Squaring the terms of an $\ell$th order linear recurrence

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Toufik Mansour

Department of Mathematics, Chalmers University of Technology, S-412 96 Göteborg, Sweden

toufik@math.chalmers.se

1 Introduction and the Main result

An $\ell$th order linear recurrence is a sequence in which each is a linear combination of the $\ell$ previous terms. The symbolic representation of an $\ell$th order linear recurrence defined by

$$a_n = \sum_{j=1}^{\ell} p_j a_{n-j} = p_1 a_{n-1} + p_2 a_{n-2} + \cdots + p_\ell a_{n-\ell},$$

is $(a_0, a_1, \ldots, a_{\ell-1}; p_1, \ldots, p_\ell))_{n \geq 0}$, or briefly $(a_n)_{n \geq 0}$, where the $p_i$ are constant coefficients, with given $a_j = c_j$ for all $j = 0, 1, \ldots, \ell - 1$, and $n \geq \ell$; in such a context, $(a_n)_{n \geq 0}$ is called an $\ell$-sequence.

In the case $\ell = 2$, this sequence is called Horadam’s sequence and was introduced, in 1965, by Horadam [4, 5], and it generalizes many sequences (see [1, 6]). Examples of such sequences are the Fibonacci numbers $(F_n)_{n \geq 0}$, the Lucas numbers $(L_n)_{n \geq 0}$, and the Pell numbers $(P_n)_{n \geq 0}$, when one has the following initial conditions: $p_1 = p_1 = c_1 = 1$, $c_0 = 0$; $p_1 = p_2 = c_1 = 1$, $c_0 = 2$; and $p_1 = 2$, $p_2 = c_1 = 1$, $c_0 = 0$; respectively. In 1962, Riordan [8] found the generating function for powers of Fibonacci numbers. He proved that the generating function $F_k(x) = \sum_{n \geq 0} F_k^n x^n$ satisfies the recurrence relation

$$(1 - a_k x + (-1)^k x^2) F_k(x) = 1 + k x \sum_{j=1}^{[k/2]} (-1)^j \binom{k}{j} F_{k-2j}((-1)^j x)$$

for $k \geq 1$, where $a_1 = 1$, $a_2 = 3$, $a_s = a_{s-1} + a_{s-2}$ for $s \geq 3$, and $(1 - x - x^2)^{-j} = \sum_{k \geq 0} a(k^j x^{k-2j})$. Horadam [5] gave a recurrence relation for $H_k(x)$ (see also [3]). Haukkannen [2] studied linear combinations of Horadam’s sequences and the generating function of

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the ordinary product of two of Horadam’s sequences. Recently, Mansour [7] found a formula for the generating functions of powers of Horadam’s sequence. In this paper we interested in studying the generating function for squaring the terms of the \( \ell \)-sequence, that is,

\[
A_\ell(x) = A_\ell(x; c_0, \ldots, c_{\ell-1}; p_1, \ldots, p_\ell) = \sum_{n \geq 0} a_n^2(c_0, \ldots, c_{\ell-1}; p_1, \ldots, p_\ell)x^n.
\]

The main result of this paper can be formulated as follows. Let \( \Delta_\ell = (\Delta_\ell(i, j))_{0 \leq i, j \leq \ell - 1} \) be the \( \ell \times \ell \) matrix

\[
\Delta_\ell(i, j) = \begin{cases} 
1 - \sum_{s=1}^{\ell} p_j^2 x^j, & i = j = 0 \\
-2 x v_j, & i = 0 \text{ and } 1 \leq j \leq \ell - 1 \\
-p_i x^i, & 1 \leq i \leq \ell - 1 \text{ and } j = 0 \\
\delta_{i, j} - p_{i-j} x^{i-j} - p_{i+j} x^{i+j}, & 1 \leq i \leq \ell - 1 \text{ and } 1 \leq j \leq \ell - i \\
\delta_{i, j}, & 1 \leq i \leq \ell - 1 \text{ and } \ell + 1 - i \leq j \leq \ell - 1
\end{cases}
\]

where \( v_j \) is given by

\[
v_j = p_1 p_{j+1} + p_2 p_{j+2} x + \cdots + p_{\ell-j} p_\ell x^{\ell-j-1},
\]

for all \( j = 1, 2, \ldots, \ell - 1 \), we define \( p_i = 0 \) for \( i \leq 0 \), and \( \delta_{i, j} = \begin{cases} 
1, & \text{if } i = j \\
0, & \text{if } i \neq j
\end{cases} \).

Let \( \Gamma_\ell = (\Gamma_\ell(i, j))_{0 \leq i, j \leq \ell - 1} \) be the \( \ell \times \ell \) matrix

\[
\Gamma_\ell(i, j) = \begin{cases} 
x \sum_{s=0}^{\ell-1} (c_s^2 - w_{s-1}) x^j, & i = j = 0 \\
x^{i+1} \sum_{s=0}^{\ell-1-i} c_s (c_{s+i} - w_{s+i-1}) x^s, & j = 0 \text{ and } 1 \leq i \leq \ell - 1 \\
\Delta_\ell(i, j), & 0 \leq i \leq \ell - 1 \text{ and } 1 \leq j \leq \ell - 1
\end{cases}
\]

where \( w_j \) is given by

\[
w_j = p_0 c_j + p_2 c_{j-1} + \cdots + p_{j+1} c_0 = \sum_{s=1}^{j+1} p_s c_{j+1-s},
\]

for \( j = 0, 1, \ldots, \ell - 2 \) with \( w_{-1} = 0 \).

**Theorem 1.1** The generating function \( A_\ell(x) \) is given by

\[
\frac{\det(\Gamma_\ell)}{x \det(\Delta_\ell)}.
\]

The paper is organized as follows. In Section 2 we give the proof of Theorem 1.1 and in Section 3 we give some applications for Theorem 1.1.

## 2 Proofs

Let \((a_n)_{n \geq 0}\) be a sequence satisfying Relation (1) and \( \ell \) be any positive integer. We define a family \( \{f_\ell(n)\}_{d=0}^{\ell-1} \) of sequences by

\[
f_\ell(n) = a_{n-1} a_{n-1-s},
\]

for
and a family \( \{ F_d(x) \}_{d=0}^{\ell-1} \) of generating functions by
\[
F_d(x) = \sum_{n\geq 1} a_{n-1} a_{n-1-d} x^n.
\] (2)

Now we state two relations (Lemma 2.1 and Lemma 2.2) between the generating functions \( F_d(x) \) and \( F_0(x) = x A_\ell(x) \) that play the crucial roles in the proof of Theorem 1.1.

**Lemma 2.1** We have
\[
F_0(x) = F_0(x) \sum_{j=1}^{\ell} p_j^2 x^j + 2x \sum_{j=1}^{\ell-1} v_j F_j(x) + x \sum_{j=0}^{\ell-1} (c_j^2 - w_{j-1}^2) x^j.
\]

**Proof.** Since the sequence \( (a_n)_{n\geq 0} \) satisfying Relation (1) we get that
\[
a_n^2 = \sum_{j=1}^{\ell} p_j^2 a_{n-j}^2 + 2 \sum_{1 \leq i < j \leq \ell} p_i p_j a_{n-i} a_{n-j},
\]
for all \( n \geq \ell \). Multiplying by \( x^n \) and summing over \( n \geq \ell \) together with the following facts:

1. \( \sum_{n \geq \ell} a_n^2 x^n = \frac{1}{x} \sum_{n \geq \ell} f_0(n+1) x^{n+1} = \frac{1}{x} \left( F_0(x) - \sum_{j=1}^{\ell} a_{j-1}^2 x^j \right), \)
2. \( \sum_{n \geq \ell} a_{n-j}^2 x^n = \sum_{n \geq \ell} f_0(n-j+1) x^n = x^{j-1} \left( F_0(x) - \sum_{i=1}^{\ell-j} a_{i-1}^2 x^i \right), \)
3. \( \sum_{n \geq \ell} a_{n-i} a_{n-j} x^n = \frac{1}{x} \sum_{n \geq \ell+1} f_{j-i}(n-i) x^n = x^{i-1} \left( F_{j-i}(x) - \sum_{d=j-i+1}^{\ell-i} a_{d-1} a_{d-i-1} x^d \right), \)

we have that
\[
F_0(x) = F_0(x) \sum_{j=1}^{\ell} p_j^2 x^j + 2x \sum_{j=1}^{\ell-1} v_j F_j(x) + x \sum_{j=0}^{\ell-1} (c_j^2 - w_{j-1}^2) x^j
\]
\[
= F_0(x) \sum_{j=1}^{\ell} p_j^2 x^j + 2x \sum_{j=1}^{\ell-1} v_j F_j(x) + x \sum_{j=0}^{\ell-1} (c_j^2 - w_{j-1}^2) x^j.
\]

Hence, using the fact that \( a_j = c_j \) for \( j = 0, 1, \ldots, \ell - 1 \) we obtain the desired result. \( \Box \)

**Lemma 2.2** For any \( i = 1, 2, \ldots, \ell - 1, \)
\[
F_i(x) = p_i x^i F_0(x) + \sum_{j=1}^{\ell-i} (p_{n-j} x^{n-j} + p_{i+j} x^j) F_j(x) + x^{i+1} \sum_{j=0}^{\ell-1-i} c_j (c_{i+j} - w_{i+j-1}) x^j.
\]
Proof. By direct calculations we have for \( n \geq \ell + 1 \),

\[
f_i(n) = a_{n-1}a_{n-1-i} = \sum_{j=1}^{\ell} p_j a_{n-1-j}a_{n-1-i};
\]
equivalently,

\[
f_i(n) = p_1 f_{i-1}(n-1) + p_2 f_{i-2}(n-2) + \cdots + p_i f_0(n-i) + p_{i+1} f_1(n-i) + \cdots + p_{\ell} f_{\ell-i}(n-i).
\]

As in Lemma 2.1, multiplying by \( x^n \) and summing over \( n \geq \ell + 1 \) we get

\[
F\ell(x) - \sum_{j=1}^{\ell} a_{j-1}a_{n-1-i}x^j = \sum_{j=1}^{\ell} p_j x^j \left( F_{i-j}(x) - \sum_{d=\ell-j+1}^{\ell} a_{d-1}a_{d-(i-j)-1}x^d \right) + \sum_{j=1}^{\ell} p_j x^j \left( F_{j-i}(x) - \sum_{d=\ell-j+1}^{\ell} a_{d-1}a_{d-(j-i)-1}x^d \right)
\]
The rest is easy to check from the definitions. \( \Box \)

Proof. (Theorem 1.1) Using the above lemmas together with the definitions we have

\[
\Delta_k \cdot [F_0(x), F_1(x), F_2(x), \ldots, F_{\ell-1}(x)]^T = w^T,
\]

where the vector \( w \) is given by

\[
\begin{bmatrix}
x \sum_{j=0}^{\ell-1} (c_j^2 - w_j) x^j \\
x^2 \sum_{j=0}^{\ell-2} c_j (c_{j+1} - w_j) x^j \\
x^3 \sum_{j=0}^{\ell-3} c_j (c_{j+2} - w_{j+1}) x^j \\
\vdots \\
x^{\ell-1} \sum_{j=0}^{0} c_j (c_{j+\ell-1} - w_{j+\ell-2}) x^j
\end{bmatrix}.
\]

Hence, the solution of the above equation gives the generating function \( F_0(x) = \frac{\det(I_\ell)}{\det(\Delta)} \); equivalently, \( A_\ell(x) = \frac{\det(I_\ell)}{x \det(\Delta)} \), as claimed in Theorem 1.1. \( \Box \)

3 Applications

In this section we present some applications of Theorem 1.1.

Fibonacci numbers. Let \( F_{k,n} \) be the \( n \)th \( k \)-Fibonacci number which is given by

\[
F_{k,n} = \sum_{j=1}^{k} F_{k,n-j},
\]

for \( n \geq k \), with \( F_{k,0} = 0 \) and \( F_{k,j} = 1 \) for \( j = 1, 2, \ldots, k-1 \); in such a context, \( F_{2,n}, F_{3,n}, \) and \( F_{4,n} \) are usually called the \( n \)th Fibonacci numbers, tribonacci numbers, and tetranacci numbers, respectively. Using Theorem 1.1 with \( c_0 = 0 \) and

\[
c_1 = c_2 = \cdots = c_{k-1} = p_1 = p_2 = \cdots = p_k = 1
\]

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The generating function for the square of the $k$th-Fibonacci numbers gives the generating function $\sum_{n \geq 0} F_{k,n}^2 x^n$ (see Table 1).

From Table 1, for $k = 3$ we obtain

$$\sum_{n \geq 0} n P_{3,n}^2 x^n = \frac{x(1 - 2x + 2x^2 + 12x^3 + 8x^5 + 2x^6 + 4x^7 + 3x^8 + 2x^9)}{(x^3 - x^2 - x - 1)^2(x^3 + x^2 + 3x - 1)^2}.$$

**Pell numbers.** Let $P_{k,n}$ be the $n$th $k$-Pell number which is given by

$$P_{k,n} = 2P_{k,n-1} + \sum_{j=2}^{k} P_{k,n-j},$$

for $n \geq k$, with $P_{k,j} = 1$ for $j = 0, 1, \ldots, k - 1$; in such a context, $P_{2,n}$ is usually called the $n$th Pell number. Using Theorem 1.1 with $c_j = 1$ for $j = 0, 1, \ldots, k - 1$ and $p_j = 1$ for $j = 1, 2, \ldots, k$ gives the generating function $\sum_{n \geq 0} P_{k,n}^2 x^n$ (see Table 2).

\begin{table}
\begin{tabular}{|c|l|}
\hline
$k$ & The generating function $\sum_{n \geq 0} P_{k,n}^2 x^n$ \\
\hline
2 & $\frac{x(1-x)}{(1+x)(1-6x+x^2)}$ \\
3 & $\frac{x(1-x)(1-12x^2-13x^3-5x^4+4x^5)}{(1-6x-32x^2-x^3)(1-x+22x^2-x^3)}$ \\
4 & $\frac{x(1-x)(12x^2-53x^3-50x^4+3x^5-9x^6-12x^7+13x^8+9x^9)}{(1-x-8x^2-13x^3-20x^4+2x^5+14x^6+x^7-x^8-x^9)}$ \\
5 & $\frac{x(1-x)(9x^{12}+4x^{11}+13x^{10}+6x^9-20x^8-6x^7-2x^6-5x^5-14x^4-9x^3-3x^2-2x+1)}{(1-2x-4x^2-7x^3-11x^4-16x^5+4x^6+x^7+4x^8+4x^9+7x^{10}-x^{11}-x^{13}+x^{15})}$ \\
\hline
\end{tabular}
\end{table}

Table 2: The generating function for the square of the $k$th-Pell numbers

From Table 2, for $k = 2$ we have

$$\sum_{n \geq 0} n P_{2,n}^2 x^n = \frac{x(1 - 2x + 10x^2 - 2x^3 + x^4)}{(x+1)^2(x^2 - 6x + 1)^2}.$$

**References**


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