The stationary nonlinear Boltzmann equation in a Couette setting; isolated solutions and non-uniqueness.

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Abstract.
The stationary nonlinear Boltzmann equation is studied for forces including hard spheres, in a Couette setting between two coaxial, rotating cylinders with given maxwellian indata on the cylinders. The existence of isolated solutions together with a hydrodynamic limit control, is obtained using an asymptotic expansion and rest term in a frame with boundary layers. Depending on a parameter present in the boundary indata, two solutions connected at a regular turning point may coexist. Other situations with several coexisting solutions are also considered.

1 Introduction.

The asymptotic kinetic approach in a sharp mathematical form has its roots in works by Grad and Kogan in the 1960'ies (see [G1], [G2],[K] and references therein). A number of important results followed, concerning the nonlinear stationary Boltzmann equation in $\mathbb{R}^n$ in the close to equilibrium case ([G3], [Gu], [H], [UA] and others), where techniques of a general scope were used, such as contraction mappings (see also [EP]). Stationary problems in small domains were solved in a similar way (e.g. [P], [IS]), and the unique solvability of internal, stationary problems for the Boltzmann equation at large Knudsen numbers was likewise established (cf. [M1]). In [BCN1], a kinetic description of a gas between two plates at different temperatures and no mass flux was studied in the case of a small mean free path for the nonlinear stationary Boltzmann equation under diffuse reflection boundary conditions. Stationary, fully nonlinear hydrodynamic limits, were treated in the papers [ELM1-2]. Solutions to half-space problems for the Boltzmann equation play an important role as boundary layers in the study of hydrodynamic limits for such solutions to the Boltzmann equation when the mean free path tends to zero. This has been extensively studied in the linear context, using functional analytic and energy methods ([BCN2], [GP] and others).

In a perspective of asymptotic analysis and numerical studies, a wide range of

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questions of the above types have been addressed for the BGK and Boltzmann equations by the Kyoto group as well as by others (see the monograph [S] with extensive references).

Further away from equilibrium, weak compactness arguments have been employed instead of contraction mappings, and in the stationary case usually involving entropy dissipation control for the sharpest results. That was the case in the spatially $n$-dimensional Povzner and one space-dimensional Boltzmann papers [AN1], [AN2], and [AN3], to obtain stationary solutions via weak $L^1$-compactness under no other restrictions than Grad’s angular cut-off. The basic compactness argument in those cases, is not fully available for the Boltzmann equation itself in more than one space dimension. However, in the spatially $n$-dimensional case the entropy dissipation estimate still allows different but weaker control mechanisms, which also lead to existence results (see [AN4]).

There, in contrast to the earlier cases mentioned, complete results are so far only obtained when the velocities smaller in norm than some $\eta > 0$, are suppressed.

The present study is set in the close to equilibrium frame and gives a mathematically rigorous study of the stationary nonlinear Boltzmann equation between two coaxial cylinders $A$ and $B$, with maxwellian ingoing boundary values. The problem is extensively treated from a numerical and asymptotic perspective in [S], to which we also refer for a more complete discussion of some details. The boundary values and the solutions are assumed to be axially and circumferentially uniform in the space variables. Then, with $(r, \theta, z)$ and $(v_r, v_\theta, v_z)$ respectively denoting the spatial cylindrical coordinates and the corresponding velocity coordinates, a distribution function may be written as $f = f(r, v_r, v_\theta, v_z)$, and the Boltzmann equation becomes

$$v_r \frac{\partial f}{\partial r} + \frac{1}{r} N f = \frac{1}{\epsilon} Q(f, f),$$

$$r \in (r_A, r_B), \quad (v_r, v_\theta, v_z) \in \mathbb{R}^3.$$  \hspace{1cm} (1.1)

The maxwellian boundary data under study are

$$f(r_A, v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{1}{2}(-v_r^2 - (v_\theta - \epsilon v_\theta A_1)^2 - v_z^2)}, \quad v_r > 0,$$

$$f(r_B, v) = (2\pi)^{-\frac{3}{2}} \frac{1 + \omega_B}{(1 + \tau_B)^{\frac{3}{2}}} e^{-\frac{1}{2}(-\frac{1}{\tau_B}(v_r^2 + (v_\theta - \epsilon v_\theta B_1)^2 + v_z^2))}, \quad v_r < 0.$$  \hspace{1cm} (1.2)

Here

$$N f := v_r \frac{\partial f}{\partial v_r} - v_\theta v_r \frac{\partial f}{\partial v_\theta}.$$  \hspace{1cm} (1.3)
\[
\tilde{Q}(f, f)(v) := \int_{\mathbb{R}^3 \times S^2} B(v - v_\ast, \omega) (f(v') f'(v'_\ast) - f(v) f(v_\ast)) dv_\ast d\omega.
\]

The kernel \( B = |v - v_\ast|^2 b(\theta), b \in L^1_+ (S^2) \), is assumed to satisfy (3.19) below and belong to the Grad class, that is with its terms suitably majorized by the corresponding ones for the hard sphere model (cf. [M2]). Consider functions which are even in the axial velocity direction \( v_z \). Take the radii as \( r_A = 1 \), \( r_B > 1 \), and let \( \epsilon^4 \) denote the Knudsen number. The rotational velocities of the inner and outer cylinders are \( u_\theta A = \epsilon u_{\theta A1} \) and \( u_\theta B = \epsilon u_{\theta B1} \) respectively. The non-dimensional perturbed temperature and density are

\[
\tau_B = \epsilon^2 \tau_{B2}, \quad \omega_B = \frac{\epsilon^2}{1 + \epsilon^2 \tau_{B2}} \left( \frac{r_B^2 - 1}{r_B^2} u_{\theta A1} - \tau_{B2} + \Delta \right),
\]

where \( \tau_{B2} \) is given and \( \Delta \) is a parameter.

The main results of this paper are Theorem 1.1 about existence, and a positivity Theorem 5.2.

**Theorem 1.1** Assume that \( (u_{\theta A1} - u_{\theta B1} r_B)(3u_{\theta A1} + u_{\theta B1} r_B) > 0 \). There is a negative value \( \Delta_{bf} \) of the parameter \( \Delta \), such that for \( \Delta < \Delta_{bf} \) and \( 0 < \epsilon \) small enough, two isolated \( L^1 \)-solutions \( f_\epsilon^j \), \( j = 1, 2 \) of (1.1-2) coexist, and satisfy

\[
\int M^{-1} \sup_{\epsilon \in (r_A, r_B)} \| f_\epsilon^j(r, v) \|^2 dv < +\infty.
\]

The two solutions have different outward radial bulk velocities of order \( \epsilon^3 \). For fixed \( \epsilon \), they converge to the same solution when \( \Delta \) increases to \( \Delta_{bf} \). The solutions have rigorous hydrodynamic limits when \( \epsilon \to 0 \).

**Remark.** This existence result is based on a priori estimates of \( L^2 \)-type, which are uniform in \( \epsilon \). The positivity of these solutions can be shown using an extension of the present set-up with a priori \( L^q \)-estimates of the solutions for sufficiently large \( q > 2 \). That is the topic of an accompanying paper [AN6], which also takes up other aspects on the present problem. Our approach has wider applicability. In particular, as discussed in Section 6 below, analogous results hold for all cases of the two-roll problem treated in [S]. We also expect the techniques developed here, to be useful in the study of related problems, such as the Taylor-Couette set-up of [SD1], the Benard asymptotics of [SD2], and the two-component gases of [ATT]. The existence problem far from equilibrium for the two-roll system of this paper, is studied by weak type methods in the paper [AN5].

Write \( R = f_{\text{rest}} = P_0 f_{\text{rest}} + (I - R_0) f_{\text{rest}} = R_\parallel + R_\perp \), where \( R_0 \) is the projection on the hydrodynamic part, and

\[
f = M(1 + \psi + \epsilon \phi f_{\text{rest}}) \quad \text{with} \quad \psi = \sum_{j} \epsilon^j \psi^j, \quad M = (2\pi)^{-\frac{3}{2}} \exp\left(-\frac{\psi^2}{2}\right). \tag{1.4}
\]
Here \( \sum_{j=1}^{d} \epsilon^j \psi^j \) is the asymptotic expansion with certain boundary values equal to the terms of corresponding order in the \( \epsilon \)-expansions of (1.2), and based on a splitting into interior Hilbert behaviour, and various types of boundary layers. The main part of the paper is devoted to a rigorous study of the rest term \( R = f_{\text{rest}} \) using as ingoing boundary values what remains of (1.2), after the asymptotic expansion. The rest term problem is solved by a contraction mapping iteration.

Sections two and three are devoted to the asymptotic expansion, adapting the presentation in [S] to the needs of the present paper. For the convenience of the reader, the description of the asymptotic expansion for the two-roll system is fairly self-contained and includes particular details that are relevant here and in [AN6].

Section four discusses some a priori estimates for the rest term. A dual problem is first considered in full physical space with a priori estimates derived by multiplier techniques from Fourier-transformed, quite detailed information about the hydrodynamic moments. This gives sufficient information to estimate certain hydrodynamic moments of the original problem via a correspondingly detailed, duality based analysis. The rest of the hydrodynamics study requires a direct treatment, also building on a coupling with the asymptotic expansion. The estimate for the nonhydrodynamic part in \( L^2 \) is less involved and based on Green’s formula.

Section five deals with the rest term; a study of the contraction mapping, a remark about the hydrodynamic limits, and a positivity discussion. This is based on the technical results obtained in the earlier sections. The final section contains existence results for other two-roll problems, where the approach applies essentially without modifications.

2 The frame.

Write the solution of (1.1) as \( f = M(1 + \Phi) \). Then the new unknown \( \Phi(r, v_r, v_\theta) \) should be solution to

\[
\begin{align*}
\nu_r \frac{\partial \Phi}{\partial r} + \frac{1}{r} N \Phi &= \frac{1}{\epsilon^1} (\tilde{L} \Phi + \tilde{J}(\Phi, \Phi)), \\
\Phi(1, v) &= e^{\frac{1}{\epsilon^1} (v_\theta^2 - (v_\theta - \epsilon \nu_\theta A1)^2)} - 1, \quad v_r > 0, \\
\Phi(r_B, v) &= \frac{1 + \omega_B}{(1 + \tau_B)^{\frac{1}{2}}} e^{\frac{1}{\epsilon^1} (v_\theta^2 - \frac{1}{1 + \tau_B} (v_\theta^2 + (v_\theta - \epsilon \nu_\theta B1)^2 + v_r^2))} - 1, \quad v_r < 0.
\end{align*}
\]

Here \( \tilde{J} \) is the rescaled quadratic Boltzmann collision operator,

\[
\tilde{J}(\Phi, \psi)(v) := \frac{1}{2} \int_{B^d \times S^2} B(v - v_s, \omega) M(v_s) (\Phi(v') \psi(v_s) + \Phi(v_s') \psi(v'))
\]

\[
- \Phi(v_s) \psi(v) - \Phi(v) \psi(v_s)) dv_s d\omega,
\]

and \( \tilde{L} \) is this operator linearized around 1,

\[
(\tilde{L} \Phi)(v) := \int_{B^d \times S^2} B(v - v_s, \omega) M(v_s) (\Phi(v') + \Phi(v_s') - \Phi(v_s))
\]

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$$- \Phi(v) dv, dw = \tilde{K}(\Phi) - \nu \Phi.$$ 

Denote by $\Phi_A$ and $\Phi_B$ the truncations up to order $j_1$ of $\Phi(1, v), v_r > 0$ and $\Phi(r_B, v), v_r < 0$ respectively, e.g. for $j_1 = 4$

$$\Phi_A(v) = \epsilon u_{\Phi A_1} v_\theta + \epsilon^2 u_{\Phi A_2}^2 (1 - v_\theta^2) + \epsilon^3 \frac{u_{\Phi A_1}^3}{2} (-v_\theta + \frac{1}{3} v_\theta^3) + \epsilon^4 \frac{u_{\Phi A_1}^4}{4} (1 - v_\theta^2 + \frac{1}{6} v_\theta^4), \quad v_r > 0,$$

$$\Phi_B(v) = \epsilon u_{\Phi B_1} v_\theta + \epsilon^2 \left( r_B^2 - \frac{1}{2} u_{\Phi B_2}^2 u_{\Phi A_1}^2 - \frac{5}{2} r_B - \frac{1}{2} u_{\Phi B_1}^2 v_\theta^2 + \frac{1}{2} u_{\Phi B_1}^2 v_\theta^2 + \frac{1}{2} r_B v_\theta^2 \right) + \epsilon^3 (\Delta + u_{\Phi B_1} v_\theta (r_B^2 - \frac{1}{2} u_{\Phi B_2}^2 u_{\Phi A_1}^2 - \frac{7}{2} r_B - \frac{1}{2} u_{\Phi B_1}^2 v_\theta^2 + \frac{1}{2} r_B v_\theta^2)) + \epsilon^4 \left( \frac{7}{4} u_{\Phi B_1}^2 r_B + \frac{1}{8} u_{\Phi B_1}^2 (4 r_B^2 - \frac{1}{2} u_{\Phi B_2}^2 u_{\Phi A_1}^2 - 7 r_B - \frac{1}{2} u_{\Phi B_1}^2 v_\theta^2 + \frac{1}{2} r_B v_\theta^2) \right) + \Delta u_{\Phi B_1} v_\theta + \frac{1}{4} u_{\Phi B_1}^2 (r_B - \frac{1}{2} u_{\Phi A_1}^2 - 7 r_B - \frac{1}{2} u_{\Phi B_1}^2) v_\theta^2 + \frac{1}{8} r_B v_\theta^4 + \frac{1}{16} u_{\Phi B_1}^4 v_\theta^4), \quad v_r < 0.$$

Denote by $(\Phi_{A_i})_{1 \leq i \leq j}$ resp. $(\Phi_{B_i})_{1 \leq i \leq j}$, the first to j-th order terms of $\Phi_A$ resp. $\Phi_B$, with respect to $\epsilon$.

Solutions $\Phi$ will be determined as in (1.4), an approximate solution $\psi$ plus a rest term $R = f_{\text{rest}},$

$$\Phi(r, v) = \psi(r, v) + \epsilon^{j_0} R(r, v),$$

where for $j_1 = 4$

$$\psi(r, v) = \epsilon \left( \Phi_{H1}(r, v) + \Phi_{W1}(r, v) \right) + \epsilon^2 \left( \Phi_{H2}(r, v) + \Phi_{W2}(r, v) \right) + \epsilon^3 \left( \Phi_{H3}(r, v) + \Phi_{W3}(r, v) + \Phi_{K3A}(r, v) + \Phi_{K3B}(r, v) \right) + \epsilon^4 \left( \Phi_{H4}(r, v) + \Phi_{W4}(r, v) + \Phi_{K1A}(r, v) + \Phi_{K1B}(r, v) \right), \quad (2.4)$$

with

$$\int \Phi_{H1}(r, v)(1, v_r, v^2) M(v) dv = \int \Phi_{W1}(r, v)(1, v_r, v^2) M(v) dv$$

$$= \int \Phi_{H2}(r, v) M(v) dv = 0, \quad (2.5)$$

$$\lim_{r_B \to -\infty} \Phi_{W1}(r, v) = 0, \quad 1 \leq i \leq 4, \quad (2.6)$$
\[
\lim_{\varepsilon \to 0^+} \Phi_{K1A}\left(\frac{r - 1}{\varepsilon}, v\right) = 0,
\]
\[
\lim_{\varepsilon \to 0^+} \Phi_{K1B}\left(\frac{r - r_B}{\varepsilon}, v\right) = 0, \quad 3 \leq i \leq 4.
\]

Here \((\varepsilon \Phi_{H1} + \varepsilon^2 \Phi_{H2} + \varepsilon^3 \Phi_{H3} + \varepsilon^4 \Phi_{H4})(r, v)\) denotes the truncation up to fourth order of a formal expansion \(\sum_{k \geq 1} \varepsilon^k \Phi_{Hk}(r, v)\). The sum \((\varepsilon \Phi_{W1} + \varepsilon^2 \Phi_{W2})(\frac{r - r_B}{\varepsilon^B}, v)\) consists of correction terms allowing the boundary conditions to be satisfied to first and second orders. They correspond to a suction boundary layer at \(r_B\).

Supplementary boundary layers of Knudsen type, described by \(\varepsilon^3 (\Phi_{K3A}(\frac{r - 1}{\varepsilon}), v) + \Phi_{K3B}(\frac{r - r_B}{\varepsilon^B}, v)\) are required in order to have the boundary conditions satisfied at third and fourth orders. Similar expansions hold for \(j_1 > 4\) (cf. [S]), and will be used in Section 5 below for \(j_1 = 10\). The boundary values up to order \(j\) in \(\varepsilon\) are satisfied by \(\psi\), whereas \(\psi^{j+1}, \ldots, \psi^{j_1}\) (in particular \(\psi^8, \psi^9\) and \(\psi^{10}\) in Section 5 below) may be taken as the plain Hilbert expansion.

# 3 The asymptotic expansion.

We shall here give a fairly detailed discussion of the asymptotic expansion for \(j_1 = 4\). Recall (see [Di]) that \(L(v_\theta v_B \bar{B}) = -v_\theta v_r \bar{L}, \bar{L}(v_r \bar{A}) = v_r(v^2 - 5)\) for some functions \(\bar{B}(|v|)\) and \(\bar{A}(|v|)\), with \(v_\theta v_r \bar{B}(|v|)\) and \(v_r \bar{A}(|v|)\) bounded in the \((,)_M\)-norm, and let
\[
w_1 := \int v^2 \theta^2 B M dv, \quad w_2 = \int v_r(v^2 - 5)A M dv.
\]

Let \(g(\eta, v)\) be the solution to the half-space problem
\[
v_r \frac{\partial g}{\partial v_r} = Lg, \quad \eta > 0, \quad v \in \mathbb{R}^3;\\
g(0, v) = 0, \quad v_r > 0,\\
\int g(\eta, v)v_r M(v)dv = 1, \quad a.a. \eta > 0.
\]

By [BCN2] there are constants \(A, D,\) and \(E\) such that, (sub-)exponentially,
\[
\lim_{\eta \to +\infty} g(\eta, v) = A + Dv^2 + Ev_\theta + v_r.
\]

**Proposition 3.1**

Assume that
\[
(u_{\theta B1} r_B - u_{\theta A1})(u_{\theta B1} r_B + 3u_{\theta A1}) > 0,
\]
and set
\[
\Delta_{\text{self}} := -(2w_1 \frac{r_B + 1}{r_B}(A + 5D)(r_B u_{\theta B1} - u_{\theta A1})(r_B u_{\theta B1} + 3u_{\theta A1}))^{\frac{1}{2}}.
\]

For \(\Delta > \Delta_{\text{self}}\), there is no solution \(\Phi\) in the family defined in (2.4-7).

For \(\Delta = \Delta_{\text{self}}\), there is a unique solution \(\Phi\) in the family defined in (2.4-7).

For \(\Delta < \Delta_{\text{self}}\), there are two solutions \(\Phi\) in the family defined in (2.4-7).
The proof below starts by assuming that the $\sum_{k \geq 1} e^k \Phi_{Hk}$ satisfies (2.1) formally. The boundary condition (2.2) can be satisfied up to order two, but not from third order. Instead, supplementary boundary layers at third and fourth orders, $\Phi_{K3A}(\frac{r-1}{\epsilon^3}, v)$ and $\Phi_{K4A}(\frac{r-1}{\epsilon^4}, v)$ are introduced so that

$$(\epsilon \Phi_{H1} + e^2 \Phi_{H2})(r, v) + e^3 (\Phi_{H3}(r, v) + \Phi_{K3A}(\frac{r-1}{\epsilon^3}, v))$$

$$+ e^4 (\Phi_{H4}(r, v) + \Phi_{K4A}(\frac{r-1}{\epsilon^4}, v))$$

(3.3)

satisfies (2.1) and (2.2) up to fourth order. But (3.3) cannot satisfy the boundary conditions (2.3) even at first order. So $\Phi_{Wk}(\frac{r-r_B}{\epsilon}, v))_{k \geq 1}$ and $\Phi_{KKB}(\frac{r-r_B}{\epsilon^3}, v))_{k \geq 3}$ are introduced in order that

$$\sum_{k=1}^{2} e^k (\Phi_{Hk}(r, v) + \Phi_{Wk}(\frac{r-r_B}{\epsilon}, v))$$

$$+ \sum_{k=3}^{4} e^k (\Phi_{Hk}(r, v) + \Phi_{Wk}(\frac{r-r_B}{\epsilon}, v) + \Phi_{KKB}(\frac{r-r_B}{\epsilon^3}, v))$$

(3.4)

satisfies (2.1) and (2.3) up to fourth order.

Proof of Proposition 3.1. Denote by $Y = \frac{r-r_B}{\epsilon}$, and let the expansions

$\sum_{k \geq 1} e^k \Phi_{Hk}(r, v)$ and $\sum_{k \geq 1} e^k (\Phi_{Hk}(r_B, v) + \Phi_{Wk}(\frac{r-r_B}{\epsilon}, v))$ formally satisfy (2.1). Then,

$$\hat{L} \Phi_{H1} = \hat{L} \Phi_{H2} + \hat{J}(\Phi_{H1}, \Phi_{H1}) = \hat{L} \Phi_{H3} + 2 \hat{J}(\Phi_{H1}, \Phi_{H2})$$

$$= \hat{L} \Phi_{H1} + 2 \hat{J}(\Phi_{H1}, \Phi_{H3}) + \hat{J}(\Phi_{H2}, \Phi_{H2}) = 0,$$

$$v_r \frac{\partial \Phi_{Hk-1}}{\partial r} + \frac{1}{r} N \Phi_{Hk-1}$$

$$= \hat{L} \Phi_{Hk} + \sum_{j=1}^{k-1} \hat{J}(\Phi_{Hj}, \Phi_{Hk-j}), \ k \geq 5,$$

(3.5)

and

$$\hat{L} \Phi_{W1} = \hat{L} \Phi_{W2} + \hat{J}(\Phi_{W1}, 2\Phi_{H1}(r_B, .) + \Phi_{W1})$$

$$= \hat{L} \Phi_{W3} + 2 \hat{J}(\Phi_{H1}(r_B, .) + \Phi_{W1}, \Phi_{W2}) + 2 \hat{J}(\Phi_{W1}, \Phi_{H2}(r_B, .)) + Y \hat{J}(\Phi_{H1}(r_B, .))$$

$$= \hat{L} \Phi_{W4} + 2 \hat{J}(\Phi_{W3}, \Phi_{H1}(r_B, .) + \Phi_{W1}) + \hat{J}(\Phi_{W2}, \Phi_{W2} + 2 \Phi_{H2}(r_B, .))$$

$$+ 2 Y \hat{J}(\Phi_{W1}, \Phi_{H3}(r_B, .)) + 2 \hat{J}(\Phi_{W1}, \Phi_{H3}(r_B, .) + Y \Phi_{H2}(r_B, .))$$

$$\hat{J}(\Phi_{H1}(r_B, .)) + \frac{Y^2}{2} \Phi_{H1}'(r_B, .) - v_r \frac{\partial \Phi_{W1}}{\partial Y} = 0,$$

(3.6)

$$v_r \frac{\partial \Phi_{Wk-3}}{\partial Y} + \frac{1}{r_B} \sum_{i=0}^{k-5} (-1)^i \frac{Y}{r_B} \hat{J}(\Phi_{Hk-4-i}(r_B, .) + \Phi_{Wk-4-i})$$

$$= \hat{L} \Phi_{Wk} + \sum_{j=1}^{k-1} \hat{J}(2 \Phi_{Hj}(r_B, .) + \Phi_{Wj}, \Phi_{Wk-j}), \ k \geq 5,$$

(3.7)
Taking the conditions (2.5) into account, equations (3.5) are equivalent to

\[ \Phi_{H1}(r,v) = b_1(r)v_\theta, \]
\[ \Phi_{H2} = a_2 + d_2 v^2 + b_2 v_\theta + \frac{1}{2} b_1^2 v_\theta^2, \]
\[ \Phi_{H3} = a_3 + d_3 v^2 + b_3 v_\theta + c_3 v_r + b_1 d_2 v_\theta v^2 + b_1 b_2 v_\theta^2 + \frac{1}{6} b_1^3 v_\theta^3, \]
\[ \Phi_{H4} = a_4 + d_4 v^2 + b_4 v_\theta + c_4 v_r + (b_1 d_3 + b_2 d_2) v_\theta v^2 + (b_1 b_3 + \frac{1}{2} b_2^2 - \frac{1}{2} b_1^2 a_2) v_\theta^2 + b_1 c_3 v_r v_\theta + \frac{1}{2} b_1^2 b_2 v_\theta^3 + \frac{1}{2} d_2^2 v^4 + \frac{1}{24} b_1^4 v_\theta^4 + \frac{1}{2} b_1^2 d_2 v_\theta^2 v^2, \]

for some functions \( a_i(r), b_i(r), c_i(r), d_i(r), 1 \leq i \leq 4 \). Equations (3.6) have solutions if and only if the following compatibility conditions hold,

\[ \int \left( v_r \frac{\partial \Phi_{Hi}}{\partial r} + \frac{1}{r} N \Phi_{Hi} \right) (1, v^2 - 5, v_\theta, v_r) M(v) dv = 0, \quad i \geq 1. \]

They provide first-order differential equations for the functions \( a_i(r), b_i(r), c_i(r) \) and \( d_i(r), i \geq 1 \). In particular,

\[ (r b_1)' = 0, \quad (10 d_2 + b_1^2)' = 0, \]
\[ (r^2 c_3 b_2)' = w_1 r^2 (b_1' - \frac{1}{r} b_1)' + (2 w_1 - w_2) r (b_1' - \frac{1}{r} b_1), \]
\[ (a_2 + 5 d_2 + \frac{1}{2} b_1^2)' = \frac{1}{r} b_1^2, \]
\[ (a_3 + 5 d_3 + b_1 b_2)' = \frac{2}{r} b_1 b_2, \]
\[ (r c_3)' = 0, \]
\[ (a_4 + 5 d_4 + b_1 b_3 + \frac{1}{2} b_2') = \frac{1}{2} b_2 b_2 + \frac{35}{2} d_2^2 + \frac{7}{2} b_1^2 d_2, \]
\[ \frac{2}{r} (b_1 b_3 + \frac{1}{2} b_2 b_2 + \frac{1}{2} b_1^2 a_2) + \frac{1}{2 r} b_1^4 + \frac{7}{2 r} b_1^2 d_2, \]

Together with the boundary conditions (2.3) at first and second orders, this implies that

\[ b_1(r) = \frac{u_3 A_1}{r}, \quad a_2(r) = -\frac{1}{2} \frac{u_2^2 A_1}{r}, \quad b_2(r) = 0, \]
\[ d_2(r) = 10 \frac{u_3^2 A_1}{1 - \frac{1}{r^2}}, \quad c_3(r) = \frac{u_3}{r}, \]

for some constant \( u_3 \neq 0 \). Moreover,

\[ 10 u_3 d_3' + A_2 = 0, \quad (r b_3)' + A_2 = 0, \]

where \( A_i \) denotes an expression containing coefficients up to \( i \)-th order only. Taking the condition (2.6) into account, equations (3.7) are equivalent to

\[ \Phi_{W1}(Y, v) = z_1(Y) v_\theta, \quad \Phi_{W2} = x_2 + y_2 v^2 + z_2 v_\theta + (b_1(r_B) z_1 + \frac{1}{2} z_1^2) v_\theta^2, \]
\[
\Phi_{W3} = x_3 + y_3 v^2 + z_3 v \theta + t_3 v_r + (b_1 (r_B) y_2 + z_1 y_2 + z_1 d_2 (r_B)) v \theta v^2 \\
+ (b_1 (r_B) z_2 + z_1 z_2 + z_1 b_2 (r_B) + Y b'_1 (r_B) z_1) v^2 \\
+ \left( \frac{1}{2} b'_2 (r_B) z_1 + \frac{1}{2} b_2 (r_B) z^2_1 + \frac{1}{6} z^3_1 \right) v^3, \\
\Phi_{W4} = x_4 + y_k v^2 + z_1 v \theta + t_4 v_r + z'_1 v_r v \theta \tilde{B}(v) + \ldots,
\]

for some functions \(x_i(Y), y_i(Y), z_i(Y), t_i(Y), 1 \leq i \leq 4\). Equations (3.8) have solutions if and only if the following compatibility conditions hold

\[
\int \left( v_r \frac{\partial \Phi_{Wk-3}}{\partial Y} + \frac{1}{r_B} \sum_{i=0}^{k-5} (-1)^i \left( \frac{Y_1}{r_B} \right)^i N(\Phi_{Hk-4-i}(r_B, \cdot) + \Phi_{Wk-4-i}(\cdot)) (v^2 \\
- 5, v \theta) M(v) dv = 0, \quad k \geq 5,
\]

and

\[
\int \left( v_r \frac{\partial \Phi_{Wk-3}}{\partial Y} + \frac{1}{r_B} \sum_{i=0}^{k-5} (-1)^i \left( \frac{Y_1}{r_B} \right)^i N(\Phi_{Hk-4-i}(r_B, \cdot) + \Phi_{Wk-4-i}(\cdot)) (v^2 \\
- 5, v \theta) M(v) dv = 0, \quad k \geq 5.
\]

Equations (3.11) provide second-order differential equations for \(y_k\) (resp. \(z_k\)), \(i \geq 1\), depending on \(t_j, c_j, j \leq i + 1\) (resp. \(j \leq i + 2\)). In particular,

\[
w_1 z''_1 - \frac{u_3}{r_B} z'_1 = 0, \\
(x_2 + 5y_2 + b_1 (r_B) z_1 + \frac{1}{2} z^2_1)' = 0, \\
w_2 y_2' + \frac{10}{r_B} y_2' + A_1 = 0, \quad w_1 z''_2 - \frac{u_3}{r_B} z'_2 + A_1 = 0, \\
t'_3 = 0, \\
(x_3 + 5y_3 + b_1 (r_B) z_2 + z_1 z_2 + z_1 b_2 (r_B) + Y b'_1 (r_B) z_1)' = \\
\frac{1}{r_B} (2b_1 (r_B) z_1 + z^2_1), \\
w_2 y_3' + \frac{10}{r_B} y_3' + A_2 = 0, \\
w_1 z''_3 - \frac{u_3}{r_B} z'_3 + \left( (b_1 (r_B) + z_1) (c_3 (r_B) + t_5) \right)' + A_2 = 0, \\
t'_4 + \frac{1}{r_B} (t_3 + c_3 (r_B)) + c'_3 (r_B) = 0, \\
x_4 + 5y_k)' + A_3 = 0.
\]

First the equations \(t'_3 = 0\) and \(t'_4 + \frac{1}{r_B} t_3 = 0\) together with the conditions \(\lim_{-\infty} t_3 = 0, \lim_{-\infty} t_4 = 0\), imply that \(t_3 = t_4 = 0\). Then as will be seen in Lemmas 3.1-2, the introduction of Knudsen boundary layers at third and fourth orders will fix the values of \(x_i(0), y_i(0)\) and \(z_i(0), i = 3, 4\, in\, terms\, of\, c_i(r_B)\). Taking the fifth order coefficient \(c_5 (r_B) + t_5 = 0\, (resp.\, \lim_{-\infty} y_k = 0)\), this defines \(y_k\) (resp. \(z_i\)) in term of \(c_i(r_B)\). Equations
(3.13)-(3.15) provide first-order linear differential equations for \( x_i + 5y_i, i \geq 2 \). Together with the conditions \( \lim_{\eta \to -\infty} x_i = 0 \), this will fix the value of \( c_i(r_B) \), \( i = 3, 4 \), hence define \( x_i \) and \( y_i \). That will be done in Lemmas 3.1 and 3.2. □

**Lemma 3.1** Set \( \eta = \frac{r_1}{c}, \mu = \frac{r_2}{c} \). There are unique Knudsen boundary layers \( \Phi_{K3A}(\eta, v) \) and \( \Phi_{K3B}(\mu, v) \), and boundary values \( \Phi_{H3}(1, v) \) and \( \Phi_{W3}(0, v) \) such that

\[
v_r \frac{\partial \Phi_{K3A}}{\partial \eta} = \tilde{L} \Phi_{K3A}, \quad \eta > 0, \quad v \in \mathbb{R}^3,
\]

\[
\Phi_{K3A}(0, v) = \Phi_{A3}(v) - \Phi_{H3}(1, v), \quad v_r > 0,
\]

\[
\lim_{\eta \to +\infty} \Phi_{K3A}(\eta, v) = 0,
\]

and

\[
v_r \frac{\partial \Phi_{K3B}}{\partial \mu} = \tilde{L} \Phi_{K3B}, \quad \mu < 0, \quad v \in \mathbb{R}^3,
\]

\[
\Phi_{K3B}(0, v) = \Phi_{B3}(v) - \Phi_{H3}(r_B, v) - \Phi_{W3}(0, v), \quad v_r < 0,
\]

\[
\lim_{\mu \to -\infty} \Phi_{K3B}(\mu, v) = 0.
\]

The boundary layers fix the possible values of \( a_3(1), d_3(1), u_3, b_3(1) \) and \( x_3(0), y_3(0), z_3(0) \), hence complete the definitions of \( \Phi_{H3} \) and \( \Phi_{W3} \).

**Proof of Lemma 3.1.** The function

\[
\psi_{K3A}(\eta, v) := \Phi_{K3A}(\eta, v) - u_3(g - A - Dv^2 - Ev_\theta - v_r),
\]

with \( g, A, D \) and \( E \) defined in (3.1-2) and \( u_3 \) still unknown, should satisfy

\[
v_r \frac{\partial \psi_{K3A}}{\partial \eta} = \tilde{L} \psi_{K3A}, \quad \eta > 0, \quad v \in \mathbb{R}^3,
\]

\[
\psi_{K3A}(0, v) = u_3A - a_3(1) + (u_3D - d_3(1))v^2 + (u_3E - \frac{1}{2}u_{3A1} - b_3(1)v_\theta, \quad v_r > 0,
\]

\[
\lim_{\eta \to +\infty} \psi_{K3A}(\eta, v) = 0.
\]

Hence,

\[
a_3(1) = u_3A, \quad d_3(1) = u_3D, \quad b_3(1) = u_3E - \frac{1}{2}u_{3A1}, \quad \psi_{K3A} = 0,
\]

so that

\[
\Phi_{K3A} = u_3(g - A - Dv^2 - Ev_\theta - v_r).
\]

Analogously, the function

\[
\psi_{K3B}(\mu, v) := \Phi_{K3B}(\mu, v) - \frac{u_3}{r_B}(g(-\mu, -v) - A - Dv^2 + Ev_\theta + v_r),
\]

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should satisfy

\[
\nu_r \frac{\partial \psi_{K3B}}{\partial \mu} = \tilde{L} \psi_{K3B}, \quad \mu < 0, \quad v \in \mathbb{R}^d,
\]

\[
\psi_{K3B}(0, v) = \Delta - \frac{u_3}{r_B} A - a_3(r_B) - x_3(0) - \left( \frac{u_3}{r_B} D + d_3(r_B) + y_3(0) \right) v^2
\]

\[
+ \left( u_{\theta B1} \dfrac{r_B^2 - 1}{r_B^2} u_{\theta A1}^2 - \frac{7}{2} \tau_{B2} - u_{\theta B1}^2 \right) + \frac{u_3}{r_B} E - b_3(r_B) - z_3(0) \right) v_\theta, \quad v_r < 0,
\]

\[
\lim_{\mu \to -\infty} \psi_{K3B}(\mu, v) = 0.
\]

Hence,

\[
x_3(0) = \Delta - \frac{u_3}{r_B} A - a_3(r_B), \quad y_3(0) = -\frac{u_3}{r_B} D - d_3(r_B),
\]

\[
z_3(0) = -u_{\theta B1} \left( \dfrac{r_B^2 - 1}{r_B^2} u_{\theta A1}^2 - \frac{7}{2} \tau_{B2} - u_{\theta B1}^2 \right) + \frac{u_3}{r_B} E - b_3(r_B), \quad \psi_{K3B} = 0,
\]

and

\[
\Phi_{K3B}(\mu, v) = \frac{u_3}{r_B}(\mu_v - v) - A - Dv^2 + Ev_\theta + v_r.
\]

Moreover, by integration of (3.14) and (3.9) on \((-\infty, 0)\) and \((1, r_B)\) respectively,

\[
x_3(0) + 5y_3(0) = \frac{u_1}{2r_B^2 u_3} (u_{\theta B1} r_B + 3u_{\theta A1} r_B)(u_{\theta B1} r_B - u_{\theta A1}),
\]

\[
x_3(0) + 5y_3(0) = \Delta - u_3(A + 5D)(\frac{1}{r_B} + 1).
\]

And so, \(u_3\) must solve the equation

\[
u_3(A + 5D) \dfrac{r_B}{r_B} + 1 - \Delta u_3 + \frac{u_1}{2r_B^2} (3u_{\theta A1} + u_{\theta B1} r_B)(u_{\theta A1} - u_{\theta B1} r_B) = 0.
\]

(3.18)

A study of the positive roots \(u_3\) to (3.18) leads to the three cases described in Proposition 3.1 for \(\Delta\) with respect to \(\Delta_{k1f}\).

Remark. The above proof requires

\[
A + 5D \neq 0,
\]

(3.19)
a condition satisfied for hard spheres, and assumed to hold for the kernels \(B\) in this paper, together with the Grad class condition.

**Lemma 3.2** Set \(\eta = \frac{r_{-1}}{c_{\epsilon}}, \quad \mu = \frac{r-r_{+}}{c_{\epsilon}}\). There are unique Knudsen boundary layers \(\Phi_{K1A}(\eta, v)\) and \(\Phi_{K1B}(\mu, v)\), and boundary values \(\Phi_{H1}(1, v)\) and \(\Phi_{K1}(0, v)\) such that

\[
\nu_r \frac{\partial \Phi_{K1A}}{\partial \eta} = \tilde{L} \Phi_{K1A} + 2 \tilde{J}(\Phi_{H1}(1, \Phi_{K3A})), \quad \eta > 0, \quad v \in \mathbb{R}^d,
\]

\[
\Phi_{K1A}(0, v) = \Phi_{A1}(v) - \Phi_{H1}(1, v), \quad \nu_r > 0,
\]

\[
\lim_{\eta \to +\infty} \Phi_{K1A} = 0,
\]
and
\[ v_r \frac{\partial \Phi_{K4B}}{\partial \mu} = \tilde{L} \Phi_{K4B} + 2 \tilde{J} \left( \Phi_{H1}(r_B) + \Phi_{W1}(0), \Phi_{K3B} \right), \quad \mu < 0, \quad v \in \mathbb{R}^3, \]
\[ \Phi_{K4B}(0, v) = \Phi_{B4}(v) - \Phi_{H4}(r_B, v) - \Phi_{W4}(0, v), \quad v_r < 0, \]
\[ \lim_{\mu \to -\infty} \Phi_{K4B} = 0. \]

Proof of Lemma 3.2. Analogously to [BCN2], there are unique solutions \( \alpha \) and \( \beta \) to
\[ v_r \frac{\partial \alpha}{\partial \eta} = \tilde{L} \alpha + 2 \tilde{J} \left( \Phi_{H1}(1), \Phi_{K3A} \right), \quad \eta > 0, \quad v \in \mathbb{R}^3, \]
\[ \alpha(0, v) = -u_{\theta A1}(u_3 D v^2 + (u_3 E + \frac{1}{4} u_{\theta A1}^2) v^2 + u_3 v r), \quad v_r > 0, \]
\[ \int v_r \alpha(\eta, v) M(v) dv = 0, \]
and
\[ v_r \frac{\partial \beta}{\partial \eta} = \tilde{L} \beta + 2 \tilde{J} \left( \Phi_{H1}(r_B, -v) + \Phi_{W1}(0, -v), \Phi_{K3B}(-\eta, -v) \right), \quad \eta > 0, \quad v \in \mathbb{R}^3, \]
\[ \beta(0, v) = \Phi_{B4}(-v) - (\Phi_{H4}(r_B, -v) - a_4(r_B) - d_4(r_B) v^2 - b_4(v r) - \frac{u_4}{r_B} v_r) \]
\[ -(\Phi_{W4}(0, -v) - x_4(0) - y_4(0) v^2 - z_4(0) v_r), \quad v_r > 0, \]
\[ \int v_r \beta(\eta, v) M(v) dv = 0. \]

Moreover,
\[ \alpha \in \text{Ker} \tilde{L}, \quad \beta \in \text{Ker} \tilde{L}^\perp, \]
\[ \lim_{\eta \to +\infty} \alpha(\eta, v) = a_\infty + d_\infty v^2 + b_\infty v r, \quad \lim_{\eta \to +\infty} \beta(\eta, v) = r_\infty + s_\infty v^2 + t_\infty v r, \]
for some constants \( a_\infty, d_\infty, b_\infty, r_\infty, s_\infty \) and \( t_\infty \). The function
\[ \psi_{K4A}(\eta, v) := \Phi_{K4A}(\eta, v) - u_4(g - A - D v^2 - E v_r - v_r) \]
\[ - (\alpha - a_\infty - d_\infty v^2 - b_\infty v r) \]
should satisfy
\[ v_r \frac{\partial \psi_{K4A}}{\partial \eta} = \tilde{L} \psi_{K4A}, \quad \eta > 0, \quad v \in \mathbb{R}^3, \]
\[ \psi_{K4A}(0, v) = \frac{1}{8} u_{\theta A1}^2 + a_\infty + u_4 A - a_4(1) + (d_\infty + u_4 D - d_4(1)) v^2 \]
\[ + (b_\infty + u_4 E - b_4(1)) v r, \quad v_r < 0, \]
\[ \lim_{\mu \to -\infty} \psi_{K4A} = 0. \]

Hence,
\[ a_4(1) = \frac{1}{8} u_{\theta A1}^2 + a_\infty + u_4 A, \quad d_4(1) = d_\infty + u_4 D, \quad b_4(1) = b_\infty + u_4 E, \]
\[ \psi_{K4A} = 0. \]

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so that
\[
\Phi_{K41} = \alpha - a_\infty - d_\infty v^2 - b_\infty v_\theta + u_4 (g - A - Dv^2 - Ev_\theta - v_r).
\]

Analogously, the function
\[
\psi_{K4B}(\mu, v) := \Phi_{K4B}(\mu, v) - \frac{u_4}{r_B} (g(-\mu, -v) - A - Dv^2 + Ev_\theta - v_r)
\]
\[
- (\beta(-\mu, v) - r_\infty - s_\infty v^2 + t_\infty v_\theta)
\]
should satisfy
\[
v_r \frac{\partial \psi_{K4B}}{\partial \mu} = \tilde{L} \psi_{K4B}, \quad \mu < 0, \quad v \in \mathbb{R}^3,
\]
\[
\psi_{K4B}(0, v) = r_\infty + \frac{u_4}{r_B} A - a_4 (r_B) - x_4(0)
\]
\[
+ (s_\infty + \frac{u_4}{r_B} D - d_4 (r_B) - y_4(0)) v^2 - (t_\infty + \frac{u_4}{r_B} E + b_4 (r_B) + z_4(0)) v_\theta, \quad v_r < 0,
\]
\[
\lim_{\mu \to -\infty} \psi_{K4B}(\mu, v) = 0.
\]

Hence,
\[
x_4(0) = r_\infty - a_4 (r_B) + u_4 \frac{A}{r_B}, \quad y_4(0) = s_\infty - d_4 (r_B) + u_4 \frac{D}{r_B},
\]
\[
z_4(0) = t_\infty - b_4 (r_B) + u_4 \frac{E}{r_B}, \quad \psi_{K4B} = 0,
\]
so that
\[
\Phi_{K4B}(\mu, v) = \beta(-\mu, v) - r_\infty - s_\infty v^2 + t_\infty v_\theta
\]
\[
+ \frac{u_4}{r_B} (g(-\mu, v) - A - Dv^2 + Ev_\theta).
\]

Moreover, by integration of (3.15) and (3.10) on (-\infty, 0) and (1, r_B) respectively,
\[
(x_4 + 5y_4)(0) = \tilde{A}_3, \quad (a_4 + 5d_4) (r_B) = u_4 (A + 5D) + \tilde{A}_3,
\]
where \(\tilde{A}_3\) and \(\tilde{A}_3\) are given in terms of up to third order coefficients. This fixes the value of \(u_4\), hence uniquely defines \(\Phi_{K41}\) and \(\Phi_{K4B}\). \(\square\)

4 \textbf{On the control of } \(f_\perp\text{ and } f_\parallel\)

As orthonormal basis for the kernel of \(\tilde{L}\) in \(L^2_M(\mathbb{R}^3)\) we take \(\psi_0 = 1, \psi_\theta = v_\theta, \psi_r = v_r, \psi_z = v_z, \psi_4 = \frac{1}{\sqrt{6}}(\nu^2 - 3)\). Recall that in this paper all functions are even in \(v_z\). For functions \(f \in L^2_M((r_A, r_B) \times \mathbb{R}^3)\) we shall use the earlier splitting from the introduction into \(f = f_\parallel + f_\perp = P_0 f + (I - P_0) f\), where
\[
f_\parallel(r, v) = f_0 (r) - \frac{\sqrt{6}}{2} f_4 (r) + f_\theta (r) v_\theta + f_r (r) v_r + \frac{\sqrt{6}}{6} f_4 (r) v^2,
\]
\[
f_\perp(r, v) = f_4 (r) + \psi_4 (r) v^2,
\]
\[
\int M(v)(1, v, v^2) f_\perp(r, v) dv = 0, \\
\int M \psi_0 f(r, v) dv = f_0(r), \quad \int M \psi_4 f(r, v) dv = f_4(r), \\
\int M \psi_\theta f(r, v) dv = f_\theta(r), \quad \int M \psi_\phi f(r, v) dv = f_\phi(r).
\]

(The \( \psi_\theta \)-moment of \( f_\parallel \) vanishes since \( f \) is even in \( v_z \).) Define \( \tilde{\nu} := \nu \epsilon^4 \), and \( Df := v_r \frac{\partial f}{\partial r} + \frac{1}{r} N f \) with \( N \) given by (1.3). For \( 1 \leq q \leq +\infty \), denote by \( \| \cdot \|_q \) the usual Lebesgue norm and set
\[
\tilde{L}^q := \{ f; \| f \|_q := \left( \int M(v)\left( \int |f(x, v)|^q \,dx \right)^{\frac{1}{q}} \,dv \right)^{\frac{1}{q}} < +\infty \}.
\]

Due to the symmetries in the present setup, the position space may be changed from \( \mathbb{R}^2 \) with measure \( dx \), to \( \mathbb{R}^+ \) with measure \( r \,dr \). The relevant boundary space then becomes
\[
\tilde{L}^q := \{ f; \| f \|_\infty := \left( \int_{v_r \geq 0} v_r M(v) |f(r_A, v)|^2 \,dv \right)^{\frac{1}{2}} + \left( \int_{v_r < 0} |v_r| M(v) |f(r_B, v)|^2 \,dv \right)^{\frac{1}{2}} < +\infty \}.
\]

We shall also use
\[
\mathcal{W}^q_{\gamma^-}([r_A, r_B] \times \mathbb{R}^3) = \mathcal{W}^q := \{ f \text{ measurable on } [r_A, r_B] \times \mathbb{R}^3 : \tilde{\nu}^{\frac{1}{2}} f \in \tilde{L}^q, \tilde{\nu}^{-\frac{1}{2}} Df \in \tilde{L}^2, \gamma^+ f \in \tilde{L}^+ \}.
\]

**Lemma 4.1** For radial functions \( u \) in \( \mathbb{R}^2 \), define
\[
\| u \|_{q_{AB}} := \int_{r_A}^{r_B} |u| \,r \,dr, \quad \| \nabla_x u \|_{q_{AB}} := \int_{\mathbb{R}^2} |\nabla_x u| \,|x| \,dx = c \int_0^{r_B} |\partial_r u| \,r \,dr.
\]

It holds that
\[
\| u \|_{q_{AB}} \leq C \| \nabla_x u \|_q, \quad 1 \leq q < \infty.
\]

**Proof of Lemma 4.1.** The case \( q = 2 \) follows from [M2 (V.1.49)], and the case \( q = 1 \) is immediate (cf [Ma p.97]). Interpolation then gives the inequality for \( 1 \leq q \leq 2 \). From here the case \( q > 2 \) follows by a duality argument. \( \square \)

Define
\[
f_{\theta, r^2}(r) := \int M v_\phi^2 f_\perp(r, v) dv, \quad i + j \geq 2,
\]
and \( f_{\theta, r^2}(r) \) correspondingly, when there is an extra factor \( |v|^2 \) in the integrand.
Lemma 4.2. Given $f_0, f_\theta, f_r, f_A \in L^q(r_A, r_B)$, let $h$ be a solution in $\tilde{L}^q(r_A, r_B)$, $1 < q < \infty$, to the (dual) equation

$$-Dh = \frac{1}{\epsilon^4} \tilde{L}h - f_v, \quad r \in \mathbb{R}^+, \quad v \in \mathbb{R}^3,$$

(4.1)

with $f_v = f_0 \psi_0 + f_\theta \psi_\theta + f_r \psi_r + f_A \psi_A$ extended by zero outside $[r_A, r_B]$. Then

$$h_r(r_A) = h^\theta_r(r_A) = h_{r2}(r_A) = 0,$$

$$|\tilde{v}^2 h_{\perp}| [c | f_v |^2, \quad \| h_{\theta r} \|_{qAB} \leq \alpha^4 | f_v | |q|, \quad \| h_{r2} \|_{qAB} \leq \alpha^4 | f_v | |q|.$$ 

If $f_0 = 0$, then $h_r = 0$.

Proof of Lemma 4.2. The proof extensively uses a representation of $h = H + h^{(0)} + h^{(1)}$ from [M2 p.55-59], where the hydrodynamic moments $m_j$ are explicitly given. In the present setting we employ that representation with linearized collision operator $\tilde{L}$, and in particular $h^{(0)}$ including a scaling $\epsilon^{-4}$, giving $\|h\|_2 \leq \alpha^{-4} | f_v |_2$ with $c$ independent of $\epsilon$. Also notice that $h^{(0)}$ is orthogonal to $v_\psi v_r$. We may now use the inequality

$$-\int \tilde{L}hM dv dx \geq c |\tilde{v}^2 h_{\perp}|^2,$$

and Green’s formula on (4.1) to conclude that

$$|\tilde{v}^2 h_{\perp}|^{2} \leq \alpha^4 | f_v | |q| \int f_v |h|_2 M dv dx \leq c | f_v |^{2}. $$

It follows from the previous lemma that

$$\| h_{\theta r}^{(1)} \|_{qAB} \leq C \| \nabla_x h_{\theta r}^{(1)} \|_{q}.$$

The $h$-expansion satisfies

$$\nabla_x \tilde{h}_{\perp}^{(1)} = -(I - R_0) \epsilon^4 \mathcal{F} \frac{\xi}{|\xi|} l^{-1} \sum_{j=0}^{4} \psi_j v \frac{\xi}{|\xi|} \tilde{m}_j^{(0)},$$

where $\mathcal{F}$ denotes the Fourier transform with respect to the $x$-variable. Now $\frac{\xi}{|\xi|}$ is a Fourier multiplier in $L^q, 1 < q < \infty$ by Mikhlin’s theorem [Mi]. The representation of $\tilde{m}_j^{(0)}$ gives

$$\| \nabla_x h_{\theta r}^{(1)} \|_q \leq C \epsilon^4 \| f_v \|_q.$$

This proves
\[ \| h_{\theta r}^{(1)} \|_{q, AB} \leq C \varepsilon^4 \| f_l \|_q. \]

Multiplying the equation for \( H \) with \( M v_\theta \) and integrating, gives an equation implying that \( H_{\theta r} = \frac{\partial \phi}{\partial r} \). But \( \phi_{\theta r} = 0 \) since \( H_{\theta r} \in L^q_{\text{loc}} \). This concludes

the proof of the estimate for \( h_{\theta r} \). The estimate in the lemma for \( h_{r,2} \) is similarly proved. Using the equations for \( h_r, h_{\theta r}, h_{r,2} \) in \( L^q_{\text{loc}}(0, r_A) \), it follows that \( h_r(r_A) = h_{\theta r}(r_A) = h_{r,2}(r_A) = 0 \). From here, if \( f_0 = 0 \), then \( (r h_r)' = 0 \), hence \( r h_r = r_A h_r(r_A) = 0 \). The proof of the lemma is complete. \( \square \)

**Lemma 4.3** Let \( 2 \leq q < +\infty \), and let \( F \) be a solution in \( \mathcal{W}^{2-} \) to

\[ DF = \frac{1}{\varepsilon^4} (\tilde{L} F + \varepsilon u \tilde{J}(F, v_\theta) + g), \quad F_{|\partial \Omega^+} = f_b, \quad (4.2) \]

for \( g = g_\perp \). The following estimates hold for small enough \( \varepsilon > 0 \):

\[ | \tilde{v}^{\frac{1}{2}} F_{\perp} |_2 \leq c (| \tilde{v}^{\frac{1}{2}} g |_2 + \varepsilon | F_l |_2 + \varepsilon^2 | \tilde{v} \frac{1}{2} f_b |_\infty ), \quad (4.3) \]

\[ | \tilde{v}^{\frac{1}{2}} F |_\infty \leq c (| \tilde{v}^{\frac{1}{2}} g |_\infty + \varepsilon^{-\frac{3}{4}} | \tilde{v} \frac{1}{2} F |_q + | \tilde{v} \frac{1}{2} f_b |_\infty ). \quad (4.4) \]

**Proof of Lemma 4.3.** The estimate (4.3) for \( g = g_\perp \) and \( u = 0 \) follows from Green’s formula. The inclusion of \( c u \tilde{J}(F, v_\theta) \) to \( g \), adds \( \varepsilon | F_{\perp} |_2 \) which for \( \varepsilon \) small enough, is incorporated in the left hand side, and a term \( \alpha | F |_2 \).

We now turn to the estimate (4.4), again for \( g = g_\perp \). Employing [M2 p.101], \( F \) can for \( u = 0 \) be written as

\[ F = U_{\varepsilon} K' \frac{1}{\varepsilon^4} U_{\varepsilon} K' F + Z_1 F + Z_2 g + Z_3 W_{c \gamma}^+ F, \quad (4.5) \]

where

\[ | \tilde{v}^{\frac{1}{2}} U_{\varepsilon} K' \frac{1}{\varepsilon^4} U_{\varepsilon} K' F |_\infty \leq c_0 \varepsilon^{-\frac{3}{4}} | \tilde{v} \frac{1}{2} F |_q, \]

\[ | \tilde{v}^{\frac{1}{2}} U_{\varepsilon} K' \frac{1}{\varepsilon} U_{\varepsilon} K' F |_q \leq c_0 \varepsilon^{-\frac{3}{4}} | \tilde{v} \frac{1}{2} F |_2, \]

\[ | \tilde{v}^{\frac{1}{2}} Z_1 F |_q \leq c \delta | \tilde{v} \frac{1}{2} F |_q, \quad | \tilde{v}^{\frac{1}{2}} Z_2 g |_q \leq c | \tilde{v}^{\frac{1}{2}} g |_q, \]

\[ | \tilde{v}^{\frac{1}{2}} Z_3 W_{c \gamma}^+ F |_\infty \leq c | f_b |_\infty . \quad (4.6) \]

Using (4.5), (4.6) with \( \delta \) small enough gives (4.4). For \( \varepsilon \) small enough, the addition of \( \varepsilon u \tilde{J}(F, v_\theta) \) to \( g \) does not change the result. \( \square \)

**Lemma 4.4** Let \( 2 \leq q < +\infty \), and let \( F \) be a solution in \( \mathcal{W}^{2-} \) to (4.2) for \( g = g_\perp \). The hydrodynamic part \( F_{\parallel} \) of \( F \) can be split as \( F_{\parallel} = F_l + \frac{\phi_{\theta r}}{\varepsilon} v_r \), where \( F_r = \frac{\phi_{\theta r}}{\varepsilon} \). For small enough \( \varepsilon > 0 \)

\[ | F_l |_q \leq c \left( | F_{\perp} |_q + | \tilde{v}^{\frac{1}{2}} g |_q \right), \quad (4.7) \]

\[ \| F_r \|_q \leq c (\varepsilon^3 \| F_0 \|_q + | F_{\perp} |_q + \frac{1}{\varepsilon} \| \int g v_\theta v_r B(|v|) M dv \|_1 ). \]
Proof of Lemma 4.4. By interpolation, it is enough to prove the lemma when $q$ is an even integer. With $F$ a solution to (4.2), multiplying that equation with $M$ and integrating over $\mathbb{R}^3_0$, leads to

$$(rF_r)' = 0,$$

i.e. $F_r(r) = \frac{c_r}{r}$ for some constant $c_r$. In the same way a multiplication with $v_\theta M$ and integration over $\mathbb{R}^3_0$ leads to

$$F_{\theta r}(r) = \frac{c_{\theta r}}{r^2},$$

for some constant $c_{\theta r}$.

Multiplying equation (4.2) with $M v_r v_\theta \overline{B}(|v|)$ and integrating on $\mathbb{R}^3$, it follows that

$$\left(\frac{w_1 F_\theta}{r} + \frac{F_{\theta r^2 B}}{r^2} - \frac{F_{\theta^2 B} - 3 F_{\theta r^2 B}}{r^2}\right) + \frac{1}{r^3} \left(\frac{c_r}{r^2} + \frac{C_r}{r} w_1\right)$$

$$+ \epsilon \int \overline{J}(F_\perp, uv_\theta) v_\theta v_r \overline{B}(|v|) M dv + \int g v_\theta v_r \overline{B}(|v|) M dv \right). \quad (4.8)$$

Integrate (4.8) on $(r_1, r_2)$ for any $(r_1, r_2) \subset (r_A, r_B)$. Then integrate the new equality with respect to both variables $(r_1, r_2)$, on $(r_1, r_2) \in (r_A, \frac{r_A + r_B}{2}) \times (\frac{r_A + r_B}{2}, r_B)$ and on $(r_1, r_2) \in (r_A, \frac{3r_A + r_B}{4}) \times (\frac{3r_A + r_B}{4}, r_B)$. This provides a $2 \times 2$ invertible linear system in the variables $(\frac{c_r}{C_r}, \frac{C_r}{c_r})$ in terms of integrals of $F_{1, \theta}$, $F_{1, \perp}$ and $g$, giving

$$|c_r| \leq c(\epsilon^3 \| F_\theta \|_q + |F_\perp|_q + \frac{1}{\epsilon} \int g v_\theta v_r \overline{B}(|v|) M dv \|_1).$$

$$|c_{\theta r}| \leq c(\epsilon^4 \| F_\theta \|_q + \epsilon |F_\perp|_q + \| \int g v_\theta v_r \overline{B}(|v|) M dv \|_q).$$

The estimate for $F_r$ follows from here. To the solution $F$ of (4.2) with boundary value $f_B$, add a sum $S = C_\theta v_\theta + C_0 \psi_0 + C_1 \psi_1$. Choose the constants so that the three zero-conditions at $r_B$,

$$(f_0 + \sqrt{\frac{2}{3}} f_1 + f_2)(r_B) = (w_1 f_\theta + f_{\theta r^2 B})(r_B) =$$

$$(\sqrt{6} f_1 - 2 f_0 + C_1 f_{\theta^2 A} + 2 f_{\theta^2})(r_B) = 0,$$ \quad (4.9)

hold for $f = F + S$. Here $C_1$ is defined in (4.15) below. Notice that $f_r = F_r$, $f_\perp = F_\perp$ and that (4.8) together with the estimate for $c_r$ are satisfied also
with \( f \) replacing \( F \). Lemma 4.2 with \( f_\parallel = f_\theta \psi_\theta \) gives \((r^2 h_\theta r)' = r^2 f_\theta \) and \( h_\theta r \parallel q_{AB} \leq C \varepsilon^4 \parallel f_\theta \parallel_q \). Using (4.8) for \( f \), it follows that

\[
\begin{align*}
(r^2 h_\theta r(\frac{w_1 f_\theta}{r} + \frac{f_\theta r^2 B}{r} q^{-1} = \frac{r^3}{w_1} (\frac{w_1 f_\theta}{r} + \frac{f_\theta r^2 B}{r}) q + \\
- \frac{r^2}{w_1} f_\theta r^2 B(\frac{w_1 f_\theta}{r} + \frac{f_\theta r^2 B}{r}) q^{-1} + (q - 1) h_\theta r(\frac{w_1 f_\theta}{r} + \frac{f_\theta r^2 B}{r}) q^{-2} (f_\theta r^3 B - 3 f_\theta r^2 B) \\
+ \frac{1}{\varepsilon r} c_\delta u_{\theta r} + \frac{1}{\varepsilon} \int J(f_{\parallel, uv_\theta}) v_\theta v_r \tilde{B}(|v|) M \, dv \\
+ r \int g v_\theta v_r \tilde{B}(|v|) M \, dv)
\end{align*}
\]

(4.10)

In undifferentiated form, the left hand side vanishes at \( r_A \) since \( h_\theta r(r_A) = 0 \), and also at \( r_B \) because of (4.9). Using Hölder’s inequality and the \( c_\delta \)-estimate, it follows for \( \delta > 0 \) that

\[
\| w_1 f_\theta + f_\theta r^2 B \|_q \leq C \left( \frac{1}{\delta} \| f_{\parallel} \|_q + \delta \| f_\theta \|_q + \frac{1}{\delta} \| \tilde{v} - \frac{1}{2} \tilde{g} \|_q \right),
\]

and so

\[
\| f_\theta \|_q \leq C \left( \| f_{\parallel} \|_q + \| \tilde{v} - \frac{1}{2} \tilde{g} \|_q \right). \tag{4.11}
\]

From here the estimate for \( C_\theta \), hence \( F_\theta \), can be obtained as follows. Using \( h \) of Lemma 4.2, this time with \( f_\parallel = F_\theta \psi_\theta \) and \((r^2 h_\theta r)' = r^2 F_\theta \), together with (4.8) for \( F \), again gives a version of (4.10). Integrating this from \( r_A \) to \( r \) and then integrating over an interval in \( r \), gives an estimate for \( C_\theta \) implying

\[
\| C_\theta \| \leq C \left( \frac{1}{\delta} \| f_{\parallel} \|_q + \frac{1}{\delta} \| \tilde{v} - \frac{1}{2} \tilde{g} \|_q \right). \tag{4.12}
\]

The estimate for \( (\sqrt{6} f_4 - 2 f_0) \) starts from Lemma 4.2 with \( f_\parallel = f_0 \psi_0 + f_4 \psi_4 \) which gives \( h_{\theta 2} \parallel q_{AB} \leq C \varepsilon^4 \| f_0 \psi_0 + f_4 \psi_4 \|_q \) and \((r h_{\theta 2})' = \sqrt{6} r f_4 - 2 r f_0 \). Multiply equation (4.2) with \( \tilde{A}(|v|) v_r \tilde{M} \) respectively with \( v_r \tilde{M} \) and integrate. It follows that

\[
\begin{align*}
\left(k_4 f_4 + f_{\theta 2} \tilde{A}\right)' &= - \frac{1}{r} f_{\theta 2} \tilde{A} - f_\theta \tilde{A} + \\
&+ \frac{1}{\varepsilon r} c_\delta \left( f_{\theta 2} + \epsilon u \int v_r \tilde{A} \tilde{M} j(f_{\parallel, uv_\theta}) + \int g v_r \tilde{A}(|v|) M \, dv \right), \tag{4.13}
\end{align*}
\]

and

\[
(f_0 + \sqrt{2} f_4 + f_{\theta 2})' = \frac{f_{\theta 2} - f_\theta}{r}, \tag{4.14}
\]

where \( k_4 = \int v_r^2 \psi_4 \tilde{A} \tilde{M} \, dv \). Set

\[
C_1 = \frac{\sqrt{6}}{k_4}. \tag{4.15}
\]
Then (4.13-14) give
\[
(\sqrt{6} f_4 - 2 f_0 + C_1 f_{r^2} + 2 f_r^2)' = -\frac{1}{r} \left( C_1 f_{r^2} - C_1 f_{\theta^2} - 2 f_{\theta^2} + 2 f_r^2 \right) \\
+ \frac{C_1}{e^t} \left( f_r + c u \int v_r A M \tilde{J}(f_\perp, v_\theta) + \int g v_r A |v| M dv \right).
\] (4.16)

Similarly to the \( f_\theta \)-estimate above, it follows that
\[
\| \sqrt{6} f_4 - 2 f_0 \|_q \leq c \left( \frac{1}{\delta} \| f_\perp \|_q + \delta \| f_0 \psi_0 + f_4 \psi_4 \|_q + \frac{1}{\delta} \| \tilde{\nu}^{-\frac{1}{2}} g \|_q \right). (4.17)
\]

We next multiply (4.14) with \( q \left( f_0 + \sqrt{\frac{2}{3}} f_4 + f_r^2 \right)^{q-1} \) and integrate on \((r, r_B)\), then on \((r_A, r_B)\) to obtain
\[
\| f_0 \|_q \leq c \left( \frac{1}{\delta} \| f_\perp \|_q + \delta \| f_0 \psi_0 + f_4 \psi_4 \|_q + \frac{1}{\delta} \| \tilde{\nu}^{-\frac{1}{2}} g \|_q \right). \quad (4.18)
\]

The estimates for \( F_0, F_4, C_0, \) and \( C_4 \) can now be computed similarly to \( F_\theta \) and \( C_\theta \). Using \( h \) of Lemma 4.2, this time with \( f_\parallel = F_0 \psi_0 + F_4 \psi_4 \) and \((r h_{r^2})' = \sqrt{\delta} r F_4 - 2 r F_0\), together with (4.16) for \( F \), as previously gives an estimate for \( \sqrt{6} C_4 - 2 C_0 \),
\[
| \sqrt{6} C_4 - 2 C_0 | \leq C \left( \frac{1}{\delta} \| f_\perp \|_q + \frac{1}{\delta} \| \tilde{\nu}^{-\frac{1}{2}} g \|_q \right). \quad (4.19)
\]

An estimate for \( C_4 \) alone will use the same \( h_{r^2} \)-equation, but this time with \( f_\parallel = F_4 \psi_4 \), and start from
\[
(r h_{r^2} (f_0 + \sqrt{\frac{2}{3}} F_4 + f_r^2)', \quad (f_0 + \sqrt{\frac{2}{3}} F_4 + f_r^2) \sqrt{\delta} r F_4 + \frac{f_\parallel - f_{\theta^2}}{r} h_{r^2}.
\]

After suitable integrations with respect to \( r dr \), and invoking the previous estimates for \( f \), we obtain the following estimate for \( C_4 \),
\[
| C_4 | \leq C \left( \frac{1}{\delta} \| f_\perp \|_q + \frac{1}{\delta} \| \tilde{\nu}^{-\frac{1}{2}} g \|_q \right). \quad (4.20)
\]

The estimate for \( F_\parallel \) follows from (4.11), (4.12) and (4.17-20),
\[
| F_\parallel |_q \leq c \left( \| \tilde{\nu}^{-\frac{1}{2}} g \|_q + \| f_\perp \|_q \right).
\]

The proof of the lemma is complete. \( \square \)
Lemma 4.5 Let $u \in \mathbb{R}$, $g = g \perp$, $v^{-\frac{1}{2}}g \in \tilde{L}^\alpha$, $f_b \in L^+$, $2 \leq q < \infty$ be given. For small enough $\epsilon > 0$, there exists a unique solution $F \in \mathcal{W}^{q-}$ to

$$DF = \frac{1}{\epsilon^T}(\tilde{L}F + \epsilon u \tilde{J}(F, \psi) + g), \quad F_{/\partial \Omega^+} = f_b. \quad (4.21)$$

Proof of Lemma 4.5. By [M2 p68-69] there is a unique solution $F \in \mathcal{W}^{2-}$ for $u = 0$. Adding $\frac{\partial}{\partial \psi} \tilde{J}(F - f_r \psi, \psi)$ to the right hand side of (4.2) for $u = 0$, the equation is still solvable. This follows by an iteration scheme 'with the previous iterate in the $J$-term'. The scheme is contractive for small $\epsilon$ by the estimates in Lemmas 4.3-4. Notice that for $\epsilon$ fixed, the component $f_r = \tilde{f}_r$ of the iterates only is bounded. That requires a sub-sequence extraction to get convergence also for those components of the iterates. Hence the uniqueness is not immediate, but it follows from the previous estimates directly applied to the homogeneous equation. Next, adding $\frac{\partial}{\partial \psi} \tilde{J}(f_r \psi, \psi) = \frac{\partial}{\partial \psi} \tilde{J}(\tilde{f}_r \psi)$ is a compact perturbation, so the index is conserved. Again the previous estimates applied to the homogeneous equation imply uniqueness. We have thus proved the lemma for $u \neq 0$ and $q = 2$. Using [M2 p100], finally the result for a general $q$ follows from the result for $q = 2$. That completes the proof of the lemma. \[\square\]

5 The rest term.

In this section we discuss the rest term, when $(u_{\theta,A1} - u_{\theta,B1} r_B)(3 u_{\theta,A1} + u_{\theta,B1} r_B) > 0$ and $\Delta \leq \Delta_{Bf}$. Denote by $\psi$ the asymptotic expansion for $j_1 = 10$ (cf the end of Section 2),

$$\psi(r, v) = \sum_{i=1}^{10} \epsilon^i \psi_i.$$ 

The aim is to prove that there exists a rest term $R$, such that

$$f = M(1 + \psi + \epsilon^6 R)$$

is a solution to (1.1-2) with $M^{-1} f \in L^\infty$. Such a function $R$ would then be a solution to

$$DR = \frac{1}{\epsilon^6} \left( \tilde{L}R + 2 \tilde{J}(R, \psi) + \epsilon^6 \tilde{J}(R, R) + l \right), \quad (5.1)$$

where

$$l = \frac{1}{\epsilon^6} \left( \tilde{L} \psi + \tilde{J}(\psi, \psi) - \epsilon^4 D(\psi) \right).$$

We assume that the boundary value $f_b$ is satisfied by $M(1 + \psi)$ up to order seven in $\epsilon$, so that the boundary values for $R$ are of order two. We remark that constants that need not be precised in the asymptotics, will be 'compensated to correct value' by the rest term, as can be seen from the uniqueness.
arguments for the rest term below. Notice that $\psi^j$ can be constructed so that

$$D \sum_{i=1}^{j} \epsilon^{i} \psi^i = (I - H)D \sum_{i=1}^{j} \epsilon^{i} \psi^i,$$

hence correspondingly for $l$.

Let the sequence $(R^n)_{n \in \mathbb{N}}$ be defined by $R^0 = 0$, and

$$DR^{n+1} = \frac{1}{\epsilon^4} \left( LR^{n+1} + 2\hat{J}(R^{n+1}) + \sum_{i=1}^{5} \epsilon^{i} \psi^i + g^n \right),$$

$$R^{n+1}(1, v) = R_A(v), \quad v_r > 0, \quad R^{n+1}(r_B, v) = R_B(v), \quad v_r < 0. \quad (5.3)$$

In (5.2-3)

$$g^n := 2\hat{J}(R^n, \sum_{i=6}^{10} \epsilon^{i} \psi^i) + e^6 \hat{J}(R^n, R^n) + l,$$

$$\epsilon^6 R_A(v) := \epsilon^6 u_{\epsilon A_1 v_A - \frac{2}{\tau} u_{\psi_0 A_1} v_\psi} - 1 - \sum_{i=1}^{10} \epsilon^{i} \psi^i(r_A, v), \quad v_r > 0,$$

$$\epsilon^6 R_B(v) := \frac{1 + \omega_B}{(1 + \tau_B)} e^\frac{4}{2} \left( v^2 - \frac{2}{\tau_B} u_{\psi_0 A_1} v_\psi - (v_A - \epsilon u_{\psi_0 A_1})^2 + v^2 \right) - 1$$

$$- \sum_{i=1}^{10} \epsilon^{i} \psi^i(r_B, v), \quad v_r < 0.$$

The lemmas 4.3-5 also hold for this boundary value problem with analogous proofs, and the solutions are well defined when $\epsilon$ is small. We observe that $g^n = g^n_{\perp}$,

$$\int g^n Mdv = \int Mdv = 0, \quad \partial_r(r \int R^n u_r Mdv) = r \int Mdv = 0. \quad (5.4)$$

We now discuss existence based on the rest term iteration scheme (5.2-3).

**Theorem 5.1** For $\epsilon > 0$ small enough, there is a unique sequence $(R^n)$ of solutions to (5.2-3) in the set $X := \{ R; | \tilde{\nu}^2 R |_{\infty} \leq K, | \tilde{\nu}^2 R |_{2} \leq K \}$, for some constant $K > 0$. The sequence converges to an isolated solution of

$$DR = \frac{1}{\epsilon^4} \left( LR + 2\hat{J}(R, \psi) + e^6 \hat{J}(R, R) + l \right),$$

$$R(1, v) = R_A(v), \quad v_r > 0, \quad R(r_B, v) = R_B(v), \quad v_r < 0. \quad (5.6)$$

**Proof of Theorem 5.1.** In the case $n = 0$ and $g^0 = l$, notice that $l$ in (5.2) is of order five, and the boundary values in (5.3) are of order two in $\epsilon$. It follows from Lemmas 4.3-4 for $g^0 = l = l_{\perp}$, that uniformly with respect to $0 < \epsilon < \epsilon_0$,

$$| \tilde{\nu}^2 R^1 |_{2} \leq C \epsilon^4, \quad | \tilde{\nu}^2 R^1 |_{\infty} \leq C. \quad (5.7)$$

For $n \in \mathbb{N}$ in the equation (5.2) for $(R^{n+1} - R^n)$, the term $l$ has disappeared, $g^n = g^n_{\perp}$, and the incoming boundary values are zero. Writing $R^{n+1} = R^1 +$
\[ \sum_{j=1}^{n} (R^{n+1} - R^{j}) \text{ and using Lemmas 4.3-4, it follows that} \]
\[
\begin{align*}
| \hat{\nu}^{\frac{1}{2}} (R^{n+1} - R^n) |_2 + | \hat{\nu}^{\frac{1}{2}} (R^n - R^{n-1}) |_\infty & \leq C \epsilon \left( | \hat{\nu}^{\frac{1}{2}} (R^n - R^{n-1}) |_2 + | \hat{\nu}^{\frac{1}{2}} (R^n - R^{n-1}) |_\infty \right), \\
| \hat{\nu}^{\frac{1}{2}} R^{n+1} |_2 + | \hat{\nu}^{\frac{1}{2}} R^{n+1} |_\infty & \leq C \sum_{n=0}^{N} (C \epsilon)^j \leq K,
\end{align*}
\]
for all \( n \in \mathbb{N} \). And so, for \( \epsilon \) small enough, \( (\hat{\nu}^{\frac{1}{2}} R^n) \) converges in \( L^2 \) and \( L^\infty \) to some \( \hat{\nu}^{\frac{1}{2}} R \) with
\[
\begin{align*}
DR &= \frac{1}{\epsilon^2} \left( LR + 2J(R, \psi) + \epsilon^6 J(R, R) + l \right), \\
R(1, v) &= R_A(v), \quad v_r > 0, \quad R(r_B, v) = R_B(v), \quad v_r < 0.
\end{align*}
\]
(5.8) (5.9)

The contraction mapping argument guarantees that these solutions are isolated.
\[ \square \]

It follows from the previous proof that the hydrodynamic moments converge to solutions of the corresponding leading orders limiting fluid (Hilbert) equations, when \( \epsilon \) tends to zero.

Suppose \( f \) satisfies (5.10), (5.11) below. Proving that \( f^- = 0 \) will imply that \( f \) is a non-negative solution to (1.1-2). Using an \( L^q \)-version of Lemmas 4.3 with \( q \) large, we may extend the present contraction mapping set-up, and prove that any solution of this paper coincides with such a non-negative solution. That is left for an accompanying paper [AN6] in preparation about the problem (1.1-2), and where \( j_0 \) may be taken as four.

**Theorem 5.2** Let \( \Omega \) be a bounded set in \( \mathbb{R}^3 \), and \( f_b \) a non-negative function defined on the ingoing boundary \( \partial \Omega^+ \). If a function \( f \) such that \( M^{-1} f \in L^\infty (\Omega \times \mathbb{R}^3) \) satisfies
\[
\begin{align*}
v \cdot \nabla_x f &= Q(f^+, f^+) - ML(M^{-1} f^-), \quad (x, v) \in \Omega \times \mathbb{R}^3, \\
f &= f_b, \quad \partial \Omega^+, \quad (5.10)
\end{align*}
\]
then \( f^- = 0 \) and \( f = f^+ \) solves the boundary value problem
\[
\begin{align*}
v \cdot \nabla_x f &= Q(f, f), \quad \Omega \times \mathbb{R}^3, \\
f &= f_b, \quad \partial \Omega^+, \quad (5.11)
\end{align*}
\]

Proof of Theorem 5.2 The function \( F = M^{-1} f \) satisfies
\[
\begin{align*}
v \cdot \nabla_x F &= J(F^+, F^+) - L(F^-), \quad F = M^{-1} f_b, \quad \partial \Omega^+. 
\end{align*}
\]

Define \( J^+ \) and \( J^- \) by \( J(\varphi, \varphi) = J^+ (\varphi, \varphi) - J^- (\varphi, \varphi) \), where
\[
\begin{align*}
J^+(\varphi, \varphi)(v) &:= \int |v - v_*| |^3 b(\theta) M_* \varphi' \varphi_* d\omega, \\
J^-(\varphi, \varphi)(v) &:= \varphi(v) \int |v - v_*| |^3 b(\theta) M_* \varphi_* d\omega.
\end{align*}
\]

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Also, \( F^- \) satisfies
\[
-\nu \cdot \nabla_x F^- = \chi_{F^- \neq 0}(J^+(F^+; F^+) - L(F^-)),
\]
\[
F^- = 0, \quad \partial \Omega^+.
\] (5.12)

Multiplying (5.12) by \(-MF^-\), integrating on \(\Omega \times \mathbb{R}^3\) and using that
\[
- \int MF^- \chi_{F^- \neq 0} L(F^-) \, dv = - \int MF^- L(F^-) \, dv \geq c \int M \nu \mid (I - P_0)F^- \mid^2 \, dv,
\]
implies that
\[
\frac{1}{2} \int_{\partial \Omega^-} |v \cdot n| M(F^-)^2 + c \int_{\Omega \times \mathbb{R}^3} M \nu \mid (I - P_0)F^- \mid^2 \leq - \int MF^- \chi_{F^- \neq 0} J^+(F^+, F^+) \leq 0.
\]

It follows that with \(\partial \Omega^-\) the outgoing boundary,
\[
F^- = 0 \text{ on } \partial \Omega^- , \quad L(F^-) = 0.
\]

And so, \( F^- \) satisfies
\[
F^- = 0, \quad \partial \Omega^- \cup \partial \Omega^+, \quad v \cdot \nabla_x F^- \leq 0.
\]

This implies that \( F^- \) is identically zero. \( \square \)

6 Comments and remarks.

As mentioned in the introduction, the approach holds without change for the other cases of asymptotic expansion in the two-roll setup that are discussed in [S]. The following example illustrates the treatment in the upcoming types of situations.

Consider the equation (1.1) under the scaling
\[
v_r \frac{\partial f}{\partial r} + \frac{1}{r} N f = \frac{1}{\epsilon m} \tilde{Q}(f, f),
\]
\[
(6.1)
\]
for \( m = 3 \) and without the coupling \( \omega_B = \frac{\omega_B}{1 + \epsilon^2 \tau B_2} \left( \frac{r_B^2 - 1}{r_B^2} a_{\omega A1}^2 - \tau B_2 + \Delta \epsilon \right) \)
between the boundary parameters in (1.2). Assume the cylinders rotate in the same direction and that \( 1 < P_{SB2}/[(r_B^2 - 1)a_{\omega B1}^2] < (u_{\omega A1}/u_{\omega B1}r_B)^2 \). This guarantees an asymptotic expansion with positive (as well as one with negative) second order radial velocity \( u_{\omega B2} \), and one with third order radial velocity. For the positive one, take the asymptotic expansion \( \psi \) of Section 5 up to order nine, and the rest term \( R \) of order five in \( \epsilon \). The rest term analysis proceeds as in Section 5 and gives the following result.
Theorem 6.1 For $0 < \epsilon$ small enough, there is an isolated $L^1$-solution $f_\epsilon$ of the equation (6.1) for $m = 3$ with boundary conditions (1.2) and with positive second order radial velocity $u_{r,H2}$, for which

$$
\int M^{-1} \sup_{r \in [r_A, r_B]} \left| f_\epsilon(r, v) \right|^2 \, dv < +\infty.
$$

The hydrodynamic moments converge to solutions of the corresponding leading order (second order in $\epsilon$ for the radial velocity) limiting fluid equations, when $\epsilon$ tends to zero.

Also for the negative second order radial velocity $u_{r,H2}$, a second isolated solution can be obtained in the same way.

For the case of third order radial velocity, again take the asymptotic expansion $\psi$ in Section 5 up to order nine, and the rest term $R$ of order five in $\epsilon$. The rest term analysis proceeds as before.

Theorem 6.2 For $0 < \epsilon$ small enough, there is an isolated $L^1$-solution $f_\epsilon$ of the equation (6.1) for $m = 3$ with boundary conditions (1.2) and third order radial velocity $u_{r,H3}$ for which

$$
\int M^{-1} \sup_{r \in [r_A, r_B]} \left| f_\epsilon(r, v) \right|^2 \, dv < +\infty.
$$

The hydrodynamic moments converge to solutions of the corresponding leading order (third order in $\epsilon$ for the radial velocity moment) limiting fluid equations, when $\epsilon$ tends to zero.

Theorem 6.1-2 in particular demonstrate that three separate solutions to (1.1-2) coexist for these parameter values.

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