A convolution-thresholding scheme for the Willmore flow

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1 Introduction

Let Σ be a smooth, closed surface in $\mathbb{R}^3$. The Willmore functional for this surface is defined by

$$\mathcal{W}(\Sigma) = \int_\Sigma H^2 dA, \tag{1}$$

where H is the mean curvature of Σ and dA is the induced area measure. We refer to [17] for the general disscusion of this functional as well as description of some stationary points. The variation of this intergal for a perturbation ϕ of the surface along the normal is (see [17])

$$\delta\mathcal{W} = \int_\Sigma \phi\left(\Delta H + 2H\left(H^2 - K\right)\right) dA, \tag{2}$$

where K is the Gaussian curvature of Σ. The Willmore flow for Σ is defined as an evolution of the surface when each point of it moves with the normal velocity

$$v = \Delta H + 2H\left(H^2 - K\right). \tag{3}$$

The mathematical properties of this type of surface evolution attracted much attention during the last years. Recent mathematical results in this field include [16], [8], [7], [10] and [12]. A numerical scheme to track the evolution in $\mathbb{R}^3$ for axisymmetric surfaces was proposed in [11].

The purpose of this study is to develop a simple convolution-thresholding scheme for tracking such evolutions of a two dimensional surfaces in $\mathbb{R}^3$. Convolution-thresholding schemes have proved to be a useful tool for the numerics of the surface evolution [5], [6], [14], [15], [4] and image processing [3], [2], [13]. Furthermore, the convolution structure of the method allows a numerically efficient implementation of the method.

This paper is organized as follows. In section 2 we consider the graph of a smooth function and explicitly expressing its geometrical properties (i.e. H, K and ΔH) by the derivatives of the function. We use these elementary relations in the section 3, where the asymptotics for the convolution of an indicator function of a subset of $\mathbb{R}^3$ with a smooth, compactly supported kernel. We show that the third term of this asymptotics is proportional to $\Delta H + 2H\left(H^2 - K\right)$ and therefore can be efficiently used to construct a convolution-thresholding schemes for the Willmore flow. Several numerical examples of the flow are presented in the section 4.

2 Geometric properties of a smooth graph

Let us consider the graf Σ of a smooth function $f : \mathbb{R}^2 \mapsto \mathbb{R}$ such that $\partial f / \partial u_i \left(0,0\right) = f_i = 0$ for $i = 1, 2$. We express the mean curvature H, the Gaussian curvature K, the Laplace-Beltrami operator of the mean curvature ΔH of Σ at the origin in terms of f and its derivatives.

Consider a mapping $\mathbf{x} : \mathbb{R}^2 \mapsto \mathbb{R}^3$ by $\mathbf{x}\left(u_1, u_2\right) = \left(u_1, u_2, f\left(u_1, u_2\right)\right)$. Denote $\mathbf{x}_i = \partial \mathbf{x} / \partial u_i$ for $i = 1..2$. The outer unit normal $\mathbf{N}$ of Σ is given by

$$\mathbf{N} = \frac{\mathbf{x}_1 \times \mathbf{x}_2}{\left|\mathbf{x}_1 \times \mathbf{x}_2\right|}. \tag{4}$$

The coefficients of the first fundamental form are $g_{ij} = \left\langle \mathbf{x}_i, \mathbf{x}_j \right\rangle$ and the second fundamental form have coefficients $h_{ij} = -\left\langle \mathbf{N}_i, \mathbf{x}_j \right\rangle$, where $\mathbf{N}_i = \partial \mathbf{N} / \partial u_i$. The mean curvature is

$$H = \frac{1}{2} \sum_{i,j} g^{ij} h_{ij}, \tag{5}$$

where g^{ij} denotes the elements of the matrix inverse to g. Setting $u_1 = u_2 = 0$ we get

$$H = \frac{1}{2}\left(f_{11}(0,0) + f_{22}(0,0)\right). \tag{6}$$

In order to calculate ΔH we introduce the Christoffel sybmols

$$\Gamma_{ij}^{k} = \frac{1}{2}\sum_{h} g^{hk}\left(\partial g_{ih}/\partial u_j + \partial g_{jh}/\partial u_i - \partial g_{ij}/\partial u_h\right). \tag{7}$$

The covariant derivatives of H are

$$\nabla_i H = H_i = \frac{\partial H}{\partial u_i} \tag{8}$$

$$\nabla_i \nabla_j H = H_{ij} - \sum_k \Gamma_{ji}^k H_k \tag{9}$$

The Laplacian of H can be written as

$$\Delta H = \sum_{i,j} g^{ij} \nabla_i \nabla_j H. \tag{10}$$

After substituting (4) - (9) into (10) we get the following equality at the origin

$$\Delta H = \frac{1}{2}\left(\Delta\left(\Delta f\right) - 2H\left(12H^2 - 8K\right)\right), \tag{11}$$

where K is the Gaussian curvature, namely

$$K = \det\left(g^{ij} h_{ij}\right) = f_{11}(0,0)\, f_{22}(0,0) - f_{12}^2(0,0). \tag{12}$$

3 Asymptotics of the convolution

In this section we consider a compact subset C of $\mathbb{R}^3$ with a smooth boundary ∂C. We study the geometric properties of ∂C by considering the following convolution

$$M = \chi_C \star \rho_{t^1/4}, \tag{13}$$

where χ_C is the characteristic function of C , $\rho_{t^\alpha}(x) = \rho\left(|x|^2 / t^{2\alpha}\right) / \left(t^\alpha\right)^3$ and $\rho : (0,\infty) \mapsto [0,\infty)$ is smooth with a comact support (or exponentially decreasing) normalized by $\int_{R^3} \rho dx = 1$.

We pick a point $\mathbf{p} \in \partial C$ associate a unit normal $\mathbf{N}$ to it, and calculate M in the point $\mathbf{p} + \mathbf{N}vt$. Here $v \in \mathbb{R}$. Taking $t \mapsto 0$ we expand M into a power series in t.

In order to get the power series for (13) at $O = \mathbf{p} + \mathbf{N}vt$ it is convenient to choose O as the origin and Oz-axis parallel to $\mathbf{N}$. Then $\mathbf{p} = (0, 0, vt)$ and the boundary ∂C can be represented as a graph of a smooth function $\gamma : \mathbb{R}^{n-1} \mapsto \mathbb{R}$ in some neibourhood $\mathcal{S}$ of O.

$$\partial C = \left\{ (x, y, \gamma (x, y)) : (x, y) \in \mathcal{S} \subset \mathbb{R}^2 \right\} \tag{14}$$

Furthermore,

$$\begin{aligned} \gamma (x, y) &= vt + f (x, y) & (15) \\ \gamma (0, 0) &= vt, \text{ hence } f (0, 0) = 0 & (16) \\ \nabla \gamma (0, 0) &= \nabla f (0, 0) = 0. & (17) \end{aligned}$$

The expression for M at O now can be written as follows

$$\begin{aligned} M (O) &= \int_{\mathbb{R}^3} \chi_C (x) \, \rho_{t^{1/4}} (x) \, dx = \\ &= \int_C \frac{\rho \left((x^2 + y^2 + z^2) / t^{1/2} \right)}{t^{3/4}} dx dy dz = \\ &= \frac{1}{2} + \int_{\mathbb{R}^2} \mathcal{I} (t, x, y) \, dx dy + O \left(e^{-t^{-1/2}} \right) \end{aligned}$$

where

$$\mathcal{I} (t, x, y) = \int_0^{g(t,x,y)} \rho \left((x^2 + y^2 + z^2) \right) dz \tag{18}$$

with

$$g(t, x, y) = \frac{\gamma \left(t^{1/4} x, t^{1/4} y \right)}{t^{1/4}}. \tag{19}$$

We expand this integral into a power series in t at the point $t_0 = 0$. First we observe that $g(0, x, y) = 0$ and calculate some derivatives of $g(t, x, y)$ with respect to t at $t = 0$:

$$\begin{aligned} \frac{\partial g}{\partial t} (0, x, y) &= \frac{1}{2} y^2 \, f_{22}(0, 0) + x \, y \, f_{12}(0, 0) + \frac{1}{2} x^2 \, f_{11}(0, 0) & (20) \\ \frac{\partial^2 g}{\partial t^2} (0, x, y) &= \frac{1}{3} y^3 \, f_{222}(0, 0) + x \, y^2 \, f_{122}(0, 0) \end{aligned}$$

$$
\begin{aligned}
& + \quad x^2\, y\, f_{112}(0,0) + \frac{1}{3} x^3\, f_{111}(0,0) && (21)\\
\frac{\partial^3 g}{\partial t^3}(0,x,y) &= 6\,v + \frac{1}{4} y^4\, f_{2222}(0,0) + x\, y^3\, f_{1222}(0,0) + \frac{3}{2}\, x^2\, y^2\, f_{1122}(0,0) \\
& + \quad x^3\, y\, f_{1112}(0,0) + \frac{1}{4} x^4\, f_{1111}(0,0). && (22)
\end{aligned}
$$

Then we substitute these expressions into the expansion

$$
\begin{aligned}
& \mathcal{I}(t,x,y) = && (23)\\
& \rho(x^2+y^2)\,\frac{\partial g}{\partial t}(0,x,y)\, t^{1/4} + \frac{1}{2}\rho(x^2+y^2)\,\frac{\partial^2 g}{\partial t^2}(0,x,y)\,(0,x,y)\, t^{1/2} + && (24)\\
& + \frac{1}{6}\left(2\,\rho'(x^2+y^2)\,\frac{\partial g}{\partial t}(0,x,y)^3 + \rho(x^2+y^2)\,\frac{\partial^3 g}{\partial t^3}(0,x,y)\right) t^{3/4} + O(t) && (25)
\end{aligned}
$$

integrate over x and y and get

$$
\begin{aligned}
M(O) &= \frac{1}{2} + \frac{t^{1/4}\pi N_3}{2}\,(f_{11}+f_{22}) + \frac{t^{3/4}\pi}{64}\,[128 N_1 v + && (26)\\
& - \quad N_5\,(f_{22}+f_{11})\,\left(5\, {f_{22}}^2 + 12\, {f_{12}}^2 - 2\, f_{22}\, f_{11} + 5\, {f_{11}}^2\right) - && (27)\\
& - \quad 2\,\,(f_{2222} + 2\, f_{1122} + f_{1111})] + O\left(t^{5/4}\right), && (28)
\end{aligned}
$$

where all derivatives of f are calculated at the origin and $N_i = \int_0^\infty r^i \rho\,(r^2)\, dr$. Recalling (6),(12) and (11) we arrive at

$$
\begin{aligned}
M(O) &= \frac{1}{2} + \frac{t^{1/4}\pi N_3}{2} H + && (29)\\
& + \quad \frac{t^{3/4}\pi}{16}\left[32 N_1 v + N_5\left(\Delta H + 2H\left(H^2 - K\right)\right)\right] + O\left(t^{5/4}\right). && (30)
\end{aligned}
$$

Consider now $\rho_{at^{1/4}}$ with $a > 0$ and calculate $M_a = \chi_C \star \rho_{at^{1/4}}$ to get

$$
\begin{aligned}
M_a(O) &= \frac{1}{2} + \frac{a t^{1/4}\pi N_3}{2} H + && (31)\\
& + \quad \frac{t^{3/4}\pi}{16}\left[\frac{32 N_1 v}{a} + a^3 N_5\left(\Delta H + 2H\left(H^2 - K\right)\right)\right] + O\left(t^{5/4}\right) && (32)
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{a\,(2aM - 2M_a + 1 - a)}{\pi\,(1-a^2)} = && (33)\\
&= t^{3/4}\left(4 N_1 v - a^2 N_5/8\left(\Delta H + 2H\left(H^2 - K\right)\right)\right) + O\left(t^{5/4}\right). && (34)
\end{aligned}
$$

Now if C is given, we calculate M and M_a at each point and set $C_1 = \{\mathbf{x} \in \mathbb{R}^3 : 2aM - 2M_a + 1 - a \geq 0\}$. According to the above asymptotics, $\partial C_1 = \{\mathbf{p} + vt\mathbf{N} : \mathbf{p} \in \partial C\}$, where

$$v = \frac{a^2 N_5}{32N_1} \left(\Delta H + 2H\left(H^2 - K\right)\right) + O\left(t^{1/2}\right). \tag{35}$$

4 Numerical implemantation and examples

Suppose the initial surface Σ_0 is the boundary of a smooth compact set $C = C_0 \subset \mathbb{R}^3$. Denote the Willmore flow of the Σ_0 by $\Sigma(t)$ Using the above asymptotics we construct surfaces for time moments $t_i = i\Delta t$, $i = 1, 2, ...$ as follows.

1. Convolution i.e. construction of functions

$$M_{a_k}(x) = \chi_{C_i} \star \rho_{a_k \Delta t^{1/4}}(x) \text{ for } k = 1, 2 \text{ with } a_1 = 1 \text{ and } a_2 < 1. \tag{36}$$

2. Thresholding i.e. localisation of the next position of the surface

$$\partial C_{i+1} = \left\{x \in \mathbb{R}^3 : 2a_2 M_{a_1}(x) - 2M_{a_2}(x) + a_1 - a_2 = 0\right\}. \tag{37}$$

In our implementation we use a modification of so called Marching Cubes algorithm for extracting an isosurface. A similar algorithm was proposed in [9] and first applied for the mean curvature flow calculations in [14]. The algorithm creates an adaptive spatial discretization of C. In doing so, we significantly reduce the number of grid points. Besides that, the accurate piecewise polynomial approximation of the ∂C can be arranged.

We use Fourier series to calculate the convolutions M_{a_k}. Numerical aspects of similar computations in the case of the mean curvature evolution ave been presented by Ruuth in [14] and [4].

In order to compute Fourier coefficients of χ_C given on a non uniform grid the unequally spaced approximate fast Fourier transform algorithm [1] is used.

The numerical cost of this transform algorithm combined with the Marching Cubes procedure is [14] $O\left(m^n N_p + N_f^n \log\left(N_f\right)\right)$, where m is a constant depending on a desired accuracy in the calculation of the Fourier coefficients (in case $m = 23$ the accuracy is comparable with the machine truncation

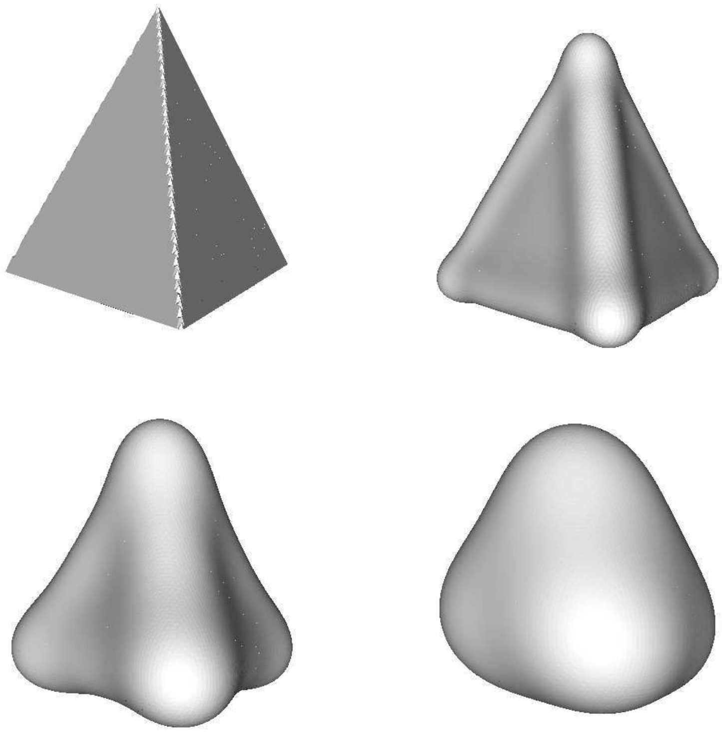

Figure 1: The evolution of a piramide: timestep is 10^{-8}, step numbers 0, 10, 110 and 610.

error), N_f is a number of the Fourier modes along each axis and N_p is the number of nodes in the grid.

The framework of the convolution thresholding method allows to consider not only initially smooth surfaces but also singular ones.

Gradient flows for more complicated functionals depending for instance on the area of the surface and the volume of C can be also treated within the same scope of ideas.

An evolution of an initially non-smooth surfaces is depicted on the Fig. 1, 2.

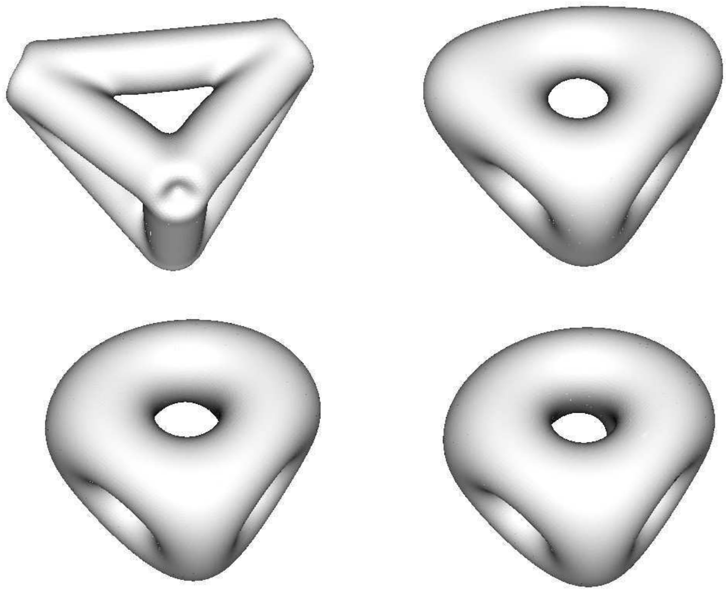

Figure 2: The evolution of a non-convex surface: timestep is 10^{-8}, step numbers 0, 80, 480 and 1080.

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