On Solutions to the Linear Boltzmann Equation for Granular Gases

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Abstract

This paper considers the time- and space-dependent, linear Boltzmann equation with general boundary conditions in the case of inelastic (granular) collisions. First mild L^1-solutions are constructed as limits of iterate functions. Then boundedness of all higher velocity moments are obtained. Finally the question of convergence to equilibrium is studied, using a general H-theorem for a relative entropy functional.

1 Introduction

The linear Boltzmann equation is frequently used for mathematical modelling in physics, (e.g. for discribing the neutron distribution in reactor physics, cf. [1]–[4]). In our earlier papers [5]–[11] we have studied the linear Boltzmann equation for a function $f(\mathbf{x}, \mathbf{v}, t)$, representing the distribution of particles with masses m colliding elastically and binary with other particles with masses m_* and with given (known) distribution function $Y(\mathbf{x}, \mathbf{v}_*)$. The purpose of this paper is to generalize our earlier results to the case of inelastic collisions for granular gases. In recent years a significant interest has been focused on the study of kinetic models for granular flows, see e.g. [12]–[14], whose papers study the non-linear Boltzmann equation for granular gases, (mostly in the case of hard sphere collisions).

So we will study collisions between particles with mass m and particles with mass m_*, such that momentum is conserved,

$$m\mathbf{v} + m_*\mathbf{v}_* = m\mathbf{v}' + m_*\mathbf{v}'_*, \tag{1.1}$$

where $\mathbf{v}, \mathbf{v}_*$ are velocities before and $\mathbf{v}', \mathbf{v}'_*$ are velocities after a collision.

In the elastic case, where also kinetic energy is conserved, one finds that the velocities after a binary collision terminate on two concentric spheres, so all velocities $\mathbf{v}'$ lie on a sphere around the center of mass, $\bar{\mathbf{v}} = (m\mathbf{v}+m_*\mathbf{v}_*)/(m+m_*)$, with radius $\frac{m_*}{m+m_*}|\mathbf{v}-\mathbf{v}_*|$, and all velocities $\mathbf{v}'_*$ lie on a sphere with the same center $\bar{\mathbf{v}}$ and with radius $\frac{m}{m+m_*}|\mathbf{v}-\mathbf{v}_*|$, cf. Figure 1 in [5].

In the granular, inelastic case we assume the following relation between the relative velocity components normal to the plane of contact of the two particles,

$$\mathbf{w}' \cdot \mathbf{u} = -\alpha(\mathbf{w} \cdot \mathbf{u}), \tag{1.2}$$

where α is a constant, $0 < \alpha \leq 1$, and $\mathbf{w} = \mathbf{v} - \mathbf{v}_*, \mathbf{w}' = \mathbf{v}' - \mathbf{v}'_*$ are the relative velocities before and after the collision, and $\mathbf{u}$ is a unit vector in the direction of impact, $\mathbf{u} = (\mathbf{v}' - \mathbf{v})/|\mathbf{v}' - \mathbf{v}|$. Then we find that $\mathbf{v}' = \mathbf{v}'_\alpha$ lies on the line between $\mathbf{v}$ and $\mathbf{v}'_1$, where $\mathbf{v}'_1$ is the postvelocity in the case of elastic collision, i.e. with $\alpha = 1$, and $\mathbf{v}'_* = \mathbf{v}'_{*\alpha}$ lies on the (parallel) line between $\mathbf{v}_*$ and $\mathbf{v}'_{*1}$.

Now it follows that the following relations hold for the velocities in the granular, inelastic case,

$$\begin{aligned} \mathbf{v}' &= \mathbf{v} - (\alpha+1)\frac{m_*}{m+m_*}(\mathbf{w} \cdot \mathbf{u}) \cdot \mathbf{u}, \\ \mathbf{v}'_* &= \mathbf{v}_* + (\alpha+1)\frac{m}{m+m_*}(\mathbf{w} \cdot \mathbf{u}) \cdot \mathbf{u}, \end{aligned} \tag{1.3}$$

where $\mathbf{w} \cdot \mathbf{u} = w\cos\theta, w = |\mathbf{v} - \mathbf{v}_*|$, if the unit vector $\mathbf{u}$ is given in spherical coordinates,

$$\mathbf{u} = (\sin\theta\cos\zeta, \sin\theta\sin\zeta, \cos\theta), \tag{1.4}$$

$0 \leq \theta \leq \pi/2$, $0 \leq \zeta < 2\pi$.

By (1.3) we get for the relative velocity after collision, $\mathbf{w}' = \mathbf{v}' - \mathbf{v}'_*$, that

$$\mathbf{w}' = \mathbf{w} - (\alpha+1)(\mathbf{w} \cdot \mathbf{u}) \cdot \mathbf{u}, \tag{1.5}$$

and we also find (for $\mathbf{w}' = \mathbf{w}'_\alpha$) that

$$|\mathbf{w}'_\alpha| = |\mathbf{w}|\sqrt{\sin^2\theta + \alpha^2\cos^2\theta}. \tag{1.6}$$

Furthermore, the change of kinetic energy ΔE in a binary granular collision can be calculated by

$$\begin{aligned} 2\Delta E &\equiv m|\mathbf{v}'|^2 + m_*|\mathbf{v}'_*|^2 - m|\mathbf{v}|^2 - m_*|\mathbf{v}_*|^2 = \\ &= -(1-\alpha^2)\frac{mm_*}{m+m_*}w^2\cos^2\theta. \end{aligned} \tag{1.7}$$

One can also see that all velocities $\mathbf{v}'$ and $\mathbf{v}'_*$ terminate on two different spheres (with different centers, if $\alpha \neq 1$), cf. [12]–[14],

$$\begin{aligned}\mathbf{v}' &= \bar{\mathbf{v}} + \frac{(1-\alpha)m_*}{2(m+m_*)}\mathbf{w} + \frac{(1+\alpha)m_* w}{2(m+m_*)}\boldsymbol{\pi},\\ \mathbf{v}'_* &= \bar{\mathbf{v}} - \frac{(1-\alpha)m}{2(m+m_*)}\mathbf{w} - \frac{(1+\alpha)mw}{2(m+m_*)}\boldsymbol{\pi},\end{aligned} \tag{1.8}$$

where $\bar{\mathbf{v}} = (m\mathbf{v} + m_*\mathbf{v}_*)/(m+m_*)$, $\mathbf{w} = \mathbf{v} - \mathbf{v}_*$, $w = |\mathbf{w}|$, and $\boldsymbol{\pi}$ is a unit vector.

Moreover, if we change notations, and let $'\mathbf{v}, '\mathbf{v}_*$ be the velocities before, and $\mathbf{v}, \mathbf{v}_*$ the velocities after a binary inelastic collision, then by (1.2) and (1.3), cf. [12]–[14],

$$\begin{aligned}'\mathbf{v} &= \mathbf{v} - \frac{(\alpha+1)m_*}{\alpha(m+m_*)}(\mathbf{wu}) \cdot \mathbf{u},\\ '\mathbf{v}_* &= \mathbf{v}_* + \frac{(\alpha+1)m}{\alpha(m+m_*)}(\mathbf{wu}) \cdot \mathbf{u}.\end{aligned} \tag{1.9}$$

In the following sections of this paper we give in Section 2 some preliminaries on the linear (space- and time-dependent) Boltzmann equation. Then in Section 3 solutions are constructed as limits of iterate functions, and in Section 4 boundedness of higher velocity moments are studied. Finally in Section 5 we give an H-theorem for a (general) relative entropy functional, and also use this theorem to study the problem of convergence, when time goes to infinity.

2 Preliminaries

We consider the time-dependent transport equation for a distribution function $f(\mathbf{x}, \mathbf{v}, t)$, depending on a space-variable $\mathbf{x} = (x_1, x_2, x_3)$ in a bounded convex body D with (piecewise) C^1-boundary $\Gamma = \partial D$, and depending on a velocity variable $\mathbf{v} = (v_1, v_2, v_3) \in V = \mathbb{R}^3$ and a time-variable $t \in \mathbb{R}_+$. Then the linear Boltzmann equation (in the case of no external forces) is in the strong form

$$\frac{\partial f}{\partial t}(\mathbf{x}, \mathbf{v}, t) + \mathbf{v} \cdot \mathrm{grad}_{\mathbf{x}} f(\mathbf{x}, \mathbf{v}, t) = (Qf)(\mathbf{x}, \mathbf{v}, t), \tag{2.1}$$

$\mathbf{x} \in D, \mathbf{v} \in V = \mathbb{R}^3, t \in \mathbb{R}_+$, supplemented by initial data

$$f(\mathbf{x}, \mathbf{v}, 0) = f_0(\mathbf{x}, \mathbf{v}), \mathbf{x} \in D, \mathbf{v} \in V. \tag{2.2}$$

The collision term can, in the case of inelastic (granular) collision, be written, cf. [12]–[14], and also [1]–[11],

$$\begin{aligned}(Qf)(\mathbf{x}, \mathbf{v}, t) = \iint_{V\Omega} &[J_\alpha(\theta, w) Y(\mathbf{x}, '\mathbf{v}_*) f(\mathbf{x}, '\mathbf{v}, t) - \\ &- Y(\mathbf{x}, \mathbf{v}_*) f(\mathbf{x}, \mathbf{v}, t)] B(\theta, w) d\mathbf{v}_* d\theta d\zeta,\end{aligned} \tag{2.3}$$

with $w = |\mathbf{v}-\mathbf{v}_*|$, where $Y \geq 0$ is a known distribution, and $B \geq 0$ is given by the collision process, and finally J_α is a factor depending on the granular process, (and giving mass-conservation, if the gain and the loss integrals converges separately). Furthermore, $'\mathbf{v},' \mathbf{v}_*$ in (2.3) are the velocities before and $\mathbf{v}, \mathbf{v}_*$ the velocities after the binary collision, cf. equation (1.9), and $\Omega = \{(\theta, \zeta) : 0 \leq \theta < \hat{\theta}, 0 \leq \zeta < 2\pi\}$ is the impact plane.

If the collision term is written in a weak form with a testfunction $g = g(\mathbf{v})$, then we (formally) get

$$
\begin{aligned}
(Qf, g) &\equiv \int_V (Qf)(\mathbf{x}, \mathbf{v}, t) g(\mathbf{v}) d\mathbf{v} = \\
&= \iiint_{VV\Omega} [g(\mathbf{v}') - g(\mathbf{v})] f(\mathbf{x}, \mathbf{v}, t) Y(\mathbf{x}, \mathbf{v}_*) B(\theta, |\mathbf{v} - \mathbf{v}_*|) d\mathbf{v} d\mathbf{v}_* d\theta d\zeta,
\end{aligned}
\tag{2.4}
$$

where $\mathbf{v}'$ is the velocity after collision.

In the following of this paper we will study the angular cut-off-case with $\hat{\theta} < \pi/2$. Then the gain and the loss term in (2.3) can be separated

$$
(Qf)(\mathbf{x}, \mathbf{v}, t) = (Q^+ f)(\mathbf{x}, \mathbf{v}, t) - (Q^- f)(\mathbf{x}, \mathbf{v}, t), \tag{2.5}
$$

where we can write

$$
\begin{aligned}
(Q^+ f)(\mathbf{x}, \mathbf{v}, t) &= \iint_{V\Omega} J_\alpha(\theta, w) Y(\mathbf{x},' \mathbf{v}_*) f(\mathbf{x},' \mathbf{v}, t) B(\theta, w) d\mathbf{v}_* d\theta d\zeta = \\
&= \int_V K_\alpha(\mathbf{x},' \mathbf{v} \to \mathbf{v}) f(\mathbf{x},' \mathbf{v}, t) d' \mathbf{v},
\end{aligned}
\tag{2.6}
$$

and

$$
(Q^- f)(\mathbf{x}, \mathbf{v}, t) = L(\mathbf{x}, \mathbf{v}) f(\mathbf{x}, \mathbf{v}, t) \tag{2.7}
$$

with the collision frequency

$$
L(\mathbf{x}, \mathbf{v}) = \iint_{V\Omega} Y(\mathbf{x}, \mathbf{v}_*) B(\theta, w) d\mathbf{v}_* d\theta d\zeta, w = |\mathbf{v} - \mathbf{v}_*|. \tag{2.8}
$$

In the case of a non-absorbing body we have the following relation

$$
L(\mathbf{x}, \mathbf{v}) = \int_V K_\alpha(\mathbf{x}, \mathbf{v} \to \mathbf{v}') d\mathbf{v}'. \tag{2.9}
$$

For hard sphere collisions the function $B(\theta, w)$ can be written, cf. [1]–[4] and [12]–[14],

$$
B(\theta, w) = \text{const} \cdot w \sin\theta \cos\theta, w = |\mathbf{v} - \mathbf{v}_*|. \tag{2.10}
$$

Another physically interesting case is that with inverse k-th power collision forces

$$B(\theta, w) = b(\theta)w^{\gamma}, \gamma = \frac{k-5}{k-1} \tag{2.11}$$

with hard forces for $k > 5$, Maxwelllian for $k = 5$, and soft forces for $3 < k < 5$.

The factor J_α in the gain term can in the hard sphere case be calculated and found to be proportional to α^{-2}, cf. [12]–[14].

Furthermore, the equation (2.1) is in the space-dependent case supplemented with (general) boundary conditions

$$\begin{aligned} &f_-(\mathbf{x}, \mathbf{v}, t) = \int_V \frac{|\mathbf{n}\tilde{\mathbf{v}}|}{|\mathbf{n}\mathbf{v}|} R(\mathbf{x}, \tilde{\mathbf{v}} \to \mathbf{v}) f_+(\mathbf{x}, \tilde{\mathbf{v}}, t) d\tilde{\mathbf{v}}, \\ &\mathbf{n}\mathbf{v} < 0, \mathbf{n}\tilde{\mathbf{v}} > 0, \mathbf{x} \in \Gamma = \partial D, t \in \mathbb{R}_+ \end{aligned} \tag{2.12}$$

where $\mathbf{n} = \mathbf{n}(\mathbf{x})$ is the unit outward normal at $\mathbf{x} \in \Gamma = \partial D$. The function $R \geq 0$ satisfies (in the non-absorbing boundary case)

$$\int_V R(\mathbf{x}, \tilde{\mathbf{v}} \to \mathbf{v}) d\mathbf{v} \equiv 1, \tag{2.13}$$

and f_- and f_+ represent the ingoing and outgoing trace functions corresponding to f. In the specular reflection case the function R is represented by a Dirac measure, $R(\mathbf{x}, \tilde{\mathbf{v}} \to \mathbf{v}) = \delta(\mathbf{v} - \tilde{\mathbf{v}} + 2\mathbf{n}(\mathbf{n}\tilde{\mathbf{v}}))$, and in the diffuse reflection case $R(\mathbf{x}, \tilde{\mathbf{v}} \to \mathbf{v}) = |\mathbf{n}\mathbf{v}| W(\mathbf{x}, \mathbf{v})$ with some given function $W \geq 0$, (e.g. Maxwellian function).

Now, using differentiation along the characteristics, the equation (2.1) can formally be written, cf. (2.5)-(2.9),

$$\begin{aligned} &\frac{d}{dt}(f(\mathbf{x} + t\mathbf{v}, \mathbf{v}, t)) = \int_V K_\alpha(\mathbf{x} + t\mathbf{v}, '\mathbf{v} \to \mathbf{v}) f(\mathbf{x} + t\mathbf{v}, '\mathbf{v}, t) d'\mathbf{v} \\ &- L(\mathbf{x} + t\mathbf{v}, \mathbf{v}) f(\mathbf{x} + t\mathbf{v}, \mathbf{v}, t). \end{aligned} \tag{2.14}$$

Let $t_b \equiv t_b(\mathbf{x}, \mathbf{v}) = \inf_{\tau \in \mathbb{R}_+}\{\tau : \mathbf{x} - \tau\mathbf{v} \notin D\}$, and $\mathbf{x}_b \equiv \mathbf{x}_b(\mathbf{x}, \mathbf{v}) = \mathbf{x} - t_b\mathbf{v}$. Here t_b represents the time for a particle going with velocity $\mathbf{v}$ from the boundary point $\mathbf{x}_b$ to the point $\mathbf{x}$.

Then we have the following *mild form* of our equation

$$f(\mathbf{x}, \mathbf{v}, t) = \bar{f}(\mathbf{x}, \mathbf{v}, t) + \int_0^t (Qf)(\mathbf{x} - \tau\mathbf{v}, \mathbf{v}, t - \tau) d\tau, \tag{2.15}$$

where

$$\bar{f}(\mathbf{x}, \mathbf{v}, t) = \begin{cases} f_0(\mathbf{x} - t\mathbf{v}, \mathbf{v}),\ 0 \leq t \leq t_b, \\ f(\mathbf{x}_b, \mathbf{v}, t - t_b),\ t > t_b, \end{cases} \tag{2.16}$$

and also the *exponential form*

$$f(\mathbf{x},\mathbf{v},t)=\bar{f}(\mathbf{x},\mathbf{v},t)\exp[-\int_0^t L(\mathbf{x}-s\mathbf{v},\mathbf{v})ds]+$$
$$+\int_0^t \exp[-\int_0^\tau L(\mathbf{x}-s\mathbf{v},\mathbf{v})ds]\int_V K_\alpha(\mathbf{x}-\tau\mathbf{v},'\mathbf{v}\to\mathbf{v})f(\mathbf{x}-\tau\mathbf{v},'\mathbf{v},t-\tau)d'\mathbf{v}d\tau. \tag{2.17}$$

Remark. One finds that f is a mild solution (2.15) if and only if f satisfies the exponential form (2.17) of the equation, (cf. our earlier papers and also the classical result by di Perna-Lions). For a proof we (among others) use that

$$\frac{d}{dt}(t_b(\mathbf{x}+t\mathbf{v},\mathbf{v}))\equiv 1. \tag{2.18}$$

3 Construction of solutions

We construct mild L^1-solutions to our problem as limits of iterate functions f^n, when $n\to\infty$. Let first $f^{-1}(\mathbf{x},\mathbf{v},t)\equiv 0$ for all $\mathbf{x},\mathbf{v}\in\mathbb{R}^3, t\in\mathbb{R}_+$. Then define, for given f^{n-1} the next iterate f^n, first at the ingoing boundary (using the appropriate boundary condition), and then inside D and at the outgoing boundary (using the exponential form of the equation),

$$\begin{aligned}&f_-^n(\mathbf{x},\mathbf{v},t)=\int_V\frac{|\mathbf{n}\tilde{\mathbf{v}}|}{|\mathbf{n}\mathbf{v}|}R(\mathbf{x},\tilde{\mathbf{v}}\to\mathbf{v})f_+^{n-1}(\mathbf{x},\tilde{\mathbf{v}},t)d\tilde{\mathbf{v}},\\&\mathbf{n}\mathbf{v}<0,\mathbf{x}\in\Gamma=\partial D,\mathbf{v}\in V=\mathbb{R}^3,t\in\mathbb{R}_+,\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}&f^n(\mathbf{x},\mathbf{v},t)=\bar{f}^n(\mathbf{x},\mathbf{v},t)\exp[-\int_0^t L(\mathbf{x}-s\mathbf{v},\mathbf{v})ds]+\\&+\int_0^t\exp[-\int_0^\tau L(\mathbf{x}-s\mathbf{v},\mathbf{v})ds]\int_V K_\alpha(\mathbf{x}-\tau\mathbf{v},'\mathbf{v}\to\mathbf{v})\cdot\\&\cdot f^{n-1}(\mathbf{x}-\tau\mathbf{v},'\mathbf{v},t-\tau)d'\mathbf{v}d\tau,\\&\mathbf{x}\in D\cup\Gamma_+(\mathbf{v}),\mathbf{v}\in V=\mathbb{R}^3,t\in\mathbb{R}_+,\end{aligned} \tag{3.2}$$

where

$$\bar{f}^n(\mathbf{x},\mathbf{v},t)=\begin{cases}f_0(\mathbf{x}-t\mathbf{v},\mathbf{v}),0\le t\le t_b\\ f_-^n(\mathbf{x}-t_b\mathbf{v},\mathbf{v},t-t_b),t>t_b.\end{cases} \tag{3.3}$$

Let also $f^n(\mathbf{x},\mathbf{v},t)\equiv 0$ for $\mathbf{x}\in\mathbb{R}^3\setminus D$.

Now we get a monotonicity lemma, which is essential in the following and which can be proved by induction.

Lemma 3.1.

$$\begin{aligned} &f^n(\mathbf{x},\mathbf{v},t) \geq f^{n-1}(\mathbf{x},\mathbf{v},t), \\ &\mathbf{x}\in D, \mathbf{v}\in V, t\in\mathbb{R}_+, n\in\mathbb{N}. \end{aligned} \tag{3.4}$$

Remark. The iterate function $f^n(\mathbf{x},\mathbf{v},t)$ represents the distribution of particles undergone at most n collisions (inside D or at the boundary $\Gamma=\partial D$) in the time interval $(0,t)$.

Using differentiation along the characteristics with (2.18), we get by (3.2) that

$$\begin{aligned} &\frac{d}{dt}[f^n(\mathbf{x}+t\mathbf{v},\mathbf{v},t)] = \int_V K_\alpha(\mathbf{x}+t\mathbf{v},{}'\mathbf{v}\to\mathbf{v}) \cdot \\ &\cdot f^{n-1}(\mathbf{x}+t\mathbf{v},{}'\mathbf{v},t)d'\mathbf{v} - L(\mathbf{x}+t\mathbf{v},\mathbf{v})f^n(\mathbf{x}+t\mathbf{v},\mathbf{v},t). \end{aligned} \tag{3.5}$$

Now integrating (3.5), it follows by (2.9) and Green's formula that

$$\begin{aligned} &\iint_{DV} f^n(\mathbf{x},\mathbf{v},t)d\mathbf{x}d\mathbf{v} + \int_0^t \iint_{\Gamma V} f_+^n(\mathbf{x},\mathbf{v},\tau)|\mathbf{n}\mathbf{v}|d\mathbf{v}d\Gamma d\tau = \\ &= \iint_{DV} f_0(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v} + \int_0^t \iint_{\Gamma V} f_-^n(\mathbf{x},\mathbf{v},\tau)|\mathbf{n}\mathbf{v}|d\mathbf{v}d\Gamma d\tau + \\ &+ \int_0^t \iint_{DV} L(\mathbf{x},\mathbf{v})[f^{n-1}(\mathbf{x},\mathbf{v},\tau) - f^n(\mathbf{x},\mathbf{v},\tau)]d\mathbf{x}d\mathbf{v}d\tau, \end{aligned} \tag{3.6}$$

where by (2.13) and (3.1)

$$\int_V f_-^n(\mathbf{x},\mathbf{v},\tau)|\mathbf{n}\mathbf{v}|d\mathbf{v} = \int_V f_+^{n-1}(\mathbf{x},\mathbf{v},\tau)|\mathbf{n}\mathbf{v}|d\mathbf{v}. \tag{3.7}$$

So by Lemma 3.1 and (3.6) with (3.7), it follows that (for all $t>0$)

$$\iint_{DV} f^n(\mathbf{x},\mathbf{v},t)d\mathbf{x}d\mathbf{v} \leq \iint_{DV} f_0(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v} < \infty. \tag{3.8}$$

Then Levi's theorem (on monotone convergence) gives existence of mild (defined by (2.15)) L^1-solutions $f(\mathbf{x},\mathbf{v},t)=\lim_{n\to\infty} f^n(\mathbf{x},\mathbf{v},t)$ to our problem with granular gases (almost in the same way as for the elastic collision case). Furthermore, if $L(\mathbf{x},\mathbf{v})f(\mathbf{x},\mathbf{v},t)\in L^1(D\times V)$, then we get equality in (3.8) for the limit function f, giving mass conservation together with uniqueness in the relevant function space, cf. [6] and also [3],

$$\iint_{DV} f(\mathbf{x},\mathbf{v},t)d\mathbf{x}d\mathbf{v} = \iint_{DV} f_0(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v}, t\in\mathbb{R}_+. \tag{3.9}$$

In summary, we have for granular gases the following existence and uniqueness theorem for solutions to our time-dependent linear Boltzmann equation with general boundary conditions.

Theorem 3.2. *Assume that K_α, L and R are non-negative, measurable functions, such that (2.9) and (2.13) hold, and $L(\mathbf{x},\mathbf{v}) \in L^1_{\text{loc}}(D \times V)$.*

a) *Then for every $f_0 \in L^1(D \times V)$ there exists a mild L^1-solution $f(\mathbf{x},\mathbf{v},t)$ to the problem (2.1)–(2.3), (2.5)–(2.9) with (2.12), (2.13), satisfying the corresponding inequality in (3.9).*

b) *Moreover, if $L(\mathbf{x},\mathbf{v})f(\mathbf{x},\mathbf{v},t) \in L^1(D \times V)$, then the trace of the function f satisfies the boundary condition (2.12) for a.e. $(\mathbf{x},\mathbf{v}) \in \Gamma \times V$. Furthermore, mass-conservation, giving equality in (3.6), holds together with uniqueness in the relevant L^1-space.*

Remarks.

1) The statement in Theorem 3.2(b) on existence of traces follows e.g. from Proposition 3.3, Chapter XI in [3].

2) The assumption $Lf \in L^1(D \times V)$ is for instance satisfied for the solution f in the case of inverse power collision forces, cf. (2.11), together with e.g. specular boundary reflections. This follows from a statement on global boundedness (in time) of higher velocity moments, cf. Theorem 4.1 and Corollary 4.2 in the next section.

3) The results above can easily be generalized to a case, where the coefficient α in relation (1.2) depends on the space-variable, i.e. with $\alpha = \alpha(\mathbf{x})$. Furthermore one can also handle a case with granular (inelastic) collisions at the boundary (with generalized specular reflections) by a relation $\mathbf{nv} = -\alpha_b \cdot \mathbf{n}\tilde{\mathbf{v}}$ with a coefficient $\alpha_b = \alpha_b(\mathbf{x}), \mathbf{x} \in \Gamma = \partial D$; see the discussion after the equations (2.12), (2.13), and cf. also ref. [14].

4 On boundedness of higher velocity moments

In this section we first study some velocity estimates, and then use these results to prove boundedness of higher velocity moments in the case of inverse power collision forces together with e.g. specular boundary conditions. These results generalize our earlier statements to the granular inelastic case. The following proposition on the difference of the squares of velocities after and before collision is an analogue of Proposition 1.1 in [5], and can be proved in (almost) the same way.

Proposition 4.1. *Let $\mathbf{v}$ and $\mathbf{v}'_\alpha(\theta,\zeta)$ be the velocities for a particle with mass m before and after a binary granular collision with a particle, having the corresponding velocities $\mathbf{v}_*$ and $\mathbf{v}'_*$ and mass m_*, such that (1.1) and (1.2) hold. Then* *the following estimate holds, (for $0 < \alpha \leq 1, 0 \leq \theta < \pi/2, 0 \leq \zeta < 2\pi$),*

$$
\begin{aligned}
&|\mathbf{v}'_\alpha(\theta, \zeta)|^2 - |\mathbf{v}|^2 \leq \\
&\leq 2(\alpha + 1)\rho_* w \cos\theta [3|\mathbf{v}_*| - \rho|\mathbf{v}|\cos\theta],
\end{aligned}
\tag{4.1}
$$

with $w = |\mathbf{v} - \mathbf{v}_*|, \rho_* = m_*/(m + m_*), \rho = m/(m + m_*)$.

Now we can prove that the following type of estimate holds also in the inelastic granular collision case, (analogously to the elastic case, cf. Proposition 1.2 in [5]).

Proposition 4.2. *Suppose $\mathbf{v}$ and $\mathbf{v}'_\alpha$ are velocities as in Proposition 4.1. Then for all $\sigma > 0$, there are positive constants K_1 and K_2 (depending on σ, m, m_* and α), such that*

$$
\begin{aligned}
&(1 + |\mathbf{v}'_\alpha(\theta)|^2)^{\sigma/2} - (1 + |\mathbf{v}|^2)^{\sigma/2} \leq \\
&\leq K_1(w\cos\theta)(1 + |\mathbf{v}_*|)^{\max(1, \sigma - 1)}(1 + |\mathbf{v}|^2)^{(\sigma-2)/2} - \\
&- K_2(w\cos^2\theta)(1 + |\mathbf{v}|^2)^{(\sigma-1)/2}.
\end{aligned}
\tag{4.2}
$$

In the rest of this section we assume inverse power collision forces with the function $B(\theta, w)$ given by (2.11).

To get higher velocity estimates for our solution f (given by Theorem 3.2 (a)), we start from equation (3.5), i.e. the differentiated mild form (along the characteristics) for the iterate function f^n, and multiply this equation by $(1 + v^2)^{\sigma/2}$, where $v = |\mathbf{v}|, \sigma > 0$. Then

$$
\begin{aligned}
&\frac{d}{dt}[(1 + v^2)^{\sigma/2} f^n(\mathbf{x} + t\mathbf{v}, \mathbf{v}, t)] = \\
&= \int_V K_\alpha(\mathbf{x} + t\mathbf{v}, {}'\mathbf{v} \to \mathbf{v})(1 + v^2)^{\sigma/2} f^{n-1}(\mathbf{x} + t\mathbf{v}, {}'\mathbf{v}, t) d{}'\mathbf{v} - \\
&- L(\mathbf{x} + t\mathbf{v}, \mathbf{v})(1 + v^2)^{\sigma/2} f^n(\mathbf{x} + t\mathbf{v}, \mathbf{v}, t).
\end{aligned}
\tag{4.3}
$$

Now integrating (4.3), it follows by Green's formula that

$$
\begin{aligned}
&\iint_{DV} (1 + v^2)^{\sigma/2} f^n(\mathbf{x}, \mathbf{v}, t) d\mathbf{x} d\mathbf{v} + \int_0^t \iint_{\Gamma V} (1 + v^2)^{\sigma/2} f_+^n(\mathbf{x}, \mathbf{v}, \tau)|\mathbf{n}\mathbf{v}| d\mathbf{v} d\Gamma d\tau = \\
&= \iint_{DV} (1 + v^2)^{\sigma/2} f_0(\mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v} + \int_0^t \iint_{\Gamma V} (1 + v^2)^{\sigma/2} f_-^n(\mathbf{x}, \mathbf{v}, \tau)|\mathbf{n}\mathbf{v}| d\mathbf{v} d\Gamma d\tau + \\
&+ \int_0^t \iiint_{DVV} K_\alpha(\mathbf{x}, {}'\mathbf{v} \to \mathbf{v})(1 + v^2)^{\sigma/2} f^{n-1}(\mathbf{x}, {}'\mathbf{v}, \tau) d\mathbf{x} d\mathbf{v} d{}'\mathbf{v} d\tau - \\
&- \int_0^t \iint_{DV} L(\mathbf{x}, \mathbf{v})(1 + v^2)^{\sigma/2} f^n(\mathbf{x}, \mathbf{v}, \tau) d\mathbf{x} d\mathbf{v} d\tau,
\end{aligned}
\tag{4.4}
$$

where all integrals exist inductively. Let

$$M_\sigma(t) = \iint_{DV} (1+v^2)^{\sigma/2} f^n(\mathbf{x},\mathbf{v},t)d\mathbf{x}d\mathbf{v} \tag{4.5}$$

be the velocity moment and take the derivative, using (4.4) and (2.9) (cf. also (2.4)). Then

$$\begin{aligned} M'_\sigma(t) = &\iint_{\Gamma V} (1+v^2)^{\sigma/2} f^n_-(\mathbf{x},\mathbf{v},t)|\mathbf{nv}|d\mathbf{v}d\Gamma - \\ - &\iint_{\Gamma V} (1+v^2)^{\sigma/2} f^n_+(\mathbf{x},\mathbf{v},t)|\mathbf{nv}|d\mathbf{v}d\Gamma + \\ + &\iiint_{DVV} K_\alpha(\mathbf{x},\mathbf{v}\to\mathbf{v}')[(1+(v')^2)^{\sigma/2} f^{n-1}(\mathbf{x},\mathbf{v},t) - (1+v^2)^{\sigma/2} f^n(\mathbf{x},\mathbf{v},t)] \cdot \\ \cdot\, & d\mathbf{x}d\mathbf{v}d\mathbf{v}', \end{aligned} \tag{4.6}$$

where $f^{n-1}(\mathbf{x},\mathbf{v},t) \leq f^n(\mathbf{x},\mathbf{v},t)$ by Lemma 3.1. For the boundary terms in (4.6) we assume a "non-heating boundary" with

$$R(\mathbf{x},\tilde{\mathbf{v}}\to\mathbf{v}) \equiv 0 \text{ for } |\mathbf{v}| > |\tilde{\mathbf{v}}|. \tag{4.7}$$

This assumption is for instance satisfied for specular reflections. Then by (2.13) and Lemma 3.1

$$\begin{aligned} &\int_V (1+v^2)^{\sigma/2} f^n_-(\mathbf{x},\mathbf{v},t)|\mathbf{nv}|d\mathbf{v} \leq \\ &\leq \int_V (1+v^2)^{\sigma/2} f^n_+(\mathbf{x},\mathbf{v},t)|\mathbf{nv}|d\mathbf{v}. \end{aligned} \tag{4.8}$$

So we have, by (4.6), (4.8) together with (2.11), that

$$\begin{aligned} M'_\sigma(t) \leq &\iiiint_{DVV\Omega} b(\theta)w^\gamma Y(\mathbf{x},\mathbf{v}_*) f^n(\mathbf{x},\mathbf{v},t) \cdot \\ &\cdot [(1+(v')^2)^{\sigma/2} - (1+v^2)^{\sigma/2}] d\mathbf{x}d\mathbf{v}d\mathbf{v}_*d\theta d\zeta. \end{aligned} \tag{4.9}$$

Here we use the estimate in Proposition 4.2 together with some elementary inequality

$$-w^{\gamma+1} \leq (1+v_*)^{\gamma+1} - 2^{-1}(1+v^2)^{(\gamma+1)/2}, -1<\gamma<1. \tag{4.10}$$

And we get that there exist constants $C_1, C_2, C_0 > 0$, such that by (4.9)

$$M'_\sigma(t) \leq C_1 M_{\sigma+\gamma-1}(t) + C_2 M_{\sigma-1}(t) - C_0 M_{\sigma+\gamma}(t), \tag{4.11}$$

where we have assumed that the functions Y and b satisfy the following conditions.

$$\int_V (1+v_*)^{\gamma+\max(2,\sigma)} \sup_{\mathbf{x}\in D}(Y(\mathbf{x},\mathbf{v}_*))d\mathbf{v}_* < \infty, \tag{4.12}$$

$$\int_V \inf_{\mathbf{x}\in D}(Y(\mathbf{x},\mathbf{v}_*))d\mathbf{v}_* > 0, \tag{4.13}$$

and

$$0 < \int_0^{\pi/2} b(\theta)d\theta < \infty \tag{4.14}$$

Now, for hard and Maxwellian collision forces, $0 \le \gamma = (k-5)/(k-1) < 1$, i.e. $k \ge 5$, it follows from (4.11), where $-M_{\sigma+\gamma}(t) \le -M_\sigma(t)$, that

$$\frac{d}{dt}(e^{C_0 t}M_\sigma(t)) \le C_3 e^{C_0 t} M_{\sigma-\delta}(t), \tag{4.15}$$

with $\delta = 1-\gamma > 0$ and some constant $C_3 > 0$.

Integrating (4.15) and using for $\sigma = \delta > 0$, that the total mass $M_0(t)$ is bounded, we get that $M_\sigma(t)$ is globally bounded (in time) for $\sigma = \delta$. Then we can continue in the same way for $\sigma = 2\delta, 3\delta, \dots$ up to $\sigma = \sigma_0$, if

$$\iint_{DV} (1+v^2)^{\sigma_0/2} f_0(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v} < \infty. \tag{4.16}$$

Finally, let $n \to \infty$ and use Lemma 3.1.

Then the following theorem for higher velocity moments holds for our solution in the granular inelastic case.

Theorem 4.1. *Assume that the collision function $B(\theta, w)$ is given for (hard or Maxwellian) k-th power forces by (2.11) and (4.14) with $k \ge 5$, i.e. $\gamma \ge 0$, and suppose that the function $Y(\mathbf{x},\mathbf{v}_*)$ satisfies (4.12), (4.13). Let the boundary relation (2.12) be given for a "non-heating" boundary by (4.7), (e.g. specular reflection). Then the higher velocity moments, belonging to the mild solution $f = f_\alpha(\mathbf{x},\mathbf{v},t)$ in Theorem 3.2 (a), are all bounded (globally in time),*

$$\begin{gathered}\iint_{DV} (1+v^2)^{\sigma/2} f_\alpha(\mathbf{x},\mathbf{v},t)d\mathbf{x}d\mathbf{v} \le C_\sigma < \infty, \\ 0 < \alpha \le 1, t > 0, 0 < \sigma \le \sigma_0,\end{gathered} \tag{4.17}$$

if $(1+v^2)^{\sigma_0/2} f_0(\mathbf{x},\mathbf{v}) \in L^1(D \times V)$.

Remark. Also for soft collision forces, $3 < k < 5$, (i.e. $-1 < \gamma < 0$), one gets analogously a result on boundedness of higher moments,

$$\int_0^t \iint_{DV} (1+v^2)^{\sigma/2} f_\alpha(\mathbf{x},\mathbf{v},\tau) d\mathbf{x} d\mathbf{v} d\tau \leq$$
$$\leq C_\sigma(1+t) \iint_{DV} (1+v^2)^{\sigma+|\gamma|} f_0(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v}. \tag{4.18}$$

We will now finish this section by proving that our solution f satisfies the assumption $Lf \in L^1(D \times V)$, giving existence of traces at the boundary together with uniqueness and mass-conservation; cf. Theorem 3.2 (b) and Remark 2 in Section 3. This result holds for both hard and soft inverse collision forces.

Corollary 4.2. *The solution $f = f_\alpha(\mathbf{x},\mathbf{v},t)$ in Theorem 3.2 (a) satisfies*

$$L(\mathbf{x},\mathbf{v}) f(\mathbf{x},\mathbf{v},t) \in L^1(D \times V), t \in \mathbb{R}_+, \tag{4.19}$$

if a) in the hard force case $\gamma = (k-5)/(k-1) \geq 0$, the assumptions of theorem 4.1 are satisfied together with

$$(1+v_*)^\gamma \sup_{\mathbf{x}\in D}(Y(\mathbf{x},\mathbf{v}_*)) \in L^1(V), \tag{4.20}$$

$$(1+v^2)^{\gamma/2} f_0(\mathbf{x},\mathbf{v}) \in L^1(D \times V), \tag{4.21}$$

and if b) in the soft force case, $-1 < \gamma = (k-5)/(k-1) < 0$,

$$\sup_{\mathbf{x}\in D}(Y(\mathbf{x},\mathbf{v}_*)) \in L^1(V) \cap L^\infty(V). \tag{4.22}$$

Proof. We estimate the collision frequency as follows

a)

$$L(\mathbf{x},\mathbf{v}) = \iint_{V\Omega} b(\theta)|\mathbf{v}-\mathbf{v}_*|^\gamma Y(\mathbf{x},\mathbf{v}_*) d\mathbf{v}_* d\theta d\zeta \leq$$
$$\leq \iint b(\theta)(1+v^2)^{\gamma/2}(1+v_*)^\gamma Y(\mathbf{x},\mathbf{v}_*) d\mathbf{v}_* d\theta d\zeta \leq$$
$$\leq \text{const.} (1+v^2)^{\gamma/2}$$

and then we use Theorem 4.1.

b) For soft forces, the collision frequency is bounded,

$$L(\mathbf{x},\mathbf{v}) = \iint_{V\Omega} b(\theta) w^\gamma Y(\mathbf{x},\mathbf{v}_*) d\mathbf{v}_* d\theta d\zeta =$$
$$= \int_\Omega b(\theta)[\int_{w<1} w^\gamma Y(\mathbf{x},\mathbf{v}_*) d\mathbf{v}_* + \int_{w\geq 1} w^\gamma Y(\mathbf{x},\mathbf{v}_*) d\mathbf{v}_*] d\theta d\zeta$$
$$\leq 2\pi \int_0^{\pi/2} b(\theta) d\theta [\sup_{\mathbf{x},\mathbf{v}_*}(Y(\mathbf{x},\mathbf{v}_*)) \int_{w<1} w^\gamma dw + \int_V \sup_{\mathbf{x}}(Y(\mathbf{x},\mathbf{v}_*)) d\mathbf{v}_*]$$
$$\leq \text{constant},$$

so $\iint Lf d\mathbf{x} d\mathbf{v} \leq \text{const.} \iint f_0 d\mathbf{x} d\mathbf{v}$.

□

5 On a (relative) entropy theorem

In this section we give generalizations (to the granular inelastic case) of our old results on entropy and convergence to equilibrium.

Let $\phi = \phi(z), \mathbb{R}_+ \to \mathbb{R}$, be a *convex* C^1-function, and assume that there exists a stationary mild solution $F = F_\alpha(\mathbf{x}, \mathbf{v})$ to the linear Boltzmann equation with boundary conditions. Then a general (relative) H-functional $H_F^\phi(f)$ for the solution $f = f_\alpha(\mathbf{x}, \mathbf{v}, t)$ can be defined by

$$H_F^\phi(f)(t) = \iint_{DV} \phi\Big(\frac{f(\mathbf{x},\mathbf{v},t)}{F(\mathbf{x},\mathbf{v})}\Big) F(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v}. \tag{5.1}$$

This functional is generalization of the usual (negative) relative entropy functional with $\phi(z) = z\log z, z = f/E$, $E = E(\mathbf{v})$ Maxwellian (used for the non-linear Boltzmann equation).

Then we can formulate the following (general) H-theorem for our solution (in the granular case).

Theorem 5.1. *Let $f = f_\alpha(\mathbf{x}, \mathbf{v}, t)$ be the mild solution given in Theorem 3.2 (a,b), and let $F = F_\alpha(\mathbf{x}, \mathbf{v}) > 0$ be a corresponding stationary mild solution with the same total mass $\iint F d\mathbf{x}d\mathbf{v} = \iint f_0 d\mathbf{x}d\mathbf{v}$. If $H_F^\phi(f_0)$ exists for a given convex C^1-function ϕ, $\mathbb{R}_+ \to \mathbb{R}$, then the relative H-functional $H_F^\phi(f)(t)$ in (5.1) exists for $t > 0$ and is non-increasing in time. Moreover*

$$H_F^\phi(f)(t) + \int_0^t P_F^\phi(f)(\tau) d\tau \le H_F^\phi(f_0), \tag{5.2}$$

where

$$\begin{aligned} P_F^\phi(f)(t) \equiv & \iiint_{DVV} d\mathbf{x} d\mathbf{v} d'\mathbf{v} K_\alpha(\mathbf{x},'\mathbf{v} \to \mathbf{v}) F(\mathbf{x},'\mathbf{v}) \cdot \\ & \cdot \Big\{ \phi\Big(\frac{f(\mathbf{x},'\mathbf{v},t)}{F(\mathbf{x},'\mathbf{v})}\Big) - \phi\Big(\frac{f(\mathbf{x},\mathbf{v},t)}{F(\mathbf{x},\mathbf{v})}\Big) - \\ & \Big[\frac{f(\mathbf{x},'\mathbf{v},t)}{F(\mathbf{x},'\mathbf{v})} - \frac{f(\mathbf{x},\mathbf{v},t)}{F(\mathbf{x},\mathbf{v})}\Big] \phi'\Big(\frac{f(\mathbf{x},\mathbf{v},t)}{F(\mathbf{x},\mathbf{v})}\Big) \Big\} \ge 0. \end{aligned} \tag{5.3}$$

Proof. (Sketch) For a formal proof of the H-theorem use differentiation (along the characteristics) to get

$$\begin{aligned} & \frac{d}{dt}\Big[F(\mathbf{x}+t\mathbf{v},\mathbf{v}) \phi\Big(\frac{f(\mathbf{x}+t\mathbf{v},\mathbf{v},t)}{F(\mathbf{x}+t\mathbf{v},\mathbf{v})}\Big)\Big] = \\ & = \Big[\phi'\Big(\frac{f}{F}\Big)\frac{df}{dt} + \Big(\phi\Big(\frac{f}{F}\Big) - \frac{f}{F}\phi'\Big(\frac{f}{F}\Big)\Big)\frac{dF}{dt}\Big](\mathbf{x}+t\mathbf{v},\mathbf{v},t), \end{aligned}$$

where

$$\begin{aligned} \frac{df}{dt} = & \int_V K_\alpha(\mathbf{x}+t\mathbf{v},'\mathbf{v} \to \mathbf{v}) f(\mathbf{x}+t\mathbf{v},'\mathbf{v},t) d'\mathbf{v} - \\ & - L(\mathbf{x}+t\mathbf{v},\mathbf{v}) f(\mathbf{x}+t\mathbf{v},\mathbf{v},t), \end{aligned}$$

and

$$\frac{dF}{dt} = \int_V K_\alpha(\mathbf{x}+t\mathbf{v},'\mathbf{v}\to\mathbf{v})F(\mathbf{x}+t\mathbf{v},'\mathbf{v})d'\mathbf{v} - L(\mathbf{x}+t\mathbf{v},\mathbf{v})F(\mathbf{x}+t\mathbf{v},\mathbf{v}).$$

Integration $\int\int\int \ldots d\mathbf{x}d\mathbf{v}d\tau$, Green's identity, and some changes of variables, using (2.9), give (5.2) with (5.1), (5.3). For the boundary terms we here use an inequality of Darrozes-Guiraud type, cf. [2], p. 115–118, and also [11]

$$\int_V F(\mathbf{x},\mathbf{v})\phi\Big(\frac{f(\mathbf{x},\mathbf{v},t)}{F(\mathbf{x},\mathbf{v})}\Big)(\mathbf{nv})d\mathbf{v} \geq 0,$$

if f and F satisfy (2.12), ϕ is convex, and $\mathbf{n}=\mathbf{n}(\mathbf{x})$ is the outward unit normal. We also see that $P_F^\phi(f)(t) \geq 0$ in (5.3) because (for convex functions ϕ)

$$\phi(b) - \phi(a) \geq (b-a)\phi'(a).$$

For a strict proof of the H-theorem 5.1, with our solution given by Theorem 3.2 (a,b), we will use iterate functions $f^n_{k,j}(\mathbf{x},\mathbf{v},t), n=0,1,2,\ldots$, corresponding to some cut-off in the initial function (for $k,j=1,2,3,\ldots$),

$$f_{0;k,j}(\mathbf{x},\mathbf{v}) = \frac{1}{j}F(\mathbf{x},\mathbf{v}) + \min(f_0(\mathbf{x},\mathbf{v}), kF(\mathbf{x},\mathbf{v})).$$

Here it follows (by induction) that (for $t>0$)

$$1/j \leq f^n_{k,j}(\mathbf{x},\mathbf{v},t)/F(\mathbf{x},\mathbf{v}) \leq k+1.$$

Then, analogously to the formal proof, we differentiate the iterate functions (along the characteristics) and integrate. Next, let $n\to\infty$ (and use dominate convergence), and then let $k,j\to\infty$ (using a.o. Fatou's lemma, and the lower semi-continuous property for convex functions).

□

Remarks.

1) The results in Theorem 5.1 hold analogously if $F = F_\alpha(\mathbf{x},\mathbf{v},t) > 0$ is any non-stationary (i.e. time-dependent) mild solution to our problem (e.g. with another initial function $\mathcal{F}_0$).

2) One fundamental question in kinetic theory concerns the large time behavior of the distribution function $f=f(\mathbf{x},\mathbf{v},t)$; in particular, the problem on convergence to a stationary solution when time goes to infinity. We can (easily) generalize our earlier results, (cf. [7], [8]), on weak and strong convergence to equilibrium to the granular, inelastic case, and we get for instance the following result:

Let $f = f_\alpha(\mathbf{x}, \mathbf{v}, t)$ be the mild solution (e.g. given by Theorem 3.2 (a,b) and Theorem 5.1) to the linear Boltzmann equation with kernel $Y = Y(\mathbf{v}_*)$ and a general collision function $B = B(\theta, w)$, (including both soft and hard inverse forces), together with general boundary conditions, and assume that there exists a stationary mild solution $F = F_\alpha(\mathbf{x}, \mathbf{v}) > 0$, (with total mass $\|F\| = \|f_0\|$). Then, for every initial function $f_0(\mathbf{x}, \mathbf{v}) \in L^1(D \times V)$, the solution $f = f_\alpha(\mathbf{x}, \mathbf{v}, t)$ converges strongly in L^1 when $t \to \infty$ to the stationary solution $F_\alpha(\mathbf{x}, \mathbf{v})$, i.e.

$$\lim_{t\to\infty} \Big(\iint_{DV} |f_\alpha(\mathbf{x}, \mathbf{v}, t) - F_\alpha(\mathbf{x}, \mathbf{v})| d\mathbf{x} d\mathbf{v} \Big) = 0.$$

For a proof of this statement, one can use the method from our earlier paper [7] (in the detailed balance case), and change the Maxwellian function E to the more general function $F = F(\mathbf{x}, \mathbf{v})$. First one proves weak convergence to equilibrium, using (among others) our new H-theorem with the convex function $\phi(z) = (z-1)^2, z = f/F$. Then the strong convergence result follows by proving a translation continuity property for our solution; for details, see [7].

3) The inequality (5.2) with (5.3) in Theorem 5.1 can also be used to prove uniqueness of stationary solutions. Because if we (for instance) take $\phi(z) = (z-1)^2, z = \tilde{F}/F$, where $\tilde{F}$ is another stationary solution, then

$$P_F^\phi(\tilde{F}) = \iint_{DVV} K_\alpha(\mathbf{x}, '\mathbf{v} \to \mathbf{v}) F(\mathbf{x}, '\mathbf{v}) \cdot \\ \cdot \Big| \frac{\tilde{F}(\mathbf{x}, '\mathbf{v})}{F(\mathbf{x}, '\mathbf{v})} - \frac{\tilde{F}(\mathbf{x}, \mathbf{v})}{F(\mathbf{x}, \mathbf{v})} \Big|^2 d\mathbf{x} d\mathbf{v} d'\mathbf{v} = 0,$$

and it follows (as in the elastic case) that $\tilde{F}(\mathbf{x}, \mathbf{v}) = F(\mathbf{x}, \mathbf{v})$.

4) In Theorem 5.1 we have assumed the existence of a stationary (equilibrium) solution to our problem. This question has been studied (and partly solved) in our earlier papers, but a general existence result for the linear stationary Boltzmann equation is not yet received; cf. also the L^1-result in [15] for velocities bounded away from zero, and cf. [16].

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