THE GENERATING FUNCTION OF TWO-STACK SORTABLE PERMUTATIONS BY DESCENTS IS REAL-ROOTED

PETTER BRÄNDÉN

ABSTRACT. Bóna has conjectured that, for fixed $n \geq 0$ and $t \geq 1$, the generating function of t-stack sortable permutations of length n by descents is real-rooted. The conjecture is known to be true for $t = 1$ and $t = n - 1$. Here we prove it for $t = 2$.

1. INTRODUCTION

Let $W_t(n, k)$ be the number of t-stack sortable permutations in the symmetric group, $\mathcal{S}_n$, with k descents. Recently Bóna [1] showed that for fixed n and t the numbers $W_t(n, k)$ form a log-concave sequence, that is,

$$W_t(n, k)^2 \geq W_t(n, k-1)W_t(n, k+1),$$

for $1 \leq k \leq n-2$. Let $W_{n,t}(x) = \sum_{k=0}^{n-1} W_t(n, k)x^k$. A sufficient condition on a sequence to be log-concave is that the corresponding polynomial is real-rooted. When $t = n - 1$ and $t = 1$ we get the Eulerian and the Narayana polynomials respectively. These are known to be real-rooted and Bóna conjectures that the same is true for general t. In what follows we will prove the conjecture for $t = 2$.

Let W be the set of finite words on $\mathbb{N}$ without repetitions. If w is any nonempty word we may write it as the concatenation $w = LnR$ where n is the greatest letter of w and L and R are the subwords to the left and right of n respectively. The stack-sorting operation $s : W \to W$ may be defined recursively by

$$s(w) = \begin{cases} w, & \text{if } w \text{ is the empty word,} \\ s(L)s(R)n, & \text{if } w = LnR \text{ is nonempty.} \end{cases}$$

The *stack sortable* permutations in $\mathcal{S}_n$ are the permutations which are mapped by s to the identity permutation. Similarly, a permutation is called *t-stack sortable* if $s^t(\pi)$ is the identity permutation.

2. STACK SORTABLE PERMUTATIONS AND JACOBI POLYNOMIALS

The number of stack sortable permutations of length n with k descents are known [9] to be the famous Narayana numbers [10, 11]

$$W_1(n, k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}.$$

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The Narayana polynomials $W_{n,1}(x)$ are known to have real zeros. A simple proof of this fact is obtained by expressing $W_{n,1}(x)$ in terms of Jacobi polynomials. Recall the definition of the *hypergeometric function* ${}_2F_1$:

$$ {}_2F_1\left(a,b;c;z\right)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_nz^n}{(c)_nn!}, $$

where $(\alpha)_0=1$ and $(\alpha)_n=\alpha(\alpha+1)\cdots(\alpha+n-1)$ when $n\geq 1$. The Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ can be expressed in the following two ways [8, Page 254]:

$$ P_n^{(\alpha,\beta)}(x)=\frac{(1+\alpha)_n}{n!}{}_2F_1\left(-n,1+\alpha+\beta+n;1+\alpha;\frac{1-x}{2}\right), \quad (1) $$

$$ P_n^{(\alpha,\beta)}(x)=\frac{(1+\alpha)_n}{n!}\left(\frac{x+1}{2}\right)^n{}_2F_1\left(-n,-\beta-n;1+\alpha;\frac{x-1}{x+1}\right). \quad (2) $$

Rewriting $W_1(n+1,k)$ we end up with

$$ W_{n+1,1}(x)={}_2F_1\left(-n,-n-1;2;x\right), $$

which by (2) gives

$$ W_{n+1,1}(x)=\frac{1}{n+1}(1-y)^nP_n^{(1,1)}\left(\frac{1+y}{1-y}\right). $$

Since the Jacobi polynomials are orthogonal when $\alpha,\beta>-1$ we know that $W_{n,1}(x)$ is real- and simple-rooted and that the zeros of $W_{n,1}(x)$ strictly interlace the zeros of $W_{n+1,1}(x)$, that is, $\{W_{n+1,1}(x)\}_{n=0}^{\infty}$ form a *Sturm sequence.*

The numbers $W_2(n,k)$ are surprisingly hard to determine despite of their compact and simple form. It was recently shown that

$$ W_2(n,k)=\frac{(n+k)!(2n-k-1)!}{(k+1)!(n-k)!(2k+1)!(2n-2k-1)!}. $$

See [2, 5, 6, 7] for proofs and more information on 2-stack sortable permutations.

A sequence of real numbers $\Gamma=\{\gamma_k\}_{k=0}^n$ is called an *n-sequence (of the first kind)* if for any real-rooted polynomial $f=a_0+a_1x+\cdots+a_nx^n$ of degree at most n the polynomial $\Gamma(f):=a_0\gamma_0+a_1\gamma_1+\cdots+a_n\gamma_nx^n$ is real-rooted. There is a simple algebraic characterisation of n-sequences [4]:

Theorem 1. *Let $\Gamma=\{\gamma_k\}_{k=0}^n$ be a sequence of real numbers. Then Γ is an n-sequence of the first kind if and only if $\Gamma[(x+1)^n]$ is real-rooted with all its zeros of the same sign.*

We need the following lemma:

Lemma 2. *Let n be a positive integer and r a non-negative real number. Then $\Gamma=\{\binom{-n-r}{k}\}_{k=0}^n$ is an n-sequence.*

Proof. Let $r > 0$. Then

$$\begin{aligned}\Gamma[(x+1)^n] &= \sum_{k=0}^{n} \binom{-n-r}{k}\binom{n}{k}x^k \\ &= {}_2F_1\left(-n, n+r; 1; x\right) \\ &= P_n^{(0,r-1)}(1-2x),\end{aligned}$$

where the last equality follows from (1). Since the Jacobi polynomials are known to have all their zeros in $[-1,1]$ when $\alpha, \beta > -1$ we have that $\Gamma[(x+1)^n]$ has all its zeros in $[0,1]$. The case $r = 0$ follows by continuity when we let r tend to zero from above. □

From the case $r = 0$ in Lemma 2 and the identity

$$\sum_{k=0}^{n} \binom{2n-k-1}{n-1}\binom{n}{k}x^k = (-1)^n \sum_{k=0}^{n} \binom{-n}{k}\binom{n}{k}(-x)^{n-k},$$

it follows that $\binom{2n-k-1}{n-1}$ is an n-sequence.

Theorem 3. *For all $n \geq 0$ the polynomial $W_{2,n}(x)$, which records 2-stack sortable permutations by descents, is real-rooted.*

Proof. We may write $W_2(n,k)$ as

$$W_2(n,k) = \frac{\binom{2n-k-1}{n-1}\binom{n+k}{n-1}\binom{2n}{2k+1}}{n^2\binom{2n}{n}}.$$

A well known result on real-rooted polynomials reads as follows: If $\sum_i a_i x^i$ is a polynomial having only real non-positive zeros then so is the polynomial $\sum_i a_{ki} x^i$, where k is any positive integer. For a proof see [3, Theorem 3.5.4]. Applying this result to the polynomial $x(1+x)^{2n}$ we see that $\sum_k \binom{2n}{2k+1}x^k$ is real-rooted. Now,

$$\sum_{k=0}^{n-1} \binom{n+k}{n-1}\binom{2n}{2k+1}x^k = \sum_{k=0}^{n-1} \binom{2n-k-1}{n-1}\binom{2n}{2k+1}x^{n-1-k},$$

which by the discussion after Lemma 2 is real-rooted. Another application of Lemma 2 gives that $W_{n,2}(x)$ is real-rooted. □

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Matematik, Chalmers tekniska högskola och Göteborgs universitet, S-412 96 Göteborg, Sweden

E-mail address: branden@math.chalmers.se