Packing patterns into words

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Abstract

We define the packing density on words and find the packing densities of several types of patterns with repeated letters allowed.

A string 213322 contains three subsequences 233, 133, 122 each of which is *order-isomorphic* (or simply *isomorphic*) to the string 122, i.e. ordered in the same way as 122. In this situation we call the string 122 a *pattern*.

Herb Wilf first proposed the systematic study of pattern containment in his 1992 address to the SIAM meeting on Discrete Mathematics. However, several earlier results on pattern containment exist, for example, those by Knuth [11] and Tarjan [15].

Most results on pattern containment actually deal with *pattern avoidance*, in other words, enumerate or consider properties of strings over a totally ordered alphabet which avoid a given pattern or set of patterns. Knuth [11] found that, for any $\pi \in S_3$, the number of n-permutations avoiding π is C_n, the nth Catalan number. Later, Simion and Schmidt [13] determined the number the number of permutations in S_n simultaneously avoiding any given set of patterns $\Pi \subseteq S_3$. Burstein [4] extended this result to the number of strings with repeated letters avoiding any set of patterns $\Pi \subseteq S_3$. Burstein and Mansour [5] considered forbidden patterns with repeated letters.

There is considerably less research on other aspects of pattern containment, specifically, on packing patterns into strings over a totally ordered alphabet (but see [1, 3, 12, 14]). In fact, all pattern packing except the one in [14] (later generalized in [1]) dealt with packing permutation patterns into permutations (i.e. strings without repeated letters). In this paper, we generalize the packing statistics and results to patterns over strings with repeated letters and relate them to the corresponding results on permutations.

1 Preliminaries

Let $[k] = \{1, 2, \ldots, k\}$ be our canonical totally ordered alphabet on k letters, and consider the set $[k]^n$ of n-letter words over $[k]$. We say that a pattern $\pi \in [l]^m$ *occurs* in $\sigma \in [k]^n$, or that σ *contains* the pattern π, if there is a subsequence of σ order-isomorphic to π.

Given a word $\sigma \in [k]^n$ and a set of patterns $\Pi \subseteq [l]^m$, let $\nu(\Pi, \sigma)$ be the total number of occurrences of patterns in Π (Π-patterns, for short) in σ. Obviously, the largest possible number of Π-occurrences in σ is $\binom{n}{m}$, when each subsequence of length m of σ is an occurrence of a Π-pattern. Define

$$\mu(\Pi, k, n) = \max\{\, \nu(\Pi, \sigma) \mid \sigma \in [k]^n\},$$

$$d(\Pi, \sigma) = \frac{\nu(\Pi, \sigma)}{\binom{n}{m}},$$

$$\delta(\Pi, k, n) = \frac{\mu(\Pi, k, n)}{\binom{n}{m}} = \max\{\, d(\Pi, \sigma) \mid \sigma \in [k]^n\},$$

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respectively, the maximum number of Π-patterns in a word in $[k]^n$, the probability that a subsequence of σ of length m is an occurrence of a Π-pattern, and the maximum such probability over words in $[k]^n$. We want to consider the asymptotic behavior of $\delta(\Pi, k, n)$ as $n \to \infty$ and $k \to \infty$.

Proposition 1.1 *If $n > m$, then $\delta(\Pi, k, n) \leq \delta(\Pi, k, n-1)$ and $\delta(\Pi, k, n) \geq \delta(\Pi, k-1, n)$.*

PROOF. The proof of Proposition 1.1 in [1] also applies to the first inequality in our proposition as well, since possible repetition of letters is irrelevant here. To see that the second inequality is true, note that increasing k, i.e. allowing more letters in our alphabet, can only increase $\mu(\Pi, k, n)$, and hence, $\delta(\Pi, k, n)$. □

The greatest possible number of distinct letters in a word σ of length n is n, which implies that $\mu(\Pi, k, n) = \mu(\Pi, n, n)$ for $k \geq n$, and hence, $\delta(\Pi, k, n) = \delta(\Pi, n, n)$ for $k \geq n$. Therefore,

$$\delta(\Pi, n, n) = \lim_{k\to\infty} \delta(\Pi, k, n).$$

We also have $\delta(\Pi, n, n) = \delta(\Pi, n+1, n) \geq \delta(\Pi, n+1, n+1)$, so $\delta(\Pi, n, n)$ is non-increasing and nonnegative, so there exists

$$\delta(\Pi) = \lim_{n\to\infty} \delta(\Pi, n, n) = \lim_{n\to\infty} \lim_{k\to\infty} \delta(\Pi, k, n).$$

We call $\delta(\Pi)$ the *packing density* of Π.

Obviously, there are two double limits. Since $0 \leq \delta(\Pi, k, n) \leq 1$, it immediately follows that there exists

$$\delta(\Pi, k) = \lim_{n\to\infty} \delta(\Pi, k, n) \in [0, 1]$$

and that $\{\delta(\Pi, k) \mid k \in \mathbb{N}\}$ is nondecreasing as $k \to \infty$. Hence, there exists

$$\delta'(\Pi) = \lim_{k\to\infty} \delta(\Pi, k) = \lim_{k\to\infty} \lim_{n\to\infty} \delta(\Pi, k, n).$$

It is easy to see that $\delta'(\Pi) \leq \delta(\Pi)$. Naturally, one wishes to determine when $\delta'(\Pi) = \delta(\Pi)$. In this paper, we will provide a sufficient condition for this equality.

The set $[k]^n$ is finite, so for each k and n, there is a string $\sigma(\Pi, k, n) \in [k]^n$ such that $d(\Pi, \sigma(\Pi, k, n)) = \delta(\Pi, k, n)$. To find $\delta(\Pi)$, we will need to find $\delta(\Pi, k, n)$, hence maximal Π-containing permutations $\sigma(\Pi, k, n)$ are of interest to us, especially, their asymptotic shape as $n \to \infty$ and $k \to \infty$.

Example 1.2 Let $\Pi = \{c_m\}$, where c_m is a constant string of m 1's. Then, clearly, $\sigma(\Pi, k, n) = c_n$ and $d(c_m, c_n) = 1$ for $n \geq m$, so $\delta(c_m, k, n) = 1$ for $n \geq m$, and hence $\delta'(c_m) = \delta(c_m) = 1$ for any $m \geq 1$.

Example 1.3 Let $\Pi = \{id_m\}$, where id_m is the identity permutation of S_m. Then $\sigma(id_m, n, n) = id_n$, so $d(id_m, id_n) = 1$, $\delta(id_m, n, n) = 1$ and $\delta(id_m) = 1$.

Determining $\delta'(id_m)$ is a bit harder. Consider a permutation $\tau \in [k]^n$. Deleting the all 1's of σ and inserting them at the left end of τ can only increase the number of occurrences of id_m. Call the resulting permutation τ_1. Then $d(id_m, \tau) \leq d(id_m, \tau_1)$. Similarly, deleting all 2's of τ_1 and inserting them immediately after the (initial) block of 1's can only increase the the number of occurrences of id_m. Iterating this procedure k times, we see that $\sigma(id_m, k, n)$ must be an nondecreasing string of digits in $[k]$. Let n_i be the number of digits i in $\sigma(id_m, k, n)$, then $\mu(id_m, k, n) = \nu(id_m, \sigma(id_m, k, n)) = n_1 n_2 \ldots n_k$ and $n_1 + n_2 + \cdots + n_k = n$. To maximize the above product we need $n_1 = n_2 = \cdots = n_k = \frac{n}{k}$. (More exactly, [12] shows that we should choose for n_i's to be such integers that $|n_i - \frac{n}{k}| < 1$ and $|n_1 + \cdots + n_r - \frac{rn}{k}| < 1$ for each $r = 1, 2, \ldots, k$.) It follows that

$$\delta(id_m, k, n) \sim \frac{\binom{k}{m}\left(\frac{n}{k}\right)^m}{\binom{n}{m}}$$

(where $a_n \sim b_n$ means $\lim_{n\to\infty} a_n/b_n = 1$), so $\delta(id_m, k) = \frac{\binom{k}{m}}{k^m}$, and thus $\delta'(id_m) = 1$ as expected.

Packing density was initially defined for patterns in permutations. Therefore, we must show that the packing density on permutations agrees with the packing density on words.

Theorem 1.4 *Let $\Pi \subseteq S_m$ be a set of permutation patterns, then*

$$\delta(\Pi) = \lim_{n\to\infty} \frac{\max\{\, \nu(\Pi,\sigma) \mid \sigma \in S_n\}}{\binom{n}{m}},$$

i.e. the packing density of Π on words is equal to that on permutations.

PROOF. It is enough to prove that

$$\mu(\Pi, n, n) = \max\{\, \nu(\Pi,\sigma) \mid \sigma \in S_n\},$$

in other words, that there is a permutation in S_n among the maximal Π-containing words in $[n]^n$. Consider any maximal Π-containing word $\sigma \in [n]^n$. Let n_i be the multiplicity of the letter i in σ. Let i_j denote the jth occurrence of the letter i, and consider the map $f : [n]^n \to S_n$ induced by the map $i_j \mapsto \sum_{r=1}^{i} n_r - j + 1$. Since all letters of each pattern in Π are distinct, Π occurs in $f(\sigma)$ at least at the same positions Π occurs in σ, so $\nu(\Pi, f(\sigma)) \geq \nu(\Pi,\sigma)$. The rest is easy. □

Apart from computing packing densities of patterns, we would also like to determine which patterns have equal packing densities, which ones are asymptotically more packable than others, etc. For example, it is easy to see that the packing density is invariant under the usual symmetry operations on $[l]^m$: *reversal* $r : \tau(i) \to \tau(m-i+1)$ and *complement* $c : \tau(i) \to l - \tau(i) + 1$, (packing density is also invariant under inverse $i : \tau \to \tau^{-1}$ when packing permutations into permutations). The operations r and c generate D_2, while r, c, i generate D_4. Patterns which can be obtained from each other by a sequence of symmetry operations are said to belong to the same *symmetry class.*

Example 1.5 The symmetry class representatives of patterns in $[3]^3$ are $123, 132, 112, 121, 111$. We know that $\delta(111) = 1 = \delta(123)$. Galvin, Kleitmann and Stromquist (independently, unpublished, see chronology in [12]) showed that $\delta(132) = 2\sqrt{3} - 3 \approx 0.4641$. Thus, we only need to determine the packing densities of 112 and 121 to completely classify patterns of length 3.

Price [12] extended Stromquist's results [14] to packing a single pattern $\pi = 1m(m-1)\dots 2$ and handled other single patterns such as 2143. Since we will also be concerned mostly with singleton sets of patterns $\Pi = \{\pi\}$, we will write $\delta(\pi)$ for $\delta(\{\pi\})$, etc.

Price's results deal with patterns of specific type, the so-called *layered* patterns.

Definition 1.6 A *layered* pattern is a strictly increasing sequence of strictly decreasing substrings. These substrings are called the *layers* of σ.

For example, $\widehat{1}\widehat{2}\widehat{3}, \widehat{1}\widehat{32}, \widehat{21}\widehat{3}, \widehat{321}$ are layered, with layers denoted by hats, while $312, 231$ are non-layered.

In fact, note that the union of symmetry classes of layered patterns consists of exactly the permutations avoiding patterns in the symmetry classes of $1342, 1423, 2413$.

In [14], Stromquist proved a theorem (later generalized in [1]) on packing layered patterns into permutations.

The inductive proof of this theorem defines a permutation (or a poset) π to be *layered on top* (or *LOT*) if any of its maximal elements is greater than any non-maximal element. The set of these maximal elements is called the *final layer* of π (even if π is not necessarily layered).

Proposition 1.7 *Let Π be a multiset of LOT permutations (not necessarily all distinct or of the same length). Then there is an LOT permutation σ^* which maximizes the expression*

$$\nu(\Pi,\sigma) = \sum_{\pi\in\Pi} a_\pi \nu(\pi,\sigma), \quad a_\pi \geq 0. \tag{1.1}$$

Furthermore, if the final layer of every $\pi \in \Pi$ has size greater than 1, then every such σ^* is LOT.

Applying this proposition inductively, [1], following [14], obtains

Theorem 1.8 *Let Π be a multiset of layered permutations. Then there is a layered permutation σ^* which maximizes the expression (1.1). Furthermore, if all the layers of every $\pi \in \Pi$ have size greater than 1, then every such σ^* is layered.*

Following [1, 12], we will also define the *ℓ-layer packing density* $\delta_\ell(\Pi)$ for sets of layered permutations Π as the packing density of Π among the permutations with at most ℓ layers. It was shown in both of the above papers that $\delta(\Pi) = \lim_{\ell\to\infty} \delta_\ell(\Pi)$.

2 Monotone patterns

The easiest type of patterns with repeated letters are those whose letters are nondecreasing (or non-increasing) from left to right. By analogy with layered patterns, we will consider nondecreasing patterns.

Theorem 2.1 *Let $\Pi \in [l]^m$ be a set of nondecreasing patterns π and let $a_i(\pi)$ be the number of i's in π. For each $\pi \in \Pi \subseteq [l]^m$, let $\hat\pi \in S_m$ be the layered pattern with layer lengths $(a_1(\pi), \dots, a_l(\pi))$, and let $\hat\Pi = \{\hat\pi \mid \pi \in \Pi\}$. Then $\delta(\Pi, k) = \delta_k(\hat\Pi)$ and $\delta'(\Pi) = \delta(\Pi) = \delta(\hat\Pi)$.*

PROOF. There is an natural bijection between nondecreasing patterns on k letters and layered patterns with k layers. If π is a nondecreasing pattern with layer lengths $(a_1(\pi), \dots, a_l(\pi))$, then the map f of Theorem 1.4 induced by the map $i_j \mapsto \sum_{r=1}^{i} a_r(\pi) - j + 1$ (where i_j is the jth i from left) maps π to $\hat\pi \in S_m$. Clearly, f^{-1} is induced by a map with takes each element in the ith layer (the ith basic subsequence, in general) to integer i. □

Example 2.2 Using the results of Price [12], we obtain $\delta(112) = \delta(\widehat{2}\widehat{13}) = 2\sqrt{3} - 3$, $\delta(1122) = \delta(\widehat{21}\widehat{43}) = 3/8$. More generally, for $k \geq 2$,

$$\delta(\underbrace{1\dots1}_{k}2) = ka(1-a)^{k-1}, \quad \text{where} \quad 0 < a < 1, \quad ka^{k+1} - (k+1)a + 1 = 0.$$

Similarly, for $r, s \geq 2$,

$$\delta(\underbrace{1\dots1}_{r}\underbrace{2\dots2}_{s}) = \delta(\underbrace{1\dots1}_{r}\underbrace{2\dots2}_{s}, 2) = \binom{r+s}{r,\, s} \frac{r^r s^s}{(r+s)^{r+s}}.$$

Using the results of Albert et al. [1], we also find that $\delta(1123) = \delta(1233) = \delta(1243) = 3/8$, $\delta(\{122, 112\}) = \delta(\{132, 213\}) = 3/4$.

3 Weakly layered patterns

Again, by analogy with layered permutations, we define weakly layered strings as follows.

Definition 3.1 A string $\pi \in [l]^m$ is *weakly layered* if it is a concatenation of a nondecreasing sequence of non-increasing substrings. In other words, $\pi = \pi_1 \dots \pi_r$, where π_i are non-increasing, and $\pi_1 \leq \dots \leq \pi_r$ (that is any letter of π_i is not greater than any letter of π_j if $i \leq j$). Substrings π_i maximal with respect to these properties are called the layers of π.

It follows that the consecutive layers of a weakly layered pattern may have at most one letter value in common, for example, $\widehat{1}\widehat{21}$, $\widehat{21}\widehat{2}$, $\widehat{1}\widehat{321}$, $\widehat{1}\widehat{2}\widehat{32}$, $\widehat{21}\widehat{32}$, $\widehat{22111}\widehat{332}$. However, 1231 is not weakly layered.

Theorem 3.2 *If Π is a set of weakly layered patterns none of which contains a layer of length 1, then for each n and k, all maximal Π-containing strings in $[k]^n$ are weakly layered.*

PROOF. If f is an operation as in Theorems 1.4 and 2.1 and π is weakly layered, then $f(\pi)$ is layered. Let $f(\Pi) = \{f(\pi) \mid \pi \in \Pi\}$. It is easy to see that if σ is a maximal Π-containing string, then $f(\sigma)$ is a maximal $f(\Pi)$-containing string. If such σ is non-weakly layered, then it contains a pattern 231 or a pattern 312, hence, 231 or 312 also occurs in $f(\sigma)$, so $f(\sigma)$ is non-layered. But by Theorem 2.2 of [1], $f(\sigma)$ must be layered, contradicting our assumption. Thus, every maximal Π-containing string σ is weakly layered. □

Conjecture 3.3 *If Π is a set of weakly layered patterns, then $\delta'(\Pi) = \delta(\Pi)$ and among maximal Π-containing strings in $[k]^n$, there is one which is weakly layered.*

Note that some maximal Π-containing strings in the above conjecture may not be weakly layered. For example, 12121 is a maximal 121-containing string in $[2]^5$.

We will now find the packing density of some specific weakly layered patterns.

Theorem 3.4 $\delta(121) = \sqrt{3} - 3/2 = \frac{1}{2}\delta(112) = \frac{1}{2}\delta(213)$.

PROOF. We will begin with the pattern $\pi = 121$. Let $\sigma = \sigma(n,k)$ be a maximal 121-containing string in $[k]^n$. Without loss of generality, we can assume the smallest letter of $\sigma(n,k)$ is 1, next smallest letter is 2, etc. It is easy to see that σ should begin and end with 1.

Let σ contain n_1 1's. Let $a > 1$ be a letter in σ and m_a and b_a be the numbers of 1's to the left and to the right of a, respectively. Then $m_a + b_a = n_1$, and a occurs in $m_a b_a \leq \left\lfloor \frac{n_1^2}{4} \right\rfloor$ patterns 121 in σ which involve the letter 1. The equality certainly occurs for each a when all the 1's of σ are at the beginning or at the end of σ. Consequently, $\sigma = 1 \ldots 1\sigma_2 1 \ldots 1$, where σ_2 is a string on letters 2 and greater, is maximal 121-containing. Note that σ_2 is also maximal 121-containing.

Following Price [12], we will find the asymptotic ratio $\alpha = \lim_{n\to\infty} \frac{n_1}{n}$. Then it is easy to see that if n_r is the number of letters r in σ, we must have $\lim_{n\to\infty} \frac{n_r}{n} = \alpha(1-\alpha)^{r-1}$.

Since all the 1's of σ are at the beginning or at the end of σ, it is easy to see that half of them should be in the initial block of 1's and the other half, in the terminal block of 1's. Therefore, we have

$$d(121,\sigma) = \max_{0 \leq n_1 \leq n} \left(d(121,\sigma_2) + \left\lfloor \frac{n_1^2}{4} \right\rfloor (n - n_1) \right)$$

Now the same calculations as in [12, Theorem 5.2] yield

$$\delta(121) = \frac{3}{2} \max_{\alpha \in [0,1]} \frac{\alpha^2(1-\alpha)}{1-(1-\alpha)^3},$$

so $\alpha = (3-\sqrt{3})/2$, $1-\alpha = (\sqrt{3}-1)/2$, and $\delta(121) = \sqrt{3} - 3/2$. □

Here is the complete inventory of packing densities of 3-letter patterns by symmetry class.

Symmetry class	111	112	121	132	123
Packing density	1	$2\sqrt{3}-3$	$\frac{2\sqrt{3}-3}{2}$	$2\sqrt{3}-3$	1

4 Generalized patterns

Generalized patterns were introduced by Babson and Steingrímsson [2] and allow the requirement that some adjacent letters in a pattern be adjacent in its occurrences in an ambient string as well. For example, an occurrence of a generalized pattern 21-3 in a permutation $\pi = a_1 a_2 \cdots a_n$ is a subsequence $a_i a_{i+1} a_j$ of π such that $a_{i+1} < a_i < a_j$. Clearly, in the new notation, classical patterns are those with all hyphens, such as 1-3-2.

Notation 4.1 This notation (introduced in [2]) may be a little confusing since classical patterns (the ones with all hyphens) were previously written the same way as the generalized patterns with all adjacent letters (i.e. with no hyphens). From now on, we will use the generalized pattern notation. However, if we consider subword patterns (those with no hyphens), we may write π_g for a generalized pattern π without hyphens where the context allows for ambiguity.

As with the classical patterns, considered in the earlier sections, most papers on generalized patterns deal with pattern avoidance. For example, Claesson [8] and Claesson and Mansour [9] considered the number of permutations avoiding one or two generalized patterns with one hyphen. Burstein and Mansour [6] looked at the same problem with repeated letters allowed in both in the pattern and the ambient string. Elizalde and Noy [10] and Burstein and Mansour [7] considered generalized patterns without hyphens, i.e. with all consecutive letters adjacent.

Here we consider packing generalized patterns into words.

If $\pi \in [l]^m$ is a generalized pattern with b blocks of consecutive letters (i.e. $b-1$ hyphens), then it is easy to see by considering the positions of the first letters of the blocks of π that the maximum possible number of times π can occur in $\sigma \in [k]^n$ is

$$\binom{n-m+b}{b} \sim \frac{n^b}{b!} \quad \text{as } n \to \infty$$

(this yields $\binom{n}{m}$ when $b = m$, i.e. when π is a classical pattern).

In fact, this maximum is achieved when π is a *constant* generalized pattern, i.e. any of the generalized patterns obtained from the constant strings $11\ldots1$ by inserting hyphens at arbitrary positions (possibly, none). Obviously, maximal π-containing strings are the constant strings of length n. Thus, any set of constant generalized patterns has packing density 1. Similarly, any set Π of hyphenated identity generalized patterns has $\delta(\Pi) = 1$.

Given a set of generalized patterns with b blocks, $\Pi \subseteq [l]^m$, we define the packing density of Π similarly to that of a set of classical patterns. We will use the same notation as in Section 1 for the generalized patterns.

It is not hard to see that the analog of Theorem 1.4 holds for generalized patterns as well.

Theorem 4.2 *Let $\Pi \subseteq S_m$ be a set of generalized permutation patterns, then the packing density of Π on words is equal to that on permutations.*

PROOF. The same argument as in Theorem 1.4 shows that among maximal Π-containing strings in $[n]^n$ there is one that has no repeated letters. □

4.1 Generalized patterns without hyphens

The maximal number of occurrences of a generalized pattern in $[l]^m$ without hyphens (i.e. with $b = 1$ blocks) is $\binom{n-m+1}{1} = n - m + 1 \sim n$ as $n \to \infty$.

Theorem 4.3 *Let $\pi \in [l]^m$ be a nonconstant, nonidentity monotone generalized pattern without hyphens in which each letter i occurs m_i times. Let $M_\pi = \max(m_1, \ldots, m_l)$. Then $\delta(\pi) = \delta'(\pi) = 1/M_\pi$.*

PROOF. Let $\sigma \in [k]^n$ be a word with maximal π-containing word, then it is easy to see that σ has the form $\sigma = \sigma'\sigma' \cdots \sigma'\sigma''$, where $\sigma' = 11\ldots122\ldots2\ldots(k-1)(k-1)\ldots(k-1)k\ldots k$ such that every letter $1, 2, \ldots\ldots, k-1$ appears M_π times, k appears m_k times, and σ'' is a prefix of σ'. Hence, if $n' = n - \text{length}(\sigma'')$ (so $n - M_\pi(k-1) - m_k < n' \le n$), then

$$\frac{\frac{n'}{M_\pi(k-1)+m_k}(k-1)}{n-m+1} \le \frac{\mu(\pi, n, k)}{n-m+1} \le \frac{\frac{n-m_k}{M_\pi(k-1)+m_k}(k-1)}{n-m+1}.$$

Therefore, $\delta(\pi) = \delta'(\pi) = 1/M_\pi$. □

Theorem 4.4 *Let $\pi = (\phi_1, \ldots, \phi_s) \in [l]^m$ be any s-layered generalized pattern without hyphens such that $s > 1$. Let $M_\pi = \max_{1 \le j \le s} |\phi_j|$. Then $\delta(\pi) = \delta'(\pi) = 1/M_\pi$.*

PROOF. The same mapping as in Theorem 2.1 shows that our π has the same packing density as the corresponding monotone generalized pattern without hyphens of Theorem 4.3. □

Corollary 4.5 *Let $\pi_1 = 11\ldots12_g \in [2]^m$ and $\pi_2 = 1m(m-1)\ldots2_g \in [m]^m$, then $\delta(\pi_1) = \delta'(\pi_1) = 1/(m-1)$ and $\delta(\pi_2) = \delta'(\pi_2) = 1/(m-1)$.*

For instance, $\delta(112_g) = \delta'(112_g) = 1/2$, $\delta(132_g) = \delta'(132_g) = 1/2$, $\delta(123_g) = \delta'(123_g) = 1$.

4.2 Generalized patterns with one hyphen

The maximal number of occurrences of a generalized pattern in $[l]^m$ with one hyphen (i.e. with $b = 2$ blocks) is $\binom{n-m+2}{2} \sim n^2/2$ as $n \to \infty$.

Proposition 4.6 $\delta(11\text{-}2) = \delta'(11\text{-}2) = 1$.

PROOF. Let $\sigma \in [k]^n$ be a maximal (11-2)-containing word, then σ is a monotone nondecreasing string in which letter i occurs n_i times, $n_1 + \cdots + n_k = n$. Then $\mu(11\text{-}2, n, k) = \max\{\sum_{i=1}^{k} (n_i - 1)(n_{i+1} + \cdots + n_k) \ : \ n_1 + \cdots + n_k = n\}$. From here, it is not difficult to determine that $\mu(11\text{-}2, n, k) \sim n^2/2$ as $n \to \infty$. Choose n_i's to be such integers that $|n_i - \frac{n}{k}| < 1$ and $|n_1 + \cdots + n_r - \frac{rn}{k}| < 1$ for each $r = 1, 2, \ldots, k$. Then

$$\mu(11\text{-}2, n, k) \sim \left(\frac{n}{k}\right)^2 \binom{k}{2},$$

out of $\binom{n-1}{2}$ maximum possible occurrences, and the result follows. □

Proposition 4.7 $\delta(12\text{-}1) = \delta'(12\text{-}1) = 1/3$.

PROOF. Let $\sigma \in [k]^n$ be a word with maximum occurrences of 12-1, then $\sigma = 1212 \cdots 1211..1 \in [2]^n$ where the string 12 occurs in α exactly d times. So $\mu(12\text{-}1, n, k) = \max_{1 \le d \le n}(d(d-1)/2 + d(n-2d))$, and the maximum occurs at $d \sim n/3$. The rest is easy to check. □

Proposition 4.8 $\delta(12\text{-}3) = \delta(21\text{-}3) = 1$.

PROOF. For pattern 12-3, consider the identity permutation. For pattern 21-3, consider the layered permutation with layers of equal length. □

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