A formula for the generating functions of powers of Horadam's sequence

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1 Introduction and the Main result

The second-order linear recurrence sequence $(w_n(a,b;p,q))_{n\geq 0}$, or briefly $(w_n)_{n\geq 0}$, is defined by

$$w_{n+2} = pw_{n+1} + qw_n, \tag{1}$$

with given $w_0 = a, w_1 = b$ and $n \geq 0$. This sequence was introduced, in 1965, by Horadam [3, 4], and it generalizes many sequences (see [1, 5]). Examples of such

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sequences are Fibonacci numbers sequence $(F_n)_{n\geq 0}$, Lucas numbers sequence $(L_n)_{n\geq 0}$, and Pell numbers sequence $(P_n)_{n\geq 0}$, when one has $p = q = b = 1$, $a = 0$; $p = q = b = 1$, $a = 2$; and $p = 2$, $q = b = 1$, $a = 0$; respectively. In this paper we interested in studying the generating function for powers of Horadam's sequence, that is, $\mathcal{H}_k(x; a, b, p, q) = \mathcal{H}_k(x) = \sum_{n\geq 0} w_n^k x^n$.

In 1962, Riordan [7] found the generating function for powers of Fibonacci numbers. He proved that the generating function $\mathcal{F}_k(x) = \sum_{n\geq 0} F_n^k x^n$ satisfies the recurrence relation

$$(1 - a_k x + (-1)^k x^2)\mathcal{F}_k(x) = 1 + kx \sum_{j=1}^{[k/2]} (-1)^j \frac{a_{kj}}{j} \mathcal{F}_{k-2j}((-1)^i x)$$

for $k \geq 1$, where $a_1 = 1$, $a_2 = 3$, $a_s = a_{s-1} + a_{s-2}$ for $s \geq 3$, and $(1 - x - x^2)^{-j} = \sum_{k\geq 0} a_{kj} x^{k-2j}$. Horadam [4] gave a recurrence relation for $\mathcal{H}_k(x)$ (see also [6]). Recently, Haukkanen [2] studied linear combinations of Horadam's sequences and the generating function of the ordinary product of two of Horadam's sequences. The main result of this paper can be formulated as follows.

Let $\Delta_k = (\Delta_k(i,j))_{1\leq i,j\leq k} = \Delta_k(p,q)$ be the $k \times k$ matrix

$$\begin{pmatrix}
1 - p^k x - q^k x^2 & -xp^{k-1}q^1\binom{k}{1} & -xp^{k-2}q^2\binom{k}{2} & \cdots & -xp^2q^{k-2}\binom{k}{k-2} & -xpq^{k-1}\binom{k}{k-1} \\
-p^{k-1}x & 1 - xp^{k-2}q^1\binom{k-1}{1} & -xp^{k-3}q^2\binom{k-1}{2} & \cdots & -xpq^{k-2}\binom{k-1}{k-2} & -xq^{k-1}\binom{k-1}{k-1} \\
-p^{k-2}x & -xp^{k-3}q^1\binom{k-2}{1} & 1 - xp^{k-4}q^2\binom{k-2}{2} & \cdots & -xq^{k-2}\binom{k-2}{k-2} & 0 \\
-p^{k-3}x & -xp^{k-4}q^1\binom{k-3}{1} & -xp^{k-5}q^2\binom{k-3}{2} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
-p^2x & -xpq^1\binom{2}{1} & -xq^2\binom{2}{2} & \cdots & 1 & 0 \\
-p^1x & -xq^1\binom{1}{1} & 0 & \cdots & 0 & 1
\end{pmatrix},$$

and let $\delta_k = \delta_k(p, q, a, b)$ be the $k \times k$ matrix

$$\begin{pmatrix} a^k + g_k x & -xp^{k-1}q^1\binom{k}{1} & -xp^{k-2}q^2\binom{k}{2} & \cdots & -xp^2q^{k-2}\binom{k}{k-2} & -xpq^{k-1}\binom{k}{k-1} \\ g_{k-1}x & 1 - xp^{k-2}q^1\binom{k-1}{1} & -xp^{k-3}q^2\binom{k-1}{2} & \cdots & -xpq^{k-2}\binom{k-1}{k-2} & -xq^{k-1}\binom{k-1}{k-1} \\ g_{k-2}x & -xp^{k-3}q^1\binom{k-2}{1} & 1 - xp^{k-4}q^2\binom{k-2}{2} & \cdots & -xq^{k-2}\binom{k-2}{k-2} & 0 \\ g_{k-3}x & -xp^{k-4}q^1\binom{k-3}{1} & -xp^{k-5}q^2\binom{k-3}{2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ g_2 x & -xpq^1\binom{2}{1} & -xq^2\binom{2}{2} & \cdots & 1 & 0 \\ g_1 x & -xq^1\binom{1}{1} & 0 & \cdots & 0 & 1 \end{pmatrix},$$

where $g_j = (b^j - a^j p^j)a^{k-j}$ for all $j = 1, 2, \ldots, k$.

Theorem 1.1 *The generating function $\mathcal{H}_k(x)$ is given by* $\dfrac{\det(\delta_k)}{\det(\Delta_k)}$.

The paper is organized as follows. In Section 2 we give the proof of Theorem 1.1 and in Section 3 we give some applications for Theorem 1.1.

2 Proofs

Let $(w_n)_{n\geq 0}$ be a sequence satisfying Relation (1) and k be any positive integer. We define a family $\{A_{k,d}\}_{d=1}^{k}$ of generating functions by

$$A_{k,d}(x) = \sum_{n\geq 0} w_n^{k-d} w_{n+1}^{d} x^{n+1}. \tag{2}$$

Now we introduce two relations (Lemma 2.1 and Lemma 2.2) between the generating functions $A_{k,d}(x)$ and $\mathcal{H}_k(x)$ that play the crucial roles in the proof of Theorem 1.1.

Lemma 2.1 *For any $k \geq 1$,*

$$(1 - p^k x - q^k x^2)\mathcal{H}_k(x) - x \sum_{j=1}^{k-1} \binom{k}{j} p^{k-j} q^j A_{k,k-j}(x) = a^k + x(b^k - a^k p^k).$$

Proof. Using the binomial theorem we get

$$w_{n+2}^k = (pw_{n+1} + qw_n)^k = p^k w_{n+1}^k + \sum_{j=1}^{k-1} \binom{k}{j} p^{k-j} q^j w_{n+1}^{k-j} w_n^j + q^k w_n^k.$$

Multiplying by x^{n+2} and summing over all $n \geq 0$ with using Definition (2) we have

$$\mathcal{H}_k(x) - b^k x - a^k = p^k x(\mathcal{H}_k(x) - a^k) + x \sum_{j=1}^{k-1} \binom{k}{j} p^{k-j} q^j A_{k,k-j}(x) + q^k x^2 \mathcal{H}_k(x),$$

as requested. □

Lemma 2.2 *For any $k - 1 \geq d \geq 1$,*

$$A_{k,d}(x) - a^{k-d} b^d x = p^d x(\mathcal{H}_k(x) - a^k) + x \sum_{j=1}^{d} \binom{d}{j} p^{d-j} q^j A_{k,k-j}(x).$$

Proof. Using the binomial theorem we have

$$w_n^{k-d} w_{n+1}^d = w_n^{k-d}(pw_n + qw_{n-1})^d = w_n^{k-d} \sum_{j=0}^{d} \binom{d}{j} p^{d-j} q^j w_n^{d-j} w_{n-1}^j.$$

Multiplying by x^{n+1} and summing over all $n \geq 1$ we get

$$A_{k,d}(x) - a^{k-d} b^d x = p^d x(\mathcal{H}_k(x) - a^k) + x \sum_{j=1}^{d} \binom{d}{j} p^{d-j} q^j A_{k,k-j}(x),$$

as requested $\square$

Proof. (Theorem 1.1) By using the above lemmas together with definitions we get

$$\Delta_k \cdot [\mathcal{H}_k(x), A_{k,k-1}(x), A_{k,k-2}(x), \ldots, A_{k,1}(x)]^{\mathrm{T}} = v_k,$$

where v_k is given by

$$\left[a^k + x(b^k - a^k p^k),\ (a^1 b^{k-1} - p^{k-1} a^k)x,\ (a^2 b^{k-2} - p^{k-2} a^k)x, \ldots,\ (a^{k-1} b^1 - p^1 x a^k)x\right]^{\mathrm{T}}.$$

Hence, the solution of the above equation gives the generating function $\mathcal{H}_k(x) = \frac{\det(\delta_k)}{\det(\Delta_k)}$, as claimed in Theorem 1.1. $\square$

3 Applications

In this section we present some applications for Theorem 1.1.

Fibonacci numbers. If $a = 0$ and $p = q = b = 1$, then Theorem 1.1 for $k = 1, 2, 3, 4, 5, 6$ yields Table 1.

k	The generating function $\mathcal{H}_k(x; 0, 1, 1, 1)$
1	$\frac{x}{1-x-x^2}$
2	$\frac{x(1-x)}{(1+x)(1-3x+x^2)}$
3	$\frac{x(1-2x-x^2)}{(1+x-x^2)(1-4x-x^2)}$
4	$\frac{x(1+x)(1-5x+x^2)}{(1-x)(1+3x+x^2)(1-7x+x^2)}$
5	$\frac{x(1-7x-16x^2+7x^3+x^4)}{(1-x-x^2)(1+4x-x^2)(1-11x-x^2)}$
6	$\frac{x(1-x)(1-11x-64x^2-11x^3+x^4)}{(1+x)(1-3x+x^2)(1+7x+x^2)(1-18x+x^2)}$

Table 1. The generating function for the powers of Fibonacci numbers

Lucas numbers. If $a = 2$ and $p = q = b = 1$, then Theorem 1.1 for $k = 1, 2, 3, 4, 5, 6$ yields Table 2.

k	The generating function $\mathcal{H}_k(x; 2, 1, 1, 1)$
1	$\frac{2-x}{1-x-x^2}$
2	$\frac{4-7x-x^2}{(1+x)(1-3x+x^2)}$
3	$\frac{8-13x-24x^2+x^3}{(1+x-x^2)(1-4x-x^2)}$
4	$\frac{16-79x-164x^2+76x^3+x^4}{(1-x)(1+3x+x^2)(1-7x+x^2)}$
5	$\frac{32-255x-1045x^2+960x^3+235x^4-x^5}{(1-x-x^2)(1+4x-x^2)(1-11x-x^2)}$
6	$\frac{64-831x-5940x^2+11155x^3+5485x^4-716x^5-x^6}{(1+x)(1-3x+x^2)(1+7x+x^2)(1-18x+x^2)}$

Table 2. The generating function for the powers of Lucas numbers

Pell numbers. If $a = 0$, $b = q = 1$ and $p = 2$, then Theorem 1.1 for $k = 1, 2, 3, 4, 5, 6$ yields Table 3.

k	The generating function $\mathcal{H}_k(x; 0, 1, 2, 1)$
1	$\frac{x}{1-2x-x^2}$
2	$\frac{x(1-x)}{(1+x)(1-6x+x^2)}$
3	$\frac{x(1-4x-x^2)}{(1+2x-x^2)(1-14x-x^2)}$
4	$\frac{x(1+x)(1-14x+x^2)}{(1-x)(1+6x+x^2)(1-34x-x^2)}$
5	$\frac{x(1-38x-130x^2+38x^3+x^4)}{(1-2x-x^2)(1-82x-x^2)(1+14x-x^2)}$
6	$\frac{x(1-x)(1-104x-1210x^2-104x^3+x^4)}{(1+x)(1+34x+x^2)(1-6x+x^2)(1-198x+x^2)}$

Table 3. The generating function for the powers of Pell numbers

Chebyshev polynomials of the second kind. If $a = 1$, $b = p = 2t$ and $q = -1$, then Theorem 1.1 for $k = 1, 2, 3, 4, 5, 6$ yields Table 4.

More generally, if applying Theorem 1.1 for $k = 1, 2, 3, 4$, then we get the following corollary.

Corollary 3.1 *Let $k = 1, 2, 3, 4$. Then the generating function $\mathcal{H}_k(x)$ is given by $\frac{\mathcal{A}_k(x)}{\mathcal{B}_k(x)}$ where*

k	The generating function $\mathcal{H}_k(x;1,2t,2t,-1)$
1	$\frac{1}{1-2tx+x^2}$
2	$\frac{1+x}{(1-x)((1+x)^2-4xt^2)}$
3	$\frac{1+4tx+x^2}{(1-2tx+x^2)(1+2t(3-4t^2)x+x^2)}$
4	$\frac{(1+x)((1-x)^2+12t^2x)}{(1-x)((1+x)^2-4t^2x)(16t^2(1-t^2)x+(1-x)^2))}$
5	$\frac{1-6tx+2x^2+32t^3x+96t^4x^2+32t^3x^3-32t^2x^2-6x^3t+x^4}{(1+2t(3-4t^2)x+x^2)(1-2tx+x^2)(1-8t^3(4t^2-5)x-10tx+x^2)}$
6	$\frac{(1+x)(x^4+80t^4x^3-24x^3t^2-2x^2-480t^4x^2+640t^6x^2+88t^2x^2+80t^4x-24t^2x+1)}{(1-x)((1+x)^2-4t^2x)((1-x)^2+16t^2(1-t^2)x)((1+x)^2-4t^2(4t^2-3)^2x)}$

Table 4. The generating function for the powers of Chebyshev polynomials of the second kind

$$\mathcal{A}_1(x) = a + x(b - ap),$$

$$\mathcal{A}_2(x) = (a^2 + xb^2)(xq - 1)a^2 + a^2p^2x(xq + 1) - 2x^2pqab,$$

$$\mathcal{A}_3(x) = (a^3 + b^3x - a^3p^3x)(1 - q^3x^2) - 2xpq(a^3 + b^3x) - x^2a^3p^4q + 3ab^2x^2p^2q$$
$$+3ab^2x^3pq^3 - 3a^2bx^3p^2q^3 + 3a^2bx^2pq^2 - 3p^2x^2a^3q^2,$$

$$\mathcal{A}_4(x) = a^4 + (b^4 - a^4(p^4 + 3p^2q + q^2))x - q(5qa^4p^4 + b^4q + a^4q^3 + a^4p^6 + 7q^2a^4p^2$$
$$-6qb^2a^2p^2 - 4b^3ap^3 - 4q^2ba^3p + 3b^4p^2)x^2 + q^3(-8qba^3p^3 - 3b^4p^2 + a^4q^3$$
$$+5qa^4p^4 - 6b^2a^2p^4 - b^4q + a^4p^6 - 4q^2ba^3p + 8b^3ap^3 + 4q^2a^4p^2 + 4qb^3ap)x^3$$
$$+q^6(ap - b)^4x^4$$

and

$$\mathcal{B}_1(x) = 1 - px - x^2q,$$

$$\mathcal{B}_2(x) = (1 + xq)(p^2x - (xq - 1)^2),$$

$$\mathcal{B}_3(x) = (1 + pqx - q^3x^2)(1 - 3pqx - p^3x - q^3x^2),$$

$$\mathcal{B}_4(x) = (1 - q^2x)((1 + q^2x)^2 + p^2qx)((1 - q^2x)^2 - p^2x(p^2 + 4q)).$$

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