The stationary nonlinear Boltzmann equation in a Couette setting; L^q-solutions and positivity.

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Abstract.
This is the second part in a study of the stationary nonlinear Boltzmann equation for forces including hard spheres, in a Couette setting between two coaxial, rotating cylinders with given Maxwellian indata on the cylinders. A priori L^q-estimates are obtained, leading to isolated solutions together with a hydrodynamic limit control based on asymptotic expansions of low orders together with a rest term. A proof of the positivity of such solutions is also given.

1 Introduction.

For a general introduction see part one [AN2], which is set in the same close to equilibrium frame as the present paper, namely a stationary nonlinear Boltzmann equation in the domain Ω between two coaxial cylinders A and B with Maxwellian ingoing boundary values. The problem is extensively treated from a numerical and asymptotic perspective in [S]. The boundary values and the solutions are assumed to be axially and circumferentially uniform in the space variables. Then, with (r, θ, z) and (v_r, v_θ, v_z) respectively denoting the spatial cylindrical coordinates and the corresponding Cartesian velocity coordinates, a distribution function may be written as $f = f(r, v_r, v_\theta, v_z)$, and the Boltzmann equation becomes

$$\begin{aligned} &v_r \frac{\partial f}{\partial r} + \frac{1}{r} N f = \frac{1}{\epsilon^4} \tilde{Q}(f, f), \\ &r \in (r_A, r_B), \quad (v_r, v_\theta, v_z) \in I\!R^3. \end{aligned} \tag{1.1}$$

The ingoing Maxwellian boundary data on $\partial\Omega^+$ are

$$f(r_A, v) = (2\pi)^{-\frac{3}{2}} e^{\frac{1}{2}(-v_r^2 - (v_\theta - \epsilon u_{\theta A1})^2 - v_z^2)}, \quad v_r > 0, \tag{1.2}$$

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$$f(r_B,v)=(2\pi)^{-\frac{3}{2}}\frac{1+\omega_B}{(1+\tau_B)^{\frac{3}{2}}}e^{\frac{1}{2}(-\frac{1}{1+\tau_B}(v_r^2+(v_\theta-\epsilon u_{\theta B1})^2+v_z^2))},\quad v_r<0.$$

Here

$$\begin{aligned} Nf &:= v_\theta^2\frac{\partial f}{\partial v_r}-v_\theta v_r\frac{\partial f}{\partial v_\theta}, \\ \tilde{Q}(f,f)(v) &:= \int_{I\!R^3\times S^2} B(v-v_*,\omega)(f(v')f(v'_*)-f(v)f(v_*))dv_*d\omega. \end{aligned} \tag{1.3}$$

The kernel $B=|v-v_*|^\beta b(\theta)$, $b\in L^1_+(S^2)$, is assumed to satisfy (2.19) below and belong to the Grad class, that is with its terms suitably majorized by the corresponding ones for the hard sphere model (cf [M]). Consider functions which are even in the axial velocity direction v_z. Take the radii as $r_A=1$, $r_B>1$, and let ϵ^4 denote the Knudsen number. The rotational velocities of the inner and outer cylinders are taken as $u_{\theta A}=\epsilon u_{\theta A1}$ and $u_{\theta B}=\epsilon u_{\theta B1}$ respectively. The non-dimensional perturbed temperature and density are

$$\begin{aligned} \tau_B &= \epsilon^2\tau_{B2}, \\ \omega_B &= \frac{\epsilon^2}{1+\epsilon^2\tau_{B2}}\Big(\frac{r_B^2-1}{r_B^2}u_{\theta A1}^2-\tau_{B2}+\Delta\epsilon\Big), \end{aligned} \tag{1.4}$$

where τ_{B2} is given and Δ is a parameter.

The main result of this paper is the following theorem.

Theorem 1.1 *Assume that $(u_{\theta A1}-u_{\theta B1}r_B)(3u_{\theta A1}+u_{\theta B1}r_B)>0$. There is a negative value Δ_{bif} of the parameter Δ, such that for $\Delta<\Delta_{bif}$ and $0<\epsilon$ small enough, two isolated, non-negative L^1-solutions f_ϵ^j, $j=1,2$ of (1.1-2) coexist, and satisfy*

$$\int M^{-1} supess_{r\in(r_A,r_B)}\mid f_\epsilon^j(r,v)\mid^2 dv<+\infty.$$

The two solutions have different outward radial bulk velocities of order ϵ^3. For fixed ϵ, they converge to the same solution, when Δ increases to Δ_{bif}. The solutions have rigorous leading order hydrodynamic limits when $\epsilon\to 0$.

Remark. For a study of a corresponding two-roll problem far from equilibrium, see [AN1]. The above existence result is based on a priori estimates of L^q-type, which are uniform in ϵ. The positivity of the solutions follows from a contraction mapping iteration using the L^q-estimates for q large. The approach has wider applicability. In particular, as discussed in Section 5 below, analogous results hold for all cases of the two-roll problem treated in [S]. We expect the techniques developed here, also to be useful in the study of related problems, such as the Taylor-Couette set-up of [SHD], the Bénard asymptotics of [SD], and the two-component gases of [ATT].

Write $R = f_{rest} = P_0 f_{rest} + (I - P_0) f_{rest} = R_{||} + R_{\perp}$, where P_0 is the projection on the hydrodynamic part, and

$$f = M(1 + \psi + \epsilon^{j_0} f_{rest}) \text{ with } \psi = \sum_1^{j_1} \epsilon^j \psi^j, \quad M = (2\pi)^{-\frac{3}{2}} \exp(-\frac{v^2}{2}). \tag{1.5}$$

Here $\sum_1^{j_1} \epsilon^j \psi^j$ is the asymptotic expansion with certain boundary values equal to the terms of corresponding order in the ϵ-expansions of (1.2), and based on a splitting into interior Hilbert behaviour, and boundary layers of suction and Knudsen types. The main part of the paper is devoted to a rigorous study of the rest term $R = f_{rest}$ in L^q, using as ingoing boundary values what remains of (1.2) after the asymptotic expansion. The rest term problem is solved by a contraction mapping iteration.

Section two summarizes the asymptotic discussion from [AN2]. Section three contains some a priori estimates for the rest term. A dual problem is studied for the non-hydrodynamic part in L^q for $q < 2$, and uses a strong form of the Banach-Steinhaus theorem to obtain an a priori estimate uniform in ϵ. This is then used to prove the corresponding non-hydrodynamic estimate for the original problem and $q > 2$.
Section four deals with the rest term; the contraction mapping procedure, the hydrodynamic limits, and the positivity. The final section takes up existence results for other two-roll problems, where the approach applies essentially without modifications.

2 The asymptotic frame.

Write the solution of (1.1-2) as $f = M(1 + \Phi)$. Then the new unknown $\Phi(r, v_r, v_\theta)$ should be solution to

$$v_r \frac{\partial \Phi}{\partial r} + \frac{1}{r} N\Phi = \frac{1}{\epsilon^4}(\tilde{L}\Phi + \tilde{J}(\Phi, \Phi)), \tag{2.1}$$

$$\Phi(1, v) = e^{\frac{1}{2}(v_\theta^2 - (v_\theta - \epsilon u_{\theta A1})^2)} - 1, \quad v_r > 0, \tag{2.2}$$

$$\Phi(r_B, v) = \frac{1 + \omega_B}{(1 + \tau_B)^{\frac{3}{2}}} e^{\frac{1}{2}(v^2 - \frac{1}{1+\tau_B}(v_r^2 + (v_\theta - \epsilon u_{\theta B1})^2 + v_z^2))} - 1, \quad v_r < 0. \tag{2.3}$$

Here $\tilde{J}$ is the rescaled quadratic Boltzmann collision operator,

$$\tilde{J}(\Phi, \psi)(v) := \frac{1}{2} \int_{I\!R^3 \times S^2} B(v - v_*, \omega) M(v_*)(\Phi(v')\psi(v'_*) + \Phi(v'_*)\psi(v')$$
$$-\Phi(v_*)\psi(v) - \Phi(v)\psi(v_*)) dv_* d\omega,$$

and $\tilde{L}$ is this operator linearized around 1,

$$(\tilde{L}\Phi)(v) := \int_{I\!R^3 \times S^2} B(v - v_*, \omega) M(v_*)(\Phi(v') + \Phi(v'_*) - \Phi(v_*)$$
$$-\Phi(v)) dv_* d\omega = \tilde{K}(\Phi) - \tilde{\nu}\Phi.$$

Denote by $(\Phi_{Ai})_{1\leq i\leq j}$ resp.$(\Phi_{Bi})_{1\leq i\leq j}$, the first to j-th order terms of $\Phi(r_A,v)$ resp. $\Phi(r_B,v)$, with respect to ϵ.
Solutions Φ will be determined as in (1.5), as an approximate solution ψ plus a rest term $R=f_{rest}$,

$$\Phi(r,v)=\psi(r,v)+\epsilon^{j_0}R(r,v),$$

where for $j_1=4$

$$\begin{aligned}\psi(r,v)&=\epsilon\Big(\Phi_{H1}(r,v)+\Phi_{W1}(\frac{r-r_B}{\epsilon},v)\Big)+\epsilon^2\Big(\Phi_{H2}(r,v)+\Phi_{W2}(\frac{r-r_B}{\epsilon},v)\Big)\\ &+\epsilon^3\Big(\Phi_{H3}(r,v)+\Phi_{W3}(\frac{r-r_B}{\epsilon},v)+\Phi_{K3A}(\frac{r-1}{\epsilon^4},v)+\Phi_{K3B}(\frac{r-r_B}{\epsilon^4},v)\Big)\\ &+\epsilon^4\Big(\Phi_{H4}(r,v)+\Phi_{W4}(\frac{r-r_B}{\epsilon},v)+\Phi_{K4A}(\frac{r-1}{\epsilon^4},v)+\Phi_{K4B}(\frac{r-r_B}{\epsilon^4},v)\Big)\end{aligned} \tag{2.4}$$

with

$$\begin{aligned}&\int\Phi_{H1}(.,v)(1,v_r,v^2)M(v)dv=\int\Phi_{W1}(.,v)(1,v_r,v^2)M(v)dv\\ &=\int\Phi_{H2}(.,v)v_rM(v)dv=0,\end{aligned} \tag{2.5}$$

$$\lim_{\frac{r-r_B}{\epsilon}\to-\infty}\Phi_{Wi}(\frac{r-r_B}{\epsilon},v)=0,\quad 1\leq i\leq 4, \tag{2.6}$$

$$\lim_{\frac{r-1}{\epsilon^4}\to+\infty}\Phi_{KiA}(\frac{r-1}{\epsilon^4},v)=0,\qquad \lim_{\frac{r-r_B}{\epsilon^4}\to-\infty}\Phi_{KiB}(\frac{r-r_B}{\epsilon^4},v)=0,\quad 3\leq i\leq 4. \tag{2.7}$$

Here $(\epsilon\Phi_{H1}+\epsilon^2\Phi_{H2}+\epsilon^3\Phi_{H3}+\epsilon^4\Phi_{H4})(r,v)$ denotes the truncation up to fourth order of a formal expansion $\sum_{k\geq1}\epsilon^k\Phi_{Hk}(r,v)$. The sum $(\epsilon\Phi_{W1}+\epsilon^2\Phi_{W2})(\frac{r-r_B}{\epsilon},v)$ consists of correction terms allowing the boundary conditions to be satisfied to first and second order. They correspond to a suction boundary layer at r_B. Supplementary boundary layers of Knudsen type, described by

$$\epsilon^3(\Phi_{K3A}(\frac{r-1}{\epsilon^4},v)+\Phi_{K3B}(\frac{r-r_B}{\epsilon^4},v))+\epsilon^4(\Phi_{K4A}(\frac{r-1}{\epsilon^4},v)+\Phi_{K4B}(\frac{r-r_B}{\epsilon^4},v)),$$

are required in order to have the boundary conditions satisfied at third and fourth orders.
We shall here give a fairly detailed discussion of the asymptotic expansion for $j_0=j_1=4$. Uniqueness statements in that discussion are modulo possible shifts of terms between the asymptotic expansion and rest term at fourth order. Recall (see [D]) that $\tilde{L}((|v|)v_\theta v_r\bar{B})=-v_\theta v_r$, $\tilde{L}(v_r\bar{A})=v_r(v^2-5)$ for some functions $\bar{B}(|v|)$ and $\bar{A}(|v|)$, with $v_\theta v_r\bar{B}(|v|)$ and $v_r\bar{A}(|v|)$ bounded in the $(,)_M$-norm, and let

$$w_1:=\int v_r^2v_\theta^2\bar{B}Mdv.$$

Let $g(\eta,v)$ be the solution to the half-space problem

$$
\begin{aligned}
&v_r\frac{\partial g}{\partial v_r} = Lg, \quad \eta>0, \quad v\in I\!R^3,\\
&g(0,v)=0, \quad v_r>0,\\
&\int g(\eta,v)v_r M(v)dv = 1, \quad a.a.\ \eta>0.
\end{aligned}
\tag{2.8}
$$

By [BCN], [GP] there are constants A, D, and E such that, (sub-)exponentially,

$$
\lim_{\eta\to+\infty} g(\eta,v) = A + Dv^2 + Ev_\theta + v_r. \tag{2.9}
$$

Proposition 2.1
Assume that

$$
(u_{\theta B1}r_B - u_{\theta A1})(u_{\theta B1}r_B + 3u_{\theta A1}) > 0, \quad A+5D>0,
$$

and set

$$
\Delta_{bif} := -\Big(2w_1\frac{r_B+1}{r_B^3}(A+5D)(r_Bu_{\theta B1}-u_{\theta A1})(r_Bu_{\theta B1}+3u_{\theta A1})\Big)^{\frac{1}{2}}.
$$

For $\Delta>\Delta_{bif}$, there is no solution ψ in the family defined in (2.4-7).
For $\Delta=\Delta_{bif}$, there is a unique solution ψ in the family defined in (2.4-7).
For $\Delta<\Delta_{bif}$, there are two solutions ψ in the family defined in (2.4-7).

Proof of Proposition 2.1. Denote by $Y=\frac{r-r_B}{\epsilon}$, and let the expansions $\sum_{k\geq 1}\epsilon^k\Phi_{Hk}(r,v)$ and $\sum_{k\geq 1}\epsilon^k(\Phi_{Hk}(r_B,v)+\Phi_{Wk}(\frac{r-r_B}{\epsilon},v))$ formally satisfy (2.1). Then,

$$
\begin{aligned}
&\tilde L\Phi_{H1} = \tilde L\Phi_{H2} + \tilde J(\Phi_{H1},\Phi_{H1}) = \tilde L\Phi_{H3} + 2\tilde J(\Phi_{H1},\Phi_{H2})\\
&= \tilde L\Phi_{H4} + 2\tilde J(\Phi_{H1},\Phi_{H3}) + \tilde J(\Phi_{H2},\Phi_{H2}) = 0, \qquad (2.10)\\
&v_r\frac{\partial\Phi_{Hk-4}}{\partial r} + \frac{1}{r}N\Phi_{Hk-4} = \tilde L\Phi_{Hk} + \sum_{j=1}^{k-1}\tilde J(\Phi_{Hj},\Phi_{Hk-j}), \quad k\geq 5, \qquad (2.11)
\end{aligned}
$$

and

$$
\begin{aligned}
&\tilde L\Phi_{W1} = \tilde L\Phi_{W2} + \tilde J(\Phi_{W1}, 2\Phi_{H1}(r_B,.)+\Phi_{W1})\\
&= \tilde L\Phi_{W3} + 2\tilde J(\Phi_{H1}(r_B,.)+\Phi_{W1},\Phi_{W2}) + 2\tilde J(\Phi_{W1},\Phi_{H2}(r_B,.)+Y\Phi'_{H1}(r_B,.))\\
&= \tilde L\Phi_{W4} + 2\tilde J(\Phi_{W3},\Phi_{H1}(r_B,.)+\Phi_{W1}) + \tilde J(\Phi_{W2},\Phi_{W2}+2\Phi_{H2}(r_B,.)\\
&+2Y\Phi'_{H1}(r_B,.)) + 2\tilde J(\Phi_{W1},\Phi_{H3}(r_B,.)+Y\Phi'_{H2}(r_B,.)\\
&+\frac{Y^2}{2}\Phi''_{H1}(r_B,.)) - v_r\frac{\partial\Phi_{W1}}{\partial Y} = 0, \qquad (2.12)\\
&v_r\frac{\partial\Phi_{Wk-3}}{\partial Y} + \frac{1}{r_B}\sum_{i=0}^{k-5}(-1)^i(\frac{Y}{r_B})^iN(\Phi_{Hk-4-i}(r_B,.)+\Phi_{Wk-4-i})\\
&= \tilde L\Phi_{Wk} + \sum_{j=1}^{k-1}\tilde J(2\Phi_{Hj}(r_B,.)+\Phi_{Wj},\Phi_{Wk-j}), \quad k\geq 5. \qquad (2.13)
\end{aligned}
$$

By (2.5) and (2.10), $\Phi_{H1}(r,v) = b_1(r)v_\theta$ for some function b_1, and $\Phi_{Hi}, i \geq 2$ split into a hydrodynamical part $a_i(r) + d_i(r)v^2 + b_i(r)v_\theta + c_i(r)v_r$ and a non-hydrodynamic part involving Hilbert terms of lower order than i. Equations (2.11) have solutions if and only if the following compatibility conditions hold,

$$\int \Big(v_r \frac{\partial \Phi_{Hi}}{\partial r} + \frac{1}{r} N\Phi_{Hi}\Big)(1, v^2-5, v_\theta, v_r)M(v)dv = 0, \quad i \geq 1.$$

They provide first-order differential equations for the functions $a_i(r)$, $b_i(r)$, $c_i(r)$ and $d_i(r)$, $i \geq 1$. Together with the boundary condition (2.2) at first and second orders, this fixes $\Phi_{H1}(r,v) = \frac{u_{\theta A1}}{r}v_\theta$, Φ_{H2} and $c_3(r) = \frac{u_3}{r}$, for some constant $u_3 \neq 0$. By (2.5) and (2.12), $\Phi_{W1}(Y,v) = z_1(Y)v_\theta$, for some function z_1, and $\Phi_{Wi}, i \geq 2$ split into a hydrodynamical part $x_i(Y) + y_i(Y)v^2 + z_i(Y)v_\theta + t_i(Y)v_r$ and a non-hydrodynamic part involving Hilbert terms of lower order than i. Equations (2.13) have solutions if and only if the following compatibility conditions hold,

$$\int \Big(v_r \frac{\partial \Phi_{Wk-3}}{\partial Y} + \frac{1}{r_B}\sum_{i=0}^{k-5}(-1)^i(\frac{Y}{r_B})^i N(\Phi_{Hk-4-i}(r_B,.) + \Phi_{Wk-4-i}\Big)(v^2$$
$$-5, v_\theta)M(v)dv = 0, \quad k \geq 5, \tag{2.14}$$

and

$$\int \Big(v_r \frac{\partial \Phi_{Wk-3}}{\partial Y} + \frac{1}{r_B}\sum_{i=0}^{k-5}(-1)^i(\frac{Y}{r_B})^i N(\Phi_{Hk-4-i}(r_B,.) +$$
$$\Phi_{Wk-4-i}\Big)(1, v_r)M(v)dv = 0, \quad k \geq 5. \tag{2.15}$$

Equations (2.14) (resp. (2.15)) provide second-order (resp. first-order) differential equations for y_i and z_i (resp. $x_i + 5y_i$ and t_i). Together with the boundary conditions (2.3) at first and second orders, and the conditions (2.6) and (1.4), this fixes

$$\Phi_{W1}(Y,v) = (u_{\theta B1} - \frac{u_{\theta A1}}{r_B})e^{\frac{u_3 Y}{w_1 r_B}},$$

Φ_{W2} in terms of u_3, and implies that $t_3 = t_4 = 0$. Then, giving the value 0 to any coefficient of order bigger than 5 in the second-order differential equations satisfied by y_i and z_i, $3 \leq i \leq 4$ and taking into account (2.3-6) fixes the functions y_i and z_i, $3 \leq i \leq 4$ in terms of u_i. Finally the following Knudsen analysis at third and fourth orders make the first-order differential equations satisfied by $x_3 + 5y_3$ and $x_4 + 5y_4$ compatible with (2.3), (2.6) at third and fourth orders. □

Lemma 2.1 *Set* $\eta = \frac{r-1}{\epsilon^4}$, $\mu = \frac{r-r_B}{\epsilon^4}$. *There are unique Knudsen boundary layers* $\Phi_{K3A}(\eta, v)$ *and* $\Phi_{K3B}(\mu, v)$, *and boundary values* $\Phi_{H3}(1,v)$ *and* $\Phi_{W3}(0,v)$, *such that*

$$\begin{aligned} &v_r \frac{\partial \Phi_{K3A}}{\partial \eta} = \tilde{L}\Phi_{K3A}, \quad \eta > 0, \quad v \in I\!R^3, \\ &\Phi_{K3A}(0,v) = \Phi_{A3}(v) - \Phi_{H3}(1,v), \ v_r > 0, \\ &\lim_{\eta \to +\infty} \Phi_{K3A}(\eta, v) = 0, \end{aligned} \tag{2.16}$$

and

$$
\begin{aligned}
&v_r \frac{\partial \Phi_{K3B}}{\partial \mu} = \tilde{L}\Phi_{K3B}, \quad \mu < 0, \quad v \in I\!R^3, \\
&\Phi_{K3B}(0,v) = \Phi_{B3}(v) - \Phi_{H3}(r_B,v) - \Phi_{W3}(0,v), \ v_r < 0, \\
&\lim_{\mu \to -\infty} \Phi_{K3B}(\mu,v) = 0.
\end{aligned} \tag{2.17}
$$

The boundary layers fix the possible values of $a_3(1)$*,* $d_3(1)$*,* u_3*,* $b_3(1)$ *and* $x_3(0)$*,* $y_3(0)$*,* $z_3(0)$*, hence complete the definitions of* Φ_{H3} *and* Φ_{W3}*.*

Proof of Lemma 2.1. For a proof of Lemma 2.1, we refer to [AN2]. Recall that the analysis implies that u_3 is a solution to the equation

$$
u_3^2(A+5D)\frac{r_B+1}{r_B} - \Delta u_3 + \frac{w_1}{2r_B^2}(3u_{\theta A1} + u_{\theta B1}r_B)(u_{\theta A1} - u_{\theta B1}r_B = 0 \tag{2.18}
$$

A study of the positive roots u_3 to (2.18) leads to the three cases described in Proposition 2.1 for Δ with respect to Δ_{bif} . That proof requires

$$
A + 5D \neq 0, \tag{2.19}
$$

a condition satisfied for hard spheres, and assumed to hold for the kernels B of this paper. □

Lemma 2.2 *Set* $\eta = \frac{r-1}{\epsilon^4}$*,* $\mu = \frac{r-r_B}{\epsilon^4}$*. There are unique Knudsen boundary layers* $\Phi_{K4A}(\eta,v)$ *and* $\Phi_{K4B}(\mu,v)$*, and boundary values* $\Phi_{H4}(1,v)$ *and* $\Phi_{W4}(0,v)$ *such that*

$$
\begin{aligned}
&v_r \frac{\partial \Phi_{K4A}}{\partial \eta} = \tilde{L}\Phi_{K4A} + 2\tilde{J}(\Phi_{H1}(1), \Phi_{K3A})), \quad \eta > 0, \ v \in I\!R^3, \\
&\Phi_{K4A}(0,v) = \Phi_{A4}(v) - \Phi_{H4}(1,v), \quad v_r > 0, \\
&\lim_{\eta \to +\infty} \Phi_{K4A} = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&v_r \frac{\partial \Phi_{K4B}}{\partial \mu} = \tilde{L}\Phi_{K4B} + 2\tilde{J}(\Phi_{H1}(r_B) + \Phi_{W1}(0), \Phi_{K3B}), \quad \mu < 0, \ v \in I\!R^3, \\
&\Phi_{K4B}(0,v) = \Phi_{B4}(v) - \Phi_{H4}(r_B,v) - \Phi_{W4}(0,v), \quad v_r < 0, \\
&\lim_{\mu \to -\infty} \Phi_{K4B} = 0.
\end{aligned}
$$

Proof of Lemma 2.2. For a proof of Lemma 2.2 we refer to [AN2]. The fourth order Knudsen boundary layers fix the possible values of $a_4(1)$, $d_4(1)$, $u_4 = r_B c_4(r_B)$ and $x_4(0)$, $y_4(0)$, $z_4(0)$, hence complete the definitions of Φ_{H4} and Φ_{W4}. □

Lemma 2.3 *Denote by* $\bar{l} := \frac{1}{\epsilon^4}\Big(\tilde{L}\psi + \tilde{J}(\psi,\psi) - \epsilon^4 D\psi\Big)$*. Then,*

$$
\Big(\int M(v) \sup_{r\in(1,r_B)} |\ \bar{l}(r,v)\ |^2 \, dv\Big)^{1/2}
$$

is of order one with respect to ϵ*.*

Proof of Lemma 2.3. By definition of ψ,

$$\frac{\epsilon^2}{2}\bar{l} = \tilde{J}(\Phi_{H1}-\Phi_{H1}(r_B),\Phi_{W1})$$
$$+\epsilon\Big(\tilde{J}(\Phi_{H1}-\Phi_{H1}(r_B),\Phi_{W2})+\tilde{J}(\Phi_{W1},\Phi_{H2}-\Phi_{H2}(r_B)-Y\Phi'_{H1}(r_B)\Big)$$
$$+\epsilon^2\Big(\tilde{J}(\Phi_{W3},\Phi_{H1}-\Phi_{H1}(r_B))+\tilde{J}(\Phi_{W2},\Phi_{H2}-\Phi_{H2}(r_B)-Y\Phi'_{H1}(r_B))$$
$$+\tilde{J}(\Phi_{W1},\Phi_{H3}-\Phi_{H3}(r_B)-Y\Phi'_{H2}(r_B)-\frac{Y^2}{2}\Phi''_{H1}(r_B))$$
$$+\tilde{J}(\Phi_{K3A},\Phi_{H1}-\Phi_{H1}(1)+\Phi_{W1})+\tilde{J}(\Phi_{K3B},\Phi_{H1}-\Phi_{H1}(r_B)+\Phi_{W1}-\Phi_{W1}(0))\Big)+O(\epsilon^3).$$

Hence

$$\bar{l}=\epsilon\tilde{J}\Big(\gamma_1(r)Y^3\Phi_{W1}+\gamma_2(r)Y^2\Phi_{W2}+\gamma_3(r)Y\Phi_{W3}+\gamma_4(r)Y\Phi_{K3A}+\gamma_5(r)Y\Phi_{K3B},v_\theta\Big)$$
$$+\tilde{J}(\Phi_{K3A},\Phi_{W1})+O(\epsilon),$$

where $(\gamma_i)_{1\leq i\leq 5}$ are bounded functions in r. The announced bound follows from the sub-exponential decrease of Φ_{K3A} with $\eta=\frac{r-1}{\epsilon^4}$. □

3 On the control of $f_\perp$ and $f_{||}$

As orthonormal basis for the kernel of $\tilde{L}$ in $L^2_M(I\!R^3)$ we take $\psi_0=1,\psi_\theta=v_\theta,\psi_r=v_r,\psi_z=v_z,\psi_4=\frac{1}{\sqrt{6}}(v^2-3)$. Recall that in this paper all functions are even in v_z. For functions $f\in L^2_M([r_A,r_B]\times I\!R^3)$ we shall use the earlier splitting into $f=f_{||}+f_\perp=P_0f+(I-P_0)f$, such that

$$f_{||}(r,v)=f_0(r)-\frac{\sqrt{6}}{2}f_4(r)+f_\theta(r)v_\theta+f_r(r)v_r+\frac{\sqrt{6}}{6}f_4(r)v^2,$$
$$\int M(v)(1,v,v^2)f_\perp(r,v)dv=0,$$
$$\int M\psi_0 f(r,v)dv=f_0(r),\ \int M\psi_4 f(r,v)dv=f_4(r),$$
$$\int M\psi_\theta f(r,v)dv=f_\theta(r),\ \int M\psi_r f(r,v)dv=f_r(r).$$

(The ψ_z-moment of $f_{||}$ vanishes since f is even in v_z.) Define $\tilde{\nu}:=\nu\epsilon^4$, and $Df:=v_r\frac{\partial f}{\partial r}+\frac{1}{r}Nf$ with N given by (1.3). For $1\leq q\leq+\infty$, denote by $\|\ .\ \|_q$ the usual Lebesgue norm, and set

$$\tilde{L}^q:=\{f;\mid f\mid_q:=\Big(\int M(v)(\int\mid f(x,v)\mid^q dx)^{\frac{2}{q}}dv\Big)^{\frac{1}{2}}<+\infty\}.$$

Due to the symmetries in the present setup, the position space may be changed from $I\!R^2$ with measure dx, to $I\!R^+$ with measure rdr. The relevant boundary space becomes

$$L^+:=\{f;\mid f\mid_\sim=\Big(\int_{v_r>0}v_rM(v)\mid f(r_A,v)\mid^2dv\Big)^{\frac{1}{2}}+$$
$$\Big(\int_{v_r<0}\mid v_r\mid M(v)\mid f(r_B,v)\mid^2dv\Big)^{\frac{1}{2}}<+\infty\}.$$

We shall also use

$$\mathcal{W}^{q-}([r_A,r_B]\times I\!R^3)=\mathcal{W}^{q-}:=\{f;\nu^{\frac{1}{2}}f\in\tilde{L}^q,\nu^{-\frac{1}{2}}Df\in\tilde{L}^q,\gamma^+f\in L^+\}$$

The following propositions were proved in part one [AN2].

Proposition 3.1 *Let $u\in I\!R$, $g=g_\perp$, $\tilde{\nu}^{-\frac{1}{2}}g\in\tilde{L}^q$, $f_b\in L^+$, $2\leq q<\infty$ be given. For small enough $\epsilon>0$, there exists a unique solution $F\in\mathcal{W}^{q-}$ to*

$$DF=\frac{1}{\epsilon^4}(\tilde{L}F+\epsilon u\tilde{J}(F,v_\theta)+g),\quad F_{/\partial\Omega^+}=f_b, \tag{3.1}$$

where the boundary data are given on the ingoing boundary $\partial\Omega^+$.

Proposition 3.2 *Let $2\leq q<+\infty$, and let F be the solution in $\mathcal{W}^{q-}$ to (3.1) for $g=g_\perp$. The hydrodynamic part $F_{||}$ of F can be split as $F_{||}=F_{I||}+\frac{c_r}{r}v_r$, where $F_r=\frac{c_r}{r}$. For small enough $\epsilon>0$*

$$|\,F_{I||}\,|_q\leq c\Big(\,|\,F_\perp\,|_q+|\,\tilde{\nu}^{-\frac{1}{2}}g\,|_q\,\Big), \tag{3.2}$$

$$\|\,F_r\,\|_q\leq c(\epsilon^3\,\|\,F_\theta\,\|_q+|\,F_\perp\,|_q+\frac{1}{\epsilon}\,\|\int gv_\theta v_r\bar{B}(|v|)Mdv\,\|_1). \tag{3.3}$$

We shall also need the following $\tilde{L}^q$ estimate for the non-hydrodynamic part $F_\perp$.

Proposition 3.3 *Let $2\leq q\leq+\infty$ and let F be the solution in $\mathcal{W}^{q-}$ to (3.1) for $g=g_\perp$. The following estimates hold for small enough $\epsilon>0$;*

$$|\,\tilde{\nu}^{\frac{1}{2}}F_\perp\,|_q\leq c(|\,\tilde{\nu}^{-\frac{1}{2}}g\,|_q+\epsilon\,|\,F_rv_r\,|_q+\epsilon^2\,|\,f_b\,|_\sim),\quad q<\infty, \tag{3.4}$$

$$|\,\tilde{\nu}^{\frac{1}{2}}F_\perp\,|_\infty\leq c(|\,\tilde{\nu}^{-\frac{1}{2}}g\,|_\infty+\epsilon\,|\,F_{||}\,|_\infty+\epsilon^2\,|\,f_b\,|_\sim),$$

$$|\,\tilde{\nu}^{\frac{1}{2}}F\,|_\infty\leq c(|\,\tilde{\nu}^{-\frac{1}{2}}g\,|_\infty+\epsilon^{-\frac{8}{q}}\,|\,F\,|_q+|\,f_b\,|_\sim). \tag{3.5}$$

The estimate (3.5) also holds when g has a non-vanishing hydrodynamic component $g_{||}$.

Proof of Proposition 3.3. The estimate (3.5) was proved in part one ([AN2]) under the restriction $g=g_\perp$. The same proof holds without this restriction, so we shall only discuss (3.4).
The mapping from $\tilde{\nu}^{\frac{1}{2}}\tilde{L}^q\times L^+$ into $\mathcal{W}^{q-}$ given by $(g,f_b)\to F$, with F the solution to (3.1), is continuous and bijective by [M, Ch 6.1]. The analysis in [M] is carried out for $2\leq q\leq\infty$.
Invoking a duality argument, similar results then follow for $1\leq q<2$, in particular for the dual problem to (3.1) with $u=0$;

$$-Dh=\frac{1}{\epsilon^4}(\tilde{L}h-H_\perp),\quad h_{/\partial\Omega^-}=h_b, \tag{3.6}$$

where the boundary data $h_b = 0$ are given on the outgoing boundary $\partial\Omega^-$. Here corresponding to (3.6), the mapping T_ϵ from $\tilde{\nu}^{\frac{1}{2}}\tilde{L}^q_\perp$ into $\tilde{\nu}^{-\frac{1}{2}}\tilde{L}^q \times L^+$ given by $T_\epsilon(H_\perp) = (h_\perp, \epsilon^2\gamma^+h)$, is continuous for $1 \leq q \leq 2$ and ϵ close to zero. Moreover, T_ϵ is for $q = 2$ equicontinuous with respect to ϵ close to zero, as follows from multiplying (3.6) with h, using Green's formula and the spectral inequality,

$$-\int h\tilde{L}hMdvdx \geq c \mid \tilde{\nu}^{\frac{1}{2}}h_\perp \mid_2^2 .$$

Below that equicontinuity with respect to ϵ from the case $q = 2$, will also be needed for $1 \leq q < 2$. But $h_\perp$ belongs for $q < 2$ both to $\tilde{L}^q$ and $\tilde{L}^2$ whenever $H_\perp$ is in $\tilde{L}^2$. The $\tilde{L}^q$-norm of $\tilde{\nu}^{\frac{1}{2}}h_\perp$ in the case $1 \leq q < 2$, is bounded by a constant times the corresponding $\tilde{L}^2$-norm. That makes the $\tilde{L}^q$-norm of $\tilde{\nu}^{\frac{1}{2}}h_\perp$ and the L^+ norm of $\epsilon^2\gamma^+h$ uniformly bounded with respect to ϵ close to zero, for a residual set of elements $\tilde{\nu}^{-\frac{1}{2}}H_\perp$ in $(I-P_0)\tilde{L}^q$. The Banach-Steinhaus theorem applies, so the norms of T_ϵ are uniformly bounded for ϵ close to zero. Hence the following estimate holds for (3.6) in the case of zero outgoing boundary data, $0 < \epsilon$ close to zero, and $1 \leq q \leq 2$,

$$\mid \tilde{\nu}^{\frac{1}{2}}h_\perp \mid_q + \epsilon^2 \mid \gamma^+h \mid_\sim \leq c_q \mid \tilde{\nu}^{-\frac{1}{2}}H_\perp \mid_q . \tag{3.7}$$

Here c_q is uniformly bounded for $1 \leq q \leq 2$. Turning to the estimate (3.4) involving F, $g = g_\perp$, and f_b, for $q = 2$ it essentially follows from Green's formula, as was shown in part one. For $q > 2$, set $u = 0$ in (3.1) to start with, use the dual results (3.6-7) for q' when $1 \leq q' < 2$, and take for $2 < q < \infty$

$$H = H_\perp = (I - P_0)[\tilde{\nu} \mid F_\perp \mid^{q-2} F_\perp(\int_{r_A}^{r_B} \mid F_\perp \mid^q rdr)^{-\frac{q-2}{q}}].$$

That leads to the following equation for $D(hF)$,

$$\int MD(hF)rdrdv = \frac{1}{\epsilon^4}(\int MFH_\perp rdrdv + \int Mg_\perp hrdrdv).$$

Recalling (3.7) and the definition of $H_\perp$,

$$\begin{aligned}
&\mid \tilde{\nu}^{\frac{1}{2}}F_\perp \mid_q^2 \leq \mid \tilde{\nu}^{-\frac{1}{2}}g_\perp \mid_q \mid \tilde{\nu}^{\frac{1}{2}}h_\perp \mid_{q'} + \epsilon^4 \mid f_b \mid_\sim \mid \gamma^+h \mid_\sim \\
&\leq c\Big(\mid \tilde{\nu}^{-\frac{1}{2}}g_\perp \mid_q \mid \tilde{\nu}^{-\frac{1}{2}}H_\perp \mid_{q'} + \epsilon^2 \mid f_b \mid_\sim \mid \tilde{\nu}^{-\frac{1}{2}}H_\perp \mid_{q'} \Big) \\
&\leq \frac{c}{\delta} \mid \tilde{\nu}^{-\frac{1}{2}}g_\perp \mid_q^2 + \delta \mid \tilde{\nu}^{\frac{1}{2}}F_\perp \mid_q^2 + \frac{c\epsilon^4}{\delta} \mid f_b \mid_\sim^2 .
\end{aligned}$$

Here c is independent of q, so we may also take the limit when $q \to \infty$. The estimate (3.4) follows. The inclusion of $\epsilon uJ(F, v_\theta)$ to g, adds $u\epsilon \mid F_\perp \mid_q$ which is incorporated in the left hand side, and a term $u\epsilon \mid F_{||} \mid_q$, which in turn is estimated using (3.2-3) for $q < \infty$. □

4 The rest term.

In this section we discuss the rest term, when $(u_{\theta A1}-u_{\theta B1}r_B)(3u_{\theta A1}+u_{\theta B1}r_B)>0$ and $\Delta\leq\Delta_{bif}$. Denote by $\bar{\chi}=\chi_{|v|<\frac{1}{\sqrt{\epsilon}}}$ and by ψ the approximate solution(s) from Section 2,

$$\psi(r,v)=\sum_{i=1}^{4}\epsilon^i\psi^i.$$

Except for the term $z_1'v_\theta v_r\bar{B}(|v|)$ in ψ^4, each ψ^i for $1\leq i\leq 4$ is a polynomial of order i in the v-variable, with bounded coefficients in the r-variable. If we remove $z_1'v_\theta v_r\bar{B}(|v|)$ from ψ^4, denoting the result by $\bar{\psi}^4=\psi^4-z_1'v_\theta v_r\bar{B}(|v|)$, and for convenience set $\bar{\psi}^j=\psi^j$ for $j=1,2,3$, then it holds that for ϵ small enough,

$$1+\bar{\chi}\bar{\psi}=1+\bar{\chi}\Big(\sum_{i=1}^{4}\epsilon^i\bar{\psi}^i\Big)\geq 0,$$

and also for any q, that the $\tilde{L}^q$-norm of $(1-\bar{\chi})\bar{\psi}$ is of any desired (high) order in ϵ. The aim is to prove that there exists a rest term R, such that

$$f=M(1+\bar{\chi}\bar{\psi}+\epsilon^4R)$$

is a nonnegative solution to (1.1-2) with $M^{-1}f\in\tilde{L}^\infty$. Such a function R would then be a solution to

$$DR=\frac{1}{\epsilon^4}\Big(\tilde{L}R+2\tilde{J}(R,\bar{\chi}\bar{\psi})+\epsilon^4\tilde{J}(R,R)+l\Big),\tag{4.1}$$

where

$$l=\frac{1}{\epsilon^4}\Big(\tilde{L}(\bar{\chi}\bar{\psi})+\tilde{J}(\bar{\chi}\bar{\psi},\bar{\chi}\bar{\psi})-\epsilon^4D(\bar{\chi}\bar{\psi})\Big).$$

Notice that for $j=1,...,4$, ψ^j can be constructed so that $D\psi^j=(I-P_0)D\psi^j$, hence that the corresponding $l=l_\perp$. In Section 2, ψ was constructed so that the lowest order term in l contains an ϵ-factor of order one. After the above change to $\bar{\psi}^4$, this still 'nearly' holds. The only new term of a different type appearing, is $\tilde{L}(z_1'v_\theta v_r\bar{B})=-z_1'v_\theta v_r$. This term is non-hydrodynamic and of ϵ-order one in L^1, due to the factor $\exp(\frac{c_3}{w_1r_B}\frac{r-r_B}{\epsilon})$ in z_1' (cf Section 2). Denote by $u(r)=b_1(r)+z_1(\frac{r-r_B}{\epsilon})$ the coefficient of v_θ in the first-order term of ψ, $\psi^1(r,v)=u(r)v_\theta$. It is uniformly bounded with respect to r in $(1,r_B)$, since $\frac{c_3}{w_1r_B}$ is positive. Let the sequences $(R^n)_{n\in I\!N}$ and $(\bar{R}^n)_{n\in I\!N}$ be defined by $R^0=\bar{R}^0=0$, and

$$DR^{n+1}=\frac{1}{\epsilon^4}\Big(\tilde{L}R^{n+1}+2\epsilon u\tilde{J}(\bar{R}^{n+1},\bar{\chi}v_\theta)+g^n\Big),\tag{4.2}$$

$$R^{n+1}(1,v)=R_A(v),\ v_r>0,\quad R^{n+1}(r_B,v)=R_B(v),\ v_r<0.\tag{4.3}$$

In (4.2-3)

$$g^n := 2\tilde{J}(\bar{R}^n, \bar{\chi}\bar{\psi}) - 2\tilde{J}(\bar{R}^n, \bar{\chi}\epsilon\bar{\psi}^1) + \epsilon^4 \tilde{J}(\bar{R}^n, \bar{R}^n) + l,$$

$$\epsilon^4 R_A(v) := e^{\epsilon u_{\theta A1} v_\theta - \frac{\epsilon^2}{2} u^2_{\theta A1} v^2_\theta} - 1 - \bar{\chi}\bar{\psi}(r_A, v), \quad v_r > 0,$$

$$\epsilon^4 R_B(v) := \frac{1+\omega_B}{(1+\tau_B)^{\frac{3}{2}}} e^{\frac{1}{2}\left(v^2 - \frac{1}{1+\tau_B}(v_r^2 + (v_\theta - \epsilon u_{\theta B1})^2 + v_z^2)\right)} - 1 - \bar{\chi}\bar{\psi}(r_B, v), \quad v_r < 0,$$

and

$$\bar{R}^n(r,v) = R^n(r,v) \ \text{ when } \ \epsilon^4 R^n(r,v) \geq -\Big(1 + \bar{\chi}\sum_{i=1}^{4} \epsilon^i \bar{\psi}^i(r,v)\Big),$$

$$\bar{R}^n(r,v) = -\frac{1}{\epsilon^4}\Big(1 + \bar{\chi}\sum_{i=1}^{4} \epsilon^i \bar{\psi}^i(r,v)\Big) \ \text{ otherwise.}$$

The solutions are well defined, since the proof in part one [AN2] of Proposition 3.1, can be extended to the case with $\bar{R}^{n+1}$ instead of R^{n+1} in the $\tilde{J}$-term of (4.2). We observe that

$$\int g^n M dv = \int l M dv, \quad \partial_r (r \int R^n v_r M dv) = r \int l M dv. \tag{4.4}$$

We now discuss the existence problem for the rest term iteration scheme (4.2-3).

Proposition 4.1 *For $2 \leq q < \infty$ and $\epsilon > 0$ small enough, there is a unique sequence (R^n) of solutions to (4.2-3) in the set $X := \{R; | \tilde{\nu}^{\frac{1}{2}} R |_q \leq K\}$, for some constant K. The sequence converges in $\tilde{L}^q$ to an isolated solution of*

$$DR = \frac{1}{\epsilon^4}\Big(\tilde{L}R + \epsilon^4 \tilde{J}(\bar{R}, \bar{R}) + 2\tilde{J}(\bar{R}, \bar{\chi}\bar{\psi}) + l\Big), \tag{4.5}$$

$$R(1,v) = R_A(v), \ v_r > 0, \quad R(r_B, v) = R_B(v), \ v_r < 0. \tag{4.6}$$

Replacing $\bar{R}^n$ by R^n in (4.2), the sequence (R^n) converges in $\tilde{L}^q$ to an isolated solution of (4.1), (4.6).

Proof of Proposition 4.1. It is obviously enough to prove the the proposition for q large and finite, below for $q \geq 16$. For an equation of type

$$Df = \frac{1}{\epsilon^4}\Big(\tilde{L}f + 2\epsilon u \tilde{J}(\bar{f}, \bar{\chi} v_\theta) + g\Big),$$

$$f(1,v) = f_b(1,v), \ v_r > 0, \quad f(r_B, v) = f_b(r_B, v), \ v_r < 0,$$

with $g = g_\perp$, it follows from Proposition 3.2 and Proposition 3.3 that the following estimates hold,

$$| f_{I||} |_q \leq c\Big(| f_\perp |_q + | \tilde{\nu}^{-\frac{1}{2}} g |_q \Big), \tag{4.7}$$

$$\| f_r \|_q \leq c(\epsilon^3 \| f_\theta \|_q + | f_\perp |_q + \frac{1}{\epsilon} \| \int g v_\theta v_r \bar{B}(|v|) M dv \|_1) \tag{4.8}$$

$$| \tilde{\nu}^{\frac{1}{2}} f_\perp |_q \leq C\Big(| \tilde{\nu}^{-\frac{1}{2}} g_\perp |_q + \epsilon | f_r v_r |_q + \epsilon^2 | \tilde{\nu}^{\frac{1}{2}} f_b |_\sim \Big), \quad q < \infty, \tag{4.9}$$

$$| \tilde{\nu}^{\frac{1}{2}} f |_\infty \leq C\Big(| \tilde{\nu}^{-\frac{1}{2}} g_\perp |_\infty + \epsilon^{-\frac{8}{q}} | f |_q + | \tilde{\nu}^{\frac{1}{2}} f_b |_\sim \Big). \tag{4.10}$$

For (4.2), (4.3) in the case $n = 0$, and taking $g^0 = l_\perp$, notice that R^1 has boundary values of order one in ϵ and that the lowest order term in l is $z_1' v_\theta v_r$. If we write $l = z_1' v_\theta v_r + \tilde{l}$, then $\tilde{l}$ has order at least one with respect to ϵ in $\tilde{L}^q$ by Lemma 2.3. In this case (4.7-9) give

$$| R_{||}^1 |_q \leq C\Big(| R_\perp^1 |_q + \frac{1}{\epsilon} | z_1' v_\theta v_r |_1 + | z_1' v_\theta v_r |_q + \frac{1}{\epsilon} | \tilde{\nu}^{-\frac{1}{2}} \tilde{l}_\perp |_q \Big). \quad (4.11)$$

Using that z_1' is of order ϵ in L^1 and generally of order $\frac{1}{q}$ in $\tilde{L}^q$, it follows for $q < \infty$ from (4.7-9) that uniformly with respect to $0 < \epsilon < \epsilon_0$,

$$| \tilde{\nu}^{\frac{1}{2}} R^1 |_q \leq C. \quad (4.12)$$

This together with (4.10) implies that $| \tilde{\nu}^{\frac{1}{2}} R^1 |_\infty$ can be estimated by a term of order $\epsilon^{-\frac{8}{q}}$.

For the corresponding estimates and $g^0 = l_{||}$, we consider the equation

$$Df = \frac{1}{\epsilon^4}\Big(\tilde{L} f + \epsilon u \tilde{J}(f, v_\theta) + l_{||} \Big) \text{ with } f_b = 0. \quad (4.13)$$

Applying Green's formula to the equation gives that

$$| \tilde{\nu}^{\frac{1}{2}} f_\perp |_2 \leq c(\delta^{-1} | l_{||} |_2 + \delta | f_{||} |_2).$$

As discussed above, $l_{||}$ is of arbitrarily high order in ϵ except for a v_θ-contribution of ϵ-order $3+\frac{1}{q}$, coming from $D(z_1' v_\theta v_r \bar{B})$. When using the approach of Proposition 3.2 in part one ([AN2]) to estimate $f_{||}$, that v_θ-contribution never appears, and modulo an $l_{||}$- term of arbitrarily high order in ϵ,

$$| \tilde{\nu}^{\frac{1}{2}} f_{||} |_2 \leq c | \tilde{\nu}^{\frac{1}{2}} f_\perp |_2,$$

hence

$$| \tilde{\nu}^{\frac{1}{2}} f_\perp |_2 \leq c | l_{||} |_2 .$$

Existence for (4.13) can now be obtained in $\mathcal{W}^{q-}$ similarly to Proposition 3.1 (cf part one [AN2]).

The proof of (3.5) in part one (cf (4.6) in [AN2]), implies that

$$| \tilde{\nu}^{\frac{1}{2}} f |_q \leq c\Big(\epsilon^{\frac{8}{q}-4} | \tilde{\nu}^{\frac{1}{2}} f |_2 + | \tilde{\nu}^{-\frac{1}{2}} l_{||} |_q \Big).$$

Hence

$$| \tilde{\nu}^{\frac{1}{2}} f |_q \leq c(\epsilon^{\frac{8}{q}-4} | l_{||} |_2 + | l_{||} |_q),$$

and (4.12) holds for $q \leq 16$. For $16 < q < \infty$ the $l_{||}$ part of R^1 is of negative order $\frac{16-q}{2q}$. For an estimate of $| \tilde{\nu}^{\frac{1}{2}} R^1 |_\infty$ with the help of $\tilde{L}^q$ when $l = l_{||}$, use (3.5) to conclude

$$| \tilde{\nu}^{\frac{1}{2}} f |_\infty \leq c\Big(\epsilon^{-\frac{8}{q}} | \tilde{\nu}^{\frac{1}{2}} f |_q + | \tilde{\nu}^{-\frac{1}{2}} l_{||} |_\infty \Big) \leq c\Big(\epsilon^{-4} | l_{||} |_2 + | l_{||} |_\infty \Big).$$

It follows that for $16 < q \leq \infty$, the $l_{||}$ part of R^1 estimated in $\tilde{L}^q$, gives a contribution of order $\epsilon^{\frac{16-q}{2q}}$.

For $n \in I\!N$ it holds that $(R^{(n+1)} - R^n)$ has $l = 0$, $g^n = g^n_\perp$ and zero ingoing boundary values. Writing $R^{(n+1)} = R^1 + \sum_{j=1}^n (R^{(j+1)} - R^j)$, it follows from (4.7-10) that for $16 \leq q \leq \infty$

$$\begin{aligned}
&| \tilde{\nu}^{\frac{1}{2}} (\bar{R}^{(n+1)} - \bar{R}^n) |_q \leq | \tilde{\nu}^{\frac{1}{2}} (R^{(n+1)} - R^n) |_q \leq C^n \epsilon^n | R^1 |_q, \\
&| \tilde{\nu}^{\frac{1}{2}} \bar{R}^{(n+1)} |_\infty \leq | \tilde{\nu}^{\frac{1}{2}} R^{(n+1)} |_\infty \leq C | R^1 |_\infty \sum_0^n (C\epsilon)^j \leq C_1 | R^1 |_\infty,
\end{aligned}$$

for all $n \in I\!N$. And so, for ϵ small enough, (R^n), resp. $(\bar{R}^n)$ converges in $\tilde{L}^q$ for $q \geq 16$, to some R, resp. $\bar{R}$, solutions to (4.5-6). Analogously taking $\bar{R}^n = R^n$ and $\epsilon > 0$ small enough, $(R^n)_{n \in I\!N}$ converges in $\tilde{L}^q$ to some R, solution to (4.1) with ingoing boundary values (4.6). The contraction mapping argument guarantees that these solutions are isolated. □

Proposition 4.2 *Let Ω be a bounded set in $I\!R^3$, and f_b a nonnegative function defined on $\partial\Omega^+$. If a function f such that $M^{-1} f \in \tilde{L}^\infty(\Omega \times I\!R^3)$ satisfies*

$$v \cdot \nabla_x f = Q(f^+, f^+) - ML(M^{-1} f^-), \quad (x, v) \in \Omega \times I\!R^3, \tag{4.14}$$

$$f = f_b, \quad \partial\Omega^+, \tag{4.15}$$

then $f^- = 0$ and $f = f^+$ solves the boundary value problem

$$\begin{aligned}
&v \cdot \nabla_x f = Q(f, f), \quad \Omega \times I\!R^3, \\
&f = f_b, \quad \partial\Omega^+.
\end{aligned}$$

For a proof of the proposition, see part one [AN2].

Let $f = M(1 + \bar{\chi} \sum_{i=1}^4 \epsilon^i \bar{\psi}^i + \epsilon^4 R)$ where R satisfies (4.5-6). Denote by $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$. Then

$$f^+ = M(1 + \bar{\chi} \sum_{i=1}^4 \epsilon^i \bar{\psi}^i + \epsilon^4 \bar{R}),$$

and f satisfies (4.14), (4.15). Since $f^- = 0$ by Proposition 4.2, the $R's$ of (4.1) and (4.5) coincide, and this solution f to (1.1-2) is nonnegative.

It also follows from the proof that the solution f is isolated. Constants that need not be precised in the fourth order asymptotics, are 'compensated to correct value' by the fourth order rest term(, as can be seen from the uniqueness arguments for the rest term. This also holds for the third order v_θ suction term coming from a fifth order coefficient $c_5(r_B) + t_5 = 0$, since at third order the relevant term has a factor $\tilde{L}(v) = 0$). In fact the previous analysis would have proceeded without changes, had we taken for ψ^4 only its Hilbert part, and included the upcoming boundary values in those for the rest term.

It finally follows from the previous proof that the hydrodynamic moments converge to solutions of the corresponding leading order limiting fluid (Hilbert) equations, when ϵ tends to zero.

5 Comments and remarks.

As mentioned in the introduction, the approach holds without change for the other cases of asymptotic expansion in the two-roll setup that are discussed in [S]. The following example illustrates the treatment in the upcoming types of situations.
Consider the equation (1.1) under the scaling

$$\begin{aligned} & v_r \frac{\partial f}{\partial r} + \frac{1}{r} N f = \frac{1}{\epsilon^m} \tilde{Q}(f, f), \\ & r \in (r_A, r_B), \quad (v_r, v_\theta, v_z) \in I\!R^3, \end{aligned} \tag{5.1}$$

for $m = 3$ and without the coupling $\omega_B = \frac{\epsilon^2}{1+\epsilon^2 \tau_{B2}} \left(\frac{r_B^2 - 1}{r_B^2} u_{\theta A1}^2 - \tau_{B2} + \Delta\epsilon \right)$ between the boundary values. Assume the cylinders rotate in the same direction and that $1 < P_{SB2}/[(r_B^2 - 1) u_{\theta B1}^2] < (u_{\theta A1}/u_{\theta B1} r_B)^2$. This guarantees an asymptotic expansion with positive (as well as one with negative) second order radial velocity u_{rH2}, and one with third order radial velocity. For the positive one, take the asymptotic expansion ψ of Section 4 up to order three, and the rest term R of order three in ϵ. The rest term analysis proceeds as in Section 4 and gives the following result.

Theorem 5.1 *For $0 < \epsilon$ small enough, there is an isolated, positive L^1-solution f_ϵ of the equation (5.1) for $m = 3$ with boundary conditions (1.2), and with positive second order radial velocity u_{rH2}, for which*

$$\int M^{-1} supess_{r \in (r_A, r_B)} \mid f_\epsilon^j(r, v) \mid^2 dv < +\infty.$$

The hydrodynamic moments converge in L^∞ to solutions of the corresponding leading order (second order in ϵ for the radial velocity) limiting fluid equations, when ϵ tends to zero.

Also for the negative second order radial velocity u_{rH2}, a second isolated solution can be obtained in the same way.
For the case of third order radial velocity, again take the asymptotic expansion ψ in Section 4 up to order three, and the rest term R of order three in ϵ. The rest term analysis proceeds as before.

Theorem 5.2 *For $0 < \epsilon$ small enough, there is an isolated, positive L^1-solution f_ϵ of the equation (5.1) for $m = 3$ with boundary conditions (1.2), and third order radial velocity u_{rH3} for which*

$$\int M^{-1} supess_{r \in (r_A, r_B)} \mid f_\epsilon^j(r,v) \mid^2 dv < +\infty.$$

The hydrodynamic moments converge in L^∞ to solutions of the corresponding leading order (third order in ϵ for the radial velocity moment) limiting fluid equations, when ϵ tends to zero.

The proof of convergence for the radial velocity moment also requires a splitting in L^q of the type in [AN2]. Theorems 5.1-2 in particular demonstrate that three separate solutions to (1.1-2) coexist for these parameter values.

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