GENERALIZATIONS OF SOME IDENTITIES

ININVOLVING THE FIBONACCI NUMBERS

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1 Introduction

The generalized Fibonacci and Lucas numbers are defined by

\[ U_n(p, q) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n(p, q) = \alpha^n + \beta^n, \quad (1) \]

where \( \alpha = \frac{1}{2}(p + \sqrt{p^2 - 4q}) \) and \( \beta = \frac{1}{2}(p - \sqrt{p^2 - 4q}) \). Clearly, \( U_n(p, q) \) and \( V_n(p, q) \) are the usual Fibonacci and Lucas sequences \( \{ F_n \} \) and \( \{ L_n \} \) when \( p = 1 \) and \( q = -1 \).

\[ ^1 \text{Research financed by EC’s IHRP Programme, within the Research Training Network "Algebraic Combinatorics in Europe", grant HPRN-CT-2001-00272} \]
**Definition 1.1** Let \( d \geq 0 \). For any \( n \geq 0 \), we define

\[
s_d(n; p, q; k) = \sum_{j_1 + j_2 + \ldots + j_d = n} \prod_{i=1}^{d} U_{k,j_i}(p, q).
\]

For the Fibonacci numbers, Zhang [3] found the following identities:

\[
s_2(n; 1, -1; 1) = \frac{1}{5}((n - 1)F_n + 2nF_{n-1}), \quad n \geq 1, \tag{2}
\]

\[
s_3(n; 1, -1; 1) = \frac{1}{50}((5n^2 - 9n - 2)F_{n-1} + (5n^2 - 3n - 2)F_{n-2}), \quad n \geq 2, \tag{3}
\]

and when \( n \geq 3 \),

\[
s_4(n; 1, -1; 1) = \frac{1}{150}((4n^3 - 12n^2 - 4n + 12)F_{n-2} + (3n^3 - 6n^2 - 3n + 6)F_{n-3}). \tag{4}
\]

Recently, Zhao and Wang [2] extended these identities to the case of \( \{U_n(p, q)\} \) and \( \{V_n(p, q)\} \); for \( n \geq 1 \)

\[
s_2(n; p, q; k) = \frac{U_k(p, q)}{V_k^2(p, q) - 4q^k} \left( (n - 1)U_{nk}(p, q)V_k(p, q) - 2nq^kU_{(n-1)k}(p, q) \right), \tag{5}
\]

for \( n \geq 2 \),

\[
s_3(n; p, q; k) = \frac{U_{nk}(p, q)}{2[V_k^2(p, q) - 4q^k]} \left( (n - 1)(n - 2)V_k^2(p, q)U_{nk}(p, q)ight.
\]

\[
- q^kV_k(p, q)(4n^2 - 6n - 4)U_{(n-1)k}(p, q)
\]

\[
+ (4n^2 - 28n + 28(n - 3)V_k(p, q) + 80)U_{(n-2)k}(p, q) \right), \tag{6}
\]
and when \( n \geq 3 \),

\[
s_d(n; p, q; k) = \frac{U_k^2(p, q)}{q^{U_k(p, q)} - q^k} \left( V_k^2(p, q) (n - 1)(n - 2)(n - 3) U_n k(p, q) \\
- 6q^k V_k^2(p, q) (n - 2)(n - 3)(n + 1) U_{n-1} k(p, q) \\
+ 12q^2 V_k(p, q)(n - 3)(n^2 + n - 1) U_{n-2} k(p, q) \\
- 8q^3 n(n^2 - 4) U_{n-3} k(p, q) \right). \tag{7}
\]

In this paper, we extend the above conclusions. We establish an identity for the case \( s_d(n; p, q; k) \) for any \( d \geq 2 \).

### 2 Main results

We denote by \( G_k(x; p, q) \) the generating function of \( \{U_n k(p, q)\} \), that is, \( G_k(x; p, q) = \sum_{n \geq 0} U_n k(p, q) x^n \), where \( k \) is a positive integer. Clearly, by Definition 1 and the geometric formula,

\[
G_k(x; p, q) = \frac{x U_k(p, q)}{1 - V_k(p, q)x + q^kx^2}.
\]

We define \( F_k(x) = F_k(x; p, q) = \frac{G_k(x; p, q)}{x} \). Then

\[
F_k(x) = \sum_{n \geq 1} U_n k(p, q) x^{n-1} = \frac{U_k(p, q)}{1 - V_k(p, q)x + q^kx^2}. \tag{8}
\]

**Definition 2.1** Let \( a(0, d) = 4^d \) for any \( d \geq 0 \), and \( a(\ell, 0) = 0 \) for any \( \ell \geq 1 \). We define \( a(\ell, d) \) for \( \ell, d \geq 1 \) by \( a(\ell, d) = 4(\ell + 1) \cdot a(\ell, d - 1) + \ell \cdot a(\ell - 1, d - 1) \).
Using this definition we quickly generate the numbers $a(\ell, d)$; the first few of these numbers are given in Table 1.

<table>
<thead>
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<th>$d \setminus \ell$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<td>0</td>
<td>0</td>
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<td>0</td>
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<tr>
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<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>12</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
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<td>0</td>
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<td>0</td>
</tr>
<tr>
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<td>10080</td>
<td>720</td>
</tr>
</tbody>
</table>

Table 1: Values of $a(\ell, d)$ where $0 \leq \ell, d \leq 6$.

We can also use Definition 2.1 to find an explicit formula for $a(\ell, d)$.

**Lemma 2.2** For any $\ell, d \geq 0$,

$$a(\ell, d) = 4^{d-\ell} \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} (\ell + 1 - j)^d.$$
Proof. By Definition 2.1 it is easy to see that the lemma holds for \( \ell = 0 \) or \( d = 0 \).

Using induction on \( d \) and \( \ell \) we get that

\[
4(\ell + 1) \cdot a(\ell, d - 1) + \ell \cdot a(\ell - 1, d - 1)
\]

\[
= (\ell + 1)4^{d-\ell} \sum_{j=0}^{d} (-1)^{j} \binom{\ell}{j} (\ell + j - 1)^{d-1} + \ell \cdot 4^{d-\ell} \sum_{j=0}^{\ell-1} (-1)^{j} \binom{\ell-1}{j} (\ell - j)^{d-1}
\]

\[
= 4^{d-\ell} \left[ (\ell + 1) \sum_{j=0}^{\ell} (-1)^{j} \binom{\ell}{j} (\ell + j - 1)^{d-1} + \ell \sum_{j=1}^{\ell} (-1)^{j-1} \binom{\ell-1}{j-1} (\ell - j)^{d-1} \right]
\]

\[
= 4^{d-\ell} \left[ (\ell + 1)^{d} + \sum_{j=1}^{\ell} (-1)^{j} \binom{\ell}{j} (\ell + 1 - j)^{d} \right] = a(\ell, d + 1),
\]

as requested. \( \square \)

**Definition 2.3** Let \( b(1, d) = (-2)^{d-1} \) for any \( d \geq 1 \), and \( b(\ell, 1) = 0 \) for any \( \ell \geq 2 \).

We define \( b(\ell, d) \) for \( \ell, d \geq 2 \) by \( b(\ell, d) = b(\ell - 1, d - 1) - 2\ell \cdot b(\ell, d - 1) \).

Using this definition we quickly generate the numbers \( b(\ell, d) \); the first few of these numbers are given in Table 2.

<table>
<thead>
<tr>
<th>(d)</th>
<th>(\ell)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
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<td>1</td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>2</td>
<td>-2</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
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<td>3</td>
<td>4</td>
<td>-6</td>
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<td>0</td>
<td>0</td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
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<td>-720</td>
<td>260</td>
<td>-30</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 2: Values of \( b(\ell, d) \) where \( 0 \leq \ell, d \leq 6 \).

We can also use Definition 2.3 to find an explicit formula for the numbers \( b(\ell, d) \).
Lemma 2.4 For any $\ell, d \geq 1$,

$$b(\ell, d) = \frac{(-1)^{d-1}2^{d-\ell}}{(\ell - 1)!} \sum_{j=0}^{\ell-1} (-1)^j \binom{\ell - 1}{j} (j + 1)^{d-1}.$$ 

**Proof.** By Definition 2.3 it is easy to see that the lemma holds for $\ell = 1$ or $d = 1$.

Using induction on $d$ and $\ell$ we get that

\[
\begin{aligned}
b(\ell - 1, d - 1) - 2\ell \cdot b(\ell, d - 1) &= \frac{(-1)^{d-1}2^{d-\ell}}{(\ell - 2)!} \sum_{j=0}^{\ell-2} (-1)^j \binom{\ell - 2}{j} (j + 1)^{d-2} - \frac{2\ell(-1)^{d-1}2^{d-\ell-1}}{(\ell - 1)!} \sum_{j=0}^{\ell-1} (-1)^j \binom{\ell - 1}{j} (j + 1)^{d-2} \\
&= \frac{(-1)^{d-1}2^{d-\ell}}{(\ell - 1)!} \left[ \ell \sum_{j=0}^{\ell-1} (-1)^j \binom{\ell - 1}{j} (j + 1)^{d-2} - (\ell - 1) \sum_{j=0}^{\ell-2} (-1)^j \binom{\ell - 2}{j} (j + 1)^{d-2} \right] \\
&= \frac{(-1)^{d-1}2^{d-\ell}}{(\ell - 1)!} \left[ (-1)^{d-1} \ell^{d-1} + \sum_{j=0}^{\ell-2} (-1)^j \left( \binom{\ell - 1}{j} - (\ell - 1) \binom{\ell - 2}{j} \right) (j + 1)^{d-2} \right] \\
&= \frac{(-1)^{d-1}2^{d-\ell}}{(\ell - 1)!} \sum_{j=0}^{\ell-1} (-1)^j \binom{\ell - 1}{j} (j + 1)^{d-1} = b(\ell, d),
\end{aligned}
\]

as requested. \qed

Now we introduce a relation that plays the crucial role in the proof of the main result of this paper.

**Proposition 2.5** Let $d \geq 1$. The generating function $F_k(x; p, q)$ satisfies the following equation:

\[
\sum_{j=0}^{d} \left[ (4q^j)^{d-j} \left( \sum_{i=0}^{j} (-1)^i \binom{j}{i} (j + 1 - i)^d \right) \left( \frac{d}{t_k[p, q]} \right)^j F_{k+1}^j(x; p, q) \right] = \sum_{j=1}^{d} \frac{(-1)^{d-1}2^{d-j}}{(j-1)!} \left( \sum_{i=0}^{j-1} (-1)^i \binom{j-1}{i} (i + 1)^{d-1} \right) \left( V_k(p, q) - 2q^j x \right)^j F_k^j(x; p, q),
\]
where \( F_k^{(j)}(x; p, q) \) is the \( j \)th derivative with respect to \( x \) of \( F_k(x; p, q) \).

**Proof.** We define \( A = \frac{v_k^2(p, q) - 4q^k}{v_k(p, q)} \) and \( B = V_k(p, q) - 2q^kx \). Let us prove this theorem by induction on \( d \). Noticing that

\[
F_k^{(1)}(x; p, q) = \frac{(V_k(p, q) - 2q^kx)F_k(x; p, q)}{1 - V_k(p, q)x + q^kx^2},
\]

we get

\[4q^kF_k(x; p, q) + A \cdot F_k^2(x; p, q) = B \cdot F_k^{(1)}(x; p, q),\]

therefore, the theorem holds for \( d = 1 \). Now we suppose that the theorem holds for \( d \), that is,

\[
\sum_{j=0}^{d} a(j, d)q^{(d-j)k} \left( \frac{v_k^2(p, q) - 4q^k}{v_k(p, q)} \right)^j F_k^{(j+1)}(x; p, q)
\]

\[= \sum_{j=1}^{d} b(j, d)q^{(d-j)k} (V_k(p, q) - 2q^kx)^j F_k^{(j)}(x; p, q).\]

Therefore, derivative this equation with respect to \( x \) we have that

\[
\sum_{j=0}^{d} (j + 1)a(j, d)q^{(d-j)k} A^j F_k^j(x; p, q) F_k^{(1)}(x; p, q)
\]

\[= \sum_{j=1}^{d} b(j, d)q^{(d-j)k} B^j F_k^{(j+1)}(x; p, q) - \sum_{j=1}^{d} 2jq^k b(j, d)q^{(d-j)k} B^{j-1} F_k^{(j)}(x; p, q).\]

If multiplying by \( B \) and using Equation 9 then we get that

\[
\sum_{j=0}^{d} (j + 1)a(j, d)q^{(d-j)k} A^{j+1} F_k^{j+2}(x; p, q) + \sum_{j=0}^{d} 4(j + 1)a(j, d)q^{(d+1-j)k} A^j F_k^{j+1}(x; p, q)
\]

\[= \sum_{j=2}^{d+1} b(j - 1, d)q^{(d+1-j)k} B^j F_k^{(j)}(x; p, q) - \sum_{j=1}^{d} 2jb(j, d)q^{(d+1-j)k} B^j F_k^{(j)}(x; p, q),\]
equivalently,
\[
\sum_{j=0}^{d+1} (ja(j-1, d) + 4(j+1)a(j, d))q^{(d+1-j)k} A^j F_k^{j+1}(x; p, q) = \sum_{j=1}^{d+1} (b(j-1, d) - 2j b(j, d))q^{(d+1-j)k} B^j F_k^{(j)}(x; p, q),
\]

Therefore, using Definition 2.1 and Definition 2.3 we have that
\[
\sum_{j=0}^{d+1} a(j, d+1)q^{(d+1-j)k} A^j F_k^{j+1}(x; p, q) = \sum_{j=1}^{d+1} b(j, d+1)q^{(d+1-j)k} B^j F_k^{(j)}(x; p, q).
\]

Hence, using Lemma 2.2 and Lemma 2.4 we get the desired result. \(\square\)

By the above proposition, we have the main result of this paper.

**Theorem 2.6** Let \(d \geq 1\). For any \(n \geq d\),

\[
\sum_{j=0}^{d} \left[ (4q^k)^{d-j} \left( \sum_{i=0}^{j} (-1)^i \binom{j}{i} (j+1-i)^d \left( \frac{v_{i}^{2}(p,q)^{1-4q^k}}{v_{i}^{2}(p,q)} \right) \right)^j s_{j+1}(n+j-d; p, q; k) \right]
\]

\[
= \sum_{j=1}^{d} \left[ \frac{(-1)^{d-1} (2q^k)^{d-j}}{(d-1)!} \left( \sum_{i=0}^{d-j} (-1)^i \binom{j-1}{i} (i+1)^{d-1} \right) \left( \sum_{s=0}^{j} v_{d,j,s}(n)U_{[n+j-d-s]}(p, q) \binom{j}{i} \right) \right],
\]

where \(v_{d,j,s}(n) = (-2q^k)^{j-s}v_{k}^{j-s}(p, q) \prod_{i=1}^{j} (n+j-d-s-i)\).

**Proof.** If comparing the coefficients of \(x^{n-(d+1)}\) on both sides of Proposition 2.5 we get the desired result. \(\square\)

Theorem 2.6 provides a finite algorithm for finding \(s_d(n; p, q; k)\) in terms of \(U_{nk}(p, q)\) and \(V_{nk}(p, q)\), since we have to consider all \(s_j(n; p, q; k)\) for \(j = 1, 2, \ldots, d\). The algorithm has been implemented in Maple, and yields explicit results for \(1 \leq d \leq 6\). Below we present several explicit calculations.
Corollary 2.7 (see Zhao and Wang [2, Equation 9]) For any \( n \geq 1 \),

\[
s_2(n; p, q; k) = \frac{U_k(p, q)}{V_k^2(p, q)} \left( (n - 1)V_k(p, q)U_{nk}(p, q) - 2nq^kU_{(n-1)k}(p, q) \right).
\]

**Proof.** Theorem 2.6 for \( d = 2 \) yields

\[
4q^k s_1(n - 1; p, q; k) + \frac{v_k^2(p, q) - 4q^k}{v_k(p, q)} s_2(n; p, q; k) = (n - 1)V_k(p, q)U_{nk}(p, q) - 2(n - 2)q^kU_{(n-1)k}(p, q).
\]

Using the fact that \( s_1(n; p, q; k) = U_{nk}(p, q) \) we get the desired result. \( \square \)

Corollary 2.8 (see Zhao and Wang [2, Equation 10]) For any \( n \geq 2 \),

\[
s_3(n; p, q; k) = \frac{U_k^2(p, q)}{2[V_k(p, q) - 4q^k]} \left( (n - 1)(n - 2)V_k^2(p, q)U_{nk}(p, q) - 2q^k(n - 2)(2n + 1)V_k(p, q)U_{(n-1)k}(p, q) + 4q^{2k}(n - 2)(n + 2)U_{(n-2)k}(p, q) \right).
\]

**Proof.** Theorem 2.6 for \( d = 3 \) yields

\[
16q^{2k} s_1(n - 2; p, q; k) + 12q^k \frac{v_k^2(p, q) - 4q^k}{v_k(p, q)} s_2(n - 1; p, q; k) + \frac{2(v_k^2(p, q) - 4q^k)^2}{v_k(p, q)} s_3(n; p, q; k) = (n - 1)(n - 2)V_k^2(p, q)U_{nk}(p, q) - 2(n - 2)(2n - 5)q^kV_k(p, q)U_{(n-1)k}(p, q) + 4q^{2k}(n - 2)^2U_{(n-2)k}(p, q).
\]

Using Corollary 2.7 with the fact that \( s_1(n; p, q; k) = U_{nk}(p, q) \) we get the desired result. \( \square \)
Similarly, if applying Theorem 2.6 for \( d \) with using the formulas of \( s_j(n; p, q; k) \) for \( j = 1, 2, \ldots, d - 1 \), then we get the following result (in the case \( d = 4 \) see [2, Equation 11]).

**Corollary 2.9** We have

(i) For any \( n \geq 3 \),

\[
\begin{align*}
 s_4(n; p, q; k) &= \frac{U^3_{\overline{p}[p,q]}}{6(V^3_k(p,q) - 4q^k)^3} V^3_k(p,q)(n-1)(n-2)(n-3)U_{nk}(p,q) \\
& \quad - 6q^k V^2_k(p,q)(n-2)(n-3)(n+1)U_{(n-1)k}(p,q) \\
& \quad + 12q^{2k} V_k(p,q)(n-3)(n^2 + n - 1)U_{(n-2)k}(p,q) \\
& \quad - 8q^{3k} n(n^2 - 4)U_{(n-3)k}(p,q).
\end{align*}
\]

(ii) For any \( n \geq 4 \),

\[
\begin{align*}
 s_5(n; p, q; k) &= \frac{U^4_{\overline{p}[p,q]}}{4(V^3_k(p,q) - 4q^k)^4} V^4_k(p,q)(n-1)(n-2)(n-3)(n-4)U_{nk}(p,q) \\
& \quad - 4q^k V^3_k(p,q)(n-2)(n-3)(n-4)(2n + 3)U_{(n-1)k}(p,q) \\
& \quad + 12q^{2k} V^2_k(p,q)(n-3)(n-4)(2n^2 + 4n - 1)U_{(n-2)k}(p,q) \\
& \quad - 8q^{3k} V_k(p,q)(n-4)(2n+1)(2n^2 + 2n - 9)U_{(n-3)k}(p,q) \\
& \quad + 16q^{4k} (n-3)(n+1)(n+3)U_{(n-4)k}(p,q).
\end{align*}
\]
(iii) For any \( n \geq 5 \),

\[
s_6(n; p, q; k) = \frac{u_5^*(p, q)}{s_5(V_k^*(p, q) - 4q^k)} \left( V_k^5(p, q)(n - 1)(n - 2)(n - 3)(n - 4)(n - 5)U_{n,k}(p, q) - 10q^k V_k^4(p, q)(n - 2)(n - 3)(n - 4)(n - 5)(n + 2)U_{(n-1),k}(p, q) + 20q^{2k} V_k^3(p, q)(n - 3)(n - 4)(n - 5)(2n^2 + 6n + 1)U_{(n-2),k}(p, q) - 40q^{3k} V_k^2(p, q)(n - 4)(n - 5)(n + 1)(2n^2 + 4n - 9)U_{(n-3),k}(p, q) + 80q^{4k} V_k(p, q)(n - 5)(n^4 + 2n^3 - 10n^2 - 11n + 9)U_{(n-4),k}(p, q) - 32q^{5k} n(n - 4)(n - 2)(n + 4)U_{(n-5),k}(p, q) \right).
\]

From these results, it is very easy to obtain Equations 2-4. If \( k = 1 \) and \( p = -q = 1 \), then by using Corollary 2.9 together with the recurrence \( F_n = F_{n-1} + F_{n-2} \) we arrive to

\[
\sum_{a+b+c+d+e=n} F_a F_b F_c F_d F_e = \frac{1}{45.5\pi} (3(n - 1)(8n^3 - 5n^2 - 27n + 50)F_n - 20n(5n^2 - 17)F_{n-1})
\]

\[
\sum_{a+b+c+d+e+f=n} F_a F_b F_c F_d F_e F_f = \frac{1}{513\pi} ((n - 1)(5n^4 - 70n^3 + 65n^2 + 490n + 264)F_n + 2n(5n^2 + 5n^2 - 226)F_{n-1}).
\]

References


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