $\alpha>0$. Now, if considering the following problem: find $\mu^{2\alpha}_T(n;m)$ for any $\alpha>0$, $n\ge 0$, and $m\in\mathbb{Z}$? then by Theorem 1.1 the following result is true.

Theorem 5.1. *Let $T=\{x_1,x_2,\ldots,x_d\}$ be any finite set of complex numbers. The generating function $\sum_{n\ge0}\sum_{m\in\mathbb{Z}}\mu^{2\alpha}_T(n;m)x^nw^m$ is given by*

$$\sum_{n\ge1}\sum_{j_1,\ldots,j_n=1}^{d}\sum_{\substack{k_1+\cdots+k_n=\alpha\\ \ell_1+\cdots+\ell_n=\alpha}}\prod_{a=1}^{n}(x_{j_a}^{t_a}\overline{x_{j_a}}^{r_a})\binom{\alpha}{k_1,\ldots,k_n}\binom{\alpha}{\ell_1,\ldots,\ell_n}\frac{x^{n-1}w^{\left(\sum_{a=1}^{n}(a-1)(r_a-t_a)\right)}}{d^n}.$$

Moreover, $\mu^{2\alpha}_T(n;m)$ is given by

$$\frac{1}{d^{n+1}}\sum_{j_1,\ldots,j_{n+1}=1}^{d}\sum_{\substack{k_1+\cdots+k_{n+1}=\alpha\\ \ell_1+\cdots+\ell_{n+1}=\alpha\\ \sum_{s=1}^{n+1}(s-1)(\ell_s-k_s)=m}}\prod_{a=1}^{n+1}(x_{j_a}^{k_a}\overline{x_{j_a}}^{\ell_a})\binom{\alpha}{k_1,\ldots,k_{n+1}}\binom{\alpha}{\ell_1,\ldots,\ell_{n+1}}.$$

5.2. **Exact formulas.** The second of these directions is to obtain an exact formula for $\mu^{2\alpha}_T(n;m)$ for given α and m. By definitions, it is clear that $\alpha!^2\mu^{2\alpha}_T(n;m)$ is the coefficient of x^nw^m of $E^{\alpha,\alpha}_T(x,w)$ (see Section 3). Therefore, similarly as in Section 3 we get the following result.

Corollary 5.2. *Let $T=\{x_1,x_2,\ldots,x_d\}$ be any finite set of complex numbers. Then*

$$\mu^{2\alpha}_T(n;m)=\begin{cases}\frac{n+1-|m|}{d^2}A^{0,1}_TA^{1,0}_T & m\neq0,\ -n\le m\le m\\ \frac{n+1}{d}A^{1,1}_T & m=0\\ 0 & \textit{otherwise.}\end{cases}$$

This result can be extended to the case of $\mu^4_T(n)$. Since the answers become very cumbersome, we present here only the simplest case when $A^{1,0}_T=\sum_{j=1}^d x_j=0$.

Corollary 5.3. *Let* $T = \{x_1, x_2, \ldots, x_d\}$ *be any finite set of complex numbers such that* $A_T^{1,0} = \sum_{j=1}^d x_j = 0$. *Then*

$$\mu_T^{2\alpha}(n;m) = \begin{cases} \frac{n+1-|m/2|}{d^2} A_T^{0,2} A_T^{2,0} & m \neq 0,\ -2n \leq m \leq 2n,\ m \text{ even} \\ \frac{n+1}{d} A_T^{1,1} + \frac{4}{d^2}\binom{n+1}{2}(A_T^{1,1})^2 & m = 0 \\ 0 & \text{otherwise.} \end{cases}$$

5.3. **Average integrals.** The third of these directions is to consider the general case to find an explicit formula for $e_T(n,s,t,m)$. Theorem 1.1 and μ-algorithm give a complete answer for the generating function for these numbers, and explicit formula for $e_T(n,s,t,m)$. For example, the following result is true.

Corollary 5.4. *Let* $T = \{x_1, x_2, \ldots, x_d\}$ *be any finite set of complex numbers. Then for any* $n \geq 0$ *and* $m \in \mathbb{Z}$,

$$e_T(n,1,2,m) = \begin{cases} \frac{1}{d} A_T^{1,2} & 0 \leq m \leq n \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad e_T(n,2,1,m) = \begin{cases} \frac{1}{d} A_T^{2,1} & -n \leq m \leq 0 \\ 0 & \text{otherwise} \end{cases}.$$

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