GENERALIZATIONS OF SOME IDENTITIES INVOLVING THE FIBONACCI NUMBERS

March 7, 2003

Toufik Mansour [1]

Department of Mathematics, Chalmers University of Technology, S-412 96 Göteborg, Sweden

toufik@math.chalmers.se

1 Introduction

The generalized Fibonacci and Lucas numbers are defined by

$$U_n(p,q) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n(p,q) = \alpha^n + \beta^n, \tag{1}$$

where $\alpha = \frac{1}{2}(p + \sqrt{p^2 - 4q})$ and $\beta = \frac{1}{2}(p - \sqrt{p^2 - 4q})$. Clearly, $U_n(p,q)$ and $V_n(p,q)$ are the usual Fibonacci and Lucas sequences $\{F_n\}$ and $\{L_n\}$ when $p = 1$ and $q = -1$.

[1] Research financed by EC's IHRP Programme, within the Research Training Network "Algebraic Combinatorics in Europe", grant HPRN-CT-2001-00272

Definition 1.1 *Let $d \geq 0$. For any $n \geq 0$, we define*

$$s_d(n; p, q; k) = \sum_{j_1+j_2+\cdots+j_d=n} \prod_{i=1}^{d} U_{k\cdot j_i}(p, q).$$

For the Fibonacci numbers, Zhang [3] found the following identities:

$$s_2(n; 1, -1; 1) = \frac{1}{5}((n-1)F_n + 2nF_{n-1}), \quad n \geq 1, \tag{2}$$

$$s_3(n; 1, -1; 1) = \frac{1}{50}((5n^2 - 9n - 2)F_{n-1} + (5n^2 - 3n - 2)F_{n-2}), \quad n \geq 2, \tag{3}$$

and when $n \geq 3$,

$$s_4(n; 1, -1; 1) = \frac{1}{150}((4n^3 - 12n^2 - 4n + 12)F_{n-2} + (3n^3 - 6n^2 - 3n + 6)F_{n-3}). \tag{4}$$

Recently, Zhao and Wang [2] extended these identites to the case of $\{U_n(p, q)\}$ and $\{V_n(p, q)\}$; for $n \geq 1$

$$s_2(n; p, q; k) = \frac{U_k(p, q)}{V_k^2(p, q) - 4q^k} \Big((n-1)U_{nk}(p, q)V_k(p, q) - 2nq^k U_{(n-1)k}(p, q) \Big), \tag{5}$$

for $n \geq 2$,

$$\begin{aligned} s_3(n; p, q; k) \quad &= \tfrac{U_k^2(p,q)}{2(V_k^2(p,q)-4q^k)^2} \Big((n-1)(n-2)V_k^2(p, q)U_{nk}(p, q) \\ &-q^k V_k(p, q)(4n^2 - 6n - 4)U_{(n-1)k}(p, q) \\ &+(4n^2 - 28n + 28(n-3)V_k(p, q) + 80)U_{(n-2)k}(p, q) \Big), \end{aligned} \tag{6}$$

and when $n \geq 3$,

$$\begin{aligned} s_4(n;p,q;k) &= \tfrac{U_k^3(p,q)}{6(V_k^2(p,q)-4q^k)^3}\Big(V_k^3(p,q)(n-1)(n-2)(n-3)U_{nk}(p,q) \\ &-6q^kV_k^2(p,q)(n-2)(n-3)(n+1)U_{(n-1)k}(p,q) \\ &+12q^{2k}V_k(p,q)(n-3)(n^2+n-1)U_{(n-2)k}(p,q) \\ &-8q^{3k}n(n^2-4)U_{(n-3)k}(p,q)\Big). \end{aligned} \tag{7}$$

In this paper, we extend the above conclusions. We establish an identity for the case $s_d(n;p,q;k)$ for any $d \geq 2$.

2 Main results

We denote by $G_k(x;p,q)$ the generating function of $\{U_{n\cdot k}(p,q)\}$, that is, $G_k(x;p,q) = \sum_{n\geq 0} U_{n\cdot k}(p,q)x^n$, where k is a positive integer. Clearly, by Definition 1 and the geometric formula,

$$G_k(x;p,q) = \frac{xU_k(p,q)}{1 - V_k(p,q)x + q^kx^2}.$$

We define $F_k(x) = F_k(x;p,q) = \frac{G_k(x;p,q)}{x}$. Then

$$F_k(x) = \sum_{n\geq 1} U_{n\cdot k}(p,q)x^{n-1} = \frac{U_k(p,q)}{1 - V_k(p,q)x + q^kx^2}. \tag{8}$$

Definition 2.1 *Let $a(0,d) = 4^d$ for any $d \geq 0$, and $a(\ell,0) = 0$ for any $\ell \geq 1$. We define $a(\ell,d)$ for $\ell, d \geq 1$ by $a(\ell,d) = 4(\ell+1)\cdot a(\ell,d-1) + \ell \cdot a(\ell-1,d-1)$.*

Using this definition we quickly generate the numbers $a(\ell, d)$; the first few of these numbers are given in Table 1.

$d \backslash \ell$	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	4	1	0	0	0	0	0
2	16	12	2	0	0	0	0
3	64	112	48	6	0	0	0
4	256	960	800	240	24	0	0
5	1024	7936	11520	6240	1440	120	0
6	4096	64512	154112	134400	53760	10080	720

Table 1: Values of $a(\ell, d)$ where $0 \leq \ell, d \leq 6$.

We can also use Definition 2.1 to find an explicit formula for $a(\ell, d)$.

Lemma 2.2 *For any* $\ell, d \geq 0$,

$$a(\ell, d) = 4^{d-\ell} \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} (\ell + 1 - j)^d.$$

Proof. By Definition 2.1 it is easy to see that the lemma holds for $\ell = 0$ or $d = 0$. Using induction on d and ℓ we get that

$$\begin{aligned}
&4(\ell+1)\cdot a(\ell,d-1)+\ell\cdot a(\ell-1,d-1)\\
&= (\ell+1)4^{d-\ell}\sum_{j=0}^{d}(-1)^j\binom{\ell}{j}(\ell+1-j)^{d-1}+\ell\cdot 4^{d-\ell}\sum_{j=0}^{\ell-1}(-1)^j\binom{\ell-1}{j}(\ell-j)^{d-1}\\
&= 4^{d-\ell}\left[(\ell+1)\sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}(\ell+1-j)^{d-1}+\ell\sum_{j=1}^{\ell}(-1)^{j-1}\binom{\ell-1}{j-1}(\ell+1-j)^{d-1}\right]\\
&= 4^{d-\ell}\left[(\ell+1)^d+\sum_{j=1}^{\ell}(-1)^j\binom{\ell}{j}(\ell+1-j)^d\right]=a(\ell,d+1),
\end{aligned}$$

as requested. □

Definition 2.3 *Let $b(1,d) = (-2)^{d-1}$ for any $d \geq 1$, and $b(\ell,1) = 0$ for any $\ell \geq 2$. We define $b(\ell,d)$ for $\ell, d \geq 2$ by $b(\ell,d) = b(\ell-1,d-1) - 2\ell \cdot b(\ell,d-1)$.*

Using this definition we quickly generate the numbers $b(\ell,d)$; the first few of these numbers are given in Table 2.

$d\backslash\ell$	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0
2	-2	1	0	0	0	0	0
3	4	-6	1	0	0	0	0
4	-8	28	-12	1	0	0	0
5	16	-120	100	-20	1	0	0
6	-32	496	-720	260	-30	1	0

Table 2: Values of $b(\ell,d)$ where $0 \leq \ell, d \leq 6$.

We can also use Definition 2.3 to find an explicit formula for the numbers $b(\ell,d)$.

Lemma 2.4 *For any $\ell, d \geq 1$,*

$$b(\ell,d) = \frac{(-1)^{d-1}2^{d-\ell}}{(\ell-1)!}\sum_{j=0}^{\ell-1}(-1)^j\binom{\ell-1}{j}(j+1)^{d-1}.$$

Proof. By Definition 2.3 it is easy to see that the lemma holds for $\ell = 1$ or $d = 1$. Using induction on d and ℓ we get that

$$\begin{aligned}
&b(\ell-1,d-1) - 2\ell\cdot b(\ell,d-1)\\
&\quad = \tfrac{(-1)^{d-2}2^{d-\ell}}{(\ell-2)!}\sum_{j=0}^{\ell-2}(-1)^j\tbinom{\ell-2}{j}(j+1)^{d-2} - \tfrac{2\ell(-1)^{d-2}2^{d-\ell-1}}{(\ell-1)!}\sum_{j=0}^{\ell-1}(-1)^j\tbinom{\ell-1}{j}(j+1)^{d-2}\\
&\quad = \tfrac{(-1)^{d-1}2^{d-\ell}}{(\ell-1)!}\left[\ell\sum_{j=0}^{\ell-1}(-1)^j\tbinom{\ell-1}{j}(j+1)^{d-2} - (\ell-1)\sum_{j=0}^{\ell-2}(-1)^j\tbinom{\ell-2}{j}(j+1)^{d-2}\right]\\
&\quad = \tfrac{(-1)^{d-1}2^{d-\ell}}{(\ell-1)!}\left[(-1)^{d-1}\ell^{d-1} + \sum_{j=0}^{\ell-2}(-1)^j\left(\ell\tbinom{\ell-1}{j} - (\ell-1)\tbinom{\ell-2}{j}\right)(j+1)^{d-2}\right]\\
&\quad = \tfrac{(-1)^{d-1}2^{d-\ell}}{(\ell-1)!}\left[(-1)^{d-1}\ell^{d-1} + \sum_{j=0}^{\ell-2}(-1)^j\tbinom{\ell-1}{j}(j+1)^{d-1}\right]\\
&\quad = \tfrac{(-1)^{d-1}2^{d-\ell}}{(\ell-1)!}\sum_{j=0}^{\ell-1}(-1)^j\tbinom{\ell-1}{j}(j+1)^{d-1} = b(\ell,d),
\end{aligned}$$

as requested. □

Now we introduce a relation that plays the crucial role in the proof of the main result of this paper.

Proposition 2.5 *Let $d \geq 1$. The generating function $F_k(x;p,q)$ satisfies the following equation:*

$$\begin{aligned}
&\sum_{j=0}^{d}\left[(4q^k)^{d-j}\left(\sum_{i=0}^{j}(-1)^i\tbinom{j}{i}(j+1-i)^d\right)\left(\tfrac{V_k^2(p,q)-4q^k}{U_k(p,q)}\right)^j F_k^{j+1}(x;p,q)\right]\\
&\quad = \sum_{j=1}^{d}\left[\tfrac{(-1)^{d-1}(2q^k)^{d-j}}{(j-1)!}\left(\sum_{i=0}^{j-1}(-1)^i\tbinom{j-1}{i}(i+1)^{d-1}\right)\left(V_k(p,q)-2q^kx\right)^j F_k^{(j)}(x;p,q)\right],
\end{aligned}$$

where $F_k^{(j)}(x;p,q)$ *is the* j*th derivative with respect to* x *of* $F_k(x;p,q)$*.*

Proof. We define $A = \frac{V_k^2(p,q)-4q^k}{U_k(p,q)}$ and $B = V_k(p,q) - 2q^k x$. Let us prove this theorem by induction on d. Noticing that

$$F_k^{(1)}(x;p,q) = \frac{(V_k(p,q) - 2q^k x)F_k(x;p,q)}{1 - V_k(p,q)x + q^k x^2}, \tag{9}$$

we get

$$4q^k F_k(x;p,q) + A \cdot F_k^2(x;p,q) = B \cdot F_k^{(1)}(x;p,q),$$

therefore, the theorem holds for $d = 1$. Now we suppose that the theorem holds for d, that is,

$$\sum_{j=0}^{d} a(j,d)q^{(d-j)k} \left(\tfrac{V_k^2(p,q)-4q^k}{U_k(p,q)}\right)^j F_k^{j+1}(x;p,q) \\ = \sum_{j=1}^{d} b(j,d)q^{(d-j)k} \left(V_k(p,q) - 2q^k x\right)^j F_k^{(j)}(x;p,q).$$

Therefore, derivative this equation with respect to x we have that

$$\sum_{j=0}^{d} (j+1)a(j,d)q^{(d-j)k} A^j F_k^j(x;p,q) F_k^{(1)}(x;p,q) \\ = \sum_{j=1}^{d} b(j,d)q^{(d-j)k} B^j F_k^{(j+1)}(x;p,q) - \sum_{j=1}^{d} 2jq^k b(j,d)q^{(d-j)k} B^{j-1} F_k^{(j)}(x;p,q).$$

If multiplying by B and using Equation 9 then we get that

$$\sum_{j=0}^{d} (j+1)a(j,d)q^{(d-j)k} A^{j+1} F_k^{j+2}(x;p,q) + \sum_{j=0}^{d} 4(j+1)a(j,d)q^{(d+1-j)k} A^j F_k^{j+1}(x;p,q) \\ = \sum_{j=2}^{d+1} b(j-1,d)q^{(d+1-j)k} B^j F_k^{(j)}(x;p,q) - \sum_{j=1}^{d} 2jb(j,d)q^{(d+1-j)k} B^j F_k^{(j)}(x;p,q),$$

equivalently,

$$\sum_{j=0}^{d+1}(ja(j-1,d)+4(j+1)a(j,d))q^{(d+1-j)k}A^jF_k^{j+1}(x;p,q)$$
$$=\sum_{j=1}^{d+1}(b(j-1,d)-2jb(j,d))q^{(d+1-j)k}B^jF_k^{(j)}(x;p,q),$$

Therefore, using Definition 2.1 and Definition 2.3 we have that

$$\sum_{j=0}^{d+1}a(j,d+1)q^{(d+1-j)k}A^jF_k^{j+1}(x;p,q)=\sum_{j=1}^{d+1}b(j,d+1)q^{(d+1-j)k}B^jF_k^{(j)}(x;p,q).$$

Hence, using Lemma 2.2 and Lemma 2.4 we get the desired result. □

By the above proposition, we have the main result of this paper.

Theorem 2.6 *Let $d\geq 1$. For any $n\geq d$,*

$$\sum_{j=0}^{d}\left[(4q^k)^{d-j}\left(\sum_{i=0}^{j}(-1)^i\binom{j}{i}(j+1-i)^d\right)\left(\frac{V_k^2(p,q)-4q^k}{U_k(p,q)}\right)^j s_{j+1}(n+j-d;p,q;k)\right]$$
$$=\sum_{j=1}^{d}\left[\frac{(-1)^{d-1}(2q^k)^{d-j}}{(j-1)!}\left(\sum_{i=0}^{j-1}(-1)^i\binom{j-1}{i}(i+1)^{d-1}\right)\left(\sum_{s=0}^{j}v_{d,j,s}(n)U_{(n+j-d-s)k}(p,q)\binom{j}{s}\right)\right],$$

where $v_{d,j,s}(n)=(-2q^k)^sV_k^{j-s}(p,q)\prod_{i=1}^{j}(n+j-d-s-i)$.

Proof. If comparing the coefficients of $x^{n-(d+1)}$ on both sides of Proposition 2.5 we get the desired result. □

Theorem 2.6 provides a finite algorithm for finding $s_d(n;p,q;k)$ in terms of $U_{nk}(p,q)$ and $V_{nk}(p,q)$, since we have to consider all $s_j(n;p,q;k)$ for $j=1,2,\ldots,d$. The algorithm has been implemented in Maple, and yields explicit results for $1\leq d\leq 6$. Below we present several explicit calculations.

Corollary 2.7 (see Zhao and Wang [2, Equation 9]) *For any* $n \geq 1$,

$$s_2(n;p,q;k) = \frac{U_k(p,q)}{V_k^2(p,q)-4q^k}\Big((n-1)V_k(p,q)U_{nk}(p,q) - 2nq^kU_{(n-1)k}(p,q)\Big).$$

Proof. Theorem 2.6 for $d = 2$ yields

$$\begin{aligned} &4q^k s_1(n-1;p,q;k) + \tfrac{V_k^2(p,q)-4q^k}{U_k(p,q)} s_2(n;p,q;k) \\ &\qquad = (n-1)V_k(p,q)U_{nk}(p,q) - 2(n-2)q^kU_{(n-1)k}(p,q). \end{aligned}$$

Using the fact that $s_1(n;p,q;k) = U_{nk}(p,q)$ we get the desired result. □

Corollary 2.8 (see Zhao and Wang [2, Equation 10]) *For any* $n \geq 2$,

$$\begin{aligned} s_3(n;p,q;k) &= \tfrac{U_k^2(p,q)}{2(V_k^2(p,q)-4q^k)^2}\Big((n-1)(n-2)V_k^2(p,q)U_{nk}(p,q) \\ &-2q^k(n-2)(2n+1)V_k(p,q)U_{(n-1)k}(p,q) \\ &+4q^{2k}(n-2)(n+2)U_{(n-2)k}(p,q)\Big). \end{aligned}$$

Proof. Theorem 2.6 for $d = 3$ yields

$$\begin{aligned} &16q^{2k}s_1(n-2;p,q;k) + 12q^k\tfrac{V_k^2(p,q)-4q^k}{U_k(p,q)}s_2(n-1;p,q;k) + \tfrac{2(V_k^2(p,q)-4q^k)^2}{U_k^2(p,q)}s_3(n;p,q;k) \\ &= (n-1)(n-2)V_k^2(p,q)U_{nk}(p,q) - 2(n-2)(2n-5)q^kV_k(p,q)U_{(n-1)k}(p,q) \\ &+4q^{2k}(n-3)^2U_{(n-2)k}(p,q). \end{aligned}$$

Using Corollary 2.7 with the fact that $s_1(n;p,q;k) = U_{nk}(p,q)$ we get the desired result.

□

Similarly, if applying Theorem 2.6 for d with using the formulas of $s_j(n;p,q;k)$ for $j = 1,2,\ldots,d-1$, then we get the following result (in the case $d=4$ see [2, Equation 11]).

Corollary 2.9 *We have*

(i) *For any* $n \geq 3$,

$$\begin{aligned}
&s_4(n;p,q;k)\\
&= \tfrac{U_k^3(p,q)}{6(V_k^2(p,q)-4q^k)^3}\Big(V_k^3(p,q)(n-1)(n-2)(n-3)U_{nk}(p,q)\\
&-6q^kV_k^2(p,q)(n-2)(n-3)(n+1)U_{(n-1)k}(p,q)\\
&+12q^{2k}V_k(p,q)(n-3)(n^2+n-1)U_{(n-2)k}(p,q)\\
&-8q^{3k}n(n^2-4)U_{(n-3)k}(p,q)\Big).
\end{aligned}$$

(ii) *For any* $n \geq 4$,

$$\begin{aligned}
&s_5(n;p,q;k)\\
&= \tfrac{U_k^4(p,q)}{4!(V_k^2(p,q)-4q^k)^4}\Big(V_k^4(p,q)(n-1)(n-2)(n-3)(n-4)U_{nk}(p,q)\\
&-4q^kV_k^3(p,q)(n-2)(n-3)(n-4)(2n+3)U_{(n-1)k}(p,q)\\
&+12q^{2k}V_k^2(p,q)(n-3)(n-4)(2n^2+4n-1)U_{(n-2)k}(p,q)\\
&-8q^{3k}V_k(p,q)(n-4)(2n+1)(2n^2+2n-9)U_{(n-3)k}(p,q)\\
&+16q^{4k}(n-3)(n-1)(n+1)(n+3)U_{(n-4)k}(p,q)\Big).
\end{aligned}$$

(iii) *For any* $n \geq 5$,

$$
\begin{aligned}
& s_6(n;p,q;k) \\
& = \tfrac{U_k^5(p,q)}{5!(V_k^2(p,q)-4q^k)^5}\Big(V_k^5(p,q)(n-1)(n-2)(n-3)(n-4)(n-5)U_{nk}(p,q) \\
& -10q^kV_k^4(p,q)(n-2)(n-3)(n-4)(n-5)(n+2)U_{(n-1)k}(p,q) \\
& +20q^{2k}V_k^3(p,q)(n-3)(n-4)(n-5)(2n^2+6n+1)U_{(n-2)k}(p,q) \\
& -40q^{3k}V_k^2(p,q)(n-4)(n-5)(n+1)(2n^2+4n-9)U_{(n-3)k}(p,q) \\
& +80q^{4k}V_k(p,q)(n-5)(n^4+2n^3-10n^2-11n+9)U_{(n-4)k}(p,q) \\
& -32q^{5k}n(n-4)(n-2)(n+2)(n+4)U_{(n-5)k}(p,q)\Big).
\end{aligned}
$$

From these results, it is very easy to obtain Equations 2-4. If $k=1$ and $p=-q=1$, then by using Corollary 2.9 together with the recurrence $F_n = F_{n-1} + F_{n-2}$ we arrive to

$$
\begin{aligned}
& \sum_{a+b+c+d+e=n} F_aF_bF_cF_dF_e \\
& \qquad = \tfrac{1}{4!\cdot 5^4}\big(3(n-1)(8n^3-5n^2-27n+50)F_n - 20n(5n^2-17)F_{n-1}\big)
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{a+b+c+d+e+f=n} F_aF_bF_cF_dF_eF_f \\
& = \tfrac{1}{5!\cdot 5^4}\big((n-1)(5n^4-70n^3-65n^2+490n+264)F_n + 2n(5n^4+5n^2-226)F_{n-1}\big).
\end{aligned}
$$

References

[1] A.F. Horadam and Bro. J. M. Mahon, Pell and Pell-Lucas polynomials, The Fibonacci Quartely **23** (1985) 7–20.

[2] F. Zhao and T. Wang, Generalizations of some identities involving the Fibonacci numbers, *The Fibonacci Quarterly* **39** (2001) 165–167.

[3] W. Zhang, Some identities involving Fibonacci numbers, *The Fibonacci Quarterly* **35** (1997) 225–229.

2000 Mathematics Subject Classification: Primary 11B39, 11B83; Secondary 05A15