# Geometric Bounds on Certain Sublinear Functionals of Geometric Brownian Motion

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**Abstract**

Suppose $\{X_s,\ 0 \leq s \leq T\}$ is an $m$-dimensional geometric Brownian motion with drift, $\mu$ is a bounded positive Borel measure on $[0,T]$ and $\phi : \mathbb{R}^m \mapsto [0,\infty)$ is a (weighted) $l^q(\mathbb{R}^m)$-norm, $1 \leq q \leq \infty$. The purpose of this paper is to study the distribution and the moments of the random variable $Y$ given by the $L^p(\mu)$-norm, $1 \leq p \leq \infty$, of the function $s \mapsto \phi(X_s)$, $0 \leq s \leq T$. By using various geometric inequalities in Wiener space this paper gives upper and lower bounds for the distribution function of $Y$ and proves that the distribution function is log-concave and absolutely continuous on every open subset of the distributions support. Moreover, the paper derives tail probabilities, presents sharp moment inequalities and shows that $Y$ is indetermined by its moments. The paper will also discuss the so-called moment-matching method for the pricing of Asian-styled basket options.

**Key words:** geometric Brownian motion, sublinear functionals, lognormal distribution, geometric inequalities, log-concave distributions, tail probabilities, moment inequalities, moment problem, option pricing, Asian basket options.

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## 1 Introduction

Assume $(\Omega, \mathcal{F}, P)$ is a given probability space carrying an $m$-dimensional standard Brownian motion $\{W_s,\ 0 \leq s \leq T\}$. Consider the stochastic dif-

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ferential equation

$$\begin{cases} dX_s = X_s(\eta ds + CdW_s), & 0 \le s \le T, \\ X_0 = x, & x \in (0,\infty)^m, \end{cases}$$

where $C$ is a non-singular $m$ by $m$ matrix and $\eta \in \mathbb{R}^m$. Vectors in $\mathbb{R}^m$ are regarded as $m$ by 1 matrices. Moreover, multiplication of two vectors in $\mathbb{R}^m$ should be understood as coordinate-wise multiplication. Thus, the process $\{X_s,\, 0 \le s \le T\}$ is a geometric Brownian motion with drift.

Assume that $\rho \in [0,\infty)^{m-1} \times (0,\infty)$ and $1 \le q \le \infty$. Let the function $\phi_{\rho,q}$ be defined for each $x \in (0,\infty)^m$ by

$$\phi_{\rho,q}(x) = \begin{cases} \left( \sum_{i=1}^m \rho_i x_i^q \right)^{\frac{1}{q}} & \text{if } 1 \le q < \infty, \\ \max_{i=1,\dots,m} \rho_i x_i, & \text{if } q = \infty, \end{cases}$$

where $\rho_i$ and $x_i$ denote the $i$:th coordinate of $\rho$ and $x$, respectively. Moreover, suppose $\mu$ is a bounded and positive Borel measure on the interval $[0,T]$. Introduce the random variable

$$\Psi_{\mu,\rho}^{p,q}(X) = \|\phi_{\rho,q}(X_{(\cdot)})\|_{L^p(\mu)}, \quad 1 \le p \le \infty,$$

that is,

$$\Psi_{\mu,\rho}^{p,q}(X) = \begin{cases} \left( \int_0^T \phi_{\rho,q}(X_s)^p \mu(ds) \right)^{\frac{1}{p}}, & \text{if } 1 \le p < \infty, \\ \inf\left\{u\,;\, \mu\big(\{s \in [0,T]; \phi_{\rho,q}(X_s) > u\}\big) = 0\right\}, & \text{if } p = \infty. \end{cases}$$

The purpose of this paper is to investigate the law and the generalized moments of the random variable $\Psi_{\mu,\rho}^{p,q}(X)$. By a generalized moment we mean the quantity

$$M(r) = E\left[\left(\Psi_{\mu,\rho}^{p,q}(X)\right)^r\right],$$

where $r$ is a real number; if $r$ is a positive integer, we speak of a moment instead of a generalized moment.

There are various sources of interest of the random variable $\Psi_{\mu,\rho}^{p,q}(X)$. In particular in mathematical finance it is relevant in the pricing of Asian basket options ($p = q = 1$), lookback basket options ($p = \infty$, $q = 1$) and options on the maximum of several assets ($\mu$ equal to the Dirac measure at $T$ and $q = \infty$). If $\ell$ is the Lebesgue measure on $[0,T]$ and the dimension $m$ equals 1, the random variable $\Psi_{\ell,\rho}^{1,1}(X)$ is also of interest in the study of disorded systems as well as in the study of hyperbolic Brownian motion, see Yor [22]. The law of sums of lognormal random variables, which would correspond to $\mu$ equal to the Dirac measure at $T$ and $q = 1$, is of interest in geology, see Barouch et al. [2], and in radar theory, see Janos [15], to name a few areas.

Previous studies of the random variable $\Psi^{p,q}_{\mu,\rho}(X)$ have been concentrated on the one-dimensional case with $\mu$ equal to the Lebesgue measure and $p = 1$, that is, on the random variable

$$\int_0^T X_{1,s} ds, \tag{1}$$

where $\{X_{1,s},\, 0 \leq s \leq T\}$ is a one-dimensional geometric Brownian motion. Yor, and co-authors, have written a large number of articles focusing on this random variable, articles which have been collected in the monograph Yor [22]. Here Yor, among other things, describes the density of the random variable in equation (1) in terms of series of one-dimensional integrals, see Yor [22] p.43. Other results in the same direction can be found in Alili [1], Comtet et al. [8], and Dufresne [10],[11]. Moreover, Bhattacharya et al. [3] derives a partial differential equation for the density function. Explicit expressions for some of the generalized moments of the random variable in equation (1) are given in Yor [22] p. 31, Dufresne [10],[11], and Donati-Martin et al. [9]. Recently, Nikeghbali [18] has proven that the law of the random variable in equation (1) is indetermined by its moment, a question that was unsolved for a long time.

It should be mentioned that there is a large number of articles dealing with the problem of computing the expectation

$$E\big[\max(\int_0^T X_{1,s}\mu(ds) - K, 0)\big], \quad K > 0.$$

The problem appears in the pricing of Asian options. This problem is, as we can see, closely related to the problem of finding the the law of $\Psi^{1,1}_{\mu,\rho}(X)$ in the one-dimensional case. For a further discussion about this problem the reader may consult Linetsky [17], Rogers et al. [20] and the references therein.

For some results concerning the distribution and the moments of sums of lognormal random variables, see Barouch et al. [2] and Janos [15].

This paper will derive upper and lower bounds for the distribution function of $\Psi^{p,q}_{\mu,\rho}(X)$, prove that the distribution function is log-concave, and discuss conditions on $p$ and $\mu$ in order for the distribution function to be absolutely continuous. The paper will also present the asymptotic behaviour of the distribution function and give sharp inequalities for the generalized moments. Moreover, it will proven that the distribution of $\Psi^{p,q}_{\mu,\rho}(X)$ is indetermined by its moment. As we will see, this result has some consequences for the so-called moment-matching method for the pricing of Asian-styled basket options. The main tool in this paper is various geometric inequalities in Wiener space.

## 2 Upper and Lower Bounds for the Distribution Function

To begin with we will introduce some definitions that will be used throughout this text. The class $\mathcal{M}$ denotes all bounded positive measures $\mu$ on the Borel $\sigma$-algebra of $[0,T]$, where $0 < T < \infty$. The class $\mathcal{M}(0,T)$ includes all $\mu \in \mathcal{M}$ such that $\sup\{s \geq 0\,;\, \mu((s,T]) > 0\} = T$ and if $0 < t \leq T$ then $\mathcal{M}(t,T)$ consists of all $\mu \in \mathcal{M}(0,T)$ such that $\mu([0,t)) = 0$. The norm in $L^p([0,T],\mu)$ will be denoted $\|\cdot\|_{L^p(\mu)}$.

If $x \in \mathbb{R}^m$ then $x_i$ will denote the $i$:th coordinate of $x$. Let as previous the function $\phi_{\rho,q}$ be defined for each $x \in (0,\infty)^m$ by

$$\phi_{\rho,q}(x) = \begin{cases} \left(\sum_{i=1}^m \rho_i x_i^q\right)^{\frac{1}{q}} & \text{if } 1 \leq q < \infty, \\ \max_{i=1,\dots,m} \rho_i x_i, & \text{if } q = \infty, \end{cases}$$

where $\rho \in [0,\infty)^{m-1} \times (0,\infty)$. Henceforth we put $\Lambda = [0,\infty)^{m-1} \times (0,\infty)$ .

Let $X$ be a stochastic process defined by the stochastic differential equation

$$\begin{cases} dX_s = X_s(\eta ds + C dW_s), & 0 \leq s \leq T, \\ X_0 = x, & x \in (0,\infty)^m. \end{cases}$$

Here $\eta \in \mathbb{R}^m$ and $C$ is a non-singular $m$ by $m$ matrix with rows $c_1, \dots, c_m$. Vectors in $\mathbb{R}^m$ are regarded as $m$ by 1 matrices. Moreover, multiplication of two vectors in $\mathbb{R}^m$ should be understood as coordinate-wise multiplication. Let also $\sigma_m = |c_m|_2 = \max_{i=1,\dots,m} |c_i|_2$, where $|\cdot|_2$ is the Euclidean norm in $\mathbb{R}^m$.

In what follows, let the functional $\Psi^{p,q}_{\mu,\rho}$ be defined by

$$\Psi^{p,q}_{\mu,\rho}(X) = \|\phi_{\rho,q}(X_{(\cdot)})\|_{L^p(\mu)} \tag{2}$$

where $\mu \in \mathcal{M}(0,T)$, $\rho \in \Lambda$ and $1 \leq p, q \leq \infty$, that is

$$\Psi^{p,q}_{\mu,\rho}(X) = \begin{cases} \left(\int_0^T \phi_{\rho,q}(X_s)^p \mu(ds)\right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty, \\ \inf\big\{u\,;\, \mu\big(\{s \in [0,T] : \phi_{\rho,q}(X_s) > u\}\big) = 0\big\}, & \text{if } p = \infty. \end{cases}$$

For simplicity, the functional $\Psi^{p,q}_{\mu,\rho}$ will mostly be abbreviated $\Psi_{\mu,\rho}$. The law of $\Psi^{p,q}_{\mu,\rho}(X)$ will be denoted $F^{p,q}_{\mu,\rho}$, that is

$$F^{p,q}_{\mu,\rho}(s) = P\big(\Psi^{p,q}_{\mu,\rho}(X) \leq s\big), \quad s \geq 0.$$

Similarly, $F^{p,q}_{\mu,\rho}$ will mostly be written $F_{\mu,\rho}$.

As we will see, the distribution $F_{\mu,\rho}$ has some similarities with the log-normal distribution $G_\varsigma$, defined by

$$G_\varsigma(s) = \Phi\left(\frac{\ln s}{\varsigma}\right), \quad s \geq 0,\ \varsigma > 0,$$

where $\Phi$ is the standard normal distribution function. Thus, $G_\varsigma$ is the distribution function of the random variable $e^\xi$, where $\xi$ is a normal distributed random variable with mean 0 and variance $\varsigma^2$.

The next lemma will play an important role in Section 4 and 5.

**Lemma 1.** *Suppose* $0 \leq t \leq T$, $\mu \in \mathcal{M}(t,T)$, $\rho \in \Lambda$ *and* $\varsigma = \sigma_m\sqrt{T}$. *Assume* $\theta \geq 1$ *and chose* $a, b > 0$ *such that*

$$F_{\mu,\rho}(a) = G_\varsigma(b),$$

*then*

$$F_{\mu,\rho}(\theta a) \geq G_\varsigma(\theta b). \tag{3}$$

*If, in addition,* $t > 0$ *then*

$$F_{\mu,\rho}(\theta a) \leq G_\varsigma(\theta^\gamma b), \tag{4}$$

*where*

$$\gamma = \frac{\sigma_m\sqrt{T}}{\alpha\sqrt{t}} \quad \textit{with} \quad \alpha = \max_{|x|_2=1} \min_{\{i\,;\,\rho_i>0\}} \langle c_i, x\rangle > 0.$$

*Moreover, if* $0 < \theta < 1$ *then the inequalities in equation* (3) *and* (4) *are reversed.*

Equation (3) follows at once from Corollary 2 in Hörfelt [14]. However, in order to make the paper more self-contained and since the paper [14] has not been published equation (3) will be proved in this paper as well.

The proof of Lemma 1 is based on two geometric inequalities in the Wiener space. To present these inequalities we will introduce some further notation. From now on the sample space $\Omega = C_0([0,T];\mathbb{R}^m)$ consists of all functions $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$ such that, for each $i = 1, \ldots, m$, the function $\omega_i : [0,T] \mapsto \mathbb{R}$ is continuous and $\omega_i(0) = 0$. The space $\Omega$ is equipped with the norm $\|\cdot\|_{C_0}$, defined by

$$\|\omega\|_{C_0} = \max_{i=1,\ldots,m} \max_{0 \leq s \leq T} |\omega_i(s)|, \quad \omega \in \Omega.$$

The measure $P$ will henceforth denote Wiener measure on $\Omega$. Setting $W_s(\omega) = \omega(s)$, $0 \leq s \leq T$, $\omega \in \Omega$, the process $\{W_s, 0 \leq s \leq T\}$ is a standard $m$-dimensional Brownian motion with respect to $P$.

Moreover, $\mathcal{H}$ will denote the Cameron-Martin space. Here $\mathcal{H}$ consists of all functions $h = (h_1, h_2, \dots, h_m)$ such that, for each $i = 1, \dots, m$, the function $h_i : [0,T] \mapsto \mathbb{R}$ is absolutely continuous with a square integrable derivative and $h_i(0) = 0$. The space $\mathcal{H}$ is equipped with the norm $\|\cdot\|_{\mathcal{H}}$, defined by

$$\|h\|_{\mathcal{H}} = \Big(\sum_{i=1}^{m} \int_0^T \big(h_i'(s)\big)^2 ds\Big)^{\frac{1}{2}}, \quad h \in \mathcal{H}.$$

Now, let $O$ be the set of all $h \in \mathcal{H}$ such that $\|h\|_{\mathcal{H}} \le 1$. Suppose $A$ is a Borel set in $\Omega$ and $h \in O$. If

$$P(A) = \Phi(a),$$

then

$$P(A + \lambda h) \le \Phi(a + \lambda) \le P(A + \lambda O) \tag{5}$$

for each $\lambda \ge 0$. The right inequality in equation (5), i.e. $\Phi(a+\lambda) \le P(A + \lambda O)$, is a special case of the celebrated isoperimetric inequality for Gaussian measures, which was discovered independently by Borell [5] and Sudakov and Tsirelson [21]. The left inequality in equation (5), i.e. $P(A+\lambda h) \le \Phi(a+\lambda)$, is a so called shift inequality and it can be found in Kuelbs and Li [16].

Before we go on and prove Lemma 1 let us note that the process $X$ may for each $\omega \in \Omega$ and $s \in [0,T]$ be written as

$$X_s(\omega) = xe^{\tilde{\eta}s + C\omega(s)}$$

where the coordinates of $\tilde{\eta}$ are given by $\tilde{\eta}_i = \eta_i - \frac{1}{2}|c_i|_2$ and $e^x$ with $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ should be interpreted as $(e^{x_1}, \dots, e^{x_m})$.

*Proof of Lemma 1.* Firstly, suppose $\omega \in \Omega$, $h \in \mathcal{H}$, $\lambda > 0$ and note that

$$X_s(\omega) = e^{-\lambda Ch(s)} X_s(\omega + \lambda h)$$

for each $0 \le s \le T$. Thus, if $\mu \in \mathcal{M}(t,T)$ and $I = \{i;\, \rho_i > 0\}$ then

$$\Psi_{\mu,\rho}\big(X(\omega)\big) \le \big(\max_{t \le s \le T} \max_{i \in I} e^{-\lambda\langle c_i, h(s)\rangle}\big)\, \Psi_{\mu,\rho}\big(X(\omega + \lambda h)\big). \tag{6}$$

Now, to prove equation (3) assume that $h = (h_1, \dots, h_m)$. Since

$$h(s) = \int_0^s h'(u)du, \quad 0 \le s \le T,$$

the Cauchy-Schwarz inequality gives for $0 \leq s \leq T$ and $i = 1, \dots, m$

$$
\begin{aligned}
|\langle c_i, h(s)\rangle| &\leq \sigma_m \Big( \sum_{j=1}^{m} \Big( \int_0^s |h_j'(u)| du \Big)^2 \Big)^{\frac{1}{2}} \\
&\leq \sigma_m \sqrt{s} \Big( \sum_{j=1}^{m} \int_0^s \big(h_j'(u)\big)^2 du \Big)^{\frac{1}{2}} \\
&\leq \sigma_m \sqrt{T} \|h\|_{\mathcal{H}}
\end{aligned}
$$

and thus $\exp(-\lambda\langle c_i, h(s)\rangle) \leq \exp(\lambda\sigma_m\sqrt{T}\|h\|_{\mathcal{H}})$ for any $i = 1, \dots, m$ and any $0 \leq s \leq T$. By equation (6) it now follows

$$
\inf_{\|h\| \leq 1} \Psi_{\mu,\rho}\big(X(\omega + \lambda h)\big) \geq e^{-\lambda\sigma_m\sqrt{T}} \Psi_{\mu,\rho}\big(X(\omega)\big)
$$

for any $\omega \in \Omega$ and any $\lambda > 0$.

Since $\Psi_{\mu,\rho}\big(X(\cdot)\big)$ is continuous and $\{h \in \mathcal{H}; \|h\|_{\mathcal{H}} \leq 1\}$ is a separable subset of $\Omega$, the random variable $\inf_{\|h\|_{\mathcal{H}} \leq 1} \Psi_{\mu,\rho}\big(X(\cdot + \lambda h)\big)$ is Borel measurable because the infimum can be taken over a dense denumerable subset. Hence

$$
P\Big(e^{-\lambda\sigma_m\sqrt{T}} \Psi_{\mu,\rho}(X) \leq a\Big) \geq P\Big( \inf_{\|h\|_{\mathcal{H}} \leq 1} \Psi_{\mu,\rho}\big(X(\cdot + \lambda h)\big) \leq a \Big).
$$

It is given that

$$
P\Big(\Psi_{\mu,\rho}(X) \leq a\Big) = \Phi\Big(\frac{\ln b}{\varsigma}\Big)
$$

and therefore, according to the isoperimetric inequality (cf. equation (5)),

$$
P\Big( \inf_{\|h\|_{\mathcal{H}} \leq 1} \Psi_{\mu,\rho}\big(X(\cdot + \lambda h)\big) \leq a \Big) \geq \Phi\Big(\frac{\ln b}{\varsigma} + \lambda\Big)
$$

for any $\lambda > 0$. To sum up,

$$
P\Big(e^{-\lambda\sigma_m\sqrt{T}} \Psi_{\mu,\rho}(X) \leq a\Big) \geq \Phi\Big(\frac{\ln b}{\varsigma} + \lambda\Big).
$$

Set $\lambda = \frac{\ln\theta}{\sigma_m\sqrt{T}}$ and the proof of equation (3) is done.

To prove equation (4), suppose henceforth that $\mu \in \mathcal{M}(t, T)$ with $0 < t \leq T$. Consider equation (6). Substitute $\omega$ by $\omega - \lambda h$ and (subsequently replace) $h$ by $-h$ to conclude that

$$
\Psi_{\mu,\rho}\big(X(\omega + \lambda h)\big) \leq \big( \max_{t \leq s \leq T} \max_{i \in I} e^{\lambda\langle c_i, h(s)\rangle} \big) \Psi_{\mu,\rho}\big(X(\omega)\big)
$$

for every $\omega \in \Omega$, $h \in \mathcal{H}$ and $\lambda > 0$.

Let $\chi_{[0,t]}$ be the characteristic function of the interval $[0,t]$ and fix $h \in \mathcal{H}$ such that

$$h'(s) = \frac{x}{\sqrt{t}} \chi_{[0,t]}(s), \quad 0 \le s \le T,$$

where $x \in \mathbb{R}^m$ satisfies $|x|_2 = 1$ and

$$\max_{i \in I} \langle c_i, x \rangle = \min_{|y|_2=1} \max_{i \in I} \langle c_i, y \rangle.$$

Observe that $\|h\|_{\mathcal{H}} = 1$ and $\max_{i \in I} \langle c_i, x \rangle = -\alpha$, where $\alpha$ is defined as in Lemma 1. Since $h(s) = \sqrt{t} x$ for all $t \le s \le T$ it follows

$$\max_{t \le s \le T} \max_{i \in I} e^{\lambda \langle c_i, h(s) \rangle} = \max_{i \in I} e^{\lambda \sqrt{t} \langle c_i, x \rangle} = e^{-\lambda \alpha \sqrt{t}}.$$

Consequently,

$$\Psi_{\mu,\rho}\big(X(\omega + \lambda h)\big) \le e^{-\lambda \alpha \sqrt{t}} \Psi_{\mu,\rho}(X(\omega)).$$

Recall that $\|h\|_{\mathcal{H}} = 1$ and

$$P\Big(\Psi_{\mu,\rho}(X) \le a\Big) = \Phi\Big(\frac{\ln b}{\varsigma}\Big).$$

The left inequality in equation (5) implies

$$P\Big(\Psi_{\mu,\rho}\big(X(\cdot + \lambda h)\big) \le a\Big) \le \Phi\Big(\frac{\ln b}{\varsigma} + \lambda\Big),$$

for each $\lambda \ge 0$, and therefore

$$P\Big(\Psi_{\mu,\rho}(X) \le e^{\lambda \alpha \sqrt{t}} a\Big) \le \Phi\Big(\frac{\ln b}{\varsigma} + \lambda\Big). \tag{7}$$

The constant $\alpha$ is strictly greater than 0. In fact, if $A$ denotes the convex hull of the vectors $\{c_i\}_{i \in I}$ then, since $0 \notin A$, $\langle c_i, x \rangle > 0$ for all $i \in I$ if $x$ denotes the point in $A$ closest to the origin. Hence, equation (4) follows by setting $\lambda = \frac{\ln \theta}{\alpha \sqrt{t}}$ in equation (7).

The last part of Lemma 1 is obvious. □

# 3 Convexity Properties and Absolute Continuity

This section will prove that the distribution $F_{\mu,\rho}$ is log-concave and discuss conditions that implies that $F_{\mu,\rho}$ is absolutelly continuous. From now on, absolutelly continuous should be understood as absolutelly continuous with respect to the Lebesgue measure.

**Theorem 1.** *Suppose $\mu \in \mathcal{M}(0,T)$ and $\rho \in \Lambda$. The function $F_{\mu,\rho}$ is log-concave, that is*

$$F_{\mu,\rho}\big(\theta s + (1-\theta)u\big) \geq F_{\mu,\rho}(s)^{\theta} F_{\mu,\rho}(u)^{1-\theta}$$

*for all $s, u \geq 0$ and all $0 < \theta < 1$.*

*Proof.* The Wiener measure $P$ is log-concave, that is

$$P\big(\theta A + (1-\theta)B\big) \geq P(A)^{\theta} P(B)^{1-\theta}$$

for all Borel sets $A$ and $B$ in $\Omega$ and all $0 < \theta < 1$, see Borell [4]. In particular, if $\Upsilon : \Omega \mapsto \mathbb{R}$ is convex and continuous then

$$\{\Upsilon \leq \theta s + (1-\theta)u\} \supseteq \theta\{\Upsilon \leq s\} + (1-\theta)\{\Upsilon \leq u\}$$

for all $s, u \in \mathbb{R}$ and all $0 < \theta < 1$ and, accordingly from this,

$$P\big(\Upsilon \leq \theta s + (1-\theta)u\big) \geq P(\Upsilon \leq s)^{\theta} P(\Upsilon \leq u)^{1-\theta}$$

for all $s, u \in \mathbb{R}$ and all $0 < \theta < 1$. Thus, to prove Theorem 1 it remains to show that the functional $\omega \mapsto \Psi_{\mu,\rho}(X(\omega))$ is convex.

For each fixed $y \in \mathbb{R}^m$ and $k > 0$ the function $x \mapsto k\exp(\langle y, x\rangle)$ is convex and thus, if $X_s(\omega) = (X_{1,s}(\omega), \ldots, X_{m,s}(\omega))$ then $\omega \mapsto X_{i,s}(\omega)$ is convex for each fixed $i = 1, \ldots, m$ and fixed $s \in [0,T]$. The function $x_i \mapsto \phi_{\rho,q}(x_1, \ldots, x_i, \ldots, x_m)$, $1 \leq q \leq \infty$, is convex and non-decreasing for each $i = 1, \ldots, m$, which implies that $\omega \mapsto \phi_{\rho,q}(X_s(\omega))$ is convex for each $s$. The Minkowski inequality now gives that $\omega \mapsto \Psi_{\mu,\rho}(X(\omega))$ is convex and the proof is complete. □

Although the distribution $F_{\mu,\rho}$ is log-concave, the measure with distribution function $F_{\mu,\rho}$ is not log-concave for all choices of $\mu$ and $\rho$. Indeed, an absolutely continuous bounded and positive Borel measure on some open subset of $\mathbb{R}$ is log-concave if and only if the density function is log-concave, see Borell [6]. The lognormal distribution function $G_\varsigma$ has a density $g_\varsigma$ given by

$$g_\varsigma(s) = e^{-\frac{(\ln s)^2}{2\varsigma}} \frac{1}{s\sqrt{2\pi\varsigma}}, \quad s > 0.$$

It is easily seen that the function $\ln g_\varsigma$ is not concave and thus, the measure with distribution function $G_\varsigma$ is not log-concave.

The proof of the next corollary exploits an idea in Hoffman-Jørgensen et al. [13].

**Corollary 1.** *Suppose $\mu \in \mathcal{M}(0,T)$, $1 \leq p, q \leq \infty$, $\rho \in \Lambda$ and put*

$$s^* = \inf\{s \geq 0\,;\, F^{p,q}_{\mu,\rho}(s) > 0\}.$$

*The distribution $F^{p,q}_{\mu,\rho}$ is absolutely continuous on $(s^*, \infty)$. Moreover, if $\mu \in \mathcal{M}(t,T)$, $t > 0$, or if $p < \infty$ then $F^{p,q}_{\mu,\rho}$ is absolutely continuous on $[0, \infty)$.*

*Proof.* Theorem 1 gives that $\ln F^{p,q}_{\mu,\rho}(s)$ is concave for all $s > s^*$. Thus, if $s > s^*$ then the distribution $F^{p,q}_{\mu,\rho}(s)$ can be written as $\exp(\psi(s))$ for some concave function $\psi$. A concave function is absolutely continuous and hence, $F^{p,q}_{\mu,\rho}$ is absolutely continuous on $(s^*, \infty)$. It remains to establish that $F^{p,q}_{\mu,\rho}(s)$ is continuous at $s^*$ if $\mu \in \mathcal{M}(t,T)$, $t > 0$, or if $p < \infty$. Since a distribution function is right continuous this amounts to the same thing as proving that $P(\Psi^{p,q}_{\mu,\rho}(X) = s^*) = 0$ if $\mu \in \mathcal{M}(t,T)$, $t > 0$, or $p < \infty$.

Firstly, suppose either that $\mu \in \mathcal{M}(t,T)$, $t > 0$, or that $p < \infty$ and $\mu(\{0\}) = 0$. For any given $\epsilon_0 > 0$ there is an $\omega_0 \in \Omega$ such that $\Psi^{p,q}_{\mu,\rho}\big(X(\omega_0)\big) = \epsilon_0$. Define $\Upsilon(\omega) = \Psi^{p,q}_{\mu,\rho}\big(X(\omega)\big)$, the map $\Upsilon$ is continuous which yields that for each $\epsilon_1 > 0$ the set $\Upsilon^{-1}\big((-\infty, \epsilon_0 + \epsilon_1)\big)$ is an open non-empty subset of $\Omega$. The topological support of $P$ equals $\Omega$ and, hence

$$P\Big(\Upsilon^{-1}\big((-\infty, \epsilon_0 + \epsilon_1)\big)\Big) > 0$$

and therefore $s^* = 0$. But $\Upsilon(\omega) > 0$ for all $\omega \in \Omega$, which yields

$$P(\Psi^{p,q}_{\mu,\rho}(X) = s^*) = P(\Upsilon = 0) = 0.$$

Next, assume that $p < \infty$ and $\mu(\{0\}) > 0$. It is readily seen that

$$s^* \geq k = \phi_{\rho,q}(x)\, \mu(\{0\})^{\frac{1}{p}},$$

where $x = X_0$. Set $\nu(A) = \mu(A \cap (0,T])$ for every Borel set $A$ of $[0,T]$. For any $s \geq k$ it holds

$$P\big(\Psi^{p,q}_{\mu,\rho}(X) = s\big) = P\big(\Psi^{p,q}_{\nu,\rho}(X) = (s^p - k^p)^{\frac{1}{p}}\big).$$

Since $\nu(\{0\}) = 0$ the previous results implies that $P\big(\Psi^{p,q}_{\nu,\rho}(X) = s\big) = 0$ for all $s \geq 0$, and therefore $P\big(\Psi^{p,q}_{\mu,\rho}(X) = s\big) = 0$ for each $s \geq k$. In particular, $P\big(\Psi^{p,q}_{\mu,\rho}(X) = s^*\big) = 0$ and the proof is complete. □

If $\mu \in \mathcal{M}(0,T)$ then $F^{\infty,q}_{\mu,\rho}$ is not necessarily continuous at $s^*$, where $s^*$ is defined as in Corollary 1. For instance, if $\nu = \delta_0 + \delta_T$, where $\delta_s$ is the Dirac measure at $s$, then it is easily seen that $F^{\infty,q}_{\nu,\rho}$ is discontinuous at $s^*$.

# 4 Tail Probabilities

This section considers the tail probabilities for the law of $\Psi_{\mu,\rho}(X)$. To begin with we will study the upper tail probability.

In what follows we write $f(s) \sim_s g(s)$ if $f(s)/g(s) \to 1$ as $s \to \infty$. The lognormal distribution satisfies

$$\ln\big(1 - G_\varsigma(\lambda s)\big) \sim_s -\frac{1}{2}\frac{\ln^2 s}{\varsigma^2}, \tag{8}$$

for any $\lambda > 0$ and any $\varsigma > 0$. This follows at once from the well-known estimates

$$\frac{1}{\sqrt{2\pi}}\frac{s}{1+s^2}e^{-\frac{s^2}{2}} \leq 1-\Phi(s) \leq \frac{1}{\sqrt{2\pi}}\frac{1}{s}e^{-\frac{s^2}{2}}, \quad s>0,$$

and the definition of $G_\varsigma$. The next result, Theorem 2, extends this observation as well as a previous result by Janos, see [15]. Janos obtains by other methods the same result in the specal case $p=q=1$ and $\mu$ equal to a positive linear combination of Dirac measures.

**Theorem 2.** *If $\mu \in \mathcal{M}(0,T)$ and $\rho \in \Lambda$ then*

$$\ln\big(1-F_{\mu,\rho}(s)\big) \sim_s -\frac{1}{2}\frac{\ln^2 s}{\sigma_m^2 T}.$$

*Proof.* Firstly, let $a, b > 0$ be chosen such that

$$F_{\mu,\rho}(a) = G_\varsigma(b).$$

Suppose $\varsigma = \sigma_m\sqrt{T}$. From Lemma 1, equation (3), with $\theta = s/a$ it follows

$$F_{\mu,\rho}(s) \geq G_\varsigma(\frac{b}{a}s), \quad s \geq a,$$

which gives $\ln\big(1-F_{\mu,\rho}(s)\big) \leq \ln\big(1-G_\varsigma(\frac{b}{a}s)\big)$ and therefore

$$\big|\ln\big(1-F_{\mu,\rho}(s)\big)\big| \geq \big|\ln\big(1-G_\varsigma(\frac{b}{a}s)\big)\big|$$

for each $s \geq a$. Hence, if

$$l^- = \liminf_{s\to\infty}\frac{\ln\big(1-F_{\mu,\rho}(s)\big)}{\ln\big(1-G_\varsigma(s)\big)}$$

then

$$l^- \geq \liminf_{s\to\infty}\frac{\ln\big(1-G_\varsigma(\frac{b}{a}s)\big)}{\ln\big(1-G_\varsigma(s)\big)} = 1,$$

according to equation (8).

The next aim is to find an upper bound. Fix $\epsilon$ such that $0<\epsilon<T$ and define

$$\nu(A) = \mu\big(A\cap(T-\epsilon,T]\big),$$

for each Borel set $A$ of $[0,T]$. Note that $\nu \in \mathcal{M}(T-\epsilon,T)$. Moreover, set $\varrho = (0,0,\dots,0,\rho_m)$. It is evident that

$$F_{\mu,\rho}(s) \leq F_{\nu,\varrho}(s) \tag{9}$$

for each $s > 0$.

Next, because $\nu([0,T]) > 0$ there are $a, b > 0$ such that

$$F_{\nu,\varrho}(a) = G_\varsigma(b).$$

If $s \geq a$ and $\theta = s/a$, then Lemma 1, equation (4), yields

$$F_{\nu,\varrho}(s) \leq G_\varsigma\big(\frac{b}{a^\gamma}s^\gamma\big), \quad s > a,$$

with

$$\gamma = \sqrt{\frac{T}{T-\epsilon}}.$$

In view of equation (9) we find $\ln\big(1 - F_{\mu,\rho}(s)\big) \geq \ln\big(1 - G_\varsigma\big(\frac{b}{a^\gamma}s^\gamma\big)\big)$ so that

$$\left|\ln\Big(1 - F_{\mu,\rho}(s)\Big)\right| \leq \left|\ln\Big(1 - G_\varsigma\big(\frac{b}{a^\gamma}s^\gamma\big)\Big)\right|$$

for each $s \geq a$. Thus, if we set

$$l^+ = \limsup_{s\to\infty} \frac{\ln\big(1 - F_{\mu,\rho}(s)\big)}{\ln\big(1 - G_\varsigma(s)\big)}$$

we obtain

$$l^+ \leq \limsup_{s\to\infty} \frac{\ln\Big(1 - G_\varsigma\big(\frac{b}{a^\gamma}s^\gamma\big)\Big)}{\ln\big(1 - G_\varsigma(s)\big)} = \gamma^2,$$

according to equation (8).

To sum up, for every $\epsilon > 0$,

$$1 \leq l^- \leq l^+ \leq \frac{T}{T-\epsilon},$$

and the proof is done. $\square$

It is far more difficult to state any general results about the lower tail probability. For instance, if $\mu(\{0\}) > 0$ or if $p = \infty$ and $\mu \in \mathcal{M}(0,T) \setminus \bigcup_{t>0} \mathcal{M}(t,T)$ then $\inf\{s\,;\, F^{p,q}_{\mu,\rho}(s) > 0\} > 0$. However, it is possible to find upper bounds for $F_{\mu,\rho}(s)$ as $s \to 0^+$. For instance, Lemma 1 implies that $F_{\mu,\rho}(s) \leq G_\varsigma(\lambda s)$ for some $\lambda > 0$ and all sufficiently small $s > 0$. In particular, since $G_\varsigma(\lambda s) = 1 - G_\varsigma(1/\lambda s)$ equation (8) gives that there is a $k > 0$ such that

$$F_{\mu,\rho}(s) \leq e^{-k\ln^2 s} \tag{10}$$

for all sufficiently small $s > 0$.

## 5 Moment Inequalities

The purpose of this section is to derive inequalities between the generalized moments of $F^{p,q}_{\mu,\rho}$. To this end, set

$$M^{p,q}_{\mu,\rho}(r) = \int_0^\infty s^r F^{p,q}_{\mu,\rho}(ds), \quad r \in \mathbb{R}.$$

The function $M^{p,q}_{\mu,\rho}$ will often be written $M_{\mu,\rho}$. From Theorem 2 and equation (10) we can draw the conclusion that the function $M_{\mu,\rho}(r)$ is finite for all real $r$.

If $-\infty < r_0 < r_1 < \infty$ and $r_0 r_1 \neq 0$ then it is well-known that

$$M_{\mu,\rho}(r_0)^{\frac{1}{r_0}} \leq M_{\mu,\rho}(r_1)^{\frac{1}{r_1}}.$$

The main result in this section is a sharp reversed inequality.

**Theorem 3.** *Suppose $\mu \in \mathcal{M}(0,T)$ and $\rho \in \Lambda$. If $-\infty < r_0 < r_1 < \infty$ and $r_0 r_1 \neq 0$ then*

$$M_{\mu,\rho}(r_1)^{\frac{1}{r_1}} \leq C(r_0, r_1)\, M_{\mu,\rho}(r_0)^{\frac{1}{r_0}}, \tag{11}$$

*where*

$$C(r_0, r_1) = e^{\frac{1}{2}(r_1 - r_0)\sigma_m^2 T}.$$

*Moreover, there is equality in equation* (11) *if $\mu = k\delta_T$, $k > 0$, and $\rho_i = 0$ for $i = 1, \ldots, m-1$.*

*Proof.* Suppose $N_\varsigma(r) = \int_0^\infty s^r G_\varsigma(ds)$ with $\varsigma = \sigma_m \sqrt{T}$ and define a measure $\mu_0$ by $\mu_0 = k_0 \mu$, where the constant $k_0$ is given by

$$k_0 = \left( \frac{N_\varsigma(r_0)}{M_{\mu,\rho}(r_0)} \right)^{\frac{1}{r_0}}.$$

Put

$$\psi(s) = G_\varsigma(s) - F_{\mu_0,\rho}(s), \quad s \geq 0.$$

It is readily seen that

$$M_{\mu_0,\rho}(r_1) - N_\varsigma(r_1) = r_1 \int_0^\infty s^{r_1 - 1} \psi(s) ds.$$

Integration by parts implies

$$\begin{aligned} M_{\mu_0,\rho}(r_1) - N_\varsigma(r_1) =& r_1 \int_0^\infty s^{r_1 - r_0} s^{r_0 - 1} \psi(s) ds \\ =& r_1 (r_1 - r_0) \int_0^\infty s^{r_1 - r_0 - 1} \int_s^\infty u^{r_0 - 1} \psi(u) du\, ds. \end{aligned} \tag{12}$$

Note that

$$\int_0^\infty u^{r_0-1}\psi(u)du = M_{\mu_0,\rho}(r_0) - N_\varsigma(r_0) = 0 \tag{13}$$

since $M_{\mu_0,\rho}(r_0) = k_0^{r_0} M_{\mu,\rho}(r_0) = N_\varsigma(r_0)$. Next, let

$$u^* = \inf\{u > 0\,;\, \psi(u) = 0\}.$$

Lemma 1, equation (3), gives that $\psi(u) \leq 0$ for all $u > u^*$. In combination with equation (13) this yields

$$\int_s^\infty u^{r_0-1}\psi(u)du \leq 0, \quad \text{for each } s \geq 0.$$

Thus, equation (12) yields $\chi\big(M_{\mu_0,\rho}(r_1) - N_\varsigma(r_1)\big) \leq 0$, where $\chi$ denotes the sign of $r_1$. The relation $M_{\mu_0,\rho}(r_1) = k_0^{r_1} M_{\mu,\rho}(r_1)$ now implies

$$\chi\frac{M_{\mu,\rho}(r_1)}{M_{\mu,\rho}(r_0)^{\frac{r_1}{r_0}}} \leq \chi\frac{N_\varsigma(r_1)}{N_\varsigma(r_0)^{\frac{r_1}{r_0}}} \quad \text{or} \quad \frac{M_{\mu,\rho}(r_1)^{\frac{1}{r_1}}}{M_{\mu,\rho}(r_0)^{\frac{1}{r_0}}} \leq \frac{N_\varsigma(r_1)^{\frac{1}{r_1}}}{N_\varsigma(r_0)^{\frac{1}{r_0}}}.$$

Since $N_\varsigma(u) = e^{\frac{1}{2}(\varsigma u)^2}$ we have established the desired inequality. The last part of the theorem is obvious. □

# 6 The Moment Problem and the Moment-Matching Method

This section will prove that the distribution $F_{\mu,\rho}$ is indetermined by its moments and discuss the moment-matching method for the pricing of Asian basket options.

We first recall some definitions. The distribution $F_{\mu,\rho}$ is said to be *indetermined* (or *Stieltjes-indetermined*) by its moments if there exist a distribution function $F$ with support on the positive real axis such that $F \neq F_{\mu,\rho}$ and

$$M_{\mu,\rho}(k) = \int_0^\infty u^k F(du) \quad \text{for all } k \in \mathbb{N},$$

where $M_{\mu,\rho}$ is defined as in the previous section.

It is well-known that the lognormal distribution is indetermined by its moments. This result goes back to Heyde [12] who also construct a class of distribution functions with the same moments as the lognormal distribution.

Suppose $\varrho = (0, \ldots, 0, \varrho_m)$ with $\varrho_m > 0$. Nikeghbali [18] has recently proved that for any $\mu \in \mathcal{M}(0,T)$ the distribution $F^{1,1}_{\mu,\varrho}$ is indetermined by its moments. Nikeghbali proved this result using a criteria by Pakes in [19].

Namely, Theorem 5 in Pakes [19] states that if $F$ is a distribution function with support on the positive real axis such that

$$\int_u^\infty \frac{-\ln\big(1-F(s)\big)}{s^{3/2}}ds < \infty,$$

for some $u > 0$, then $F$ is indetermined by its moment.

The result by Pakes can also be applied to show that $F_{\mu,\rho}$ is indetermined by its moment. Indeed, from Theorem 2 it follows that

$$\int_u^\infty \frac{-\ln\big(1-F_{\mu,\rho}(s)\big)}{s^{3/2}}ds < \infty, \quad u > 0,$$

and thus, we have

**Corollary 2.** *Suppose $\mu \in \mathcal{M}(0,T)$ and $\rho \in \Lambda$. The distribution $F_{\mu,\rho}$ is indetermined by its moments.*

Corollary 2 has some consequences for the so-called moment-matching method for the pricing of Asian basket options. Pricing these options is equivalent to determine the expectation

$$E\big[\max(\Psi^{1,1}_{\mu,\rho}(X) - K, 0)\big],$$

where $\mu \in \mathcal{M}(0,T)$, $\rho \in \Lambda$, and $K$ is some constant. Note that the quantities $M^{1,1}_{\mu,\rho}(k)$, $k \in \mathbb{N}$, for many choices of $\mu$ easily can be computed analytically. A common approach used by practioners to estimate the price is the so-called moment-matching method, which means that the one determines a random variable $Y$ such that

$$E\big[Y^k\big] = M^{1,1}_{\mu,\rho}(k), \quad k = 1, \dots, n,$$

and then approximate

$$E\big[\max(\Psi^{1,1}_{\mu,\rho}(X) - K, 0)\big] \approx E\big[\max(Y - K, 0)\big]. \tag{14}$$

If $K > 0$, then Corollary 2 shows that even with $n = \infty$ then it is not guaranteed that there is equality in equation (14). To be more specific, there is a random variable $Y$ such that $E[Y^k] = M^{1,1}_{\mu,\rho}(k)$ for all $k \in \mathbb{N}$ and a constant $K > 0$ such that the left hand side is not equal to the right hand side in equation (14). Recall that if $Y$ and $Y'$ are two non-negative random variables with finite expectation such that $E[\max(Y - K, 0)] = E[\max(Y' - K, 0)]$ for all $K \geq 0$, then $Y$ and $Y'$ are equal in law. For other aspects on the moment-matching method, see Brigo et al. [7].

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