The stationary nonlinear Boltzmann equation in a Couette setting; isolated solutions and non-uniqueness.

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Abstract.
The stationary nonlinear Boltzmann equation is studied for forces including hard spheres, in a Couette setting between two coaxial, rotating cylinders with given maxwellian indata on the cylinders. The existence of isolated solutions together with a hydrodynamic limit control, is obtained using an asymptotic expansion and rest term in a frame with boundary layers. Depending on a parameter present in the boundary indata, two solutions connected at a regular turning point may coexist. Other situations with several coexisting solutions are also considered.

1 Introduction.

The asymptotic kinetic approach in a sharp mathematical form has its roots in works by Grad and Kogan in the 1960'ies (see [G1], [G2],[K] and references therein). A number of important results followed, concerning the nonlinear stationary Boltzmann equation in $\mathbb{R}^n$ in the close to equilibrium case ([G3], [Gu], [H], [UA] and others), where techniques of a general scope were used, such as contraction mappings (see also [EP]). Stationary problems in small domains were solved in a similar way (e.g. [P], [IS]), and the unique solvability of internal, stationary problems for the Boltzmann equation at large Knudsen numbers was likewise established (cf. [M1]). In [BCN1], a kinetic description of a gas between two plates at different temperatures and no mass flux was studied in the case of a small mean free path for the nonlinear stationary Boltzmann equation under diffuse reflection boundary conditions. Stationary, fully nonlinear hydrodynamic limits, were treated in the papers [ELM1-2].
Solutions to half-space problems for the Boltzmann equation play an important role as boundary layers in the study of hydrodynamic limits for such solutions to the Boltzmann equation when the mean free path tends to zero. This has been extensively studied in the linear context, using functional analytic and energy methods ([BCN2], [GP] and others).
In a perspective of asymptotic analysis and numerical studies, a wide range of

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questions of the above types have been addressed for the BGK and Boltzmann equations by the Kyoto group as well as by others (see the monograph [S] with extensive references).
Further away from equilibrium, weak compactness arguments have been employed instead of contraction mappings, and in the stationary case usually involving entropy dissipation control for the sharpest results. That was the case in the spatially n-dimensional Povzner and one space-dimensional Boltzmann papers [AN1], [AN2], and [AN3], to obtain stationary solutions via weak L^1-compactness under no other restrictions than Grad's angular cut-off. The basic compactness argument in those cases, is not fully available for the Boltzmann equation itself in more than one space dimension. However, in the spatially n-dimensional case the entropy dissipation estimate still allows different but weaker control mechanisms , which also lead to existence results (see [AN4]). There, in contrast to the earlier cases mentioned, complete results are so far only obtained when the velocities smaller in norm than some $\eta > 0$, are suppressed.
The present study is set in the close to equilibrium frame and gives a mathematically rigorous study of the stationary nonlinear Boltzmann equation between two coaxial cylinders A and B, with maxwellian ingoing boundary values. The problem is extensively treated from a numerical and asymptotic perspective in [S], to which we also refer for a more complete discussion of some details. The boundary values and the solutions are assumed to be axially and circumferentially uniform in the space variables. Then, with (r, θ, z) and (v_r, v_θ, v_z) respectively denoting the spatial cylindrical coordinates and the corresponding velocity coordinates, a distribution function may be written as $f = f(r, v_r, v_\theta, v_z)$, and the Boltzmann equation becomes

$$\begin{aligned} &v_r \frac{\partial f}{\partial r} + \frac{1}{r} N f = \frac{1}{\epsilon^4} \tilde{Q}(f, f), \\ &r \in (r_A, r_B), \quad (v_r, v_\theta, v_z) \in I\!R^3. \end{aligned} \tag{1.1}$$

The maxwellian boundary data under study are

$$\begin{aligned} f(r_A, v) &= (2\pi)^{-\frac{3}{2}} e^{\frac{1}{2}(-v_r^2 - (v_\theta - \epsilon u_{\theta A1})^2 - v_z^2)}, \quad v_r > 0, \\ f(r_B, v) &= (2\pi)^{-\frac{3}{2}} \frac{1 + \omega_B}{(1 + \tau_B)^{\frac{3}{2}}} e^{\frac{1}{2}(-\frac{1}{1+\tau_B}(v_r^2 + (v_\theta - \epsilon u_{\theta B1})^2 + v_z^2))}, \quad v_r < 0. \end{aligned} \tag{1.2}$$

Here

$$N f := v_\theta^2 \frac{\partial f}{\partial v_r} - v_\theta v_r \frac{\partial f}{\partial v_\theta}, \tag{1.3}$$

$$\tilde{Q}(f,f)(v) := \int_{I\!R^3\times S^2} B(v-v_*,\omega)(f(v')f(v'_*)-f(v)f(v_*))dv_*d\omega.$$

The kernel $B = |v-v_*|^\beta b(\theta)$, $b \in L^1_+(S^2)$, is assumed to satisfy (3.19) below and belong to the Grad class, that is with its terms suitably majorized by the corresponding ones for the hard sphere model (cf [M2]). Consider functions which are even in the axial velocity direction v_z. Take the radii as $r_A = 1$, $r_B > 1$, and let ϵ^4 denote the Knudsen number. The rotational velocities of the inner and outer cylinders are $u_{\theta A} = \epsilon u_{\theta A1}$ and $u_{\theta B} = \epsilon u_{\theta B1}$ respectively. The non-dimensional perturbed temperature and density are

$$\tau_B = \epsilon^2 \tau_{B2}, \quad \omega_B = \frac{\epsilon^2}{1+\epsilon^2\tau_{B2}}\Big(\frac{r_B^2-1}{r_B^2}u_{\theta A1}^2 - \tau_{B2} + \Delta\epsilon\Big),$$

where τ_{B2} is given and Δ is a parameter.

The main results of this paper are Theorem 1.1 about existence, and a positivity Theorem 5.2.

Theorem 1.1 *Assume that $(u_{\theta A1} - u_{\theta B1}r_B)(3u_{\theta A1} + u_{\theta B1}r_B) > 0$. There is a negative value Δ_{bif} of the parameter Δ, such that for $\Delta < \Delta_{bif}$ and $0 < \epsilon$ small enough, two isolated L^1-solutions f_ϵ^j, $j = 1,2$ of (1.1-2) coexist, and satisfy*

$$\int M^{-1} supess_{r\in(r_A,r_B)} \mid f_\epsilon^j(r,v) \mid^2 dv < +\infty.$$

The two solutions have different outward radial bulk velocities of order ϵ^3. For fixed ϵ, they converge to the same solution when Δ increases to Δ_{bif}. The solutions have rigorous hydrodynamic limits when $\epsilon \to 0$.

Remark. This existence result is based on a priori estimates of L^2-type, which are uniform in ϵ. The positivity of these solutions can be shown using an extension of the present set-up with a priori L^q-estimates of the solutions for sufficiently large $q > 2$. That is the topic of an accompanying paper [AN6], which also takes up other aspects on the present problem. Our approach has wider applicability. In particular, as discussed in Section 6 below, analogous results hold for all cases of the two-roll problem treated in [S]. We also expect the techniques developed here, to be useful in the study of related problems, such as the Taylor-Couette set-up of [SD1], the Benard asymptotics of [SD2], and the two-component gases of [ATT]. The existence problem far from equilibrium for the two-roll system of this paper, is studied by weak type methods in the paper [AN5].

Write $R = f_{rest} = P_0 f_{rest} + (I - P_0)f_{rest} = R_{||} + R_\perp$, where P_0 is the projection on the hydrodynamic part, and

$$f = M(1+\psi+\epsilon^{j_0}f_{rest}) \quad \text{with} \psi = \sum_1^{j_1}\epsilon^j\psi^j, M = (2\pi)^{-\frac{3}{2}}\exp{(-\frac{v^2}{2})}. \tag{1.4}$$

Here $\sum_1^{j_1} \epsilon^j \psi^j$ is the asymptotic expansion with certain boundary values equal to the terms of corresponding order in the ϵ-expansions of (1.2), and based on a splitting into interior Hilbert behaviour, and various types of boundary layers. The main part of the paper is devoted to a rigorous study of the rest term $R = f_{rest}$ using as ingoing boundary values what remains of (1.2), after the asymptotic expansion. The rest term problem is solved by a contraction mapping iteration.

Sections two and three are devoted to the asymptotic expansion, adapting the presentation in [S] to the needs of the present paper. For the convenience of the reader, the description of the asymptotic expansion for the two-roll system is fairly self-contained and includes particular details that are relevant here and in [AN6].
Section four discusses some a priori estimates for the rest term. A dual problem is first considered in full physical space with a priori estimates derived by multiplier techniques from Fourier-transformed, quite detailed information about the hydrodynamic moments. This gives sufficient information to estimate certain hydrodynamic moments of the original problem via a correspondingly detailed, duality based analysis. The rest of the hydrodynamics study requires a direct treatment, also building on a coupling with the asymptotic expansion. The estimate for the nonhydrodynamic part in L^2 is less involved and based on Green's formula.
Section five deals with the rest term; a study of the contraction mapping, a remark about the hydrodynamic limits, and a positivity discussion. This is based on the technical results obtained in the earlier sections. The final section contains existence results for other two-roll problems, where the approach applies essentially without modifications.

2 The frame.

Write the solution of (1.1) as $f = M(1+\Phi)$. Then the new unknown $\Phi(r, v_r, v_\theta)$ should be solution to

$$v_r \frac{\partial \Phi}{\partial r} + \frac{1}{r} N\Phi = \frac{1}{\epsilon^4}(\tilde{L}\Phi + \tilde{J}(\Phi, \Phi)), \tag{2.1}$$

$$\Phi(1, v) = e^{\frac{1}{2}(v_\theta^2 - (v_\theta - \epsilon u_{\theta A1})^2)} - 1, \quad v_r > 0, \tag{2.2}$$

$$\Phi(r_B, v) = \frac{1+\omega_B}{(1+\tau_B)^{\frac{3}{2}}} e^{\frac{1}{2}(v^2 - \frac{1}{1+\tau_B}(v_r^2 + (v_\theta - \epsilon u_{\theta B1})^2 + v_z^2))} - 1, \quad v_r < 0. \tag{2.3}$$

Here $\tilde{J}$ is the rescaled quadratic Boltzmann collision operator,

$$\tilde{J}(\Phi, \psi)(v) := \frac{1}{2}\int_{I\!R^3 \times S^2} B(v - v_*, \omega) M(v_*)(\Phi(v')\psi(v'_*) + \Phi(v'_*)\psi(v')$$
$$-\Phi(v_*)\psi(v) - \Phi(v)\psi(v_*)) dv_* d\omega,$$

and $\tilde{L}$ is this operator linearized around 1,

$$(\tilde{L}\Phi)(v) := \int_{I\!R^3 \times S^2} B(v - v_*, \omega) M(v_*)(\Phi(v') + \Phi(v'_*) - \Phi(v_*)$$

$$-\Phi(v))dv_*d\omega = \tilde{K}(\Phi) - \tilde{\nu}\Phi.$$

Denote by Φ_A and Φ_B the truncations up to order j_1 of $\Phi(1,v), v_r > 0$ and $\Phi(r_B,v), v_r < 0$ respectively, e.g. for $j_1 = 4$

$$\Phi_A(v) = \epsilon u_{\theta A1}v_\theta + \epsilon^2 \frac{u_{\theta A1}^2}{2}(-1+v_\theta^2)$$
$$+\epsilon^3\frac{u_{\theta A1}^3}{2}(-v_\theta + \frac{1}{3}v_\theta^3) + \epsilon^4\frac{u_{\theta A1}^4}{4}(\frac{1}{2} - v_\theta^2 + \frac{1}{6}v_\theta^4), \quad v_r > 0,$$
$$\Phi_B(v) = \epsilon u_{\theta B1}v_\theta + \epsilon^2(\frac{r_B^2-1}{2r_B^2}u_{\theta A1}^2 - \frac{5}{2}\tau_{B2} - \frac{1}{2}u_{\theta B1}^2 + \frac{1}{2}u_{\theta B1}^2 v_\theta^2 + \frac{1}{2}\tau_{B2}v^2)$$
$$+\epsilon^3(\Delta + u_{\theta B1}v_\theta(\frac{r_B^2-1}{r_B^2}u_{\theta A1}^2 - \frac{7}{2}\tau_{B2} - u_{\theta B1}^2 + \frac{1}{3}u_{\theta B1}^2 v_\theta^2 + \frac{1}{2}\tau_{B2}v^2))$$
$$+\epsilon^4(\frac{7}{4}u_{\theta B1}^2\tau_{B2} + \frac{1}{8}u_{\theta B1}^4 + \frac{27}{8}\tau_{B2}^2 - \frac{r_B^2-1}{4r_B^2}u_{\theta A1}^2 u_{\theta B1}^2 - \frac{3}{4}\frac{r_B^2-1}{r_B^2}u_{\theta A1}^2\tau_{B2} +$$
$$\Delta u_{\theta B1}v_\theta + \frac{1}{4}u_{\theta B1}^2(\frac{r_B^2-1}{r_B^2}u_{\theta A1}^2 - 7\tau_{B2})v_\theta^2 + \frac{1}{4}\tau_{B2}(\frac{r_B^2-1}{r_B^2}u_{\theta A1}^2 - 7\tau_{B2} - u_{\theta B1}^2)v^2$$
$$+\frac{1}{8}\tau_{B2}^2 v^4 + \frac{1}{24}u_{\theta B1}^4 v_\theta^4), \quad v_r < 0.$$

Denote by $(\Phi_{Ai})_{1\leq i\leq j}$ resp. $(\Phi_{Bi})_{1\leq i\leq j}$, the first to j-th order terms of Φ_A resp. Φ_B, with respect to ϵ.
Solutions Φ will be determined as in (1.4), an approximate solution ψ plus a rest term $R = f_{rest}$,

$$\Phi(r,v) = \psi(r,v) + \epsilon^{j_0}R(r,v),$$

where for $j_1 = 4$

$$\psi(r,v) = \epsilon\Big(\Phi_{H1}(r,v) + \Phi_{W1}(\frac{r-r_B}{\epsilon},v)\Big) + \epsilon^2\Big(\Phi_{H2}(r,v) + \Phi_{W2}(\frac{r-r_B}{\epsilon},v)\Big)$$
$$+\epsilon^3\Big(\Phi_{H3}(r,v) + \Phi_{W3}(\frac{r-r_B}{\epsilon},v) + \Phi_{K3A}(\frac{r-1}{\epsilon^4},v) + \Phi_{K3B}(\frac{r-r_B}{\epsilon^4},v)\Big)$$
$$+\epsilon^4\Big(\Phi_{H4}(r,v) + \Phi_{W4}(\frac{r-r_B}{\epsilon},v) + \Phi_{K4A}(\frac{r-1}{\epsilon^4},v)$$
$$+\Phi_{K4B}(\frac{r-r_B}{\epsilon^4},v)\Big), \tag{2.4}$$

with

$$\int \Phi_{H1}(.,v)(1,v_r,v^2)M(v)dv = \int \Phi_{W1}(.,v)(1,v_r,v^2)M(v)dv$$
$$= \int \Phi_{H2}(.,v)v_r M(v)dv = 0, \tag{2.5}$$
$$\lim_{\frac{r-r_B}{\epsilon}\to-\infty} \Phi_{Wi}(\frac{r-r_B}{\epsilon},v) = 0, \quad 1\leq i\leq 4, \tag{2.6}$$

$$\lim_{\frac{r-1}{\epsilon^4}\to+\infty} \Phi_{KiA}(\frac{r-1}{\epsilon^4},v)=0,$$
$$\lim_{\frac{r-r_B}{\epsilon^4}\to-\infty} \Phi_{KiB}(\frac{r-r_B}{\epsilon^4},v)=0, \quad 3\leq i\leq 4. \tag{2.7}$$

Here $(\epsilon\Phi_{H1}+\epsilon^2\Phi_{H2}+\epsilon^3\Phi_{H3}+\epsilon^4\Phi_{H4})(r,v)$ denotes the truncation up to fourth order of a formal expansion $\sum_{k\geq1}\epsilon^k\Phi_{Hk}(r,v)$. The sum $(\epsilon\Phi_{W1}+\epsilon^2\Phi_{W2})(\frac{r-r_B}{\epsilon},v)$ consists of correction terms allowing the boundary conditions to be satisfied to first and second orders. They correspond to a suction boundary layer at r_B. Supplementary boundary layers of Knudsen type, described by $\epsilon^3(\Phi_{K3A}(\frac{r-1}{\epsilon^4},v)+\Phi_{K3B}(\frac{r-r_B}{\epsilon^4},v))$ are required in order to have the boundary conditions satisfied at third and fourth orders. Similar expansions hold for $j_1>4$ (cf [S]), and will be used in Section 5 below for $j_1=10$. The boundary values up to order j in ϵ are satisfied by ψ, whereas $\psi^{j+1},...,\psi^{j_1}$ (in particular ψ^8, ψ^9 and ψ^{10} in Section 5 below) may be taken as the plain Hilbert expansion.

3 The asymptotic expansion.

We shall here give a fairly detailed discussion of the asymptotic expansion for $j_1=4$. Recall (see [D]) that $\tilde{L}(v_\theta v_r\bar{B})=-v_\theta v_r$, $\tilde{L}(v_r\bar{A})=v_r(v^2-5)$ for some functions $\bar{B}(|v|)$ and $\bar{A}(|v|)$, with $v_\theta v_r\bar{B}(|v|)$ and $v_r\bar{A}(|v|)$ bounded in the $(,)_M$-norm, and let

$$w_1:=\int v_r^2v_\theta^2\bar{B}Mdv, \quad w_2=\int v_r(v^2-5)\bar{A}Mdv.$$

Let $g(\eta,v)$ be the solution to the half-space problem

$$\begin{aligned}&v_r\frac{\partial g}{\partial v_r}=Lg, \quad \eta>0, \quad v\in I\!R^3,\\ &g(0,v)=0, \quad v_r>0,\\ &\int g(\eta,v)v_rM(v)dv=1, \quad a.a.\ \eta>0.\end{aligned} \tag{3.1}$$

By [BCN2] there are constants A, D, and E such that, (sub-)exponentially,

$$\lim_{\eta\to+\infty} g(\eta,v)=A+Dv^2+Ev_\theta+v_r. \tag{3.2}$$

Proposition 3.1
Assume that

$$(u_{\theta B1}r_B-u_{\theta A1})(u_{\theta B1}r_B+3u_{\theta A1})>0,$$

and set

$$\Delta_{bif}:=-\Big(2w_1\frac{r_B+1}{r_B^3}(A+5D)(r_Bu_{\theta B1}-u_{\theta A1})(r_Bu_{\theta B1}+3u_{\theta A1})\Big)^{\frac{1}{2}}.$$

For $\Delta>\Delta_{bif}$, there is no solution Φ in the family defined in (2.4-7).
For $\Delta=\Delta_{bif}$, there is a unique solution Φ in the family defined in (2.4-7).
For $\Delta<\Delta_{bif}$, there are two solutions Φ in the family defined in (2.4-7).

The proof below starts by assuming that the $\sum_{k\geq 1}\epsilon^k\Phi_{Hk}$ satisfies (2.1) formally. The boundary condition (2.2) can be satisfied up to order two, but not from third order. Instead, supplementary boundary layers at third and fourth orders, $\Phi_{K3A}(\frac{r-1}{\epsilon^4},v)$ and $\Phi_{K4A}(\frac{r-1}{\epsilon^4},v)$ are introduced so that

$$(\epsilon\Phi_{H1}+\epsilon^2\Phi_{H2})(r,v)+\epsilon^3(\Phi_{H3}(r,v)+\Phi_{K3A}(\frac{r-1}{\epsilon^4},v))$$
$$+\epsilon^4(\Phi_{H4}(r,v)+\Phi_{K4A}(\frac{r-1}{\epsilon^4},v)) \tag{3.3}$$

satisfies (2.1) and (2.2) up to fourth order. But (3.3) cannot satisfy the boundary conditions (2.3) even at first order. So $(\Phi_{Wk}(\frac{r-r_B}{\epsilon},v))_{k\geq 1}$ and $(\Phi_{KkB}(\frac{r-r_B}{\epsilon^4},v))_{k\geq 3}$ are introduced in order that

$$\sum_{k=1}^{2}\epsilon^k(\Phi_{Hk}(r,v)+\Phi_{Wk}(\frac{r-r_B}{\epsilon},v))$$
$$+\sum_{k=3}^{4}\epsilon^k\Big(\Phi_{Hk}(r,v)+\Phi_{Wk}(\frac{r-r_B}{\epsilon},v)+\Phi_{KkB}(\frac{r-r_B}{\epsilon^4},v) \tag{3.4}$$

satisfies (2.1) and (2.3) up to fourth order.

Proof of Proposition 3.1. Denote by $Y=\frac{r-r_B}{\epsilon}$, and let the expansions $\sum_{k\geq 1}\epsilon^k\Phi_{Hk}(r,v)$ and $\sum_{k\geq 1}\epsilon^k(\Phi_{Hk}(r_B,v)+\Phi_{Wk}(\frac{r-r_B}{\epsilon},v))$ formally satisfy (2.1). Then,

$$\begin{aligned}
&\tilde{L}\Phi_{H1}=\tilde{L}\Phi_{H2}+\tilde{J}(\Phi_{H1},\Phi_{H1})=\tilde{L}\Phi_{H3}+2\tilde{J}(\Phi_{H1},\Phi_{H2})\\
&=\tilde{L}\Phi_{H4}+2\tilde{J}(\Phi_{H1},\Phi_{H3})+\tilde{J}(\Phi_{H2},\Phi_{H2})=0,\\
&v_r\frac{\partial\Phi_{Hk-4}}{\partial r}+\frac{1}{r}N\Phi_{Hk-4}\\
&=\tilde{L}\Phi_{Hk}+\sum_{j=1}^{k-1}\tilde{J}(\Phi_{Hj},\Phi_{Hk-j}),\quad k\geq 5,
\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}
&\tilde{L}\Phi_{W1}=\tilde{L}\Phi_{W2}+\tilde{J}(\Phi_{W1},2\Phi_{H1}(r_B,.)+\Phi_{W1})\\
&=\tilde{L}\Phi_{W3}+2\tilde{J}(\Phi_{H1}(r_B,.)+\Phi_{W1},\Phi_{W2})+2\tilde{J}(\Phi_{W1},\Phi_{H2}(r_B,.)+Y\Phi'_{H1}(r_B,.))\\
&=\tilde{L}\Phi_{W4}+2\tilde{J}(\Phi_{W3},\Phi_{H1}(r_B,.)+\Phi_{W1})+\tilde{J}(\Phi_{W2},\Phi_{W2}+2\Phi_{H2}(r_B,.)\\
&+2Y\Phi'_{H1}(r_B,.))+2\tilde{J}(\Phi_{W1},\Phi_{H3}(r_B,.)+Y\Phi'_{H2}(r_B,.)\\
&+\frac{Y^2}{2}\Phi''_{H1}(r_B,.))-v_r\frac{\partial\Phi_{W1}}{\partial Y}=0,
\end{aligned}\tag{3.6}$$

$$\begin{aligned}
&v_r\frac{\partial\Phi_{Wk-3}}{\partial Y}+\frac{1}{r_B}\sum_{i=0}^{k-5}(-1)^i(\frac{Y}{r_B})^iN(\Phi_{Hk-4-i}(r_B,.)+\Phi_{Wk-4-i})\\
&=\tilde{L}\Phi_{Wk}+\sum_{j=1}^{k-1}\tilde{J}(2\Phi_{Hj}(r_B,.)+\Phi_{Wj},\Phi_{Wk-j}),\quad k\geq 5.
\end{aligned}\tag{3.7}$$

Taking the conditions (2.5) into account, equations (3.5) are equivalent to

$$\Phi_{H1}(r,v) = b_1(r)v_\theta,$$
$$\Phi_{H2} = a_2 + d_2 v^2 + b_2 v_\theta + \frac{1}{2}b_1^2 v_\theta^2,$$
$$\Phi_{H3} = a_3 + d_3 v^2 + b_3 v_\theta + c_3 v_r + b_1 d_2 v_\theta v^2 + b_1 b_2 v_\theta^2 + \frac{1}{6}b_1^3 v_\theta^3,$$
$$\Phi_{H4} = a_4 + d_4 v^2 + b_4 v_\theta + c_4 v_r + (b_1 d_3 + b_2 d_2) v_\theta v^2 + (b_1 b_3 + \frac{1}{2}b_2^2 - \frac{1}{2}b_1^2 a_2) v_\theta^2$$
$$+ b_1 c_3 v_r v_\theta + \frac{1}{2}b_1^2 b_2 v_\theta^3 + \frac{1}{2}d_2^2 v^4 + \frac{1}{24}b_1^4 v_\theta^4 + \frac{1}{2}b_1^2 d_2 v_\theta^2 v^2,$$

for some functions $a_i(r)$, $b_i(r)$, $c_i(r)$, $d_i(r)$, $1 \le i \le 4$. Equations (3.6) have solutions if and only if the following compatibility conditions hold,

$$\int \Big(v_r \frac{\partial \Phi_{Hi}}{\partial r} + \frac{1}{r}N\Phi_{Hi}\Big)(1, v^2 - 5, v_\theta, v_r)M(v)dv = 0, \quad i \ge 1.$$

They provide first-order differential equations for the functions $a_i(r)$, $b_i(r)$, $c_i(r)$ and $d_i(r)$, $i \ge 1$. In particular,

$$(rb_1)' = 0, \quad (10d_2 + b_1^2)' = 0,$$
$$(r^2 c_3 b_2)' = w_1 r^2 (b_1' - \frac{1}{r}b_1)' + (2w_1 - w_2) r (b_1' - \frac{1}{r}b_1),$$
$$(a_2 + 5d_2 + \frac{1}{2}b_1^2)' = \frac{1}{r}b_1^2, \tag{3.8}$$
$$(a_3 + 5d_3 + b_1 b_2)' = \frac{2}{r}b_1 b_2, \tag{3.9}$$
$$(rc_3)' = 0,$$
$$(a_4 + 5d_4 + b_1 b_3 + \frac{1}{2}b_2^2 - \frac{1}{2}b_1^2 a_2 + \frac{35}{2}d_2^2 + \frac{7}{2}b_1^2 d_2)'$$
$$= \frac{2}{r}(b_1 b_3 + \frac{1}{2}b_2^2 - \frac{1}{2}b_1^2 a_2) + \frac{1}{2r}b_1^4 + \frac{7}{r}b_1^2 d_2, \tag{3.10}$$
$$(rc_4)' = 0.$$

Together with the boundary conditions (2.3) at first and second orders, this implies that

$$b_1(r) = \frac{u_{\theta A1}}{r}, \quad a_2(r) = -\frac{1}{2}u_{\theta A1}^2 \frac{1}{r^2}, \quad b_2(r) = 0,$$
$$d_2(r) = \frac{1}{10}u_{\theta A1}^2 (1 - \frac{1}{r^2}), \quad c_3(r) = \frac{u_3}{r},$$

for some constant $u_3 \neq 0$. Moreover,

$$10u_3 d_3' + A_2 = 0, \quad (rb_3)' + A_2 = 0,$$

where A_i denotes an expression containing coefficients up to i-th order only. Taking the condition (2.6) into account, equations (3.7) are equivalent to

$$\Phi_{W1}(Y,v) = z_1(Y)v_\theta, \quad \Phi_{W2} = x_2 + y_2 v^2 + z_2 v_\theta + (b_1(r_B) z_1 + \frac{1}{2}z_1^2) v_\theta^2,$$

$$
\begin{aligned}
&\Phi_{W3} = x_3 + y_3 v^2 + z_3 v_\theta + t_3 v_r + (b_1(r_B) y_2 + z_1 y_2 + z_1 d_2(r_B)) v_\theta v^2 \\
&+ (b_1(r_B) z_2 + z_1 z_2 + z_1 b_2(r_B) + Y b_1'(r_B) z_1) v_\theta^2 \\
&+ (\frac{1}{2} b_1^2(r_B) z_1 + \frac{1}{2} b_1(r_B) z_1^2 + \frac{1}{6} z_1^3) v_\theta^3, \\
&\Phi_{W4} = x_4 + y_4 v^2 + z_4 v_\theta + t_4 v_r + z_1' v_r v_\theta \bar{B}(v) + ...,
\end{aligned}
$$

for some functions $x_i(Y)$, $y_i(Y)$, $z_i(Y)$, $t_i(Y)$, $1 \leq i \leq 4$. Equations (3.8) have solutions if and only if the following compatibility conditions hold

$$
\begin{aligned}
&\int \Big(v_r \frac{\partial \Phi_{Wk-3}}{\partial Y} + \frac{1}{r_B} \sum_{i=0}^{k-5} (-1)^i (\frac{Y}{r_B})^i N(\Phi_{Hk-4-i}(r_B,.) + \Phi_{Wk-4-i} \Big) (v^2 \\
&-5, v_\theta) M(v) dv = 0, \quad k \geq 5, \qquad (3.11)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int \Big(v_r \frac{\partial \Phi_{Wk-3}}{\partial Y} + \frac{1}{r_B} \sum_{i=0}^{k-5} (-1)^i (\frac{Y}{r_B})^i N(\Phi_{Hk-4-i}(r_B,.) + \\
&\Phi_{Wk-4-i} \Big) (1, v_r) M(v) dv = 0, \quad k \geq 5. \qquad (3.12)
\end{aligned}
$$

Equations (3.11) provide second-order differential equations for y_i (resp.z_i), $i \geq 1$, depending on t_j, c_j, $j \leq i+1$ (resp.$j \leq i+2$). In particular,

$$
\begin{aligned}
&w_1 z_1'' - \frac{u_3}{r_B} z_1' = 0, \\
&(x_2 + 5y_2 + b_1(r_B) z_1 + \frac{1}{2} z_1^2)' = 0, \qquad (3.13) \\
&w_2 y_2'' + \frac{10}{r_B} y_2' + A_1 = 0, \quad w_1 z_2'' - \frac{u_3}{r_B} z_2' + A_1 = 0, \\
&t_3' = 0, \\
&(x_3 + 5y_3 + b_1(r_B) z_2 + z_1 z_2 + z_1 b_2(r_B) + Y b_1'(r_B) z_1)' = \\
&\frac{1}{r_B} (2 b_1(r_B) z_1 + z_1^2), \qquad (3.14) \\
&w_2 y_3'' + \frac{10}{r_B} y_3' + A_2 = 0, \\
&w_1 z_3'' - \frac{u_3}{r_B} z_3' + \Big((b_1(r_B) + z_1)(c_5(r_B) + t_5) \Big)' + A_2 = 0, \\
&t_4' + \frac{1}{r_B} (t_3 + c_3(r_B)) + c_3'(r_B) = 0, \\
&(x_4 + 5y_4)' + A_3 = 0. \qquad (3.15)
\end{aligned}
$$

First the equations $t_3' = 0$ and $t_4' + \frac{1}{r_B} t_3 = 0$ together with the conditions $\lim_{-\infty} t_3 = 0$, $\lim_{-\infty} t_4 = 0$, imply that $t_3 = t_4 = 0$. Then as will be seen in Lemmas 3.1-2, the introduction of Knudsen boundary layers at third and fourth orders will fix the values of $x_i(0)$, $y_i(0)$ and $z_i(0)$, $i = 3, 4$, in terms of $c_i(r_B)$. Taking the fifth order coefficient $c_5(r_B) + t_5 = 0$, and $\lim_{-\infty} y_i = 0$ (resp. $\lim_{-\infty} z_i = 0$), this defines y_i (resp. z_i) in term of $c_i(r_B)$. Equations

(3.13)-(3.15) provide first-order linear differential equations for $x_i + 5y_i$, $i \geq 2$. Together with the conditions $\lim_{-\infty} x_i = 0$, this will fix the value of $c_i(r_B)$, $i = 3, 4$, hence define x_i and y_i. That will be done in Lemmas 3.1 and 3.2. $\square$

Lemma 3.1 *Set* $\eta = \frac{r-1}{\epsilon^4}$, $\mu = \frac{r-r_B}{\epsilon^4}$. *There are unique Knudsen boundary layers* $\Phi_{K3A}(\eta, v)$ *and* $\Phi_{K3B}(\mu, v)$, *and boundary values* $\Phi_{H3}(1, v)$ *and* $\Phi_{W3}(0, v)$ *such that*

$$\begin{aligned} & v_r \frac{\partial \Phi_{K3A}}{\partial \eta} = \tilde{L}\Phi_{K3A}, \quad \eta > 0, \quad v \in I\!R^3, \\ & \Phi_{K3A}(0, v) = \Phi_{A3}(v) - \Phi_{H3}(1, v), \; v_r > 0, \\ & \lim_{\eta \to +\infty} \Phi_{K3A}(\eta, v) = 0, \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} & v_r \frac{\partial \Phi_{K3B}}{\partial \mu} = \tilde{L}\Phi_{K3B}, \quad \mu < 0, \quad v \in I\!R^3, \\ & \Phi_{K3B}(0, v) = \Phi_{B3}(v) - \Phi_{H3}(r_B, v) - \Phi_{W3}(0, v), \; v_r < 0, \\ & \lim_{\mu \to -\infty} \Phi_{K3B}(\mu, v) = 0. \end{aligned} \tag{3.17}$$

The boundary layers fix the possible values of $a_3(1)$, $d_3(1)$, u_3, $b_3(1)$ *and* $x_3(0)$, $y_3(0)$, $z_3(0)$, *hence complete the definitions of* Φ_{H3} *and* Φ_{W3}.

Proof of Lemma 3.1. The function

$$\psi_{K3A}(\eta, v) := \Phi_{K3A}(\eta, v) - u_3(g - A - Dv^2 - Ev_\theta - v_r),$$

with g, A, D and E defined in (3.1-2) and u_3 still unknown, should satisfy

$$\begin{aligned} & v_r \frac{\partial \psi_{K3A}}{\partial \eta} = \tilde{L}\psi_{K3A}, \quad \eta > 0, \quad v \in I\!R^3, \\ & \psi_{K3A}(0, v) = u_3 A - a_3(1) + (u_3 D - d_3(1))v^2 \\ & +(u_3 E - \frac{1}{2}u_{\theta A1}^3 - b_3(1))v_\theta, \quad v_r > 0, \\ & \lim_{\eta \to +\infty} \psi_{K3A}(\eta, v) = 0. \end{aligned}$$

Hence,

$$a_3(1) = u_3 A, \quad d_3(1) = u_3 D, \quad b_3(1) = u_3 E - \frac{1}{2}u_{\theta A1}^3, \quad \psi_{K3A} = 0,$$

so that

$$\Phi_{K3A} = u_3(g - A - Dv^2 - Ev_\theta - v_r).$$

Analogously, the function

$$\psi_{K3B}(\mu, v) := \Phi_{K3B}(\mu, v) - \frac{u_3}{r_B}(g(-\mu, -v) - A - Dv^2 + Ev_\theta + v_r),$$

should satisfy

$$
\begin{aligned}
& v_r \frac{\partial \psi_{K3B}}{\partial \mu} = \tilde{L}\psi_{K3B}, \quad \mu < 0, \quad v \in I\!R^3, \\
& \psi_{K3B}(0,v) = \Delta - \frac{u_3}{r_B}A - a_3(r_B) - x_3(0) - (\frac{u_3}{r_B}D + d_3(r_B) + y_3(0))v^2 \\
& + \Big(u_{\theta B1}(\frac{r_B^2 - 1}{r_B^2}u_{\theta A1}^2 - \frac{7}{2}\tau_{B2} - u_{\theta B1}^2) + \frac{u_3}{r_B}E - b_3(r_B) - z_3(0)\Big)v_\theta, \quad v_r < 0, \\
& \lim_{\mu \to -\infty} \psi_{K3B}(\mu, v) = 0.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& x_3(0) = \Delta - \frac{u_3}{r_B}A - a_3(r_B), \quad y_3(0) = -\frac{u_3}{r_B}D - d_3(r_B), \\
& z_3(0) = -u_{\theta B1}(\frac{r_B^2 - 1}{r_B^2}u_{\theta A1}^2 - \frac{7}{2}\tau_{B2} - u_{\theta B1}^2) + \frac{u_3}{r_B}E - b_3(r_B), \quad \psi_{K3B} = 0,
\end{aligned}
$$

and

$$
\Phi_{K3B}(\mu, v) = \frac{u_3}{r_B}(g(-\mu, -v) - A - Dv^2 + Ev_\theta + v_r).
$$

Moreover, by integration of (3.14) and (3.9) on $(-\infty, 0)$ and $(1, r_B)$ respectively,

$$
\begin{aligned}
& x_3(0) + 5y_3(0) = \frac{w_1}{2r_B^2 u_3}(u_{\theta B1}r_B + 3u_{\theta A1}r_B)(u_{\theta B1}r_B - u_{\theta A1}), \\
& x_3(0) + 5y_3(0) = \Delta - u_3(A + 5D)(\frac{1}{r_B} + 1).
\end{aligned}
$$

And so, u_3 must solve the equation

$$
u_3^2(A + 5D)\frac{r_B + 1}{r_B} - \Delta u_3 + \frac{w_1}{2r_B^2}(3u_{\theta A1} + u_{\theta B1}r_B)(u_{\theta A1} - u_{\theta B1}r_B) = 0. \tag{3.18}
$$

A study of the positive roots u_3 to (3.18) leads to the three cases described in Proposition 3.1 for Δ with respect to Δ_{bif} . $\square$

Remark. The above proof requires

$$
A + 5D \neq 0, \tag{3.19}
$$

a condition satisfied for hard spheres, and assumed to hold for the kernels B in this paper, together with the Grad class condition.

Lemma 3.2 *Set* $\eta = \frac{r-1}{\epsilon^4}$, $\mu = \frac{r - r_B}{\epsilon^4}$. *There are unique Knudsen boundary layers* $\Phi_{K4A}(\eta, v)$ *and* $\Phi_{K4B}(\mu, v)$, *and boundary values* $\Phi_{H4}(1, v)$ *and* $\Phi_{W4}(0, v)$ *such that*

$$
\begin{aligned}
& v_r \frac{\partial \Phi_{K4A}}{\partial \eta} = \tilde{L}\Phi_{K4A} + 2\tilde{J}(\Phi_{H1}(1), \Phi_{K3A})), \quad \eta > 0, \ v \in I\!R^3, \\
& \Phi_{K4A}(0, v) = \Phi_{A4}(v) - \Phi_{H4}(1, v), \quad v_r > 0, \\
& \lim_{\eta \to +\infty} \Phi_{K4A} = 0,
\end{aligned}
$$

and

$$\begin{aligned}
&v_r \frac{\partial \Phi_{K4B}}{\partial \mu} = \tilde{L}\Phi_{K4B} + 2\tilde{J}(\Phi_{H1}(r_B) + \Phi_{W1}(0), \Phi_{K3B}), \quad \mu < 0,\ v \in I\!R^3,\\
&\Phi_{K4B}(0,v) = \Phi_{B4}(v) - \Phi_{H4}(r_B, v) - \Phi_{W4}(0,v), \quad v_r < 0,\\
&\lim_{\mu \to -\infty} \Phi_{K4B} = 0.
\end{aligned}$$

<u>Proof of Lemma 3.2.</u> Analogously to [BCN2], there are unique solutions α and β to

$$\begin{aligned}
&v_r \frac{\partial \alpha}{\partial \eta} = \tilde{L}\alpha + 2\tilde{J}(\Phi_{H1}(1), \Phi_{K3A}), \quad \eta > 0,\ v \in I\!R^3,\\
&\alpha(0,v) = -u_{\theta A1}(u_3 D v_\theta v^2 + (u_3 E + \frac{1}{4}u_{\theta A1}^3)v_\theta^2 + u_3 v_r v_\theta), \quad v_r > 0,\\
&\int v_r \alpha(\eta, v) M(v) dv = 0,
\end{aligned}$$

and

$$\begin{aligned}
&v_r \frac{\partial \beta}{\partial \eta} = \tilde{L}\beta + 2\tilde{J}(\Phi_{H1}(r_B, -v) + \Phi_{W1}(0,-v), \Phi_{K3B}(-\eta, -v)), \quad \eta > 0,\ v \in I\!R^3,\\
&\beta(0,v) = \Phi_{B4}(-v) - (\Phi_{H4}(r_B, -v) - a_4(r_B) - d_4(r_B)v^2 - b_4(r_B)v_\theta - \frac{u_4}{r_B}v_r)\\
&-(\Phi_{W4}(0,-v) - x_4(0) - y_4(0)v^2 - z_4(0)v_\theta), \quad v_r > 0,\\
&\int v_r \beta(\eta, v) M(v) dv = 0.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\alpha \in Ker\tilde{L},\ \beta \in Ker\tilde{L}^\perp,\\
&\lim_{\eta \to +\infty} \alpha(\eta, v) = a_\infty + d_\infty v^2 + b_\infty v_\theta,\ \lim_{\eta \to +\infty} \beta(\eta, v) = r_\infty + s_\infty v^2 + t_\infty v_\theta,
\end{aligned}$$

for some constants a_∞, d_∞, b_∞, r_∞, s_∞ and t_∞. The function

$$\begin{aligned}
&\psi_{K4A}(\eta, v) := \Phi_{K4A}(\eta, v) - u_4(g - A - Dv^2 - Ev_\theta - v_r)\\
&-(\alpha - a_\infty - d_\infty v^2 - b_\infty v_\theta)
\end{aligned}$$

should satisfy

$$\begin{aligned}
&v_r \frac{\partial \psi_{K4A}}{\partial \eta} = \tilde{L}\psi_{K4A}, \quad \eta > 0, \quad v \in I\!R^3,\\
&\psi_{K4A}(0,v) = \frac{1}{8}u_{\theta A1}^2 + a_\infty + u_4 A - a_4(1) + (d_\infty + u_4 D - d_4(1))v^2\\
&+(b_\infty + u_4 E - b_4(1))v_\theta, \quad v_r < 0,\\
&\lim_{\mu \to -\infty} \psi_{K4A} = 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
&a_4(1) = \frac{1}{8}u_{\theta A1}^2 + a_\infty + u_4 A, \quad d_4(1) = d_\infty + u_4 D, \quad b_4(1) = b_\infty + u_4 E,\\
&\psi_{K4A} = 0,
\end{aligned}$$

so that

$$\Phi_{K4A} = \alpha - a_\infty - d_\infty v^2 - b_\infty v_\theta + u_4(g - A - Dv^2 - Ev_\theta - v_r).$$

Analogously, the function

$$\psi_{K4B}(\mu, v) := \Phi_{K4B}(\mu, v) - \frac{u_4}{r_B}(g(-\mu, -v) - A - Dv^2 + Ev_\theta - v_r)$$
$$-(\beta(-\mu, -v) - r_\infty - s_\infty v^2 + t_\infty v_\theta)$$

should satisfy

$$v_r \frac{\partial \psi_{K4B}}{\partial \mu} = \tilde{L}\psi_{K4B}, \quad \mu < 0, \quad v \in I\!R^3,$$
$$\psi_{K4B}(0, v) = r_\infty + \frac{u_4}{r_B}A - a_4(r_B) - x_4(0)$$
$$+(s_\infty + \frac{u_4}{r_B}D - d_4(r_B) - y_4(0))v^2 - (t_\infty + \frac{u_4}{r_B}E + b_4(r_B) + z_4(0))v_\theta, \quad v_r < 0,$$
$$\lim_{\mu \to -\infty} \psi_{K4B}(\mu, v) = 0.$$

Hence,

$$x_4(0) = r_\infty - a_4(r_B) + u_4\frac{A}{r_B}, \quad y_4(0) = s_\infty - d_4(r_B) + u_4\frac{D}{r_B},$$
$$z_4(0) = t_\infty - b_4(r_B) + u_4\frac{E}{r_B}, \quad \psi_{K4B} = 0,$$

so that

$$\Phi_{K4B}(\mu, v) = \beta(-\mu, -v) - r_\infty - s_\infty v^2 + t_\infty v_\theta$$
$$+\frac{u_4}{r_B}(g(-\mu, -v) - A - Dv^2 + Ev_\theta).$$

Moreover, by integration of (3.15) and (3.10) on $(-\infty, 0)$ and $(1, r_B)$ respectively,

$$(x_4 + 5y_4)(0) = \bar{A}_3, \quad (a_4 + 5d_4)(r_B) = u_4(A + 5D) + \tilde{A}_3,$$

where $\bar{A}_3$ and $\tilde{A}_3$ are given in terms of up to third order coefficients. This fixes the value of u_4, hence uniquely defines Φ_{K4A} and Φ_{K4B}. □

4 On the control of $f_\perp$ and $f_{||}$

As orthonormal basis for the kernel of $\tilde{L}$ in $L^2_M(I\!R^3)$ we take $\psi_0 = 1, \psi_\theta = v_\theta, \psi_r = v_r, \psi_z = v_z, \psi_4 = \frac{1}{\sqrt{6}}(v^2 - 3)$. Recall that in this paper all functions are even in v_z. For functions $f \in L^2_M((r_A, r_B) \times I\!R^3)$ we shall use the earlier splitting from the introduction into $f = f_{||} + f_\perp = P_0 f + (I - P_0)f$, where

$$f_{||}(r, v) = f_0(r) - \frac{\sqrt{6}}{2}f_4(r) + f_\theta(r)v_\theta + f_r(r)v_r + \frac{\sqrt{6}}{6}f_4(r)v^2,$$

$$\int M(v)(1,v,v^2)f_\perp(r,v)dv = 0,$$
$$\int M\psi_0 f(r,v)dv = f_0(r),\ \int M\psi_4 f(r,v)dv = f_4(r),$$
$$\int M\psi_\theta f(r,v)dv = f_\theta(r),\ \int M\psi_r f(r,v)dv = f_r(r).$$

(The ψ_z-moment of $f_{||}$ vanishes since f is even in v_z.) Define $\tilde{\nu} := \nu\epsilon^4$, and $Df := v_r\frac{\partial f}{\partial r} + \frac{1}{r}Nf$ with N given by (1.3). For $1 \leq q \leq +\infty$, denote by $\| \,.\, \|_q$ the usual Lebesgue norm and set

$$\tilde{L}^q := \{f; | f |_q := \Big(\int M(v)(\int | f(x,v) |^q dx)^{\frac{2}{q}} dv\Big)^{\frac{1}{2}} < +\infty\}.$$

Due to the symmetries in the present setup, the position space may be changed from $I\!R^2$ with measure dx, to $I\!R^+$ with measure rdr. The relevant boundary space then becomes

$$L^+ := \{f; | f |_\sim = \Big(\int_{v_r>0} v_r M(v) | f(r_A,v) |^2 dv\Big)^{\frac{1}{2}} +$$
$$\Big(\int_{v_r<0} | v_r | M(v) | f(r_B,v) |^2 dv\Big)^{\frac{1}{2}} < +\infty\}.$$

We shall also use

$$\mathcal{W}^{q-}([r_A,r_B]\times I\!R^3) = \mathcal{W}^{q-} :=$$
$$\{f \text{ measurable on } [r_A,r_B]\times I\!R^3\,; \tilde{\nu}^{\frac{1}{2}} f \in \tilde{L}^q, \tilde{\nu}^{-\frac{1}{2}} Df \in \tilde{L}^q, \gamma^+ f \in L^+\}.$$

Lemma 4.1 *For radial functions u in $I\!R^2$, define*

$$\| u \|_{qAB}^q := \int_{r_A}^{r_B} | u |^q rdr, \quad \| \nabla_x u \|_q^q := \int_{I\!R^2} | \nabla_x u |^q dx = c\int_0^\infty | \partial_r u |^q rdr.$$

It holds that

$$\| u \|_{qAB} \leq C \| \nabla_x u \|_q, \quad 1 \leq q < \infty.$$

Proof of Lemma 4.1. The case $q = 2$ follows from [M2 (V.1.49)], and the case $q = 1$ is immediate (cf [Ma p.97]). Interpolation then gives the inequality for $1 \leq q \leq 2$. From here the case $q > 2$ follows by a duality argument. □

Define

$$f_{\theta^i r^j}(r) := \int M v_\theta^i v_r^j f_\perp(r,v)dv,\ i+j \geq 2,$$

and $f_{\theta^i r^j 2}(r)$ correspondingly, when there is an extra factor $| v |^2$ in the integrand.

Lemma 4.2 *Given $f_0, f_\theta, f_r, f_4 \in L^q(r_A, r_B)$, let h be a solution in $\tilde{L}^q_{r,loc}(I\!R^+ \times I\!R^3)$, $1 < q < \infty$, to the (dual) equation*

$$-Dh = \frac{1}{\epsilon^4}\tilde{L}h - f_{||}, \quad r \in I\!R^+, \quad v \in I\!R^3, \tag{4.1}$$

with $f_{||} = f_0\psi_0 + f_\theta\psi_\theta + f_r\psi_r + f_4\psi_4$ extended by zero outside $[r_A, r_B]$. Then

$$h_r(r_A) = h_{\theta r}(r_A) = h_{r2}(r_A) = 0,$$
$$| \tilde{\nu}^{\frac{1}{2}} h_\perp |_2 \leq c \mid f_{||} |_2, \quad \| h_{\theta r} \|_{qAB} \leq c\epsilon^4 \mid f_{||} |_q, \quad \| h_{r2} \|_{qAB} \leq c\epsilon^4 \mid f_{||} |_q .$$

If $f_0 = 0$, then $h_r = 0$.

<u>Proof of Lemma 4.2</u> The proof extensively uses a representation of $h = H + h^{(0)} + h^{(1)}$ from [M2 p.55-59], where the hydrodynamic moments m_j are explicitly given. In the present setting we employ that representation with linearized collision operator $\tilde{L}$, and in particular $h^{(0)}$ including a scaling ϵ^{-4}, giving $| h_{||} |_2 \leq c\epsilon^{-4} \mid f_{||} |_2$ with c independent of ϵ. Also notice that $h^{(0)}$ is orthogonal to $v_\theta v_r$. We may now use the inequality

$$-\int h\tilde{L}hMdvdx \geq c \mid \tilde{\nu}^{\frac{1}{2}} h_\perp |_2^2,$$

and Green's formula on (4.1) to conclude that

$$| \tilde{\nu}^{\frac{1}{2}} h_\perp |_2^2 \leq c\epsilon^4 \left| \int f_{||} h_{||} Mdvdx \right| \leq c \mid f_{||} |_2^2 .$$

It follows from the previous lemma that

$$\| h^{(1)}_{\theta r} \|_{qAB} \leq C \| \nabla_x h^{(1)}_{\theta r} \|_q .$$

The h-expansion satisfies

$$\hat{\nabla}_x \hat{h}^{(1)}_\perp = -(I - P_0)\epsilon^4 \frac{\xi}{\mid \xi \mid} \tilde{L}^{-1} \sum_{j=0}^{4} \psi_j v \frac{\xi}{\mid \xi \mid} \hat{m}^{(0)}_j,$$

where $\hat{.} = \mathcal{F}$ denotes the Fourier transform with respect to the x-variable. Now $\frac{\xi}{|\xi|}$ is a Fourier multiplier in $L^q, 1 < q < \infty$ by Mikhlin's theorem [Mi]. The representation of $\hat{m}^{(0)}_j$ gives

$$\| \nabla_x h^{(1)}_{\theta r} \|_q \leq C\epsilon^4 \| f_{||} \|_q .$$

This proves

$$\| h_{\theta r}^{(1)} \|_{qAB} \leq C\epsilon^4 \| f_{||} \|_q .$$

Multiplying the equation for H with Mv_θ and integrating, gives an equation implying that $H_{\theta r} = \frac{c_{\theta r}}{r^2}$. But $c_{\theta r} = 0$ since $H_{\theta r} \in \tilde{L}^q_{\text{loc}}$. That concludes the proof of the estimate for $h_{\theta r}$. The estimate in the lemma for h_{r2} is similarly proved. Using the equations for $h_r, h_{\theta r}, h_{r2}$ in $\tilde{L}^q_{\text{loc}}(0, r_A)$, it follows that $h_r(r_A) = h_{\theta r}(r_A) = h_{r2}(r_A) = 0$. From here, if $f_0 = 0$, then $(rh_r)' = 0$, hence $rh_r = r_A h_r(r_A) = 0$. The proof of the lemma is complete. □

Lemma 4.3 *Let $2 \leq q < +\infty$, and let F be a solution in $\mathcal{W}^{2-}$ to*

$$DF = \frac{1}{\epsilon^4}(\tilde{L}F + \epsilon u\tilde{J}(F, v_\theta) + g), \quad F_{/\partial\Omega^+} = f_b, \tag{4.2}$$

for $g = g_\perp$. The following estimates hold for small enough $\epsilon > 0$;

$$| \tilde{\nu}^{\frac{1}{2}} F_\perp |_2 \leq c(| \tilde{\nu}^{-\frac{1}{2}} g |_2 + \epsilon | F_{||} |_2 + \epsilon^2 | \tilde{\nu}^{\frac{1}{2}} f_b |_\sim), \tag{4.3}$$

$$| \tilde{\nu}^{\frac{1}{2}} F |_\infty \leq c(| \tilde{\nu}^{-\frac{1}{2}} g |_\infty + \epsilon^{-\frac{8}{q}} | \tilde{\nu}^{\frac{1}{2}} F |_q + | \tilde{\nu}^{\frac{1}{2}} f_b |_\sim). \tag{4.4}$$

<u>Proof of Lemma 4.3.</u> The estimate (4.3) for $g = g_\perp$ and $u = 0$ follows from Green's formula. The inclusion of $\epsilon u\tilde{J}(F, v_\theta)$ to g, adds $\epsilon \mid F_\perp \mid_2$ which for ϵ small enough, is incorporated in the left hand side, and a term $c\epsilon \mid F_{||} \mid_2$.

We now turn to the estimate (4.4), again for $g = g_\perp$. Employing [M2 p.101], F can for $u = 0$ be written as

$$F = U_\epsilon \frac{K'}{\epsilon^4} U_\epsilon \frac{K'}{\epsilon^4} F + Z_1 F + Z_2 g + Z_3 W_\epsilon \gamma^+ F, \tag{4.5}$$

where

$$\begin{aligned} &| \tilde{\nu}^{\frac{1}{2}} U_\epsilon \frac{K'}{\epsilon^4} U_\epsilon \frac{K'}{\epsilon^4} F |_\infty \leq c_\delta \epsilon^{-\frac{8}{q}} | \tilde{\nu}^{\frac{1}{2}} F |_q, \\ &| \tilde{\nu}^{\frac{1}{2}} U_\epsilon \frac{K'}{\epsilon^4} U_\epsilon \frac{K'}{\epsilon^4} F |_q \leq c_\delta \epsilon^{(\frac{8}{q}-4)} | \tilde{\nu}^{\frac{1}{2}} F |_2, \\ &| \tilde{\nu}^{\frac{1}{2}} Z_1 F |_q \leq c\delta | \tilde{\nu}^{\frac{1}{2}} F |_q, \quad | \tilde{\nu}^{\frac{1}{2}} Z_2 g |_q \leq c | \tilde{\nu}^{-\frac{1}{2}} g |_q, \\ &| \tilde{\nu}^{\frac{1}{2}} Z_3 W_\epsilon \gamma^+ F |_\infty \leq c | f_b |_\sim . \end{aligned} \tag{4.6}$$

Using (4.5), (4.6) with δ small enough gives (4.4). For ϵ small enough, the addition of $\epsilon u\tilde{J}(F, v_\theta)$ to g does not change the result. □

Lemma 4.4 *Let $2 \leq q < +\infty$, and let F be a solution in $\mathcal{W}^{q-}$ to (4.2) for $g = g_\perp$. The hydrodynamic part $F_{||}$ of F can be split as $F_{||} = F_{I||} + \frac{c_r}{r} v_r$, where $F_r = \frac{c_r}{r}$. For small enough $\epsilon > 0$*

$$\begin{aligned} &| F_{I||} |_q \leq c\Big(| F_\perp |_q + | \tilde{\nu}^{-\frac{1}{2}} g |_q \Big), \\ &\| F_r \|_q \leq c(\epsilon^3 \| F_\theta \|_q + | F_\perp |_q + \frac{1}{\epsilon} \| \int g v_\theta v_r \bar{B}(|v|) M dv \|_1). \end{aligned} \tag{4.7}$$

<u>Proof of Lemma 4.4.</u> By interpolation, it is enough to prove the lemma when q is an even integer. With F a solution to (4.2), multiplying that equation with M and integrating over $I\!R^3_v$, leads to

$$(rF_r)' = 0,$$

i.e. $F_r(r) = \frac{c_r}{r}$ for some constant c_r. In the same way a multiplication with $v_\theta M$ and integration over $I\!R^3_v$ leads to

$$F_{\theta r}(r) = \frac{c_{\theta r}}{r^2},$$

for some constant $c_{\theta r}$.

Multiplying equation (4.2) with $Mv_r v_\theta \bar{B}(|v|)$ and integrating on $I\!R^3$, it follows that

$$\begin{aligned}(\frac{w_1 F_\theta}{r} + \frac{F_{\theta r^2 \bar{B}}}{r})' - \frac{F_{\theta^3 \bar{B}} - 3F_{\theta r^2 \bar{B}}}{r^2} &= \frac{1}{r\epsilon^4}\Big(\frac{c_{\theta r}}{r^2} + \epsilon\frac{c_r}{r}uw_1 \\ +\epsilon\int \tilde{J}(F_\perp, uv_\theta)v_\theta v_r \bar{B}(|v|)Mdv &+ \int gv_\theta v_r \bar{B}(|v|)Mdv\Big). \qquad (4.8)\end{aligned}$$

Integrate (4.8) on (r_1, r_2) for any $(r_1, r_2) \subset (r_A, r_B)$. Then integrate the new equality with respect to both variables (r_1, r_2), on $(r_1, r_2) \in (r_A, \frac{r_A + r_B}{2}) \times (\frac{r_A + r_B}{2}, r_B)$ and on $(r_1, r_2) \in (r_A, \frac{3r_A + r_B}{4}) \times (\frac{r_A + 3r_B}{4}, r_B)$. This provides a 2×2 invertible linear system in the variables $(\frac{c_r}{\epsilon^3}, \frac{c_{\theta r}}{\epsilon^4})$ in terms of integrals of $F_{1,\theta}$, $F_{1,\perp}$ and g, giving

$$\begin{aligned}&| c_r | \leq c(\epsilon^3 \parallel F_\theta \parallel_q + | F_\perp |_q + \frac{1}{\epsilon} \parallel \int gv_\theta v_r \bar{B}(|v|)Mdv \parallel_1). \\ &| c_{\theta r} | \leq c(\epsilon^4 \parallel F_\theta \parallel_q + \epsilon | F_\perp |_q + \parallel \int gv_\theta v_r \bar{B}(|v|)Mdv \parallel_q).\end{aligned}$$

The estimate for F_r follows from here. To the solution F of (4.2) with boundary value f_b, add a sum $S = C_\theta r v_\theta + C_0\psi_0 + C_4\psi_4$. Choose the constants so that the three zero-conditions at r_B,

$$\begin{aligned}&(f_0 + \sqrt{\frac{2}{3}}f_4 + f_{r^2})(r_B) = (w_1 f_\theta + f_{\theta r^2 \bar{B}})(r_B) = \\ &(\sqrt{6}f_4 - 2f_0 + C_1 f_{r^2 \bar{A}} + 2f_{r^2})(r_B) = 0, \qquad (4.9)\end{aligned}$$

hold for $f = F + S$. Here C_1 is defined in (4.15) below. Notice that $f_r = F_r$, $f_\perp = F_\perp$ and that (4.8) together with the estimate for c_r are satisfied also with f replacing F. Lemma 4.2 with $f_{||} = f_\theta \psi_\theta$ gives $(r^2 h_{\theta r})' = r^2 f_\theta$ and $\| h_{\theta r} \|_{qAB} \leq C\epsilon^4 \| f_\theta \|_q$. Using (4.8) for f, it follows that

$$\Big(r^2 h_{\theta r}(\frac{w_1 f_\theta}{r} + \frac{f_{\theta r^2 \bar{B}}}{r})^{q-1}\Big)' = \frac{r^3}{w_1}(\frac{w_1 f_\theta}{r} + \frac{f_{\theta r^2 \bar{B}}}{r})^q$$
$$-\frac{r^2}{w_1} f_{\theta r^2 \bar{B}}(\frac{w_1 f_\theta}{r} + \frac{f_{\theta r^2 \bar{B}}}{r})^{q-1} + (q-1)h_{\theta r}(\frac{w_1 f_\theta}{r} + \frac{f_{\theta r^2 \bar{B}}}{r})^{q-2}\Big(f_{\theta^3 \bar{B}} - 3f_{\theta r^2 \bar{B}}$$
$$+\frac{1}{\epsilon^4}(\frac{c_{\theta r}}{r} + \epsilon w_1 u c_r + \epsilon r \int \tilde{J}(f_\perp, uv_\theta) v_\theta v_r \bar{B}(|v|) M dv$$
$$+r \int g v_\theta v_r \bar{B}(|v|) M dv)\Big). \tag{4.10}$$

In undifferentiated form, the left hand side vanishes at r_A since $h_{\theta r}(r_A) = 0$, and also at r_B because of (4.9). Using Hölder's inequality and the c_r-estimate, it follows for $\delta > 0$ that

$$\| w_1 f_\theta + f_{\theta r^2 \bar{B}} \|_q \leq c\Big(\frac{1}{\delta} \mid f_\perp \mid_q + \delta \| f_\theta \|_q + \frac{1}{\delta} \mid \tilde{\nu}^{-\frac{1}{2}} g \mid_q \Big),$$

and so

$$\| f_\theta \|_q \leq \frac{c}{\delta}\Big(\mid f_\perp \mid_q + \mid \tilde{\nu}^{-\frac{1}{2}} g \mid_q \Big). \tag{4.11}$$

From here the estimate for C_θ, hence F_θ, can be obtained as follows. Using h of Lemma 4.2, this time with $f_{||} = F_\theta \psi_\theta$ and $(r^2 h_{\theta r})' = r^2 F_\theta$, together with (4.8) for F, again gives a version of (4.10). Integrating this from r_A to r and then integrating over an interval in r, gives an estimate for C_θ^q implying

$$\mid C_\theta \mid \leq C\Big(\frac{1}{\delta} \mid f_\perp \mid_q + \frac{1}{\delta} \mid \tilde{\nu}^{-\frac{1}{2}} g \mid_q \Big). \tag{4.12}$$

The estimate for $(\sqrt{6} f_4 - 2f_0)$ starts from Lemma 4.2 with $f_{||} = f_0 \psi_0 + f_4 \psi_4$ which gives $\| h_{r2} \|_{qAB} \leq C\epsilon^4 \| f_0 \psi_0 + f_4 \psi_4 \|_q$ and $(r h_{r2})' = \sqrt{6} r f_4 - 2r f_0$. Multiply equation (4.2) with $\bar{A}(|v|) v_r M$ respectively with $v_r M$ and integrate. It follows that

$$\Big(k_4 f_4 + f_{r^2 \bar{A}}\Big)' = -\frac{1}{r}\Big(f_{r^2 \bar{A}} - f_{\theta^2 \bar{A}}\Big) +$$
$$\frac{1}{\epsilon^4}\Big(f_{r2} + \epsilon u \int v_r \bar{A} M \tilde{J}(f_\perp, v_\theta) + \int g v_r \bar{A}(|v|) M dv\Big), \tag{4.13}$$

and

$$\Big(f_0 + \sqrt{\frac{2}{3}} f_4 + f_{r^2}\Big)' = \frac{f_{\theta^2} - f_{r^2}}{r}, \tag{4.14}$$

where $k_4 = \int v_r^2 \psi_4 \bar{A} M dv$. Set

$$C_1 = \frac{\sqrt{6}}{k_4}. \tag{4.15}$$

Then (4.13-14) give

$$\Big(\sqrt{6}f_4-2f_0+C_1f_{r^2\bar{A}}+2f_{r^2}\Big)'=-\frac{1}{r}\Big(C_1f_{r^2\bar{A}}-C_1f_{\theta^2\bar{A}}-2f_{\theta^2}+2f_{r^2}\Big)$$
$$+\frac{C_1}{\epsilon^4}\Big(f_{r2}+\epsilon u\int v_r\bar{A}M\tilde{J}(f_\perp,v_\theta)+\int gv_r\bar{A}(|v|)Mdv\Big). \tag{4.16}$$

Similarly to the f_θ-estimate above, it follows that

$$\|\sqrt{6}f_4-2f_0\|_q\leq c\Big(\frac{1}{\delta}\mid f_\perp\mid_q+\delta\mid f_0\psi_0+f_4\psi_4\mid_q+\frac{1}{\delta}\mid\tilde{\nu}^{-\frac{1}{2}}g\mid_q\Big). \tag{4.17}$$

We next multiply (4.14) with $q\Big(f_0+\sqrt{\frac{2}{3}}f_4+f_{r^2}\Big)^{q-1}$ and integrate on (r,r_B), then on (r_A,r_B) to obtain

$$\|f_0\|_q\leq c\Big(\frac{1}{\delta}\mid f_\perp\mid_q+\delta\mid f_0\psi_0+f_4\psi_4\mid_q+\frac{1}{\delta}\mid\tilde{\nu}^{-\frac{1}{2}}g\mid_q\Big). \tag{4.18}$$

The estimates for F_0, F_4, C_0, and C_4 can now be computed similarly to F_θ and C_θ. Using h of Lemma 4.2, this time with $f_{||}=F_0\psi_0+F_4\psi_4$ and $(rh_{r2})'=\sqrt{6}rF_4-2rF_0$, together with (4.16) for F, as previously gives an estimate for $\sqrt{6}C_4-2C_0$,

$$\mid\sqrt{6}C_4-2C_0\mid\leq C\Big(\frac{1}{\delta}\mid f_\perp\mid_q+\frac{1}{\delta}\mid\tilde{\nu}^{-\frac{1}{2}}g\mid_q\Big). \tag{4.19}$$

An estimate for C_4 alone will use the same h_{r2}-equation, but this time with $f_{||}=F_4\psi_4$, and start from

$$\Big(rh_{r2}(f_0+\sqrt{\frac{2}{3}}F_4+f_{r^2})\Big)'=(f_0+\sqrt{\frac{2}{3}}F_4+f_{r^2})\sqrt{6}rF_4+\frac{f_{\theta^2}-f_{r^2}}{r}rh_{r2}.$$

After suitable integrations with respect to rdr, and invoking the previous estimates for f, we obtain the following estimate for C_4,

$$\mid C_4\mid\leq C\Big(\frac{1}{\delta}\mid f_\perp\mid_q+\frac{1}{\delta}\mid\tilde{\nu}^{-\frac{1}{2}}g\mid_q\Big). \tag{4.20}$$

The estimate for $F_{I||}$ follows from (4.11), (4.12) and (4.17-20),

$$\mid F_{I||}\mid_q\leq c\Big(\mid\tilde{\nu}^{-\frac{1}{2}}g\mid_q+\mid f_\perp\mid_q\Big).$$

The proof of the lemma is complete. □

Lemma 4.5 *Let $u \in I\!R$, $g = g_\perp$, $\tilde{\nu}^{-\frac{1}{2}} g \in \tilde{L}^q$, $f_b \in L^+$, $2 \leq q < \infty$ be given. For small enough $\epsilon > 0$, there exists a unique solution $F \in \mathcal{W}^{q-}$ to*

$$DF = \frac{1}{\epsilon^4}(\tilde{L}F + \epsilon u \tilde{J}(F, v_\theta) + g), \quad F_{/\partial\Omega^+} = f_b. \tag{4.21}$$

Proof of Lemma 4.5. By [M2 p68-69] there is a unique solution $F \in \mathcal{W}^{2-}$ for $u = 0$. Adding $\frac{\epsilon u}{\epsilon^4} \tilde{J}(F - f_r \psi_r, v_\theta)$ to the right hand side of (4.2) for $u = 0$, the equation is still solvable. This follows by an iteration scheme 'with the previous iterate in the $\tilde{J}$-term'. The scheme is contractive for small ϵ by the estimates in Lemmas 4.3-4. Notice that for ϵ fixed, the component $f_r = \frac{c_r}{r}$ of the iterates only is bounded. That requires a sub-sequence extraction to get convergence also for those components of the iterates. Hence the uniqueness is not immediate, but it follows from the previous estimates directly applied to the homogeneous equation. Next, adding $\frac{\epsilon u}{\epsilon^4} \tilde{J}(f_r \psi_r, v_\theta) = \frac{\epsilon u}{\epsilon^4} \tilde{J}(\frac{c_r}{r} \psi_r)$ is a compact perturbation, so the index is conserved. Again the previous estimates applied to the homogeneous equation imply uniqueness. We have thus proved the lemma for $u \neq 0$ and $q = 2$. Using [M2 p100], finally the result for a general q follows from the result for $q = 2$. That completes the proof of the lemma. $\square$

5 The rest term.

In this section we discuss the rest term, when $(u_{\theta A1} - u_{\theta B1} r_B)(3u_{\theta A1} + u_{\theta B1} r_B) > 0$ and $\Delta \leq \Delta_{bif}$. Denote by ψ the asymptotic expansion for $j_1 = 10$ (cf the end of Section 2),

$$\psi(r, v) = \sum_{i=1}^{10} \epsilon^i \psi^i.$$

The aim is to prove that there exists a rest term R, such that

$$f = M(1 + \psi + \epsilon^6 R)$$

is a solution to (1.1-2) with $M^{-1} f \in \tilde{L}^\infty$. Such a function R would then be a solution to

$$DR = \frac{1}{\epsilon^4}\Big(\tilde{L}R + 2\tilde{J}(R, \psi) + \epsilon^6 \tilde{J}(R, R) + l\Big), \tag{5.1}$$

where

$$l = \frac{1}{\epsilon^6}\Big(\tilde{L}\psi + \tilde{J}(\psi, \psi) - \epsilon^4 D(\psi)\Big).$$

We assume that the boundary value f_b is satisfied by $M(1 + \psi)$ up to order seven in ϵ, so that the boundary values for R are of order two. We remark that constants that need not be precised in the asymptotics, will be 'compensated to correct value' by the rest term, as can be seen from the uniqueness arguments for the rest term below. Notice that ψ^j can be constructed so that $D\sum_{i=1}^{j_1}\epsilon^i\psi^i = (I-P_0)D\sum_{i=1}^{j_1}\epsilon^i\psi^i$, hence correspondingly for l.

Let the sequence $(R^n)_{n\in I\!N}$ be defined by $R^0=0$, and

$$DR^{n+1} = \frac{1}{\epsilon^4}\Big(\tilde{L}R^{n+1} + 2\tilde{J}(R^{n+1}, \sum_{i=1}^{5}\epsilon^i\psi^i) + g^n\Big), \tag{5.2}$$

$$R^{n+1}(1,v) = R_A(v),\ v_r > 0, \quad R^{n+1}(r_B,v) = R_B(v),\ v_r < 0. \tag{5.3}$$

In (5.2-3)

$$g^n := 2\tilde{J}(R^n, \sum_{i=6}^{10}\epsilon^i\psi^i) + \epsilon^6\tilde{J}(R^n,R^n) + l,$$

$$\epsilon^6 R_A(v) := e^{\epsilon u_{\theta A1}v_\theta - \frac{\epsilon^2}{2}u^2_{\theta A1}v^2_\theta} - 1 - \sum_{i=1}^{10}\epsilon^i\psi^i(r_A,v), \quad v_r > 0,$$

$$\epsilon^6 R_B(v) := \frac{1+\omega_B}{(1+\tau_B)^{\frac{3}{2}}} e^{\frac{1}{2}\left(v^2 - \frac{1}{1+\tau_B}(v_r^2 + (v_\theta - \epsilon u_{\theta B1})^2 + v_z^2)\right)} - 1$$

$$-\sum_{i=1}^{10}\epsilon^i\psi^i(r_B,v), \quad v_r < 0.$$

The lemmas 4.3-5 also hold for this boundary value problem with analogous proofs, and the solutions are well defined when ϵ is small. We observe that $g^n = g^n_\perp$,

$$\int g^n M dv = \int lMdv = 0, \quad \partial_r(r\int R^n v_r M dv) = r\int lMdv = 0. \tag{5.4}$$

We now discuss existence based on the rest term iteration scheme (5.2-3).

Theorem 5.1 *For $\epsilon > 0$ small enough, there is a unique sequence (R^n) of solutions to (5.2-3) in the set $X := \{R; \mid \tilde{\nu}^{\frac{1}{2}}R\mid_\infty \le K,\ \mid \tilde{\nu}^{\frac{1}{2}}R\mid_2 \le K\}$, for some constant $K > 0$. The sequence converges to an isolated solution of*

$$DR = \frac{1}{\epsilon^4}\Big(\tilde{L}R + 2\tilde{J}(R,\psi) + \epsilon^6\tilde{J}(R,R) + l\Big), \tag{5.5}$$

$$R(1,v) = R_A(v),\ v_r > 0, \quad R(r_B,v) = R_B(v),\ v_r < 0. \tag{5.6}$$

Proof of Theorem 5.1. In the case $n=0$ and $g^0 = l$, notice that l in (5.2) is of order five, and the boundary values in (5.3) are of order two in ϵ. It follows from Lemmas 4.3-4 for $g^0 = l = l_\perp$, that uniformly with respect to $0 < \epsilon < \epsilon_0$,

$$\mid \tilde{\nu}^{\frac{1}{2}}R^1\mid_2 \le C\epsilon^4, \quad \mid \tilde{\nu}^{\frac{1}{2}}R^1\mid_\infty \le C. \tag{5.7}$$

For $n \in I\!N$ in the equation (5.2) for $(R^{n+1}-R^n)$, the term l has disappeared, $g^n = g^n_\perp$, and the ingoing boundary values are zero. Writing $R^{n+1} = R^1 +$

$\sum_{j=1}^{n}(R^{j+1}-R^{j})$ and using Lemmas 4.3-4, it follows that

$$
\begin{aligned}
&\mid \tilde{\nu}^{\frac{1}{2}}(R^{n+1}-R^{n}) \mid_{2} + \mid \tilde{\nu}^{\frac{1}{2}}(R^{n+1}-R^{n}) \mid_{\infty} \leq \\
&C\epsilon\Big(\mid \tilde{\nu}^{\frac{1}{2}}(R^{n}-R^{n-1}) \mid_{2} + \mid \tilde{\nu}^{\frac{1}{2}}(R^{n}-R^{n-1}) \mid_{\infty} \Big), \\
&\mid \tilde{\nu}^{\frac{1}{2}}R^{n+1} \mid_{2} + \mid \tilde{\nu}^{\frac{1}{2}}R^{n+1} \mid_{\infty} \leq C\sum_{0}^{n}(C\epsilon)^{j} \leq K,
\end{aligned}
$$

for all $n \in I\!N$. And so, for ϵ small enough, $(\tilde{\nu}^{\frac{1}{2}}R^{n})$ converges in $\tilde{L}^{2}$ and $\tilde{L}^{\infty}$ to some $\tilde{\nu}^{\frac{1}{2}}R$ with

$$
DR = \frac{1}{\epsilon^{4}}\Big(\tilde{L}R + 2\tilde{J}(R,\psi) + \epsilon^{6}\tilde{J}(R,R) + l\Big), \tag{5.8}
$$

$$
R(1,v) = R_{A}(v),\ v_{r} > 0, \quad R(r_{B},v) = R_{B}(v),\ v_{r} < 0. \tag{5.9}
$$

The contraction mapping argument guarantees that these solutions are isolated.
□

It follows from the previous proof that the hydrodynamic moments converge to solutions of the corresponding leading orders limiting fluid (Hilbert) equations, when ϵ tends to zero.

Suppose f satisfies (5.10), (5.11) below. Proving that $f^{-}=0$ will imply that f is a non-negative solution to (1.1-2). Using an $\tilde{L}^{q}$-version of Lemmas 4.3 with q large, we may extend the present contraction mapping set-up, and prove that any solution of this paper coincides with such a non-negative solution. That is left for an accompanying paper [AN6] in preparation about the problem (1.1-2), and where j_{0} may be taken as four.

Theorem 5.2 *Let Ω be a bounded set in $I\!R^{3}$, and f_{b} a nonnegative function defined on the ingoing boundary $\partial\Omega^{+}$. If a function f such that $M^{-1}f \in \tilde{L}^{\infty}(\Omega \times I\!R^{3})$ satisfies*

$$
v\cdot\nabla_{x}f = Q(f^{+},f^{+}) - ML(M^{-1}f^{-}), \quad (x,v) \in \Omega \times I\!R^{3}, \tag{5.10}
$$

$$
f = f_{b}, \quad \partial\Omega^{+}, \tag{5.11}
$$

then $f^{-}=0$ and $f=f^{+}$ solves the boundary value problem

$$
\begin{aligned}
&v\cdot\nabla_{x}f = Q(f,f), \quad \Omega \times I\!R^{3}, \\
&f = f_{b}, \quad \partial\Omega^{+}.
\end{aligned}
$$

Proof of Theorem 5.2 The function $F = M^{-1}f$ satisfies

$$
v\cdot\nabla_{x}F = J(F^{+},F^{+}) - L(F^{-}), \quad F = M^{-1}f_{b}, \quad \partial\Omega^{+}.
$$

Define J^{+} and J^{-} by $J(\varphi,\varphi) = J^{+}(\varphi,\varphi) - J^{-}(\varphi,\varphi)$, where

$$
\begin{aligned}
J^{+}(\varphi,\varphi)(v) &:= \int \mid v-v_{*} \mid^{\beta} b(\theta)M_{*}\varphi'\varphi'_{*}dv_{*}d\omega, \\
J^{-}(\varphi,\varphi)(v) &:= \varphi(v)\int \mid v-v_{*} \mid^{\beta} b(\theta)M_{*}\varphi_{*}dv_{*}d\omega.
\end{aligned}
$$

Also, F^- satisfies

$$
\begin{aligned}
&-v\cdot\nabla_x F^- = \chi_{F^-\neq 0}(J^+(F^+,F^+)-L(F^-)), \qquad (5.12)\\
&F^-=0, \quad \partial\Omega^+.
\end{aligned}
$$

Multiplying (5.12) by $-MF^-$, integrating on $\Omega\times I\!\!R^3$ and using that

$$
-\int MF^-\chi_{F^-\neq 0}L(F^-)dv = -\int MF^-L(F^-)dv \geq c\int M\nu \mid (I-P_0)F^- \mid^2 dv,
$$

implies that

$$
\begin{aligned}
&\frac{1}{2}\int_{\partial\Omega^-} \mid v\cdot n\mid M(F^-)^2 + c\int_{\Omega\times I\!\!R^3} M\nu \mid (I-P_0)F^- \mid^2\\
&\leq -\int MF^-\chi_{F^-\neq 0}J^+(F^+,F^+)\leq 0.
\end{aligned}
$$

It follows that with $\partial\Omega^-$ the outgoing boundary,

$$
F^- = 0 \text{ on } \partial\Omega^-, \quad L(F^-)=0.
$$

And so, F^- satisfies

$$
F^-=0,\ \partial\Omega^-\cup\partial\Omega^+, \quad v\cdot\nabla_x F^-\leq 0.
$$

This implies that F^- is identically zero. □

6 Comments and remarks.

As mentioned in the introduction, the approach holds without change for the other cases of asymptotic expansion in the two-roll setup that are discussed in [S]. The following example illustrates the treatment in the upcoming types of situations.
Consider the equation (1.1) under the scaling

$$
\begin{aligned}
&v_r\frac{\partial f}{\partial r}+\frac{1}{r}Nf = \frac{1}{\epsilon^m}\tilde{Q}(f,f), \qquad (6.1)\\
&r\in(r_A,r_B), \quad (v_r,v_\theta,v_z)\in I\!\!R^3,
\end{aligned}
$$

for $m=3$ and without the coupling $\omega_B = \frac{\epsilon^2}{1+\epsilon^2\tau_{B2}}\Big(\frac{r_B^2-1}{r_B^2}u^2_{\theta A1}-\tau_{B2}+\Delta\epsilon\Big)$ between the boundary parameters in (1.2). Assume the cylinders rotate in the same direction and that $1<P_{SB2}/[(r_B^2-1)u^2_{\theta B1}]<(u_{\theta A1}/u_{\theta B1}r_B)^2$. This guarantees an asymptotic expansion with positive (as well as one with negative) second order radial velocity u_{rH2}, and one with third order radial velocity. For the positive one, take the asymptotic expansion ψ of Section 5 up to order nine, and the rest term R of order five in ϵ. The rest term analysis proceeds as in Section 5 and gives the following result.

Theorem 6.1 *For $0 < \epsilon$ small enough, there is an isolated L^1-solution f_ϵ of the equation (6.1) for $m = 3$ with boundary conditions (1.2) and with positive second order radial velocity u_{rH2}, for which*

$$\int M^{-1} supess_{r \in (r_A, r_B)} \mid f_\epsilon(r,v) \mid^2 dv < +\infty.$$

The hydrodynamic moments converge to solutions of the corresponding leading order (second order in ϵ for the radial velocity) limiting fluid equations, when ϵ tends to zero.

Also for the negative second order radial velocity u_{rH2}, a second isolated solution can be obtained in the same way.
For the case of third order radial velocity, again take the asymptotic expansion ψ in Section 5 up to order nine, and the rest term R of order five in ϵ. The rest term analysis proceeds as before.

Theorem 6.2 *For $0 < \epsilon$ small enough, there is an isolated L^1-solution f_ϵ of the equation (6.1) for $m = 3$ with boundary conditions (1.2) and third order radial velocity u_{rH3} for which*

$$\int M^{-1} supess_{r \in (r_A, r_B)} \mid f_\epsilon(r,v) \mid^2 dv < +\infty.$$

The hydrodynamic moments converge to solutions of the corresponding leading order (third order in ϵ for the radial velocity moment) limiting fluid equations, when ϵ tends to zero.

Theorem 6.1-2 in particular demonstrate that three separate solutions to (1.1-2) coexist for these parameter values.

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