The rate at which a class of convolution transforms can tend to zero in the non-trivial case

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Abstract

In this paper results concerning how fast some well-known integral transforms, for example the Laplace transform, can tend to zero in the non-trivial case is proved. The results follows as a consequence of a general theorem, which applies to a wide class of convolution transforms. MSC classification is primarily 44A35 and secondary 44A10.

0 Introduction

Let us consider a convolution transform,

$$K * \phi(x) = \int_{-\infty}^{\infty} K(x-t)\phi(t)dt,$$

where K belong to some class of integrable functions and ϕ is a bounded and measurable function.

It is well-known that for special kernels K, the convolution transform $K * \phi(x)$ cannot tend to zero to fast when $x \to +\infty$ unless the function ϕ is trivial. This is for example true for the convolution transform associated with the Laplace transform.

In this paper we will prove a quite general result which applies to a large class of kernels, including the convolution kernels associated with the Laplace, Meijer and Weierstrass transforms.

For the Laplace transform problems of this kind has been treated for example by Hirschman ([3], [4]). The author has treated problems of this kind in ([1], [2]), and this paper is a generalisation of some results in [2].

1 Preliminaries

We use common notations for Fourier transform,

$$\hat{K}(t) = \int \exp(-itx)K(x)dx.$$

Here and throughout the text we let an unspecified region of integration be the real line R.

We will also use the concept of conjugate Young functions, and we say that F and G are a pair of conjugate Young functions if

$$F(x) = \int_0^x f(t)dt \text{ and } G(x) = \int_0^x g(t)dt$$

for some non-decreasing functions f and g, with $f(0) = g(0) = 0$ and such that

$$f(g(x)) = g(f(x)) = x, \quad 0 \le x < \infty.$$

We also let an analytic function g be of minimal order ρ if

$$|g(z)| \le C \exp(\delta |z|^\rho)$$

for any complex number z and any positive real number δ. Here and throughout the paper we let C stand for positive constants not necessarily the same each time.

Let us also introduce the class $E(M, P)$ of convolution kernels considered.

DEFINITION 1.1. By $E(M, P)$ we denote all integrable functions K which satisfy the following three conditions:

1^0 $\hat{K}(t) \neq 0$ for any real number t.

2^0 The function g such that

$$g(t) = \hat{K}(t)^{-1}$$

can be analytically continued in a region $\mathrm{Im}t > -\lambda$ for some $\lambda > 0$.

3^0 This function g satisfies an inequality

$$|g(t)| \le C \exp(M(x) - P(y))$$

for some even, non-negative and non-decreasing functions M and P and all $t = x + iy$ with $y = \mathrm{Im}t > -\lambda$.

Before stating the theorem we need a lemma.

Lemma 1.2. If $p > 0$ is a natural number and if

$$H(x) = \int \exp(ixt - t^{2p})dt, \tag{1.1}$$

then

$$H(x) \neq 0 \text{ a.e. for real values of } x,$$

and for any fixed real number $\delta > 0$

$$H(x) = O(\exp(-\delta|x|)) \text{ when } x \to \pm\infty.$$

PROOF. Make the substitutions $t = u \pm i\delta$ in the integral and then translate the line of integration using Cauchy's integral formula. This is possible since the integrand in (1.1) tends uniformly to zero when $|z| \to \infty$ in the strip $|\mathrm{Im}t| \le \delta$. Now since $H(z)$ is analytic in the corresponding open strip, $H(x) \neq 0$ a.e. for real values of x.

2 A general theorem

THEOREM 2.1. Suppose that

(2.1) $$K \in E(M,P), \text{ where for some real } q > 0 \varlimsup_{x\to\infty} M(x)x^{-q} = 0,$$

and that ϕ is a bounded and measurable function such that

(2.2) $$\psi(x) = K * \phi(x) = O(\exp(-F(x)) \text{ when } x \to \infty$$

for some Young function F with conjugate function G.

Furthermore suppose that for some real number y

(2.3) $$\varlimsup_{x\to\infty}(-sy + G(s+\varepsilon) - P(s) + M(\delta s)) < \infty$$

for any $\varepsilon > 0$ and δ such that

(2.4) $$\delta = 0 \text{ if } q \le 2 \text{ and otherwise } \delta > \tan\left(\frac{\pi}{2} - \frac{\pi}{q}\right).$$

From these conditions it follows that

$$\phi(x) = 0 \text{ a.e. for } x > y.$$

PROOF. Let the function H be defined as in Lemma 1.2 above with $2p > q$ and let

$$u(x) = H(-x)\varphi(x+y).$$

The Fourier transform of u can be rewritten in the form

(2.5) $$\hat{u}(z) = h * \varphi(y)$$

if

$$h(x) = H(x)\exp(izx).$$

If now Re $z \ge 0$ and

$$Q(x) = \frac{1}{2\pi}\int \exp(ixv)\hat{H}(v-z)g(v)dv$$

then

(2.6) $$h * \phi(y) = \psi * Q(y).$$

The equality (2.6) follows if we start with the right hand member and simplify it by changing the order of integrations (Cf. [2)] p.4). We also use that $K * Q(x) = h(x)$ a.e. since $\hat{K}(\xi)\hat{Q}(\xi) = \hat{H}(\xi - z) = \hat{h}(\xi)$.

Thus by (2.6)

$$\hat{u}(t+is) = \frac{1}{2\pi}\int(\psi(y-x)\int \exp(ixv)\hat{H}(v-t-is)g(v)dv)dx.$$

Given any $\varepsilon > 0$ we now make the substitution $v = \xi + t + i(s+\frac{\varepsilon}{2})$, where $s \geq 0$, and translate the line of integration using Cauchy's integral theorem. This is possible since the integrand is uniformly convergent to zero when $|v| \to \infty$ in any strip $|\mathrm{Im} v| \leq \rho$ if $s \geq 0$.

We then obtain that

$$\hat{u}(t+is) = \frac{1}{2\pi}\int(\psi(y-x)\exp(-(s+\frac{\varepsilon}{2})x+ixt)\int \exp(ix\xi)\hat{H}(\xi+i\frac{\varepsilon}{2})g(\xi+t+i(s+\frac{\varepsilon}{2}))d\xi)dx.$$

Now by Definition 1.1

$$\left|g(\xi+t+i(s+\frac{\varepsilon}{2}))\right| \leq C\exp(M(\xi+t)-P(s))$$

and hence

$$|\hat{u}(t+is)| \leq C\exp(-P(s))\int|\psi(y-x)|\exp(-(s+\frac{\varepsilon}{2})x)dx\int \exp(M(\xi+t))\left|\hat{H}(\xi+i\frac{\varepsilon}{2})\right|d\xi.$$

After a substitution $x = y - u$ we have that

$$|\hat{u}(t+is)| \leq C\exp(-P(s)-(s+\frac{\varepsilon}{2})y)\int|\psi(u)|\exp((s+\frac{\varepsilon}{2})u)du\int \exp(M(\xi+t))\left|\hat{H}(\xi+i\frac{\varepsilon}{2})\right|d\xi.$$

Since it is easy to see by (2.1) that

$$M(\xi+t) \leq M(\omega t) + C\xi^q + C \text{ for any } \omega > 1, \tag{2.7}$$

where C may depend on ω, we have the estimate

$$|\hat{u}(t+is)| \leq C\exp(M(\omega t)-P(s)-(s+\frac{\varepsilon}{2})y)\int|\psi(x)|\exp((s+\frac{\varepsilon}{2})x)dx.$$

Now

$$\int|\psi(x)|\exp((s+\frac{\varepsilon}{2})x)dx \leq C(\frac{2}{\varepsilon}+\int_0^\infty \exp((s+\frac{\varepsilon}{2})x-F(x))dx) \leq C\exp(G(s+\varepsilon))$$

since for any pair of conjugated Young functions F and G

$$(s+\varepsilon)x \leq F(x)+G(s+\varepsilon).$$

For any fixed $\varepsilon > 0$ and $\omega > 1$ we have now proved that

$$|\hat{u}(t+is)| \leq C\exp(-(s+\frac{\varepsilon}{2})y-P(s)+G(s+\varepsilon)+M(\omega t)) \text{ for all real } t \text{ and } s \geq 0. \tag{2.8}$$

We continue by putting

$$u = u_1 + u_2 \text{ where } u_1(x) = u(x) \text{ if } x \geq 0 \text{ and } u_1(x) = 0 \text{ elsewhere.}$$

It is trivial to see that

$$|\hat{u}_1(t+is)| \leq C \text{ if } s \leq 0 \text{ and that } |\hat{u}_2(t-is)| \leq C \text{ if } s \geq 0,$$

and thus by (2.3)

$$|\hat{u}_1(t+is)| \leq C \exp(M(\omega t)) \text{ if } s \geq 0.$$

Hence $\hat{u}_1$ is of minimal order q.

If now $q < 2$ then $\hat{u}_1$ is of minimal order 2 and since $\hat{u}_1$ is bounded on both axis in the complex plane it follows from the Phragmen-Lindelöf theorem (cf. e.g. [4] p. 177-178) that $\hat{u}_1$ is bounded in the whole complex plane.

If $q \geq 2$ it follows from formula (2.8) and condition (2.4) that $\hat{u}_1(t+is)$, for a properly chosen ω, is bounded on the lines $s = \pm\delta^{-1}t$ in the upper halfplane, and then also when $s \geq \delta^{-1}|t|$. Also in this case we can use Phragmen-Lindelöf's theorem to conclude that $\hat{u}_1$ is bounded in the complex plane.

By Liouvilles theorem it now follows that $\hat{u}_1$ is a constant and since $\hat{u}_1(is) \to 0$ when $s \to -\infty$ we can conclude that $\hat{u}_1 \equiv 0$.

From the uniqueness of the Fourier transform it follows that $u_1(x) = 0$ a.e. and hence

$$\phi(x) = 0 \text{ a.e.} \quad \text{for } x > y,$$

which was to be proved.

3 Applications for some well-known transforms

In Corollary 3.1 and 3.2 below we let K_0 be the modified Bessel function,

$$K_0(x) = \frac{1}{2}\int \exp(-x\cosh x)dt.$$

COROLLARY 3.1. Let α be a measurable and bounded function on the positive real line and let γ be a positive real number. If the Laplace transform

(3.1) $$f(s) = \int_0^\infty \exp(-st)\alpha(t)dt,$$

or if the Meijer transform

(3.2) $$f(s) = \int_0^\infty \sqrt{st}K_0(st)\alpha(t)dt,$$

where the integrals converges for all large values of s,
and if

(3.3) $$f(s) = O(\exp(-\gamma s)) \text{ when } s \to +\infty$$

then

(3.4) $$\alpha(t) = 0 \text{ a.e. for } 0 < t < \gamma.$$

PROOF. Let $s = \exp x$ and $t = \exp(-v)$ in (3.1) and (3.2). Then (3.1) transforms into the convolution transform

$$K * \phi(x) = \psi(x) = e^x f(\exp x)$$

with

$$K(x) = \exp(-e^x + x),\ \phi(x) = \alpha(e^{-x}) \text{ and } \hat{K}(t) = \Gamma(1 - it),$$

and (3.2) is transformed into

$$K * \phi(x) = \psi(x) = \exp(\frac{x}{2}) \cdot f(\exp x)$$

with

$$K(x) = e^x K_0(e^x), \phi(x) = \exp(-\frac{x}{2}) f(\exp x) \text{ and } \hat{K}(t) = 2^{t-1}\Gamma^2(\frac{1-it}{2}).$$

By use of Stirlings formula we have the same estimate in both cases,

$$|g(z)| \leq C \exp(m|x| - (y \ln(1+y) - y)).$$

We now use Theorem 2.1 with $P(s) = s\ln(1+s) - s$ and $F(x) = \gamma \exp x - x$. For the conjugate Young function we see that

$$G(s) = s\ln(1+s) - s + \ln(1+s) - s\ln\gamma$$

and then (2.3) in Theorem 2.1 is true if $y > -\ln\gamma$ and hence

$$\phi(x) = 0 \text{ a.e. for } x > -\ln\gamma.$$

This implies (3.4) and Corollary 3.1 is proved.

REMARK. For the Laplace transform we could equally well start with (3.5) below instead of (3.1) if

$$f(s) = \int_0^\infty e^{-st} d\alpha(t). \tag{3.5}$$

Here α is supposed to be of bounded variation on the positive real line with $\alpha(0) = 0$. After a partial integration in (3.5) we see that $f(s)$ still satisfies condition (3.3). Furthermore it is no restriction to suppose that α is bounded on the real line in this case, since the mere existence of (3.5) for all $s > 0$ implies that

$$f(s) = \int_\omega^\infty e^{-st} d\alpha(t) = O(e^{-\gamma s}), s \to +\infty \text{ for any } \gamma < \omega. \quad \text{(Cf [7], p.39.)}$$

In(3.5) we could then suppose that $\alpha(t) = 0$ for all large values of t without changing the problem. Hence it is no restriction to suppose that α is bounded in case of the Laplace transform in the corollary.

COROLLARY 3.2. Let α be a measurable and bounded function on the positive real line and let γ be a positive real number. If the iterated Laplace transform

$$f(s) = \int_0^\infty \frac{\exp(-st)}{t} dt \int_0^\infty \exp(-\frac{u}{t})\alpha(u)du = \int_0^\infty K_0(2\sqrt{st})\alpha(t)dt \tag{3.6}$$

fulfills the condition

$$f(s) = O(\exp(-\gamma\sqrt{s}), s \to +\infty$$

then

$$\alpha(t) = 0 \text{ a.e. in the intervall } 0 < t < 4\gamma^2. \tag{3.7}$$

PROOF. In formula (3.6) we make the substitutions $s = \exp x, u = \exp(-v)$ and then formula (3.6) transforms into a convolution transform

$$\psi(x) = K * \phi(x)$$

with

$$K(x) = \int_{-\infty}^\infty \exp(-e^{(x-u)} - e^u)du,$$
$$\phi(x) = \alpha(\exp(-x)),\ \psi(x) = e^x f(e^x) \text{ and } \hat{K}(t) = \Gamma^2(1-it).$$

This time

$$F(x) = \gamma \exp\frac{x}{2} - x,\ G(s) = 2s\ln s - 2s - 2s\ln 2\gamma \text{ and } P(s) = 2s\ln(1+s) - 2s.$$

It follows that (2.3) in Theorem 2.1 is true if

$$y > -2\ln(2\gamma)$$

and the desired result (3.7) follows from this.

COROLLARY 3.3. Suppose that ϕ is a bounded and measurable function on R and let the Weierstrass transform

$$\psi(x) = \int_{-\infty}^\infty K(x-t)\phi(t)dt, \text{ where } K(x) = \frac{1}{\sqrt{4\pi}}\exp(-\frac{x^2}{4}), \tag{3.8}$$

satisfy the condition

$$\psi(x) = O(\exp(-\gamma x^2)) \text{ when } x \to \infty \tag{3.9}$$

for some $\gamma > \frac{1}{4}$.

Then we have that

$$\phi(x) = 0 \text{ a.e. in } R.$$

PROOF. Since $\hat{K}(t) = \exp(-t^2)$ it follows that $|g(x+iy)| = \exp(x^2 - y^2)$, We must now take $q > 2$ and start with $F(x) = \gamma x^2$ and $P(s) = s^2$.

We see that

$$G(s) = \frac{s^2}{4\gamma}$$

and then

$$-sy + G(s+\varepsilon) - P(s) + M(\delta s) = -sy + \frac{(s+\varepsilon)^2}{4\gamma} - s^2 + (\delta s)^2$$

If now $\gamma > \frac{1}{4}$ we can choose ε and δ such that conditions (2.2) and (2.3) in Theorem 2.1 is fulfilled for any value of y and hence

$$\phi(x) = 0 \text{ a.e. in } R,$$

which was to be proved.

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