A Discrete Four Vertex Theorem

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Abstract

A discrete four vertex theorem is proved for a general plane polygon using a method of proof that also yields a proof, that appears to be new, for the classical four vertex theorem.

1 Introduction

The purpose of this paper is to establish a discrete four vertex theorem, also termed a four vertex theorem for polygons. We first reprove the classical four vertex theorem using a method of proof that appears to be new. Then we prove a the discrete four vertex theorem in the case of convex polygons. The result for convex polygons and the method of proof for the smooth case are used to prove our main result, (with terminology to be explained below):

Theorem 1 (Discrete Four Vertex Theorem, (DFV)) *Consider γ, a plane simple closed polygon. Assume that γ is locally regular and that the vertices are not on a circle. Let k denote the discrete curvature of γ. Then k has at least two local maxima and at least two local minima.*

Theorem 1 was proven by S. Bilinski [1, 1963] in the case of convex polygons with obtuse vertex angles.

The classical four vertex theorem essentially states that if γ is a smooth plane, simple, closed curve with curvature k, then k is either constant, in which case γ is a circle, or k has at least two local minima and two local maxima. A local extreme value for the curvature is called a vertex.

*Björn Dahlberg died on the 30th of January 1998. The results of this paper is a minor part of his post-humous work. Dalhberg realized that these ideas could be generalized in several directions, e.g. to discrete affine geometry and the six vertex theorem, see [11]. The final version was prepared by Vilhelm Adolfsson and Peter Kumlin, Dept. of Mathematics, Chalmers University of Technology and Göteborg University, SE-412 96 Göteborg, Sweden; email: kumlin@math.chalmers.se

The classical four vertex theorem was originally proven for the case of convex curves by S. Mukhopadhyaya [9, 1909], and independently by A. Kneser [8]. For proofs in the case of possibly non-convex curves, see Fog [5], Jackson [7] and Vietoris [14]. It is now a classical theorem that is proven in most introductory differential geometry books, although usually only for convex curves. For a simple proof for possibly non-convex curves, see H. Guggenheimer [6]. A sample of the large literature on the four vertex theorem and its generalisations is listed in R. Osserman [10]. For the converse of the four vertex theorem, see B. Dahlberg [4].

We introduce some terminology. Let γ be a plane closed simple polygonal curve. Let $V(\gamma)$ denote the set of its vertices $\{P_0, P_1, \dots, P_{N-1}\}$ ordered counterclockwise and with the indices counted modulo N. We let $\Omega(\gamma)$ denote the closure of the set enclosed by γ. For $P \in V(\gamma)$ let $\rho(P)$ be the unique circle passing through P and its two immediate neighbouring vertices; $\rho(P)$ is called the *discrete circle of curvature* at the vertex P. Let $\omega(P)$ be the closed disk with $\rho(P)$ as its boundary.

We define the *discrete curvature* k of γ as follows. If $P \in V(\gamma)$ is on the line segment between the two immediate neighbours of P we set $k(P) = 0$. Otherwise let $\alpha(P)$ denote the interior angle at P and let $R(P) > 0$ be the radius of $\rho(P)$. If $\alpha(P) < \pi$, we set $k(P) = \frac{1}{R(P)}$ and if $\alpha(P) > \pi$, we set $k(P) = -\frac{1}{R(P)}$. It is easy to give examples of polygons for which k has exactly one local maximum, so we have to find some additional conditions on a polygon for the four vertex theorem to hold for k.

We first make the notion of a *local extreme point* precise. A subset E of $V(\gamma)$ is said to be *connected* if E consists of a sequence of consecutive vertices. In this case E is called an *interval*. If $E \subset V(\gamma)$ is an interval of the form $E = \{P_j, P_{j+1}, \dots, P_k\}$ we let $\hat{E} = E \bigcup \{P_{j-1}, P_{k+1}\}$. Clearly $\hat{E}$ is also an interval. Let $f : V(\gamma) \to \mathbf{R}$ be a function and for $P \in V(\gamma)$ let $I(f, P)$ be the largest interval J of vertices that contains P for which $f(Q) = f(P)$ for all $Q \in J$. We will say that P is a local maximum for f if $f(Q) < f(P)$ for all $Q \in \hat{I}(f, P) \setminus I(f, P)$. Similarly, we will say that P is a local minimum for f if $f(Q) > f(P)$ for $Q \in \hat{I}(f, P) \setminus I(f, P)$. We will say that two local extreme points P_1 and P_2 are distinct if $I(f, P_1)$ and $I(f, P_2)$ are disjoint.

For $P \in V(\gamma)$ let $\mathcal{A}(P)$ be the closed convex sector with apex at P that is determined by P and its two immediate neighbours. If $k(P) = 0$, we require that $\mathcal{A}(P)$ is that half-plane determined by P and its two immediate neighbours such that $\mathcal{A}(P) \bigcap W = \Omega(\gamma) \bigcap W$ for all sufficiently small neighbourhoods W of P.

Definition 1 *We say that γ is* locally regular *if for all vertices $P \in V(\gamma)$ the centre of the circle of curvature at P is contained in $\mathcal{A}(P)$.*

We remark that by elementary geometry one has the following sufficient conditions for local regularity.

Proposition 1 *Let γ be a plane simple closed polygonal curve. If for each vertex P the interior angle $\alpha(P)$ satisfies $\frac{\pi}{2} \leq \alpha(P) \leq \frac{3\pi}{2}$, then γ is locally regular. If all edges have the same length, then γ is locally regular.*

Note also that every plane simple closed polygonal curve can be considered as regular by adding extra vertices on the line intervals connecting the original vertices. Of course the discrete curvature will be changed in this case. If however the emphasis is on the ordered set of vertices, rather than the corresponding polygonal curve, adding vertices does not seem natural and local regularity becomes substantial.

Finally, as a convenient notation, let $[P, Q]$ for two points $P, Q \in \mathbf{R}^2$, denote the line segment joining P and Q and $(P, Q) = [P, Q] \setminus \{P, Q\}$. We also consider when convenient, $\mathbf{R}^2$ identified with $\mathbf{C}$ in the usual way. With abuse of notation we write polygon for polygonal curve throughout the rest of the paper.

We state and prove the classical four vertex theorem as Theorem 4 in section 2. The method of proof involves techniques used in association with Cauchy's Lemma which we state as Theorem 2 in the same section. We prove a smooth version of Cauchy's Lemma in Theorem 3 which is used to prove the smooth four vertex theorem. We state and prove the preliminary (convex) case, Theorem 6 (CDFV), in section 3. The formulation is similar to a well-known result for smooth curves, stated as Theorem 5 at the end of section 2. The proof of our main result, Theorem 1, is in section 4.

2 Four Vertex Theorem — smooth case

We consider Cauchy's result on convex polygons, developed for his proof of the rigidity of convex polyhedrons. We only consider polygons in $\mathbf{R}^2$.

Lemma 1 *Let $\gamma = \{P_0, \ldots, P_{N+1}\}$ and $\Gamma = \{Q_0, \ldots, Q_{N+1}\}$ be two convex polygons. Let α_k, β_k denote the interior angles of γ, Γ at the k:th vertex. Assume that $\alpha_k \leq \beta_k$ for $1 \leq k \leq N$ and $|P_{k+1} - P_k| \leq |Q_{k+1} - Q_k|$ for $0 \leq k \leq N$. Then $|P_{N+1} - P_0| \leq |Q_{N+1} - Q_0|$ with equality if and only if $\alpha_k = \beta_k$ for all $k \in \{1, \ldots, N\}$.*

For a proof of this and the next theorem, Cauchy's Lemma, see [13]. We introduce some notation needed for the statement of Cauchy's Lemma. Let $\mathbf{Z}_n = \{0, 1, \dots, n-1\}$. We say that a set $I \subset \mathbf{Z}_n$ is an *interval* if there are integers a, b such that $a \le b$ and $I = \{k \bmod n : a \le k \le b\}$. Let W_n be the class of sequences $\xi = \{\xi_j\}_{j=0}^{n-1}$ such that $\Sigma_{j=0}^{n-1}\xi_j = 0$ but $\xi_k \neq 0$ for some k. If $\xi \in W_n$ then $R(\xi)$ denotes the number of distinct maximal intervals $I \subset \mathbf{Z}_n$ such that $\xi_k \ge 0$ for all $k \in I$. It is clear that $R(\xi) = R(-\xi)$ for all $\xi \in W_n$. This means that a sequence $\xi \in W_n$ has $2R(\xi)$ sign changes.

Theorem 2 (Cauchy's Lemma) *Let $\gamma = \{P_0, \dots, P_{N-1}\}$ and $\Gamma = \{Q_0, \dots, Q_{N-1}\}$ be two closed convex polygons. Let α_k, β_k be the interior angles of γ, Γ at the k:th vertex and set $\theta_k = \beta_k - \alpha_k$. Assume $|Q_{k+1} - Q_k| = |P_{k+1} - P_k|$ for $0 \le k < N$. Also assume that $\beta_k \neq \alpha_k$ for some k. Set $\theta = \{\theta_k\}_{k=0}^{N-1}$. Then $\theta \in W_N$ and has at least 4 sign changes.*

We will need the smooth version of Cauchy's Lemma. Let $I = [a, b]$ be a closed bounded interval. Let $\mathcal{F}(I)$ denote the class of one–to–one C^1-mappings $\gamma : I \to \mathbf{R}^2$ with $|\frac{d\gamma}{dt}| = 1$ and $\frac{d\gamma}{dt}$ Lipschitz continuous such that γ together with the line segment $[\gamma(a), \gamma(b)]$ bounds a convex domain. We also require that the curvature of γ is non-negative *a.e.*

Lemma 2 *Let $I = [a, b]$ and $\gamma, \Gamma \in \mathcal{F}(I)$. Let γ, Γ have the curvature k and K. Assume $k \ge K$ a.e. in I. Then $|\gamma(b) - \gamma(a)| \le |\Gamma(b) - \Gamma(a)|$ with equality if and only if $k = K$ a.e. in I.*

PROOF: Set $\dot\gamma = \frac{d\gamma}{dt}$ and $\dot\Gamma = \frac{d\Gamma}{dt}$. We begin by picking a $\rho \in (a, b)$ such that

$$\dot\gamma(\rho) = \frac{\gamma(b) - \gamma(a)}{|\gamma(b) - \gamma(a)|}.$$

We may without loss of generality assume that $\gamma(\rho) = \Gamma(\rho) = 0$ and $\dot\gamma(\rho) = \dot\Gamma(\rho) = (1, 0)$. Define α, β by the requirements that $\alpha(\rho) = \beta(\rho) = 0$, $\frac{d\alpha}{dt}(t) = k(t)$ and $\frac{d\beta}{dt}(t) = K(t)$. Then $\gamma(s) = \int_\rho^s e^{i\alpha(t)}\,dt$ and $\Gamma(s) = \int_\rho^s e^{i\beta(t)}\,dt$. Let $\gamma(t)$ and $\Gamma(t)$ have the abscissas $x(t)$ and $\xi(t)$. By convexity $0 \le \alpha(t) \le \pi$ for $t \in [\rho, b]$. Hence $0 \le \beta(t) \le \alpha(t) \le \pi$ for $t \in [\rho, b]$. If $\rho \le s \le b$ then

$$\xi(s) = \int_\rho^s \cos\beta(t)\,dt \ge \int_\rho^s \cos\alpha(t)\,dt = x(s).$$

In particular, $\xi(b) \geq x(b)$. Similarly $\xi(a) \leq x(a)$. Hence

$$|\Gamma(b) - \Gamma(a)| \geq |\xi(b) - \xi(a)| = \xi(b) - \xi(a)$$
$$\geq x(b) - x(a) = |\gamma(b) - \gamma(a)|,$$

which yields the lemma. □

We make a more precise definition of the class of smooth convex closed curves of length L. For $L > 0$ we let $\mathcal{F}_L$ be the class of C^2-mappings $\gamma : \mathbf{R} \to \mathbf{R}^2$ such that $|\frac{d\gamma}{dt}| = 1$, $\gamma(t + L) = \gamma(t)$ for all $t \in \mathbf{R}$, γ is one–to–one on $[0, L)$ and γ has non-negative curvature.

The following smooth version of Cauchy's Lemma is due to Blaschke [2] but our proof follows Bol [3].

Theorem 3 *Let $\gamma, \Gamma \in \mathcal{F}_L$ have the curvature k and K. Assume $k(s) \neq K(s)$ for some s. Set $f = k - K$. Then there are four points s_i, $s_1 < s_2 < s_3 < s_4$, $s_4 - s_1 < L$ such that*

$$\max\{f(s_2), f(s_4)\} < 0 < \min\{f(s_1), f(s_3)\}.$$

Remark: The theorem says that f has 4 distinct zeros, i.e. γ and Γ have equal curvatures in at least 4 points. Moreover f has at least two local maxima and two local minima.

PROOF: Assume the conclusion does not hold. Then we can find $a, b \in \mathbf{R}$ with $a < b < a + L$ such that $f \geq 0$ in $I_1 = [a, b]$ and $f \leq 0$ in $I_2 = [b, a + L]$. Using Lemma 2 on I_1 we see that $|\gamma(b) - \gamma(a)| \leq |\Gamma(b) - \Gamma(a)|$. Using Lemma 2 on I_2 gives $|\gamma(b) - \gamma(a)| \geq |\Gamma(b) - \Gamma(a)|$. Hence $|\gamma(b) - \gamma(a)| = |\Gamma(b) - \Gamma(a)|$ so again by Lemma 2 we have $k = K$ *a.e.* Since k and K are continuous we must have $k(s) = K(s)$ for all $s \in \mathbf{R}$. This contradiction establishes the theorem. □

We next reprove the classical four-vertex theorem. The method of proof appears to be new.

Theorem 4 (Classical Four Vertex Theorem) *Let $\gamma : S^1 \to \mathbf{R}^2$ be of class C^2 with $|\frac{d\gamma}{dt}| > 0$. Assume γ is simple and that γ is not a circle. Let k denote the curvature of γ. Then there are four consecutive points $p_i \in S^1$ such that*

$$\max\{k(p_1), k(p_3)\} < \min\{k(p_2), k(p_4)\}.$$

PROOF: We will assume that γ has been given the arc length parametrization and positive orientation so that $\int_{S^1} k\,d\sigma = 2\pi$. We also remark that if $k \geq 0$ then the result follows from Theorem 3. Put $m = \min k$ and $M = \max k$. We will now assume $m < 0$ since we have established the result otherwise. Pick $q_0, q_1 \in S^1$ such that $m = k(q_0)$ and $M = k(q_1)$. Notice that $M > 0$. Now q_0, q_1 split S^1 into two closed intervals I_0 and I_1.

Assume now that the conclusion of the theorem fails. Then $k|I_0$ and $k|I_1$ are both monotone. Let C be the smallest circle enclosing γ and let R denote its radius. C is uniquely defined. Note that $E \equiv \gamma \bigcap C$ consists of two or more components otherwise C is not minimal. Clearly if E consists of a single point then that would contradict the minimality of C. If E consists of a single component with positive length there are four consequetive points A, B, C and $\gamma(q_0)$, where $A, C, \gamma(q_0) \notin E$ and $B \in E$, with $\max(k(B), k(q_0)) < \min(k(A), k(C))$ since $\gamma \setminus E$ lies inside the circle C and thereby contradicting the assumption that k is monotone on I_0 and I_1. Hence there are two distinct points $w_0, w_1 \in S^1$ such that $\gamma(w_0)$ and $\gamma(w_1)$ belong to different components of E.

Clearly $k(w_0) \geq \dfrac{1}{R}$ and $k(w_1) \geq \dfrac{1}{R}$. Since $k(q_0) < 0$ it follows that $q_0 \notin \{w_0, w_1\}$. Now w_0, w_1 split S^1 into two closed intervals J_0, J_1. We may choose the ordering so that $q_0 \in J_1$ but $q_0 \notin J_0$.

We now claim that

$$\min_{J_0} k \geq m^*, \tag{1}$$

where $m^* = \min\{k(w_0), k(w_1)\}$. Notice that $m^* \geq \dfrac{1}{R} > 0$. We shall now prove (1). If $q_1 \in J_1$ then k is monotone on J_0, which yields (1). If $q_1 \notin J_1$ then q_1 splits J_0 into two closed intervals $J_0^{'}, J_0^{''}$ with common end points at q_1. Since $q_0 \in J_1$ we have that k is monotone on both $J_0^{'}$ and $J_0^{''}$. Since $k(q_1) \geq m^*$ we see that (1) follows in this case also.

Since $\gamma(w_0)$ and $\gamma(w_1)$ belong to different components of $\gamma \bigcap C$ we must have that there is some $w \in J_0$ such that $\gamma(w) \notin C$. We may assume that $U = \{w \in J_0 : \gamma(w) \notin C\}$ is connected since otherwise we replace J_0 with the closure of a component of U.

Let Γ be the Jordan curve of class C^1 one gets by letting $\Gamma|J_0 = \gamma|J_0$ and letting $\Gamma|J_1$ be the corresponding arc on C. Then Γ is a convex Jordan curve with bounded curvature except at the points $\Gamma|J_0 \bigcap J_1$. Let L be the length of Γ and let K denote the curvature of Γ. If σ denotes the arc length measure of Γ then $\int K\,d\sigma = 2\pi$. Since, using (1), $K \geq \dfrac{1}{R}$ we therefore have $L \leq 2\pi R$. However if $L = 2\pi R$ then Γ must be the circle C which is impossible by the construction of J_0.

Put $A = \dfrac{2\pi}{L}$ so A is the curvature of a circle C_A of length L. Setting $E = \{w \in J_0 : k(w) \geq A\}$ we have from the monotonicity properties of k that E is an interval. Let F be the closure of the complement of E in S^1. Then F is an interval also. We let a, b denote the common end points of E and F. Let l be the length of $\Gamma|E$. Let a^*, b^* be points on C_A such that the length of one of the arcs on C_A connecting a^* and b^* has length l. Applying Lemma 2 to E shows that

$$|\Gamma(b) - \Gamma(a)| \leq |a^* - b^*|.$$

Applying Lemma 2 to F gives

$$|\Gamma(b) - \Gamma(a)| \geq |a^* - b^*|.$$

Hence

$$|\Gamma(b) - \Gamma(a)| = |a^* - b^*|$$

so one more application of Lemma 2 shows that $K = A$ *a.e.* in S^1. This is impossible which establishes the theorem. □

We conclude this paragraph by formulating an analogue of the four-vertex theorem that is similar to the discrete convex version appearing in the next section.

Let γ be a Jordan curve of class C^2 in $\mathbf{R}^2$ with positive orientation and let Ω denote the closure of the set enclosed by γ. For $P \in \gamma$ let $\Sigma(P)$ denote largest circle C with $P \in C$ and the interior of C inside Ω, $n(P)$ the unit inward normal of γ at P and $k(P)$ the curvature of γ at P.

If $k(P) \neq 0$ then

$$R(P) = \frac{1}{k(P)}$$

denotes the *radius of curvature*. For $k(P) \neq 0$ we set

$$\rho(P) = \{w \in \mathbf{R}^2 : |w - z(P)| = |R(P)|\},$$

where $z(P) = P + R(P)n(P)$ and for $k(P) = 0$ we set

$$\rho(P) = \{w \in \mathbf{R}^2 : \langle w - P, n(P)\rangle = 0\}.$$

The set $\rho(P)$ is called the *circle of curvature* for γ at P. If $k(P) > 0$ we set

$$\omega(P) = \{w \in \mathbf{R}^2 : |w - z(P)| \leq R(P)\},$$

if $k(P) < 0$

$$\omega(P) = \{w \in \mathbf{R}^2 : |w - z(P)| \geq |R(P)|\}$$

and if $k(P) = 0$

$$\omega(P) = \{w \in \mathbf{R}^2 : \langle w - P, n(P) \rangle \geq 0\}.$$

The following theorem is due to Guggenheimer [6].

Theorem 5 *Let γ be a Jordan curve of class C^2 in $\mathbf{R}^2$. Assume γ is not a circle. Then there are two points $P_0, P_1 \in \gamma$, $P_0 \neq P_1$, such that $k(P_0) > 0$, $k(P_1) > 0$, $\Sigma(P_0) = \rho(P_0)$ and $\Sigma(P_1) = \rho(P_1)$. Also, there are two points $Q_0, Q_1 \in \gamma$, $Q_0 \neq Q_1$, such that both $\omega(Q_0)$ and $\omega(Q_1)$ contain Ω. Furthermore the circles $\Sigma(P_0), \Sigma(P_1), \rho(Q_0)$ and $\rho(Q_1)$ are all pairwise distinct.*

3 Discrete Four Vertex Theorem — convex case

We consider the discrete convex version of the four-vertex theorem. We have the following analogue of Theorem 5 for strictly convex polygons. Recall that a convex polygon is *strictly convex* if no three vertices lie on a line.

Theorem 6 (Convex Discrete Four-Vertex Theorem (CDFV)) *Let γ be a strictly convex polygon in $\mathbf{R}^2$. Assume that $V(\gamma)$ does not lie on a circle. Then there are $P, Q \in V(\gamma)$ such that*

$$\mathrm{Int}(\omega(P)) \bigcap V(\gamma) = \mathrm{Int}(\omega(Q)) \bigcap V(\gamma) = \emptyset.$$

Also there are $P^*, Q^* \in V(\gamma)$ such that

$$V(\gamma) \subset \omega(P^*)$$

and

$$V(\gamma) \subset \omega(Q^*).$$

Furthermore the circles $\rho(P), \rho(P^*), \rho(Q)$ and $\rho(Q^*)$ are all pairwise distinct.

From now on we let in this section, γ be a strictly convex polygon with N vertices. For $P \in V(\gamma)$ let $\mathcal{F}(P)$ denote the family of all closed disks U such that $P \in \partial U$, $V(\gamma) \subset U$ and ∂U contains at least three points in $V(\gamma)$. Similarly, let $\mathcal{G}(P)$ be the family of all open disks U such that $P \in \partial U$, $V(\gamma) \bigcap U = \emptyset$ and ∂U contains at least three points in $V(\gamma)$.

Lemma 3 *For all $P \in V(\gamma)$ we have that both $\mathcal{F}(P)$ and $\mathcal{G}(P)$ are non-empty.*

PROOF: If P and Q are two distinct points in $\mathbf{R}^2$ we let $A(P,Q)$ denote the class of closed disks U such that $\{P,Q\} \subset \partial U$. Notice that if L denotes the line through P and Q then

$$\mathbf{R}^2 \setminus (L \setminus [P,Q]) = \bigcup_{U \in A(P,Q)} U. \tag{2}$$

Assume $P \in V(\gamma)$. Since γ is strictly convex there is a closed disk W such that $V(\gamma) \subset W$ and $P \in \partial W$. Pick $Q_1 \in \partial W$ with $Q_1 \neq P$. From (2) it follows that there is a $W_1 \in A(P,Q_1)$ such that $V(\gamma) \subset W_1$ and $(\partial W_1 \setminus \{P\}) \bigcap V(\gamma) \neq \emptyset$. Let $Q_2 \neq P$ belong to $\partial W_1 \bigcap V(\gamma)$. Again using (2) we see that there is a $W_2 \in A(P,Q_2)$ such that $V(\gamma) \subset W_2$ and $(\partial W_2 \setminus \{P,Q_2\}) \bigcap V(\gamma) \neq \emptyset$. Hence $\mathcal{F}(P) \neq \emptyset$.

A similar argument shows that $\mathcal{G}(P) \neq \emptyset$. □

Remark: The proof that $\mathcal{G}(P) \neq \emptyset$ does not require that γ is strictly convex.

Lemma 4 *Let $P \in V(\gamma)$ and let $U \in \mathcal{F}(P)$. Assume $I \neq \emptyset$ is a maximal interval contained in $V(\gamma) \setminus \partial U$. If $Q \in I$ and $W \in \mathcal{F}(Q)$ then*

$$\partial W \bigcap V(\gamma) \subset \hat{I}.$$

PROOF: There is nothing to prove if $\hat{I} = V(\gamma)$. Assume therefore that $\hat{I}$ is a proper subset of $V(\gamma)$. Let $P_1, P_2 \in V(\gamma)$ be such that $P_1 \neq P_2$ and $\hat{I} = I \bigcup \{P_1, P_2\}$. Let L denote the line through P_1 and P_2. Then L divides $\mathbf{R}^2$ into two open half planes H and H^*. Assume that $Q \in H$. Then $I = H \bigcap V(\gamma)$ by convexity. Notice that $\{P_1, P_2\} \subset W$. Hence $\partial W \bigcap U \subset \overline{H}$.

In particular,

$$\partial W \bigcap V(\gamma) \subset (\partial W \bigcap U) \bigcap V(\gamma) \subset \hat{I}.$$

□

Lemma 5 *Assume $V(\gamma)$ is not contained in a circle. Then there are points $P^*, Q^* \in V(\gamma)$ such that*

$$V(\gamma) \subset \omega(P^*),$$
$$V(\gamma) \subset \omega(Q^*)$$

and

$$\rho(P^*) \neq \rho(Q^*).$$

PROOF: Assume that there exists a $P \in V(\gamma)$ and $U \in \mathcal{F}(P)$ such that $V(\gamma) \bigcap \partial U$ is not an interval. Let $I_1, \ldots, I_n$, $n \geq 2$, be the maximal non-empty intervals in $V(\gamma) \setminus \partial U$. By repeated use of Lemma 4 follows that there are points $Q_k \in I_k$, $1 \leq k \leq n$, such that if $W_k \in \mathcal{F}(Q_k)$ then $V(\gamma) \bigcap \partial W_k \subset \hat{I}_k$ and $V(\gamma) \bigcap \partial W_k$ is connected. Also, $Q_j \notin \partial W_k$ if $j \neq k$. Since $\partial W_k \bigcap V(\gamma)$ is connected there is a $Q_k^* \in \partial W_k \bigcap V(\gamma)$ such that $\omega(Q_k^*) = W_k$. If the assumption does not hold then the result is trivial. □

Lemma 6 *Let $P \in V(\gamma)$ and let $U \in \mathcal{G}(P)$. Assume $I \neq \emptyset$ is a maximal interval contained in $V(\gamma) \setminus \partial U$. If $Q \in I$ and $W \in \mathcal{G}(Q)$ then*

$$\partial W \bigcap V(\gamma) \subset \hat{I}.$$

PROOF: There is nothing to prove if $\hat{I} = V(\gamma)$. Assume $\hat{I} \neq V(\gamma)$. Suppose there is a $Q^* \in (\partial W \bigcap V(\gamma)) \setminus \hat{I}$. Let P_1 and P_2 be the end points of $\hat{I}$. Then $P_1, P_2 \in \partial U \bigcap V(\gamma)$ and $W \bigcap \{P_1, P_2\} = \emptyset$. Since $Q \in \partial W$ but $Q \notin \partial U$ neither U nor W can be contained in the other. Let L be the line through P_1 and P_2. Then L separates $\mathbf{R}^2$ into two half planes H and H^*, say $Q \in H$. Then $Q^* \in H^*$ since $\hat{I} = V(\gamma) \bigcap \overline{H}$. Since $[Q, Q^*] \bigcap L \subset (P_1, P_2)$ by the strict convexity of γ we have that $\overline{W} \bigcap L \subset [P_1, P_2]$.

Hence $\partial W \bigcap \partial U \subset \overline{H}$ so $\overline{W} \bigcap H^* \subset U$. Since U is open and $U \bigcap V(\gamma) = \emptyset$ the lemma follows by contradiction. □

Lemma 7 *Assume $V(\gamma)$ is not contained on a circle. Then there are P^*, Q^* in $V(\gamma)$ such that*

$$\text{Int}(\omega(P^*)) \bigcap V(\gamma) = \text{Int}(\omega(Q^*)) \bigcap V(\gamma) = \emptyset$$

and

$$\rho(P^*) \neq \rho(Q^*).$$

PROOF: A straight-forward modification of the proof of Lemma 5 gives the result. □

PROOF: (of Theorem 6) The theorem is a direct consequence of Lemma 5 and Lemma 7. □

Corollary 1 *Let R be a strictly convex quadrilateral with vertex set S. Assume that S is not contained in a circle. Then there are two diametrically opposite vertices P, P^* such that*

$$S \subset \omega(P)$$

and

$$S \subset \omega(P^*).$$

Furthermore, the discrete circles of curvature at the other pair of diametrically opposite vertices contain no point from S in there interiors.

PROOF: From the Convex Discrete Four-Vertex Theorem (CDFV) follows that there is a $P \in S$ such that $S \subset \omega(P)$. Let P^* be the vertex diametrically opposite to P. From Lemma 4 and the fact that $S \backslash \rho(P) = \{P^*\}$ follows that $S \subset \omega(P^*)$. The remaining part of the corollary follows from the Convex Discrete Four-Vertex Theorem. □

4 Discrete Four Vertex Theorem

We prove our main result, Theorem 1, in this section. We will need some notation and some preliminary results.

Let e be an edge of the simple closed polygon γ and let l be the line that contains e. Let M be the midpoint of e. Let h_e be that closed half-plane determined by l such that for all sufficiently small neighbourhoods W of M, one has that $h_e \bigcap W = \Omega(\gamma) \bigcap W$. Let P be an endpoint of e. If $k(P) = 0$, we set $\delta(P, e) = h_e$. If $k(P) \neq 0$, we let $\omega(P)$ denote the closed disk determined by the circle of curvature at P. We set $\delta(P, e) = \omega(P) \bigcap h_e$ if $k(P) > 0$. If $k(P) < 0$, we let $\omega^c(P)$ denote the closure of the complement of $\omega(P)$ and set $\delta(P, e) = \omega^c(P) \bigcup (\omega(P) \bigcap h_e)$.

Lemma 8 *Let γ be a simple closed polygon. Assume that γ is locally regular. Let e be an edge of γ and let P, Q be the endpoints of e. Then the following holds:*

1. *If $k(P) \geq 0$, then $\delta(P, e) \subset h_e$.*

2. *If $k(P) \leq 0$, then $h_e \subset \delta(P, e)$.*

3. *If $k(P) \leq k(Q)$, then $\delta(Q, e) \subset \delta(P, e)$.*

PROOF: The first two properties are immediate consequences of the definition of $\delta(P, e)$. Hence Property 3 holds if $k(P)k(Q) \leq 0$. Assume now that $k(P)k(Q) > 0$. Let l be the line that contains e. Since γ is locally regular and $k(P)k(Q) > 0$, it follows that the circles of curvature at P and Q have their centres on the same side of l. If $R(P), R(Q)$ denote the radii of the circles of curvature at P, Q, then $R(P) \geq R(Q)$ if $k(P) > 0$ and $R(P) \leq R(Q)$ if $k(P) < 0$, which yields the lemma. □

Lemma 9 *Let γ be a strictly convex polygon whose vertices do not fall on a circle. Assume that γ is locally regular. Then the discrete curvature of γ has at least two local maxima and at least two local minima.*

PROOF: We first remark that the discrete curvature k is not constant. It is enough to show that k has at least four local extreme points. Let $Q_0, Q_1 \in V(\gamma)$ be defined by $k(Q_0) = \min\{k(P) : P \in V(\gamma)\}$ and $k(Q_1) = \max\{k(P) : P \in V(\gamma)\}$. Assume that the conclusion fails. Then k must be monotone on the two maximal intervals in $V(\gamma) \setminus \{Q_0, Q_1\}$ determined by Q_0 and Q_1.

Let $P \in V(\gamma)$ be different from Q_1. We can find vertices $P_1, P_2, \ldots, P_N$ such that $P_1 = Q_1, P_N = P$, $k(P_{i+1}) \leq k(P_i)$ and P_iP_{i+1} is an edge for each $i \in \{1, \ldots, N-1\}$. Let e_i be the edge determined by P_i and P_{i+1} and set $h_i = h_{e_i}$, where the half-plane h_{e_i} was defined above. From the above lemma follows that $\omega(P_i) \bigcap h_i \subset \omega(P_{i+1}) \bigcap h_i$. Since $\Omega \subset h_i$ for all $i \in \{1, \ldots, N-1\}$, we have that for all $P \in V(\gamma)$

$$\omega(Q_1) \bigcap \Omega \subset \omega(P) \bigcap \Omega.$$

Let $E = \{P \in V(\gamma) : \mathrm{Int}(\omega(P)) \bigcap V(\gamma) = \emptyset\}$. Hence $\omega(P) = \omega(Q_1)$ for all $P \in E$.

This contradicts Theorem 6, which in particular says that $\{\omega(P) : P \in E\}$ consists of at least two distinct disks. This contradiction shows that k must have at least four local extreme points, which shows the lemma. □

Lemma 10 *Let ABC be a triangle and denote by $\mathcal{A}$, the closed convex sector with vertex A determined by the triangle. Let ρ be the circumcircle of the triangle. Assume that the centre of ρ belongs to $\mathcal{A}$ and assume that Δ is a closed disk containing the triangle such that $A \in \partial\Delta$. Let r, R denote the radii of Δ and ρ. Then $R \leq r$ with equality if and only if $\rho = \partial\Delta$.*

PROOF: By possibly shrinking Δ we may assume that $\partial\Delta$ passes through another vertex of the triangle, say $B \in \partial\Delta$. Let H be the closed half-plane determined by the line through A, B that contains C. Let δ be the closed disk with centre at the midpoint of $[A, B]$ that goes through A. Then $\delta \bigcap H \subset \omega \bigcap H$, where ω is the closed disk determined by ρ since the centre of ρ belongs to $\mathcal{A}$.

Let α denote the angle at A of the triangle. If $0 < \alpha \leq \pi/2$, then the centre of ρ lies inside the triangle. Therefore ρ is the smallest enclosing circle of the triangle so $R \leq r$ in this case with equality if and only if $\rho = \partial\Delta$.

We are now left with the case $\alpha > \pi/2$. We first notice that $C \notin \delta$ in this case. We claim that if z denotes the centre of Δ, then $z \in H$. For if $z \notin H$, then $\Delta \bigcap H \subset \delta \bigcap H$, which is impossible since $C \in \Delta$. Hence $z \in H$ as claimed. We are now left with the situation $A, B \in \partial\Delta \bigcap \rho$ and both Δ and ρ have their enters in H. Assuming $r < R$ implies that $\rho \bigcap \Delta \bigcap \mathrm{Int}(H) = \emptyset$, which is impossible since $C \in \Delta$. Hence $R \leq r$ with equality if and only if $\rho = \partial\Delta$. The lemma is therefore proved. $\square$

PROOF: (of Theorem 1) We begin by picking $Q_0, Q_1 \in V(\gamma)$ such that $k(Q_0) = \min\{k(P) : P \in V(\gamma)\}$ and $k(Q_1) = \max\{k(P) : P \in V(\gamma)\}$. Assume now that the conclusion fails. Then by Lemma 9 we must have

$$k(Q_0) \leq 0.$$

Furthermore, k must be monotone on the two subintervals of $V(\gamma)$ that have Q_0 and Q_1 as endpoints.

Let Δ be the smallest closed disk that contains γ and let r denote its radius. Put $E = V(\gamma) \bigcap \partial\Delta$. Then $V(\gamma) \neq E$ by the assumptions on γ. We remark that for all $P \in E$ the interior angle of γ at P is strictly less than π. From Lemma 10 it follows that $k(P) \geq \frac{1}{r}$ for all $P \in E$. Moreover, if $P \in E$ and if at least one immediate neighbour of P does not belong to E, then $k(P) > \frac{1}{r}$.

We now claim that E is not connected. To show the claim, we assume that E is an interval. Let N be the number of points in E. We must have $N \geq 2$. If $N = 2$ let $E = \{A, B\}$. Then A, B must lie on a diameter of Δ so that $|A - B| = 2r$. Since $k(A) > \frac{1}{r}$ by the above reasoning, it follows that A, B lie on a circle of radius strictly less than r, which is impossible. If $N > 2$ let $E_0 = \{P \in E : P \text{ is not an endpoint of } E\}$, then $E_0 \neq \emptyset$ and $k(P) = \frac{1}{r}$ for all $P \in E_0$. Since $k(P) > \frac{1}{r}$ for $P \in E \setminus E_0$ and $Q_0 \notin E$ but $k(Q_0) \leq 0$, it follows that k has at least two local minima, which again is incompatible with our assumption. We have therefore established that E is not connected.

Let $\mathcal{F}$ be the collection of maximal intervals that are contained in $V(\gamma) \setminus E$. Since E is not connected, $\mathcal{F}$ must contain at least two intervals. Set $F = \{P \in V(\gamma) : k(P) \leq 0\}$. It follows from the monotonicity properties of k that F is connected. Since $E \bigcap F = \emptyset$, there must exist a $J \in \mathcal{F}$ such that $F \bigcap J = \emptyset$, i.e., $k(P) > 0$ for all $P \in J$.

Let $\hat{J}$ be the union of J and its two immediate neighbours. Setting $J^* = \hat{J} \setminus J$, we have $J^* \subset E$ and $k(P) > \frac{1}{r}$ for all $P \in J^*$. Moreover, J^* consists of two distinct vertices of γ.

Denote by Γ the polygonal sub-arc of γ that has $\hat{J}$ as its vertex set. Then Γ separates Δ into two closed domains; let U be that domain that contains $\Omega(\gamma)$. Let Γ^* be that sub-arc of ∂U that has J^* as its endpoints and is contained in $\partial\Delta$. Notice that $\partial U = \Gamma \bigcup \Gamma^*$. Since $k(P) > 0$ for all $P \in J$, it follows that U is convex.

Denote by l the length of Γ^*. We claim that $l \geq \pi r$. Noticing that $J \subset \mathrm{Int}(\Delta)$, we see that if $l < \pi r$, then $\hat{J} \bigcup \Gamma^*$ is contained in a disk of radius $r_1 < r$. This is impossible since $\Omega(\gamma) \subset U$ and Δ is the smallest disk containing γ.

We now make a polygonal approximation of ∂U by selecting consecutive points $\{A_i\}_1^m$ on ∂U such that $A_i \in \partial U \setminus \Gamma$. In particular, $A_i \notin J^*$, $1 \leq i \leq m$. Set $V^* = \hat{J} \bigcup \{A_i : 1 \leq i \leq m\}$ and let γ^* be the convex polygon with vertex set V^*. Clearly γ^* is strictly convex. Let k^* denote the discrete curvature relative to γ^*.

From the monotonicity properties of k follows that

$$\min\{k(P) : P \in J\} \geq \min\{k(P) : P \in J^*\} > \frac{1}{r}.$$

Since $k(P) = k^*(P)$ for $P \in J$, we have that

$$k^*(P) > \frac{1}{r} \text{ for } P \in J.$$

For $P \in V^*$ we let $\alpha^*(P)$ be the interior angle of γ^* at P. Also we let $\omega^*(P)$ denote the closed disk whose boundary equals the circle of curvature of γ^* at P.

Since $l \geq \pi r$, we see that if A_1 and A_m have been selected sufficiently close to J^*, then $\alpha^*(P) > \pi/2$ for all $P \in J^*$. From Lemma 10 follows that $k^*(P) > \frac{1}{r}$ for $P \in J^*$. Summarising, we have therefore obtained that

$$k^*(P) > \frac{1}{r} \text{ for all } P \in \hat{J}.$$

Let $S = \{P \in V^* : V^* \subset \omega^*(P)\}$. Clearly, $\Omega(\gamma) \subset \omega^*(P)$ for all $P \in S$ so we must have

$$k^*(P) \leq \frac{1}{r} \text{ for all } P \in S.$$

Since $\omega^*(P) = \Delta$ for all $P \in V^* \setminus \hat{J}$, we have that $S = V^* \setminus \hat{J}$ and $\{\omega^*(P) : P \in S\} = \{\Delta\}$. This contradicts however Theorem 6, which says that $\{\omega^*(P) : P \in S\}$ consists of at least two distinct disks. This contradiction establishes the theorem. $\square$

References

[1] Bilinski, S., Die primitivste Form des Vierscheitelsatzes, *Glasnik Mat.-Fiz. Astronom. Drustvo Mat. Fiz. Hrvatske* , Ser. II, **18**, (1963), 85 – 93.

[2] Blaschke, H., Über Paare umfangsgleichen Eilinien, *Jahresber. Deutsch. Math.-Verein.* **48**, (1938), 69 – 74

[3] Bol, G., Über ein Satz von Cauchy, *Jahresber. Deutsch. Math.-Verein.* **48**, (1938), 74 – 76

[4] Dahlberg, B., The converse of the four vertex theorem, manuscript, 1997

[5] Fog, D., Über den Vierscheitelsatz und seine Verallgemeinerungen, *Sitzungsbereiche der Berlin Akademie der Wissenschaft*, (1933), 251–254

[6] Guggenheimer, H., On plane Minkowski geometry, *Geom. Dedicata* **12**, (1982), 371 – 381

[7] Jackson, S. B., Vertices for plane curves, *Bulletin of the American Mathematical Society* **50**, (1944), 564–578

[8] Kneser, A., Bemerkungen über die Anzahl der Extreme der Krümmung auf geschlossenen Kurven und über verwandte Fragen in einer nichteuklidischen Geometrie, *Festschrift zum 70. Geburtstag von H. Weber*, Leipzig, (1912), 170 – 180

[9] Mukhopadhyaya, S., New methods in the geometry of a plane arc, *Bull. Calcutta Math. Soc.* **1**, (1909), 31 – 37

[10] Osserman, R., The four-or-more vertex theorem, *Amer. Math. Monthly* **92**, (1985), 332 – 337

[11] Råde, J., Adolfsson, V., Andersson, R., Johansson, B. and Kumlin, P., A Six Vertex Theorem for Polygons, manuscript, 2000.

[12] Schatteman, A., A four-vertex theorem for polygons and its generalization to polytopes, *Geom. Dedicata* **34** (1990), 229 – 242

[13] Schoenberg, I. J. and Zaremba, S. K., On Cauchy's lemma concerning convex polygons, *Canad. J. Math.* **19** (1967), 1062 – 1071

[14] Vietoris, L., Ein einfacher Beweis des Vierscheitelsatzes der ebenen Kurven, *Archiv der Mathematik* **3**, (1952), 97–102

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