On the derivation of a linear Boltzmann equation from a periodic lattice gas

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1 Introduction

The linear Boltzmann equation

$$\partial_t f(x,v,t) + v\cdot\partial_x f(x,v,t) = \frac{1}{2}\int_{S^1_-} \left(f(x,v',t) - f(x,v,t)\right)|v\cdot\omega|\,d\omega \qquad (1)$$

describes the evolution of a density of particles in a medium in which the particles don't interact among themselves. The motion of each particle is described by a jump process: The speed of a particle is constant (equal to one), and also the direction is constant in exponentially distributed time intervals. At the end of such an interval, the direction jumps according to a law that corresponds to the specular reflection on a circular obstacle (with a uniformly distributed impact parameter).

This Boltzmann equation can be rigorously derived as the “Boltzmann-Grad” limit of a system with obstacles of finite size. This was done by Gallavotti [Ga1, Ga2] (but see also Spohn [Sp]) by considering obstacles of diameter ϵ whose centers are distributed in the plane according to a Poisson law with density ϵ^{-1}. A formal calculation yields a mean free path of order one, uniformly in ϵ, and Gallavotti showed that this is rigorously true, and that the limiting evolution equation is really the Boltzmann equation (1).

Quite contrary to this, Bourgain et al. [BGW, GW] showed that the corresponding scaling for a *periodic* distribution of scatterers cannot give rise to a Boltzmann equation, the reason being that the distribution of free path lengths is not exponential in that case. An asymptotic formula is given in [CG]. At a formal level, however, it can still work as was shown by Golse [G].

As a way of deriving a linear Boltzmann equation starting from a periodic distribution of scatterers, Caglioti et al [CPR] considered scatterers of diameter ϵ with centers on a rectangular lattice with parameter ϵ: in each lattice point, independently of the other lattice points, the probability of finding a scatterer is ϵ. In the limit as ϵ tends to zero, this distribution approaches a Poisson distribution, but one cannot immediately infer from that, that the dynamics of scattered particles approach a Boltzmann process.

In Caglioti et al, the convergence to the Boltzmann process is proven rigorously for a third kind of process, a "Markovian" process, in which there is an obstacle on every lattice point, but each time the test particle encounters an obstacle, there is an independent random choice: with probability $1-\epsilon$ the test particle continues as if the obstacle was not there, and with probability ϵ the particle is scattered. They then prove that in the limit all three processes are equivalent.

In this paper we consider a scaling which is intermediary between the case considered in [CPR] and in [BGW]: the scatterers still have radius ϵ, but the lattice parameter is ϵ^{ν}, where $1/2 < \nu \leq 1$. In order to achieve a proper Boltzmann-Grad limit, the probability of finding a scatterer at a lattice site must be $\epsilon^{2\nu-1}$; we have found it convenient to write $\nu = 1/(2-\delta)$ where δ is between zero (which corresponds to the purely periodic case), and one, which corresponds to the scaling in [CPR].

The technique we use is in the spirit very close to that of [CPR], and in particular we first study a "Markovian" process, in which each time a scatter is encountered a random choice is made as to whether scattering takes place or not, and then this system is shown to be equivalent (in the limit as $\epsilon \to 0$) to the system where the scatterer configuration is determined once and for all.

There is one major difference, however. When $\delta = 1$, the number of scatterers encountered along any line lies between ϵ^{-1} and $\sqrt{2}\epsilon^{-1}$ per unit time, which makes it comparatively easy to establish that in the limit, the mean free times are exponentially distributed. According to [BGW], this is false when $\delta = 0$, and actually for any δ strictly smaller than one. Excluding a small set of initial directions it is rather easy to show that if $0 < \delta < 1$, then in the limit, the *first* free time is exponentially distributed, and so the main problem is to prove that the same holds for the second flight (and the third, and so on).

In the second section of the paper, we describe i detail each of the stochastic processes, and state precisely the convergence theorems that are the main results of the paper: first that the law corresponding to the Markovian process converges in the sense of distributions, to the L^1-solution of the Boltzmann equation, and then that in the limit, all three processes (the fixed obstacle process, the Markovian process and the Boltzmann process) are equivalent.

Section 3 then contains the proof of Theorem 2; this is rather elementary, but somewhat technical. For a fixed ϵ, the probability to find a free flight, depends on the number of obstacle sites that this trajectory meets. Hence the proof relies on a rather careful estimate of the function $s(x, v, L)$, which gives the number of times that a line segment of length L, crosses obstacle sites, given that it starts from $x \in \mathbb{R}^2$ in the direction v. The relevant results are given in Section 4.

The proof of Theorem 1 relies on a stronger result from Section 5, where we prove a that the stochastic process related to the Lorentz gas converge to the process related the Boltzmann equation. More precisely, the trajectories of any one of the processes belong with probability one to a Skorokhod space, and define a measure on this space; we prove that the measure corresponding to the Lorenz gas converges to the measure corresponding to the Boltzmann equation. One essential ingredient in the proof is an estimate on the probability that a random trajectory returns to the same obstacle, which is the most technical part of Section 5

2 Three jump processes and their asymptotic equivalence

In this section we will describe the three stochastic processes that are the subject of this paper: the jump process associated to the scattering of a particle on a fixed but random set of scatterers with finite radius (the lattice gas), the "Markovian" process given by scattering on a set of obstacles with fixed positions, but where scattering takes place with a given probability, independently of possible previous encounters with the same obstacle, and finally, the jump process associated with the Boltzmann equation. Once these processes are well described, we are ready to state Theorem 1, an asymptotic equivalence between the three, and, as an important step on the way, Theorem 2, which states that the Markovian model converges to the Boltzmann equation.

2.1 The lattice gas

Much of the content of this section is borrowed from [CPR]. Let $\mathbb{Z}^2_\lambda$ be a two-dimensional lattice whose cells have size λ:

$$\mathbb{Z}^2_\lambda = \{(j_1\lambda, j_2\lambda) \mid j_i \in \mathbb{Z}, \quad i = 1,2\}.$$

and $\mathcal{C}$ be the lattice formed by the centers:

$$\mathcal{C} = \{((j_1+1/2)\lambda, (j_2+1/2)\lambda) | j_i \in \mathbb{Z}, \quad i = 1,2\}.$$

From here, the lattice parameter is set to $\lambda = \epsilon^{1/(2-\delta)}$. Next we consider an array of random variables

$$\{n_c\}_{c\in\mathcal{C}}$$

where n_c, the occupation number, is a random variable taking the value 1 or 0 with probability $p \equiv \epsilon^{\delta/(2-\delta)}$ and $1-p$ respectively, independently for all $c \in \mathcal{C}$. The "physical domain" for the problem is constructed by placing a circular obstacle (scatterer) of radius ϵ at the center of those lattice cells for which $n_c = 1$. For a given scatterer configuration $\{n_c\}_{c\in\mathcal{C}}$, the region occupied the set of scatters is

$$\Lambda_{\mathbf{c}} = \bigcup_{n_c=1} B_\epsilon(x_c)\,, \tag{2}$$

where $B_\epsilon(x_c)$ is a closed unit disc with radius ϵ and center at $c \in \mathcal{C}$. The set of all possible obstacles, $\bigcup B_\epsilon(x_c)$, is called the "obstacle sites".

Consider now a test point particle with initial position $x \in \mathbb{R}^2 \setminus \partial\Lambda_{\mathbf{c}}$, and with an initial velocity $v \in S^1$. (This means that particles are allowed to *start* inside a scatterer; of course, in the limit as ϵ goes to zero, the fractional volume of the scatterers goes to zero, and so this is only a matter of convenience). The particle then moves with constant velocity until it encounters a scatterer, i.e. when

$$t = \min\{\tau > 0 \mid x + v\tau \in \partial\Lambda_{\mathbf{c}},\ v\cdot\omega \le 0\}\,,$$

where ω is a unit normal vector pointing out from the scatterer, into $\mathbb{R}^2 \setminus \Lambda_{\mathbf{c}}$. At this point the velocity jumps according to a specular reflection, so that the new velocity v' is given by

$$v' = v - 2(v \cdot \omega)\omega\,. \tag{3}$$

We denote by $\tilde{z}_\epsilon(t) = \tilde{T}^t_\epsilon(x, v)$ the flow constructed in this way. A typical path is illustrated in Fig. 1. Note that this is well defined for all $x \in \mathbb{R}^2$, and that all the stochasticity comes from the generation of the configuration of the scatterers.

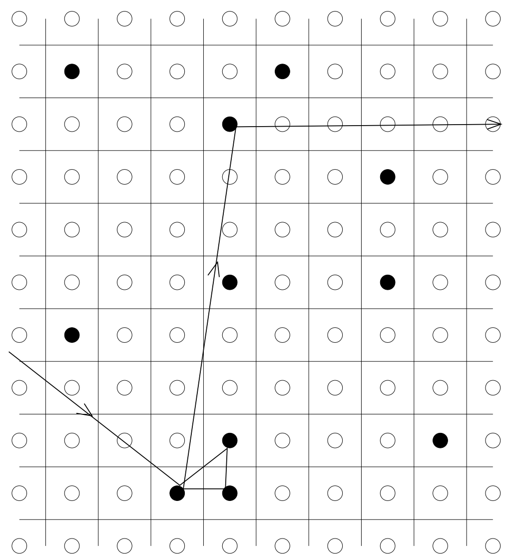

Figure 1: Typical path for the lattice gas model. The occupied obstacle sites are black; the actual occupation of a lattice site is randomly determined once and for all.

Next we consider the evolution of a density of particles, when for each particle a new configuration of scatterers is generated. If $f_0 = f_0(x, v)$ is the initial distribution density for the particle, its distribution at time $t > 0$, denoted by $\tilde{f}_\epsilon = \tilde{f}_\epsilon(x, v, t)$, is given by the formula:

$$\int_{\mathbb{R}^2 \times S^1} \tilde{f}_\epsilon(x, v, t) g(x, v)\, dxdv = \int_{\mathbb{R}^2 \times S^1} f_0(x, v)\; \tilde{\mathbb{E}}(g(\tilde{T}^t_\epsilon(x, v)))\, dxdv, \tag{4}$$

where g is any continuous function and $\tilde{\mathbb{E}}$ denotes the expectation with respect to $\{n_c\}_{c \in \mathcal{C}}$, the distribution of occupied sites. In Section 5 we shall prove:

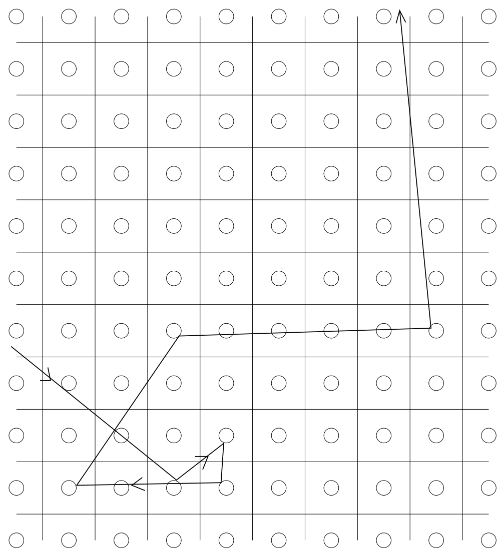

Figure 2: Typical path in the Markovian model. Note that in this case the trajectory may pass through an obstacle on which it has previously bounced, and in the same way, bounce off an obstacle that it once passed through.

Theorem 1 *Let $f_0 : \mathbb{R}^2 \times S^1 \to \mathbb{R}^+$ be the initial probability density (so that f_0 is assumed non-negative and in $L^1(\mathbb{R}^2, S^1)$ with integral one). Then, for any $t > 0$, $0 < \delta \leq 1$:*

$$\lim_{\epsilon \to 0} \tilde{f}_\epsilon(\cdot, t) = f(\cdot, t) \tag{5}$$

in $\mathcal{D}'$. The limiting function $f(\cdot, t)$ is the unique solution of the transport equation (1):

$$(\partial_t + v \cdot \nabla_x) f(x, v, t) = \frac{1}{2} \int_{S^-} \{f(x, v', t) - f(x, v, t)\} |v \cdot \omega| \, d\omega \tag{6}$$

where $S^- = \{\omega \in S^1 | \ v \cdot \omega < 0\}$, v' is the outgoing velocity after a collision with outward normal ω and in-going velocity v (see formula (3)) and $f(x, v, 0^+) = f_0(x, v)$.

2.2 The Boltzmann process

The transport equation (or linear Boltzmann equation (1)) corresponds to a stochastic process for the motion of particles. Suppose that $f(x, v, t)$ is a weak solution of the Boltzmann equation (1) with initial data $f_0(x, v)$. Then $f(x, v, t)$ induces an evolution $g = g(x, v, t)$ on a test function $g^0 \in C_0(\mathbb{R}^2 \times S^1)$ by the following formula:

$$\int_{\mathbb{R}^2 \times S^1} f_0(x, v) g(x, v, t) \, dx dv = \int_{\mathbb{R}^2 \times S^1} f(x, v, t) g^0(x, v) \, dx dv.$$

The function $g(x,v,t)$ can be expanded in a series,

$$g(x,v,t) := V^t g(x,v) = \sum_{n\geq 0} (V^t g^0)_n(x,v) = \sum_{n\geq 0} g_n(x,v,t) \tag{7}$$

where V^t is a linear semi-group. The first one is $g_0(x,v,t) = e^{-t} g^0(x+vt,v)$ and, for $n > 1$,

$$g_n(x,v,t) = (V^t g^0)_n(x,v) := e^{-t} 2^{-n} \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{n-1}}^t dt_n \int_{S^-} d\omega_1 \cdots \int_{S^-} d\omega_n \prod_{k=1}^n |\omega_k \cdot v_{k-1}| g(x_n(t), v_n).$$

Here

$$v_0 = v\,,$$
$$v_k = v'_{k-1} = v_{k-1} - 2(\omega_k \cdot v_{k-1})\omega_k\,,$$

and

$$x_n(t) = x + t_1 v + (t_2 - t_1)v_1 + \ldots (t_n - t_{n-1})v_{n-1} + (t - t_n)v_n\,.$$

This defines a stochastic process $z(t) = (x(t), v(t))$, in which $t_k - t_{k-1}$ are independent, exponentially distributed intervals between the jump times t_k. At a jump time, the particle changes velocity according to $v_{k+1} = v_k - 2(v_k \cdot \omega_k)\omega_k$, where $\omega_k \in S^-$ are randomly chosen. It is clear that

$$(x_n(t), v_n(t)) = \big(z(t)\big|\ (\text{ number of jumps in } [0,t[\ ) = n-1\big)\,, \tag{8}$$

and that $(V^t g^0)_n(x,v)$ corresponds to those particles that change velocity exactly n times in the time interval $[0,t[$.

2.3 The Markovian model

Here we consider again the periodic lattice

$$\mathcal{C} = \{((j_1 + 1/2)\lambda, (j_2 + 1/2)\lambda) \mid j_i \in \mathbb{Z}, \quad i = 1, 2\}.$$

with $\lambda = \epsilon^{1/(2-\delta)}$, but at contrast with the lattice gas, we assume that all lattice points are occupied by a circular scatterer with radius ϵ. The phase space is then

$$(\mathbb{R}^2 \setminus \Lambda_{\mathbf{c}}) \times S^1\,,$$

defined as before, but with $n_c \equiv 1$. To obtain a "mean free path" of order one, we assume that at each encounter with an obstacle, the particle performs an elastic collision with probability $p := \epsilon^{\delta/2-\delta}$ or goes ahead with probability $1-p$. After the first collision the procedure is iterated. This gives rise to a stochastic process which is Markovian, when regarded as a discrete time process

$$\mathbb{Z} \ni k \mapsto z_k \in \partial\Lambda_{\mathbf{c}} \times S^1 \,,$$

where k enumerates the instances where test-particle encounters $\partial\Lambda_{\mathbf{c}}$. However, it is *not* a Markov process in continuous time, because the time intervals between those instances are not independent. Nonetheless, we insist on calling the process $\tilde{z}_\epsilon : \mathbb{R} \to (\mathbb{R} \setminus \Lambda_{\mathbf{c}}) \times S^1$ "the Markovian model". A typical path is illustrated in Fig. 2.

The distribution density for the particle at time $t > 0$, $f_\epsilon = f_\epsilon(x, v, t)$, is given by:

$$\int_{\mathbb{R}^2\times S^1} f_\epsilon(x,v,t) g^0(x,v)\, dxdv = \int_{\mathbb{R}^2\times S^1} f_0(x,v)\mathbb{E}(g^0(T^t_\epsilon(x,v)))\, dxdv, \quad (9)$$

where $\mathbb{E}$ denotes the expectation with respect to the process z_ϵ, and where $g^0(x, v)$ is an arbitrary (continuous or smooth) function. Just like in [CPR], we can compute an exact formula for $f_\epsilon(x, v, t)$. The two observations needed are, first, that due to the reversibility of an actual scattering event (the collisions are elastic), we have

$$P^\epsilon_t(x, v|y, w) = P^\epsilon_t(y, -w|x, -v)\,,$$

where $P^\epsilon_t(x, v|y, w)$ denotes the transition probability associated with the process. This means, that though the process is irreversible, the probability of finding a certain trajectory from A to B is the same as finding the reverse trajectory from B to A. Moreover, it is easy to compute the probability of realization of a given trajectory $\Gamma^t_\epsilon(x, v)$:

$$q(\Gamma^t_\epsilon(x,v)) = p^k(1-p)^h\,, \qquad\qquad (p = \epsilon^{\delta/2-\delta}), \quad (10)$$

where k is the number of actual scattering events along the trajectory, and h is the number of times that the trajectory crosses an obstacle without scattering. In summary this gives

$$f_\epsilon(x,v,t) = \mathbb{E}[(Rf_0)(T^t_\epsilon(x,-v))] = \sum_{\Gamma^t_\epsilon(x,-v)} q(\Gamma^t_\epsilon(x,-v))(Rf_0)(\Gamma^t_\epsilon(x,-v)) \quad (11)$$

where $(Rf)(x, v) = f(x, -v)$, and where the sum is taken over all possible trajectories starting at (x, v). Clearly this is a finite sum, because there is a maximal number of obstacles in any finite time interval. A set of possible trajectories, and one realization is shown in fig. 3.

For the evolution associated to this model we prove the following theorem:

Theorem 2 *Let $f_0 : \mathbb{R}^2 \times S^1 \to \mathbb{R}^+$ be the initial probability density. Then, for any $t > 0$, $0 < \delta \leq 1$:*

$$\lim_{\epsilon\to 0} f_\epsilon(\cdot, t) = f(\cdot, t) \quad (12)$$

in $\mathcal{M}_0'$, where $f(\cdot, t) \in L^1(\mathbb{R}^2 \times S^1)$ is the unique solution of the transport equation:

$$(\partial_t + v\cdot\nabla_x) f(x,v,t) = \frac{1}{2}\int_{S^-} \{f(x,v',t) - f(x,v,t)\}|v\cdot\omega|\, d\omega\,, \quad (13)$$

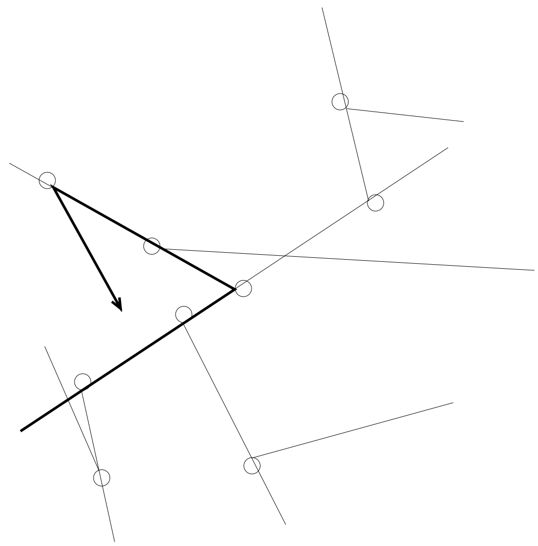

Figure 3: Possible trajectories leaving a given point, and one realization.

where $S^- = \{\omega \in S^1 | v \cdot \omega < 0\}$, v' is the outgoing velocity after a collision with outward normal ω and in-going velocity v (see formula (3)) and $f(x, v, 0^+) = f_0(x, v)$.

The proof of this result is given in Section 3. It is somewhat technical, and as a preparation, we give here some definitions related to the evolution of $z_\epsilon(t) = \big(x_\epsilon(t), v_\epsilon(t)\big)$.

Similarly to the Boltzmann process already discussed, the evolution of z_ϵ is described by a semi-group V_ϵ^t, as defined in (9):

$$g_\epsilon(x, v, t) = V_\epsilon^t g^0(x, v) := \mathbb{E}\big[g^0(T_\epsilon^t(x, v))\big]$$

This semi-group V_ϵ^t can be expanded as a sum of terms, each one taking into account the case of exactly n collisions with obstacles the given time interval in the following way:

For a fixed initial condition (x, v) and $t > 0$, let

$$S_1(t) = \big\{\tau \in (0, t) \mid x + v\tau \in \partial\Lambda_{\mathbf{c}},\ v \cdot \omega \le 0\big\}, \tag{14}$$

i.e. the set of times when a trajectory starting at x with direction v enters an obstacle, assuming that no scattering takes place, or in other words, $S_1(t)$ is the set of possible times for the first scattering event of a trajectory. Given that this first event takes place at $t_1 \in S_1(t)$, and that the outcome of the scattering gives the new velocity v_1, we can then define the set of possible times for the second scattering event, and then for the third, and so on:

$$
\begin{aligned}
S_2(t,t_1) &= \big\{\tau \in (t_1,t) \mid t_1 \in S_1(t),\ x + t_1 v + (\tau - t_1)v_1 \in \partial\Lambda_{\mathbf{c}},\ v_2\cdot\omega \le 0\big\}\\
&\quad\vdots\\
S_n(t,t_1\dots t_{n-1}) &= \big\{\tau \in (t_{n-1},t) \mid t_i \in S_i,\ i = 1\dots n-1,\\
&\qquad\qquad x + t_1 v \cdots + (\tau - t_{n-1})v_{n-1} \in \partial\Lambda_{\mathbf{c}},\ v_n\cdot\omega \le 0\big\}\\
S_{n+1}(t,t_1\dots t_n) &= \big\{\tau \in (t_n,t) \mid t_i \in S_i,\ i = 1\dots n,\\
&\qquad\qquad x + t_1 v \cdots + (\tau - t_n)v_n \in \partial\Lambda_{\mathbf{c}},\ v_{n+1}\cdot\omega \le 0\big\}.
\end{aligned}
$$

Of course all the S_n depend on the initial position, and so it would be more correct, perhaps, to write $S_n(t,t_1\dots t_{n-1};x,v)$. Given the initial position and velocity, the sets of scattering events completely determines the trajectory, because there is no other randomness in the process but the choice whether a scattering takes place or not. We denote

$$
s_n := |S_n(t,t_1\dots t_{n-1})| \tag{15}
$$

the cardinality of the set S_n, and also

$$
k^{(n)} = |S_{n+1}(t,t_1\dots t_n)| + \sum_{j=1}^{n} |S_j(t_i,t_1\dots t_{j-1})|\,,
$$

which counts the number of encounters with an obstacle which did not result in a scattering event, given that scattering did occur at at $t_1 ... t_n$. Then

$$
g_\epsilon(x,v,t) = \sum_{n\ge 0} (V_\epsilon^t g^0)_n(x,v,t) = \sum_{n\ge 0} g_{\epsilon,n}(x,v,t) \tag{16}
$$

where $g_{\epsilon,0}(x,v,t) = (1-\epsilon^{\delta/2-\delta})^{s_1} g^0(x+vt,v)$ and, for $n>1$,

$$
\begin{aligned}
g_{\epsilon,n}(x,v,t) = &\sum_{t_1\in S_1(t)} \cdots \sum_{t_n \in S_n(t,t_1\dots t_{n-1})} p^n(1-p)^{k^{(n)}}\\
&g^0(x + t_1 v + (t_2-t_1)v_1 + \dots (t_n - t_{n-1})v_{n-1} + (t-t_{n-1})v_n, v_n)\,,
\end{aligned}
$$

where $v_0 = v$ and $v_i = v'_{i-1}$ is the post collisional velocity with incoming velocity v_{i-1}, and, as before, $p = \epsilon^{\delta/2-\delta}$.

To prove Theorem 2 we shall show that, for any function $g^0 \in C_0(\mathbb{R}^2\times S^1)$ and for all $t>0$,

$$
\int_{\mathbb{R}^2\times S^1} (f_\epsilon(x,v,t) - f(x,v,t))g^0(x,v)\,dx\,dv \to 0 \qquad \text{as} \qquad \epsilon\to 0. \tag{17}
$$

3 Convergence of the Markovian model to the Boltzmann equation

From the very definition of f_ϵ and the weak definition of f,one can see that proving (17) is equivalent to showing that given the initial data $f_0(x,v)$,

$$\int_{\mathbb{R}^2\times S^1} f_0(x,v)\left(V^t g^0(x,v) - V_\epsilon^t g^0(x,v)\right)\,dxdv \to 0 \tag{18}$$

when $\epsilon \to 0$, for fixed t (but uniformly for any interval $0<t<T$, and for all $g^0(x,v)\in C_0$). That it is enough to consider test functions with compact support, follows from the fact that $\int_{\{|x|>R\}\times S^1} f(x,v,t)\,dxdv \to 0$ as $R\to\infty$ (also this holds uniformly in a bounded time interval, because of the bounded velocities), and that V_ϵ^t and V^t are bounded operators in L^∞. Moreover

$$\begin{aligned}\left|\int_{\mathbb{R}^2\times S^1} f_0(x,v)\left(V^t g^0(x,v) - V_\epsilon^t g^0(x,v)\right)\,dxdv\right| &\le \\ \|g^0\|_{L^\infty}\int_{\{f_0>M\}} f_0\,dxdv + M\|V^t g^0 - V_\epsilon^t g^0\|_{L^1}\,. \end{aligned} \tag{19}$$

The first of the terms in the right hand side go to zero as M increases to infinity (this is one point where we use in an essential way that $f_0\in L^1$). The rest of the section is devoted to proving that the second term goes to zero when $\epsilon\to 0$.

To study this second term, we rely on the semi-group property of V^t and V_ϵ^t, and that $\|V^t\|_{L^\infty} = \|V_\epsilon^t\|_{L^\infty} = 1$. The semi-groups are also bounded in L^1: $\|V^t\|_{L^1} = \|V_\epsilon^t\|_{L^1}\le 1$. Dividing the interval $[0,t]$ into N intervals gives

$$V^t g^0(x,v) - V_\epsilon^t g^0(x,v) = \sum_{j=0}^{N-1} V_\epsilon^{j\frac{t}{N}}\left(V^{\frac{t}{N}} - V_\epsilon^{\frac{t}{N}}\right)V^{(N-1-j)\frac{t}{N}} g^0(x,v)\,.$$

Hence it is enough to show that for some suitably chosen $N=N_\epsilon$,

$$N_\epsilon\,\|V^{\frac{t}{N_\epsilon}} g^0(x,v) - V_\epsilon^{\frac{t}{N_\epsilon}} g^0(x,v)\|_{L^1} \to 0 \tag{20}$$

as $\epsilon\to 0$, because then, for any $\varepsilon_0>0$, one can choose M so large that

$$\int_{\{f_0>M\}} f_0\,dxdv \le \frac{\varepsilon_0}{2\|g^0\|_{L^\infty}}\,,$$

and then take ϵ so small that $N_\epsilon\|V^{\frac{t}{N_\epsilon}} g^0(x,v) - V_\epsilon^{\frac{t}{N_\epsilon}} g^0(x,v)\|_{L^1} < \varepsilon_0/2M$. Let $\tau_\epsilon = t/N_\epsilon$. Now,

$$\begin{aligned}\left|V_\epsilon^{\tau_\epsilon} g^0(x,v) - V^{\tau_\epsilon} g^0(x,v)\right| &= \left|\sum_{n=0}^{\infty}\left(\left(V_\epsilon^{\tau_\epsilon} g^0\right)_n(x,v) - \left(V^{\tau_\epsilon} g^0\right)_n(x,v)\right)\right| \\ &\le \left|\left(V_\epsilon^{\tau_\epsilon} g^0\right)_0(x,v) - \left(V^{\tau_\epsilon} g^0\right)_0(x,v)\right| \\ &\quad + \left|\left(V_\epsilon^{\tau_\epsilon} g^0\right)_1(x,v) - \left(V^{\tau_\epsilon} g^0\right)_1(x,v)\right| \\ &\quad + \left|\sum_{n\ge 2}\left(V_\epsilon^{\tau_\epsilon} g^0\right)_n(x,v)\right| + \left|\sum_{n\ge 2}\left(V^{\tau_\epsilon} g^0\right)_n(x,v)\right| \end{aligned}$$

It is clear from the definition of $\left(V^{\tau_\epsilon} g^0\right)_n (x,v)$, that

$$\Big\| \sum_{n\geq 2} \left(V^{\tau_\epsilon} g^0\right)_n (x,v) \Big\|_{L^1} \leq \|g^0\|_{L^1} \tau_\epsilon^2 \,. \tag{21}$$

Moreover, the following propositions hold true:

Proposition 3 *Let $g^0(x,v) \in C_0(\mathbb{R}^2 \times S^1)$. Suppose that $g^0(x,v) = 0$ if $|x| > R$ (so R is the diameter of the support of g^0). Assume that $t \geq 2\epsilon^{1/(2-\delta)}$. Then*

$$\Big\| \sum_{n\geq 2} \left(V_\epsilon^t g^0\right)_n (x,v) \Big\|_{L^1} \leq C_R \|g^0\|_{L^\infty} t^2$$

Proposition 4 *Let $g^0(x,v)$ satisfy the same conditions as in Proposition 3. Assume also that $t \sim p^\alpha$, with $0 < \alpha < 1$.*

1. *If $\alpha < 1/3$, then*

$$\begin{aligned} \| \left(V_\epsilon^t g^0\right)_0 (x,v) - \left(V^t g^0\right)_0 (x,v) \| \\ \leq C_R ||g^0||_{L^\infty} t^{1+\frac{1}{4}(\frac{1}{\alpha}-3)} \sqrt{\log(1/t)} \,. \end{aligned} \tag{22}$$

2. *Let $\lambda_{g^0}(w)$ be the modulus of continuity of g^0, i.e. a function such that $|g^0(x_1,v_1) - g^0(x_2,v_2)| \leq \lambda_{g^0}(|x_1 - x_2| + |v_1 - v_2|)$ for all x_1, x_2, v_1 and v_2. There is a $\gamma > 0$ such that*

$$\begin{aligned} \| \left(V_\epsilon^t g^0\right)_1 (x,v) - \left(V^t g^0\right)_1 (x,v) \| \leq \\ C_R t \, \lambda_{g^0}(t^{\gamma/4}) + C_R ||g^0||_{L^\infty} t^{1+\gamma} \,. \end{aligned} \tag{23}$$

What this says is that, in a short time interval, the probability that a trajectory has more than two velocity jumps is very small, and that trajectories with at most one velocity jump have essentially the same distribution in the limit as ϵ go to zero.

This is enough to prove Theorem 2, because, returning to the expression (20), and using $N_\epsilon \tau = t$, we then have

$$N_\epsilon \|V_\epsilon^{\frac{t}{N_\epsilon}} g^0(x,v) - V^{\frac{t}{N_\epsilon}} g^0(x,v)\|_{L^1} \leq N_\epsilon \tau o(\tau) = t \, o(\tau) \,,$$

where $o(\tau)$ is a function converging to zero with τ. We can conclude, by taking for example $N^\epsilon = \epsilon^{-\delta/(8-4\delta)}$, which implies that the conditions for τ in Proposition 3 and Proposition 4 are satisfied. $\square$

The proofs of Proposition 3 and Proposition 4, both depend very much on Proposition 5, and its corollaries. Consider a line segment of length L starting from the point x in the direction v, and denote by $s(x,v,L)$ the number of obstacle sites that this segment crosses. Then Proposition 5 says essentially that

$$\begin{aligned} s(x,v,L) &= L\epsilon^{-\delta/(2-\delta)} + r_{D,\epsilon}(x,v,L) \qquad \text{and} \\ s(x,v,L) &\leq cL\epsilon^{-\delta/(2-\delta)} + r_{F,\epsilon}(v,L) \,, \end{aligned}$$

where the error terms r may be very large, but are small with respect to $L\epsilon^{-\delta/(2-\delta)}$ after integration over v; the second error one actually bounded, but at the cost of the constant c in front of $L\epsilon^{-\delta/(2-\delta)}$.

Proof of Proposition 3: From (16), we have

$$\begin{aligned}&\left|(V_\epsilon^t g^0)(x,v,t) - (V_\epsilon^t g^0)_0(x,v,t) - (V_\epsilon^t g^0)_1(x,v,t)\right| \\ &= \left|\sum_{n=2}^{\infty} P(n_{co}=n)\,\mathbb{E}\big[g(T_\epsilon^t(x,v))\;\big|n_{co}=n\big]\right|, \end{aligned} \tag{24}$$

where $P(n_{co}=n)$ is the probability that there are exactly n_{co} scattering events in the trajectory, and where $\mathbb{E}\big[A \;\big|\; B\big]$ denotes the conditional expectation of A given B. The right hand side of (24) is then bounded by

$$\sup_{(y,w)\in\mathbb{R}^2\times S^1} |g(y,w)| \sum_{n=2}^{\infty} P(n_{co}=n)\,, \tag{25}$$

where actually $P(n_{co}=n)$ depends on the initial value (x,v). It remains to estimate

$$\int_{A\times S^1} P(n_{co}\geq n)\,dx\,dv\,, \tag{26}$$

where A is a subset of $\mathbb{R}^2$ sufficiently large to contain the support of g.

The probability that there are at least two scattering events is

$$\begin{aligned}\sum_{n\geq 2} P(n_{co}=n) &= P(n_{co}\geq 1)\sum_{k=1}^{s_1} P(\text{scattering at } k)\, P(n_{co}\geq 2|k) \\ &= \sum_{k=1}^{s_1} p(1-p)^{k-1}(1-(1-p)^{s_2})\,; \end{aligned} \tag{27}$$

in this expression, s_1 is the number of obstacle sites crossed by a segment of length t starting in the direction v from the point x, $p(1-p)^{k-1}$ is the probability that the first scattering event takes place exactly at the k-th encounter with an obstacle, and s_2 is the number of encounters with obstacles at the second lap, *given the starting position for the first lap, the direction, and the number* k. This corresponds to the number of crossed obstacle sites along a segment of length $t-t_1$, starting at point x_1 in the direction v_1. Note that t_1, x_1 and v_1 are all well defined, given the random number k and the starting position x and v.

The probability p is supposed to be small (and actually converge to zero), so one can assume that $e^{-\frac{3}{2}} \leq (1-p)^{1/p}$, for all p sufficiently small. Then

$$1-(1-p)^{s_2} < 1-e^{-\frac{3}{2}ps_2}\,. \tag{28}$$

Next we write $b_k = p(1-p)^{k-1}/\left(1-(1-p)^{s_1}\right)$, so that $\sum_{k=1}^{s_1} b_k = 1$. Because $1-e^{-\frac{3}{2}\tau}$ is concave in τ, one can then use Jensen's inequality to deduce that

$$\sum_{k=1}^{s_1} b_k(1-e^{-\frac{3}{2}ps_2}) \leq \left(1-e^{-\frac{3p}{2}\sum_{k=1}^{s_1} b_k s_2}\right)\,. \tag{29}$$

Integrating over A while keeping the initial direction v fixed, and again using the Jensen inequality gives the estimate ($|A|$ denotes the area of the set A)

$$\int_A \sum_{n=2}^{\infty} P(n_{co}=n)\,dx \;\leq\; |A|\frac{1}{|A|}\int_A \left(1-(1-p)^{s_1}\right)\left(1-e^{-\frac{3p}{2}\sum_{k=1}^{s_1} b_k s_2}\right)dx$$
$$\leq\; |A|\sup_{x\in A}\left(1-(1-p)^{s_1}\right)\left(1-e^{-\frac{3p}{2}\frac{1}{|A|}\int_A \sum_{k=1}^{s_1} b_k s_2\,dx}\right). \tag{30}$$

There is no loss of generality in assuming that A is rectangular, with sides aligned with the direction v, and we can then choose a coordinate system so that $x=(y,z)$ where $y=x\cdot v$ (recall that $|v|=1$). Then $A=I_1\times I_2$ with $y\in I_1$ and $z\in I_2$, and the exponent in (30) becomes

$$\frac{3p}{2}\frac{1}{|A|}\int_A \sum_{k=1}^{s_1} b_k s_2\,dx = \frac{3p}{2}\frac{1}{|I_1|\,|I_2|}\int_{I_1}\int_{I_2}\sum_{k=1}^{s_1} b_k s_2\,dz\,dy\,. \tag{31}$$

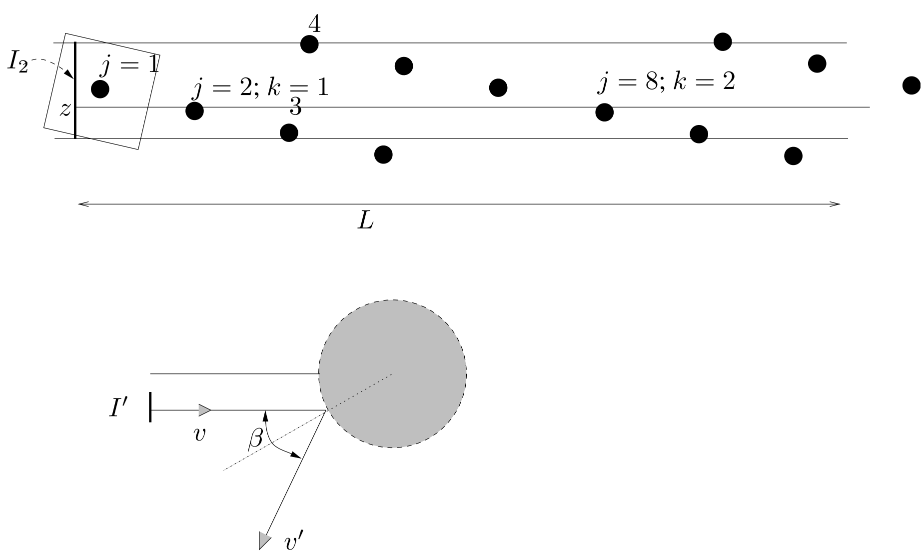

Figure 4: The figure shows a strip J_t with t in an interval of length L and $z\in I_2$

In what follows, we shall consider the integral with respect to z (see fig. 4, which shows a strip of this domain, with length $L=t$ and width $|I|$, where I denotes a segment of constant y). In the figure, all scatterers whose centers belong to the strip are enumerated with the symbol j, and $c_j\in\mathbb{R}^2$ denotes the center of the j-th scatterer. Similarly, z_j then denotes the second component of c_j in this coordinate system. Suppose now that a line segment starting from $z\in I$ in the direction v is scattered on the j-th obstacle, and that the scattering angle is β. Then the relation

$$z-z_j=\epsilon\sin\beta/2 \tag{32}$$

holds. Let $k = k(\beta, j)$ denote the number of scatterers that this line segment crosses on the way from I to the scatterer, and write $z = z(\beta, j)$. The other way around, one can follow a line from $z \in I$, and count the number of crossed obstacles, and stop on the k^{th} one along the trajectory. This identifies uniquely an obstacle j and a scattering angle β, and so there is a one to one correspondence between a couple (z, k) and (β, j). If we denote the exact time of scattering against the k^{th} obstacle along the way by $t_1 = t_1(z, k)$, and the point at which this takes place by $x_1 = x_1(z, k)$, then the expression for s_2, including all variables, in (31) is

$$s_2 = s(x_1(z,k), \beta(z,k), t - t_1(z,k))\,,$$

where $s(x, v', t)$, in general, is the number of times a segment of length t starting at x in the direction v' (the direction *after* the scattering, which is identified with the scattering angle β in a natural way). The inner integral in (31) then becomes

$$\frac{p}{|I_2|}\int_{I_2}\sum_{k=1}^{s_1} b_k s(x_1(z,k), \beta(z,k), t - t_1(z,k))\,dz \le \frac{p}{|I_2|}\int_{I_2}\sum_{k=1}^{s_1} b_k \sup_x s(x, \beta(z,k), t)\,dz\,.$$

because s is increasing in t. Using the identification of (z, k) with (β, j), and (32), one can then write the integral and sum as

$$\int_0^{2\pi} \frac{p\epsilon}{|I_2|}\sum_{c_j\in J_t} b_{k(\beta,j)} \sup_x s(x,\beta,t)\frac{1}{2}\cos(\beta/2)\,d\beta\,, \tag{33}$$

where $J_t = [y, y+t[\times I$ is a strip of length t as denoted in the figure. Let

$$\bar{s}_\epsilon(\beta,t) = \sup_x s(x,\beta,t)\,, \qquad \text{and}$$
$$B_\epsilon(\beta,t) = \frac{\epsilon}{|I|}\frac{1}{2}\cos(\beta/2)\sum_{c_j\in J_t} b_{k(\beta,j)}\,.$$

We then use Proposition 6 with $L = t$, to see that $\bar{s}_\epsilon(\beta, t) \le c_1 t \epsilon^{-\delta/(2-\delta)} + r_{F,\epsilon}(\beta, t)$, where

$$\int_0^{2\pi} |r_{F,\epsilon}(\beta,t)|\,d\beta \le c_3\,,$$

and where $|r_{F,\epsilon}(\beta, t)| \le c_2 t \epsilon^{-1/(2-\delta)}$, for some constants c_i. By the very construction, $\int_0^{2\pi} B_\epsilon(\beta, t)\,d\beta = 1$ independently of t (just carry out the sum and integral in (31) without the function s_2; for this to be exact, one has to define the $b_{k(\beta,j)}$ to be zero when this corresponds to $z \notin I$). It is also clear that

$$B_\epsilon(\beta,t) \le \frac{\epsilon}{|I|}\#\{c_j\in J_t\}\,,$$

where $\#\{\cdot\}$ denotes the cardinality of a set. Here, with a rough estimate, if $t > 2\epsilon^{1/(2-\delta)}$, then $\#\{c_j \in J_t\} \le 2t|I|\epsilon^{-2/(2-\delta)}$, and then

$$B_\epsilon(\beta,t) \le Ct\epsilon^{-\delta/(2-\delta)} = Ct/p\,.$$

Thus the expression (33) is bounded by

$$
\begin{aligned}
p\int_0^{2\pi} & B_\epsilon(\beta,t)\bar{s}_\epsilon(\beta,t)\,d\beta \leq \\
&\leq p\int_0^{2\pi} B_\epsilon(\beta,t)c_1 t\epsilon^{-\delta/(2-\delta)}\,d\beta + p\,\sup_\beta B_\epsilon(\beta,t)\int_0^{2\pi}|r_{F,\epsilon}(\beta,t)|\,d\beta \\
&\leq c_1 t + Ct\int_0^{2\pi}|r_{F,\epsilon}(\beta,t)|\,d\beta \leq Ct\,. \qquad (34)
\end{aligned}
$$

Similarly, we estimate the factor $(1-(1-p)^{s_1})$ in (30) as

$$
1-(1-p)^{s_1} \leq 1-e^{-\frac{3}{2}ps_1} \leq 1-e^{-C(t+p\,r_{F,\epsilon}(v,t))}\,.
$$

Now $1-e^{-(y_1+y_2)} \leq \min(1,y_1)+\min(1,y_2)$, and hence

$$
\begin{aligned}
&\Big(1-e^{-C(t+p\,r_{F,\epsilon}(v,t))}\Big)\Big(1-e^{-Ct}\Big) \\
&\qquad\leq C\Big(t+\min(1,p\,r_{F,\epsilon}(v,t))\Big)\min(1,t)\,. \qquad (35)
\end{aligned}
$$

We conclude the proof by integrating over v, and inserting the relations between ϵ, t and p:

$$
\begin{aligned}
\int_{A\times S^1} P(n_{co}\geq n)\,dx\,dv &\leq C_R\int_{S^1}\Big(t+\min(1,p\,r_{F,\epsilon}(v,t))\Big)\min(1,t)\,dv \\
&\leq C_R t^2\,;
\end{aligned}
$$

the only condition needed is that $t > 2\epsilon^{1/(2-\delta)}$. $\square$

Proof of Proposition 4: The idea of the proof of this proposition is similar to the the corresponding one in [CPR], though here we make direct use of the counting lemma from Section 4, just in the proof of Proposition (3).
Proof of (22):

$$
\begin{aligned}
g_0(x,v,t) &= \left(V^t g^0\right)_0(x,v) = e^{-t}g^0(x+vt,v), \qquad \text{and} \\
g_{\epsilon,0}(x,v,t) &= \left(V_\epsilon^t g^0\right)_0(x,v) = (1-p)^{s_1(x,v,t)}g^0(x+vt,t)\,,
\end{aligned}
$$

and so

$$
\begin{aligned}
\|g_0(\cdot,\cdot,t)-g_{\epsilon,0}(\cdot,\cdot,t)\|_{L^1} &\leq \|g^0\|_{L^\infty}\int_{A\times S^1}\left|e^{-t}-(1-p)^{s_1(x,v,t)}\right|dxdv \\
&= \|g^0\|_{L^\infty}\,e^{-t}\int_{A\times S^1}\left|1-e^t(1-p)^{s_1(x,v,t)}\right|dx\,dv\,, \qquad (36)
\end{aligned}
$$

where, like before, $A\subset\mathbb{R}^2$ contains the support of g^0. Next,

$$
e^t(1-p)^{s_1(x,v,t)} = e^{t+\log(1-p)s_1}\,,
$$

and using Corollary 5.1 below, we find that $|s_1 - t/p| \leq r_{1,\epsilon}(v,t)$, where on the average $r_{1,\epsilon}(v,t)$ is small. Moreover, $\log(1-p)+p < Cp^2$, and so

$$|t + \log(1-p)s_1| \leq C\left(p\, r_{1,\epsilon}(x,v,t) + pt\right).$$

Now, with $t \sim p^\alpha$, (which means that $p \sim t^{1/\alpha}$, and $\epsilon \sim t^{(2-\delta)/\delta\alpha}$), the estimate on $r_{1,\epsilon}(v,t)$ from Corollary 5.1 implies that

$$p\int_{S^1} \left|\, r_{1,\epsilon}(x,v,t)\,\right| dv < c_1 t^{1/\delta\alpha} + c_2\left(t^{1+1/\alpha}\log\left(t^{1-1/\delta\alpha}\right)\right)^{1/2}$$

Then, for small t,

$$\text{meas}(\{v \in S^1 \big|\; p\,|r_{D,\epsilon}| > t^{1+\gamma}\}) \leq C\sqrt{\log(1/t)}\, t^{\frac{\alpha+1}{2\alpha}-1-\gamma},$$

for some constant C. On the complementary set,

$$|t + \log(1-p)s_1| \leq C\Big(t^{1+\gamma} + t^{1+1/\alpha}\Big),$$

and so the integral in (36) is smaller than

$$C|A|\left(t^{1+\gamma} + t^{1+1/\alpha}\right) + C\sqrt{\log(1/t)}\, t^{\frac{\alpha+1}{2\alpha}-1-\gamma}.$$

If $\alpha < 1/3$, then the estimate (22) follows by taking $\gamma = \frac{1}{4}(\frac{1}{\alpha}-1)$.

Proof of the estimate (23): We need to compare

$$g_1(x,v,t) = \left(V^t g^0\right)_1(x,v) = e^{-t}\int_0^t \int_{S^-} g^0(x+vt_1+(t-t_1)v',v')\frac{|(v,\omega)|}{2}\, d\omega\, dt_1, \tag{37}$$

and

$$g_{\epsilon,1}(x,v,t) = \left(V_\epsilon^t g^0\right)_1(x,v) = \sum_{t_1\in S_1(t)} p(1-p)^{s_1+s_2} g^0(x+t_1 v+(t-t_1)v_1, v_1), \tag{38}$$

where $S_1(t)$ are the instances where a trajectory encounters an obstacle site, as defined in (14); the corresponding points of encounter, and deflected velocity are denoted x_1 and v_1. Also, $s_1(x,v,t)$ and $s_2(x,v,t;t_1)$ are the number of encounters with obstacles sites on the first and second lap of the trajectory, as described before.

Consider first (37). The integral over ω can be parameterized by the scattering angle β, or equivalently by $z = \sin(\beta/2)$, and so it can be written

$$e^{-t}\frac{1}{2}\int_0^t \int_{-1}^1 g^0(x+vt_1+(t-t_1)v',v')\, dz\, dt_1.$$

We then write the argument $X(t_1,z;x,v,t)$, and $v'=v'(z;v)$. For the moment v, x and t are considered as parameters, and they will only be written out when needed. We cut the integral into pieces, and write

$$e^{-t}\frac{1}{2}\sum_{j=1}^{j_0}\int_{T_j}\sum_{m=-m_0+1}^{m_0}\int_{Z_m} g^0(X(t_1,z),v'(z))\,dz\,dt_1\,, \tag{39}$$

where $T_j=[\frac{t(j-1)}{j0},\frac{tj}{j0}[$ and $Z_m=[\frac{m-1}{m_0},\frac{m}{m_0}[$. The position $X(t_1,z_1;x,v,t)$ corresponding to the endpoints of the intervals, are denoted

$$X_{j,m}(x,v)=x+(t(j-1)/j_0)v+(t-t(j-1)/j_0)\,v'\big((m-1)/m_0,v\big)\,.$$

In any one of the subsets of the integral (39), we have $0\le t_1-\frac{t(j-1)}{j0}\le t/j_0$, and $|v'(z)-v'(\frac{m-1}{m_0})|<2\sqrt{1/m_0}$. Also, obviously, $|v-v'|\le 2$, and so

$$|X(t_1,z)-X_{j,m}|=2t/j_0+2\sqrt{1/m_0}\,.$$

Because g^0 is continuous and compactly supported, it is uniformly continuous, which means that there is a function $\lambda_{g^0}(w)$ which tends to 0 as w tends to 0, such that $|g^0(x_1,v_1)-g^0(x_2,v_2)|\le\lambda_{g^0}(|x_1-x_2|+|v_1-v_2|)$. Then,

$$\begin{aligned}g_1(x,v,t)=&e^{-t}\frac{1}{2}\frac{t}{j_0}\frac{1}{m_0}\sum_{j=1}^{j_0}\sum_{m=1-m_0}^{m_0}g^0(X_{j,m},v'\big((m-1)/m_0\big))\\&+\tilde g_{1,m_0,j_0}(x,v,t)\,,\end{aligned} \tag{40}$$

where

$$\tilde g_{1,m_0,j_0}(x,v,t)\le t\lambda_{g^0}(2t/j_0+2\sqrt{1/m_0})\,.$$

Similarly we write

$$\begin{aligned}g_{\epsilon,1}(x,v,t)=p\sum_{j=1}^{j_0}\sum_{m=1-m_0}^{m_0}\sum_{t_1\in S_1(t)}&\big(1-p\big)^{s_1+s_2}\\&\times\mathbb{1}_{t_1\in T_j}\mathbb{1}_{z_1\in Z_m}\;\;g^0\big(X(t_1,z_1;x,v,t),v'(z_1;v)\big)\end{aligned}$$

where $z=z(t_1)=\sin(\beta)/2)$, and $\beta(t_1)$ is the deflection angle between v and the outgoing velocity $v_1(t_1)$. Just as in the continuous case, one can replace $X(t_1,z_1)$ by $X_{j,m}$, and use the uniform continuity of g^0. With

$$\begin{aligned}\tilde g_{\epsilon,m_0,j_0}(x,v,t)&\le p\,s(x,v,t)\lambda_{g^0}(2t/j_0+2\sqrt{1/m_0})\\&\le C\big(t+p\,r_{F,\epsilon}(v,t)\big)\lambda_{g^0}(2t/j_0+2\sqrt{1/m_0})\,,\end{aligned}$$

one can write

$$\begin{aligned} g_{\epsilon,1}(x,v,t) = p \sum_{j=1}^{j_0} \sum_{m=1-m_0}^{m_0} & g^0(X_{j,m}, v'((m-1)/m_0)) \\ & \times \sum_{t_1 \in S_1(t)} \mathbb{1}_{t_1 \in T_j} \mathbb{1}_{z_1 \in Z_m} \big(1-p\big)^{s_1+s_2} \\ & + \tilde{g}_{\epsilon,m_0,j_0}(x,v,t)\,. \end{aligned} \tag{41}$$

It remains to compare the sums in (40) and (41). For this we use again Proposition 5 and its corollaries, to see that

$$\begin{aligned} \big|t + \log(1-p)s(x,v,t_1) + & s(x_1,v_1,t-t_1)\big| \\ & \le p\, r_{1,\epsilon}(v,t_1) + p\, r_{1,\epsilon}(v_1,t-t_1) \end{aligned}$$

and so, the sums differ by at most

$$\begin{aligned} te^{-t}\,\|g^0\|_{L^\infty}\; & \frac{1}{2m_0j_0} \sum_{j,m} \Big|1 - \frac{2p\,m_0j_0}{t} \sum_{t_1\in S_1(t)} \mathbb{1}_{t_1\in T_j} \mathbb{1}_{z_1\in Z_m} e^{t+\log(1-p)(s_1+s_2)}\Big| \\ \le & te^{-t}\,\|g^0\|_{L^\infty} \left(\frac{1}{2m_0j_0} \sum_{j,m} \Big|1 - \frac{2p\,m_0j_0}{t} \#\big\{t_1 \in S_1(t) \bigm| t_1\in T_j\,, z_1 \in Z_m\big\}\Big| \right. \\ & \left. + \frac{p}{t} \sum_{t_1\in S_1(t)} \big|1 - e^{t+\log(1-p)(s_1+s_2)}\big| \right) \end{aligned}$$

The cardinality of the set in the first sum, is given by Corollary 5.2 with $L = t/j_0$ and $\kappa = 1/2m_0$, so that each of the terms in the sum is bounded by

$$\Big|1 - \frac{2p\,m_0j_0}{t}\Big(\frac{t\epsilon^{-\delta/(2-\delta)}}{2m_0j_0} + r_{1,\epsilon}(v,t/j_0,1/2m_0)\Big)\Big| \le \frac{2p\,m_0j_0}{t} r_{1,\epsilon}(v,t/j_0,1/2m_0)$$

Summing all the terms, we find that

$$\begin{aligned} \|g_1(\cdot,\cdot,t) - g_{\epsilon,1}(\cdot,\cdot,t)\|_{L^1} \le & \, C\,|A|\,t\,\lambda_{g^0}(2t/j_0 + 2\sqrt{1/m_0}) \\ & + C\,|A| \int_{S^1} p\, r_{F,\epsilon}(v,t)\,dv\,\lambda_{g^0}(2t/j_0 + 2\sqrt{1/m_0}) \\ & + C\,|A| \|g^0\|_{L^\infty} p\, m_0 j_0 \int_{S^1} r_{1,\epsilon}(v,t/j_0,1/2m_0)\,dv \\ & + e^{-t}\,\|g^0\|_{L^\infty} \int_{A\times S^1} \sum_{t_1\in S_1(t)} p\,\Big|1 - e^{-p\big(r_{1,\epsilon}(v,t_1) + r_{1,\epsilon}(v_1,t-t_1)\big)}\Big|\,dx\,dv\,. \end{aligned} \tag{42}$$

We assume as before, that $t \sim p^\alpha$ for some $\alpha < 1$, and moreover, we set $m_0 \sim \epsilon^{-\gamma_1}$ and $j_0 \sim \epsilon^{-\gamma_2}$, for some positive numbers γ_1 and γ_2. Then after integrating, the second term in (42) is absorbed by the first one. By Corollary 5.2, the third term is bounded by

$$
\begin{aligned}
&|A|\,\|g^0\|_{L^\infty}\Big(c_1 m_0 j_0 \epsilon^{1/(2-\delta)} + p\, m_0 j_0 c_2\Big(\frac{t}{m_0 j_0}\epsilon^{-\delta/(2-\delta)}\log\Big(t\epsilon^{-1/(2-\delta)}/j_0\Big)\Big)^{1/2}\Big)\\
&\le C|A|\,\|g^0\|_{L^\infty}\Big(m_0 j_0 \epsilon^{1/(2-\delta)} + c_2\,(m_0 j_0 t\, p\,)^{1/2}\Big(\log\Big(t\epsilon^{-1/(2-\delta)}/j_0\Big)\Big)^{1/2}\Big)\\
&\le C|A|\,\|g^0\|_{L^\infty}\Big(t^{\frac{1}{\alpha\delta}-\gamma_1-\gamma_2} + t^{\frac{1}{2}\left(1+\frac{1}{\alpha}-\gamma_1-\gamma_2\right)}\big(\log(1/t)\big)\Big)\,.
\end{aligned}
$$

Here any choice of $\alpha < 1$ makes it possible to choose γ_1 and γ_2 so that this term is smaller than $t^{1+\gamma}$ for $\gamma = \gamma_1 + \gamma_2$.

The last term in (42) is estimated in a different way. First of all, the factor $e^{-p\left(\,r_{1,\epsilon}(v,t_1)+r_{1,\epsilon}(v_1,t-t_1)\right)}$ is bounded by $e^t < 2$, say, for small t, simply because $s_1 + s_2 \ge 0$. We keep the same relations between t, p, m_0 and j_0 as before. By Corollary 5.1,

$$
\int_{S^1} p\,|r_{1,\epsilon}(v,t_1)|\,dv \le C\,t^{1/\alpha}\log(1/t)\,,
$$

and so, for any $\gamma < 1/\alpha$,

$$
\text{meas}\big\{v \mid p\,|r_{1,\epsilon}(v,t_1)| > t^{\gamma}\,\big\} \le C\,t^{\frac{1}{\alpha}-\gamma}\log(1/t)\,.
$$

Also, just like in the proof of Proposition 3,

$$
\int_{A\times S^1}\sum_{t_1\in S_1(t)} p\,\left|r_{1,\epsilon}(v_1,t-t_1)\,\right|dx\,dv \;\le C_R\Big(t^{1/\alpha\delta} + t^{\frac{1}{2}(1+\frac{1}{\alpha})}\sqrt{\log(1/t)}\Big)\,;
$$

this follows like in equation (34), by replacing $r_{F,\epsilon}$ with $r_{1,\epsilon}$, and then integrating over the remaining space variable and over v. Then (this is again the Tjebychev inequality)

$$
\int_{A\times S^1}\sum_{t_1\in S_1(t)} \mathbb{1}_{p\,\left|r_{1,\epsilon}(v_1,t-t_1)\,\right|\ge t^{\gamma}}\,dx\,dv \;\le C_R\Big(t^{\frac{1}{\alpha\delta}-\gamma} + t^{\frac{1}{2}(1+\frac{1}{\alpha})-\gamma}\sqrt{\log(1/t)}\Big)\,.
$$

This all means that the last term in (42) is bounded by

$$
\begin{aligned}
C\|g^0\|_{L^\infty}\left|1-e^{t^{\gamma}}\right| &\int_{A\times S^1} p\sum_{t_1\in S_1(t)} dx\,dv\\
&\quad + C_R\|g^0\|_{L^\infty}\big(t^{\frac{1}{\alpha\delta}-\gamma} + t^{\frac{1}{2}(1+\frac{1}{\alpha})-\gamma}\sqrt{\log(1/t)}\big)\\
&\le C_R\|g^0\|_{L^\infty}\Big(t^{1+\gamma} + t^{\frac{1}{\alpha\delta}-\gamma} + t^{\frac{1}{2}(1+\frac{1}{\alpha})-\gamma}\sqrt{\log(1/t)}\Big)\,,
\end{aligned}
$$

and so we can conclude by choosing $\gamma > 0$ suitably. And so all the terms in (42) go to zero faster than t, when $t \to 0$. $\square$

4 Counting encounters with obstacle sites

In this section we compute a formula that gives the number of times that a trajectory of length L starting at a given point $x \in \Omega_\epsilon$ and in a given direction $v \in S^1$ meets an obstacle. This is calculated in a very classical way, using the Fourier series, and we refer to [D, BGW] similar estimates. Setting the starting point at the edge of a lattice cell results in an error of at most one, and this will be insignificant in the end. We now refer to Figure 5. The line segment of length L is assumed to start at a point y_0 along the left side of the lattice cell, and we assume that the v meets the horizontal line with angle α. There is no loss of generality in assuming that $0 \leq \alpha < \pi/4$. As in the figure we denote y_1 the point at which the line intersects the next cell (modulo the cell size $\epsilon^{1/(2-\delta)}$), and so on, for $\{y_k\}_{k=1}^M$, where $M = \lfloor L\cos(\alpha)/\epsilon^{1/(2-\delta)} \rfloor$. Clearly the number of times that the line segment crosses the scatterer is the same as the number of y_k:s that are in the segment I, the oblige projection of the scatterer on the left side of the cell. We can then write an almost exact formula for $s(x,v,L)$, the number of times that the trajectory crosses a scatterer (we assume here that M is an even number):

$$s(x,v,L) = \sum_{k=0}^{M} \mathbb{1}_I(y_k) \quad \big(\pm 1\big) \;=\; \sum_{k=-M/2}^{M/2} \mathbb{1}_I(y_{k+M/2}) \quad \big(\pm 1\big) \tag{43}$$

where $\mathbb{1}_I$ denotes the characteristic function of the interval I, and y_k is given by the formula

$$y_k = y_0 + k\epsilon^{1/(2-\delta)}\tan(\alpha) \qquad \mathrm{mod} \quad \epsilon^{1/(2-\delta)}\,. \tag{44}$$

A first observation is that the average of $s(x,v,L)$ over x is independent of v: For any set $A \subset \mathbb{R}^2$,

$$\frac{1}{|A|}\int_A s(x,v,L)\,dx = \sum_{k=0}^{M-1} \frac{1}{|A|}\int_A \mathbb{1}_I(y_k)\,dx = M|I| \qquad \pm 1\,, \tag{45}$$

that is

$$\frac{1}{|A|}\int_A s(x,v,L)\,dx = \frac{L\cos(\alpha)}{\epsilon^{1/(2-\delta)}} \cdot \frac{\epsilon}{\epsilon^{1/(2-\delta)}\cos(\alpha)} = L\epsilon^{-\delta/(2-\delta)} \quad \pm 2\,. \tag{46}$$

Obviously this can't hold uniformly for all x; for $\alpha = 0$, for example, the value one finds is either $s(x,v,L) = 0$ or $s(x,v,L) = L\epsilon^{-1/(2-\delta)}$ depending on x. To compute a more precise estimate for a given x, we change scale so as to make the lattice size one, and and make a translation so that $\mathbb{1}_I(y)$ looks like in figure 5. The support of $\mathbb{1}_I(y)$ is then an interval of length $\epsilon^{(1-\delta)/(2-\delta)}$. In the following we will also replace the characteristic function $\mathbb{1}_I(y)$ by a regularized version, which we write

$$\Psi_\epsilon(y) = \Psi\Big(\frac{y}{\epsilon^{(1-\delta)/(2-\delta)}}\Big)\,, \tag{47}$$

where Ψ is a smooth function which approximates the characteristic function for $[-\frac{1}{2},\frac{1}{2}]$. The regularization can be chosen to give an arbitrarily good approximation, either from below or from above. As in [D, BGW] we make use of the Fourier series for Ψ_ϵ when estimating the sum in (43). Writing

$$\hat{\Psi}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i\xi x}\Psi(x)\,dx \qquad\qquad \xi \in \mathbb{R}\,,$$

and then

$$\hat{\Psi}_{\epsilon,\alpha}(\xi) = \frac{\epsilon^{(1-\delta)/(2-\delta)}}{\cos\alpha}\,\hat{\Psi}\left(\frac{\epsilon^{(1-\delta)/(2-\delta)}}{\cos\alpha}\xi\right) \qquad \xi \in \mathbb{Z}\,. \tag{48}$$

The sum is then

$$\begin{aligned} s(x,v,L) &= \sum_{k=-\frac{M}{2}}^{\frac{M}{2}} \sum_{\xi=-\infty}^{\infty} \hat{\Psi}_{\epsilon,\alpha}(\xi) e^{2\pi i(y_0+\frac{M}{2}\tan(\alpha)+k\tan(\alpha))\xi} \\ &= (M+1)\hat{\Psi}_{\epsilon,\alpha}(0) + r_{D,\epsilon}(x,v,L)\,, \end{aligned} \tag{49}$$

where

$$\begin{aligned} r_{D,\epsilon}(x,v,L) &= \sum_{k=-\frac{M}{2}}^{\frac{M}{2}} \sum_{\xi\neq 0} \hat{\Psi}_{\epsilon,\alpha}(\xi) e^{2\pi i(y_0+\frac{M}{2}\tan(\alpha)+k\tan(\alpha))\xi} \\ &= \sum_{\xi\neq 0} \hat{\Psi}_{\epsilon,\alpha}(\xi) e^{2\pi i(y_0+\frac{M}{2}\tan(\alpha))\xi} \left(\sum_{k=-\frac{M}{2}}^{\frac{M}{2}} e^{2\pi i k\tan(\alpha)\xi} \right). \end{aligned} \tag{50}$$

The factor within parenthesis in the last member is nothing but the Dirichlet kernel $D_M(w) = \frac{\sin(\pi(M+1)w)}{\sin(\pi w)}$, evaluated at the point $w = \tan(\alpha)\xi$; it follows that

$$\begin{aligned} |r_{D,\epsilon}(x,v,L)| &\leq \left| \sum_{\xi\neq 0} \hat{\Psi}_{\epsilon,\alpha}(\xi) e^{2\pi i(y_0+\frac{M}{2}\tan(\alpha))\xi} D_M(\tan\alpha\xi) \right| \\ &\leq \sum_{\xi\neq 0} |\hat{\Psi}_{\epsilon,\alpha}(\xi)|\ \ |D_M(\tan\alpha\xi)|\ . \end{aligned} \tag{51}$$

Because of (48), if Ψ is sufficiently smooth, then for any integer a, there is a constant C_a such that

$$|\hat{\Psi}_{\epsilon,\alpha}(\xi)| \leq s\frac{C_a}{1+|\xi s|^a}\,,$$

where $s = \frac{\epsilon^{(1-\delta)/(2-\delta)}}{\cos(\alpha)}$, and then the sum is bounded independently of ϵ and α whenever $a > 1$:

$$\sum_{\xi\neq 0} |\hat{\Psi}_{\epsilon,\alpha}(\xi)| \leq s\sum_{\xi\in\mathbb{Z}} \frac{C_a}{1+|s\xi|^a} \leq \int_0^{\infty} \frac{sC_a}{1+|s\xi|^a}\,d\xi < C_a\,. \tag{52}$$

Proposition 5 *For a given $x \in \mathbb{R}^2$, $v \in S^1$, and $L > 0$, let $s(x, v, L)$ be the number of times a line segment of length L, starting at x in the direction v, crosses an obstacle site (see also formula (43)). Let Ψ be a smooth approximation of the characteristic function $\mathbb{1}_{[-1/2,1/2]}$ (see fig. 5). Then*

$$s(x, v, L) = L\epsilon^{-\delta/(2-\delta)} + A_\epsilon + BL\epsilon^{-\delta/(2-\delta)}(\Psi(0) - 1) + r_{D,\epsilon}(x, v, L).$$

Here $|A_\epsilon| \leq 2\epsilon^{(1-\delta)/(2-\delta)}$, $|B| \leq 2$. Moreover, there is a constant C_Ψ, depending only on the regularization of Ψ, such that

$$\int_{S^1} \sup_x |r_{D,\epsilon}(x, v, L)|\, dv \leq C_\Psi \log\left(L\epsilon^{-1/(2-\delta)}\right). \tag{53}$$

Proof: By dividing the circle into eight octants, it is possible to reduce the problem to integrating over $0 \leq \alpha \leq \pi/4$, and thus the computations leading to (49) and (50) are valid. Doing the change of variable $\tau = \tan(\alpha)$ gives

$$\begin{aligned} &\int_0^{\pi/4} \sup_x |r_{D,\epsilon}(x, v, L)|\, d\alpha \\ &\qquad \leq \sup_{0\leq\alpha\leq\frac{\pi}{4}} \left(\sum_{\xi\neq 0} |\hat{\Psi}_{\epsilon,\alpha}(\xi)|\right) \sup_{\xi\neq 0} \int_0^1 |D_M(\tau\xi)|\, \frac{1}{1+\tau^2}\, d\tau\,. \end{aligned} \tag{54}$$

The first sum is bounded by a constant C_a, as we have seen above, and the integral of the Dirichlet kernel is itself bounded by

$$\int_0^1 |D_M(\tau\xi)|\, \frac{1}{1+\tau^2}\, d\tau = \frac{1}{|\xi|} \int_0^{|\xi|} |D_M(\tau)|\, \frac{1}{1+\left(\frac{\tau}{|\xi|}\right)^2}\, d\tau \leq C\log(M)\,;$$

the last estimate can be found e.g. in [Ed]. The result then follows, because $M = L\epsilon^{-1/(2-\delta)} \pm 1$. $\square$

Corollary 5.1 *Let $s(x, v, L)$ be defined as in Proposition 5. Then there is a function $r_{1,\epsilon}(v, L)$, and constants c_1 and c_2 such that*

$$\left| s(x, v, L) - L\epsilon^{-\delta/(2-\delta)} \right| \leq r_{1,\epsilon}(v, L)\,,$$

where

$$\int_{S^1} |r_{1,\epsilon}(v, L)|\, dv \leq c_1\epsilon^{(1-\delta)/(2-\delta)} + c_2\left(L\epsilon^{-\delta/(2-\delta)}\log\left(L\epsilon^{-1/(2-\delta)}\right)\right)^{1/2}.$$

Proof: Take $\Psi = (2\epsilon_1)^{-1}\mathbb{1}_{[-\frac{1}{2}-\epsilon_1,\frac{1}{2}+\epsilon_1]} * \mathbb{1}_{[-\epsilon_1,\epsilon_1]}$, i.e the convolution of the characteristic functions of two intervals. Then $\Psi \geq \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}$, and

$$\hat{\Psi}(\xi) = \frac{\sin(\xi(\frac{1}{2}+\epsilon_1))}{\pi\xi}\frac{\sin(\epsilon_1\xi)}{2\epsilon_1\pi\xi}\,.$$

With this choice of Ψ (which gives an *upper* bound for the number of crossed obstacles; replacing ϵ_1 by $-\epsilon_1$ in the first characteristic function gives, with exactly the same estimates, a lower bound), one thus has

$$\hat{\Psi}(0) = 1 + \mathcal{O}(\epsilon_1)\,,$$

and, for an absolute constant C,

$$|\hat{\Psi}(\xi)| \le \frac{C}{\epsilon_1}\frac{1}{1+|\xi|^2}\,.$$

For any choice of $\epsilon_1 > 0$, the equations (48) and (49) give

$$\int_{S^1} \sup_x \left| s(x,v,L) - L\epsilon^{-\delta/(2-\delta)} \right| dv \\ \le c_1 \epsilon^{(1-\delta)/(2-\delta)} + \tilde{c} L \epsilon^{-\delta/(2-\delta)} \epsilon_1 + \frac{c_3}{\epsilon_1} \log\left(L\epsilon^{-1/(2-\delta)}\right) \quad (55)$$

where the c's are fixed (not very large) constants. The result now follows by choosing ϵ_1 optimally. □

Corollary 5.2 *Let $s(x,v,L,\gamma)$ be defined as in Proposition 5, except that only those encounters with obstacles sites are counted, which fall into a subinterval I' of the crossection (see fig 4). Assume that the length of the interval is $\kappa\epsilon$, where $\kappa < 1$. Then*

$$\left| s(x,v,L,\kappa) - \kappa L \epsilon^{-\delta/(2-\delta)} \right| \le r_{1,\epsilon}(v,L,\kappa),$$

where

$$\int_{S^1} |r_{1,\epsilon}(v,L,\kappa)|\, dv \le c_1 \epsilon^{(1-\delta)/(2-\delta)} + c_2 \left(L\kappa\epsilon^{-\delta/(2-\delta)} \log\left(L\epsilon^{-1/(2-\delta)}\right)\right)^{1/2}.$$

Proof: All that changes from before, is that equation (48) is replaced by

$$\hat{\Psi}_{\epsilon,\alpha}(\xi) = \frac{\kappa\epsilon^{(1-\delta)/(2-\delta)}}{\cos\alpha} \hat{\Psi}\left(\frac{\kappa\epsilon^{(1-\delta)/(2-\delta)}}{\cos\alpha}\xi\right) \qquad \xi \in \mathbb{Z}\,.$$

Then all calculations can be carried out as before, to obtain the result. □

Proposition 6 *Let $s(x,v,L)$ be defined as in Proposition 5. Then there are constants c_1, c_2 and c_3, and a function $r_{F,\epsilon}(v,L)$, such that*

$$s(x,v,L) \le c_1 L \epsilon^{-\delta/(2-\delta)} + r_{F,\epsilon}(v,L)\,,$$

and where

$$|r_{F,\epsilon}(v,L)| \le c_2 L \epsilon^{-1/(2-\delta)} \qquad \textit{and} \qquad \int_{S^1} |r_{F,\epsilon}(v,L)|\, dv \le c_3\,.$$

Proof: Starting at equation (43), we first note that

$$\sum_{k=-M/2}^{M/2} \mathbb{1}_I(y_{k+M/2}) \leq \frac{2}{M+1} \sum_{j=-M/2}^{M/2} \sum_{k=-M/2}^{M/2} \mathbb{1}_I(y_{j+k+M/2}) \,,$$

which then changes (49) and (50) into

$$s(x,v,t) \leq 2(M+1)\hat{\Psi}_{\epsilon,\alpha}(0) + r_{F,\epsilon}(x,v,L)\,, \qquad \text{and}$$

$$r_{F,\epsilon}(x,v,L) = \sum_{\xi\neq 0} \hat{\Psi}_{\epsilon,\alpha}(\xi) e^{2\pi i (y_0 + \frac{M}{2}\tan(\alpha))\xi}\ \frac{2}{M+1}\Big(\sum_{k=-\frac{M}{2}}^{\frac{M}{2}} e^{2\pi i k \tan(\alpha)\xi}\Big)^2 \,.$$

What was before the Dirichlet kernel is here the Féjér kernel:
$F_M(w) = \frac{1}{M+1}\frac{\sin^2(\pi w(M+1))}{\sin^2(\pi w)}$, and the result follows in exactly the same way as before, because $0 \leq F_M \leq (M+1)$ and $\int_0^1 F_M(w)\,dw = 1$. $\square$

Remark. The estimates in Proposition 5 and Proposition 5.1 are considerably easier here than the ones carried out in [BGW], because here we are interested averages over free path lengths (or rather the inverse of the free path lengths) rather than their maxima. And this is one of the fundamental reasons why the main result of this paper, the convergence of the billiard dynamics towards a Boltzmann equation, holds here while it fails in [BGW].

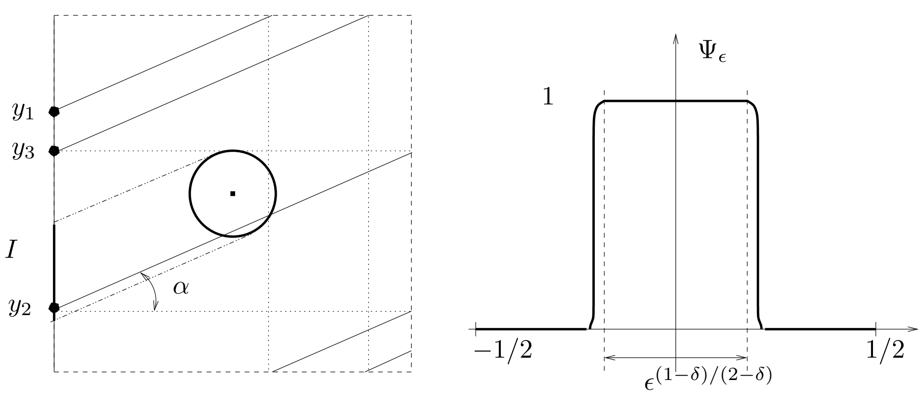

Figure 5: A line, step by step covering the torus; the obstacle radius is ϵ, the size of the torus $\epsilon^{1/2-\delta}$, and the width of the rectangle is in general much smaller than ϵ. The figure to the right shows a smooth approximation from above, of the characteristic function

5 Asymptotic equivalence of the stochastic processes

We have previously described three stochastic processes, $\tilde{z}_\epsilon(t)$, the process coming from the "diluted Lorentz gas", $z_\epsilon(t)$, the Markovian process, and finally $z(t)$, the jump process which is associated with the Boltzmann equation. In this section, we shall see that with probability one, each of these processes belong to the Skorokhod space $D_{[0,T]}(\mathbb{R}^2 \times S^1)$, that each process induces a measure $\tilde{\mu}_\epsilon$, μ_ϵ, and μ on $D_{[0,T]}(\mathbb{R}^2 \times S^1)$, and that each of μ_ϵ and $\tilde{\mu}_\epsilon$ converge to μ when $\epsilon \to 0$. Theorem 1 is a direct consequence of the statement that $\tilde{\mu}_\epsilon \to \mu$ as $\epsilon \to 0$.

We begin with some basic definitions, and then the proof that $\mu_\epsilon \to \mu$.

The Skorokhod space is the space of right continuous functions with left limits (*càdlàg*):

$$\begin{aligned} D_{[0,T]}(\mathbb{R}^2 \times S^1) = \big\{ z : [0,T] \to \mathbb{R}^2 \times S^1 \big| \forall t \in [0,T] z(t) = \lim_{s \to t^+} z(s) \;; \\ z(T) = \lim_{t \to T^-} z(s) \;; \\ \forall t \in [0,T], \exists z(t^-) = \lim_{s \to t^-} z(s) \; \big\} \,, \end{aligned}$$

equipped with the distance

$$\begin{aligned} d_S(x,y) = \inf_{\lambda \in \Lambda} \big\{ \sup_{t \in [0,T]} \| x(t) - y(\lambda(t)) \|_{\mathbb{R}^2 \times S^1} + \sup_{t \in [0,T]} |t - \lambda(t)| \big\} \,, \\ \Lambda = \big\{ \lambda \in C([0,T]) : t > s \Rightarrow \lambda(t) > \lambda(s), \lambda(0) = 0, \lambda(T) = T \big\} \,. \end{aligned}$$

It is clear that all the three processes considered here belong to $D_{[0,T]}(\mathbb{R}^2 \times S^1)$ with probability one. A time $t^* \in [0,T]$ is called a jumping time for z if $\lim_{t \to t^{*-}} z(t) \neq \lim_{t \to t^{*+}} z(t)$; it is enough to verify that with probability one, any one of these processes have only finitely many jumping times. Actually, when $\delta = 0$, which was considered in [CPR], it can happen that a trajectory is trapped in the corner between two obstacles for the Lorentz model, and bounce infinitely many times in a short time interval, and then an argument is needed to show that this happens with zero probability, if the initial data is taken from an initial distribution $f_0 \in L^1(\mathbb{R}^2 \times S^1)$; as soon as $\delta > 0$, this is impossible.

We consider now the Boltzmann process $z(t)$, where the initial data $z(0)$ is distributed according to $f_0 \in L^1(\mathbb{R}^2 \times S^1)$. This induces a measure on $D_{[0,T]}(\mathbb{R}^2 \times S^1)$, which first is defined on cylindrical, continuous functions $F : D_{[0,T]}(\mathbb{R}^2 \times S^1) \to \mathbb{R}$, i.e. functions of the form $F(z) = F_n(z(t_1), z(t_2), \dots z(t_n))$, where $F_n \in C\big((\mathbb{R}^2 \times S^1)^n\big)$, and where $0 \leq t_1 < t_2 < \dots < t_n \leq T$ is any sequence of times. For such functions a measure μ is defined by,

$$\begin{aligned} \int F(z) \, \mu(dz) = \int f_0(z_0) P_{t_1,0}(z_1 \big| z_0) P_{t_2,t_1}(z_2 \big| z_1) \cdots P_{t_n,t_{n-1}}(z_n \big| z_{n-1}) \\ \times F(z_1, z_2, \dots, z_n) dz_0 dz_1 \cdots dz_n \,, \end{aligned}$$

where $P_{t_n,t_{n-1}}(z_n \big| z_{n-1})$ is the probability of a transition from the state z_1 to the state z_2 in the interval from t_1 to t_2. The measure is then extended to all continuous functions $F : D_{[0,T]}(\mathbb{R}^2 \times S^1) \to \mathbb{R}$.

Exactly the same construction is valid for all the three processes considered here, and we denote by $\tilde{\mu}_\epsilon$, μ_ϵ, and $\tilde{P}_\epsilon$ and P_ϵ the corresponding measures and transition probabilities. Moreover, we write μ^* etc. in equations that are true for all of these processes.

We now wish to prove that the process z_ϵ converges to z as $\epsilon \to 0$, in the sense that the corresponding measures converge:

Proposition 7 *For each continuous function $F : D_{[0,T]}(\mathbb{R}^2 \times S^1)$,*

$$\lim_{\epsilon \to 0} \int F(z) \mu_\epsilon(dz) \to \int F(z) \mu(dz) \,. \tag{56}$$

Proof: All the processes considered here belong with probability one to $D_{[0,T]}(\mathbb{R}^2 \times S^1)$. We equip $\mathbb{R}^2 \times S^1$ with the metric $d(z_1, z_2) := \min(\|z_1 - z_2\|_{\mathbb{R}^2 \times S^1}, 1)$. First we recall a result from [GS, p. 431], which adapted to our case says that for such processes, if

1. the marginal distributions of $z_\epsilon(t)$ converge to the marginal distribution of $z(t)$, and

2. there is a constant C, such that for all $\epsilon > 0$, and all choices of $0 \le t_1 < t_2 < t_3 \le T$,

$$\mathbb{E}\big[d\big(z_\epsilon(t_1), z_\epsilon(t_2)\big) d\big(z_\epsilon(t_2), z_\epsilon(t_3)\big)\big] \le C(t_3 - t_1)^2 \,, \tag{57}$$

then for all continuous functional $\phi : D_{[0,T]}(\mathbb{R}^2 \times S^1) \to \mathbb{R}$, the distribution of $\phi(z_\epsilon)$ converges to the distribution of $\phi(z)$, which is exactly the statement of the proposition. (Note that (57) is a stronger statement than the condition required in [GS]).

That the one dimensional marginals converge is essentially the content of Theorem 2. To see that (56) holds for cylindrical functions that factorize as

$$F(z) = \prod_{i=1}^n F_i\big(z(t_i)\big) \,, \qquad F_i \in C_0^\infty(\mathbb{R}^2 \times S^1)$$

one can do very much as in the proof of Theorem 2. We have

$$\begin{aligned}
\int \mu^*(dz) F(z) &= \int f_0(z_0) P^*_{t_1}(z|z_1) P^*_{t_2 - t_1}(z_1|z_2) \dots P^*_{t_n - t_{n-1}}(z_{n-1}|z_n) \\
&\qquad \times F_1(z_1) F_2(z_2) \dots F_n(z_n) dz_0 dz_1 \dots dz_n \\
&= \int \big[V_*^{t_1} F_1 V_*^{t_2 - t_1} F_2 \dots F_2 V_*^{t_n - t_{n-1}} F_n\big](z) f_0(z_0)\, dz_0 \,.
\end{aligned}$$

Let

$$\begin{aligned}
G_\epsilon^n &= V_\epsilon^{t_1} F_1 V_\epsilon^{t_2 - t_1} F_2 \dots F_{n-1} V_\epsilon^{t_n - t_{n-1}} F_n \\
G^n &= V^{t_1} F_1 V^{t_2 - t_1} F_2 \dots F_{n-1} V^{t_n - t_{n-1}} F_n
\end{aligned}$$

and recall from Section 2 that

1. $\|V_*^t F\|_{L_1} \leq \|F\|_{L_1}$ (both semi-groups are contractive) ,
2. $\|(V_\epsilon^t - V^t)F\| \leq Ct\, o(\tau)$,
3. $G_*^n = V_\epsilon^{t_1} F_1 G_*^{n-1}$,$G_*^{n-1} = V_\epsilon^{t_2-t_1} F_2 G_*^{n-2}$,$\ldots$.

Thus we obtain the bound

$$\begin{aligned}
&\Big| \int \mu_\epsilon(dz)F(z) - \int \mu(dz)F(z) \Big| \leq \\
&\quad \leq (\sup_i \|F_i\|_\infty)\|G_\epsilon^{n-1} - G^{n-1}\|_{L_1} + C_1\, t_1\, o(\tau) \leq \ldots \\
&\quad \leq \tilde{C} t\, o(\tau)\,.
\end{aligned}$$

The convergence on the set of general cylindrical functions is then follows by a density argument.

It remains to check that (57) holds. This can be done exactly like in [CPR]:

$$\begin{aligned}
d(z_\epsilon(t_1), z_\epsilon(t_2))d(z_\epsilon(t_2), z_\epsilon(t_3)) &\leq \|z_\epsilon(t_1) - z_\epsilon(t_2)\|_{\mathbb{R}\times S^1}\|z_\epsilon(t_2) - z_\epsilon(t_3)\|_{\mathbb{R}\times S^1} \\
&\leq \|x_\epsilon(t_1) - x_\epsilon(t_2)\|_{\mathbb{R}^2}\|x_\epsilon(t_2) - x_\epsilon(t_3)\|_{\mathbb{R}^2} \\
&\quad + \|x_\epsilon(t_1) - x_\epsilon(t_2)\|_{\mathbb{R}^2}\|v_\epsilon(t_2) - v_\epsilon(t_3)\|_{S^1} \\
&\quad + \|v_\epsilon(t_1) - v_\epsilon(t_2)\|_{S^1}\|x_\epsilon(t_2) - x_\epsilon(t_3)\|_{\mathbb{R}^2} \\
&\quad + \|v_\epsilon(t_1) - v_\epsilon(t_2)\|_{S^1}\|v_\epsilon(t_2) - v_\epsilon(t_3)\|_{S^1}.
\end{aligned}$$

If there is at least one jumping time $\bar{t} \in (t_1, t_2)$, then $\|v_\epsilon(t_1) - v_\epsilon(t_2)\|_{S^1} = O(1)$, and if there are at least two jumping times $\bar{t}_1 \in (t_1, t_2)$ and $\bar{t}_2 \in (t_2, t_3)$, then $\|v_\epsilon(t_1) - v_\epsilon(t_2)\|_{S^1}\|v_\epsilon(t_2) - v_\epsilon(t_3)\|_{S^1} = O(1)$. We denote by $\chi_1(t_1, t_2)$ and $\chi_2(t_1, t_2, t_3)$ the characteristic functions of the sets

$$\begin{aligned}
A_1(t_1, t_2) &= \{z \in D_{[0,T]}(\mathbb{R}^2 \times S^1) : \exists t_s \in (t_1, t_2) \;\; s.t. \;\; v(t_s^-) \neq v(t_s^+))\} \\
A_2(t_1, t_2, t_3) &= \{z \in D_{[0,T]}(\mathbb{R}^2 \times S^1) : \exists t_{s_1} \in (t_1, t_2), t_{s_2} \in (t_2, t_3) \\
&\qquad\qquad\qquad\qquad \text{s.t.} \;\; v(t_{s_i}^-) \neq v(t_{s_i}^+))\}\,.
\end{aligned}$$

Because

$$\begin{aligned}
\|x_\epsilon(t_i) - x_\epsilon(t_{i+1})\|_{\mathbb{R}^2} &\leq |t_i - t_{i+1}| \\
\|v_\epsilon(t_i) - v_\epsilon(t_{i+1})\|_{S^1} &\leq 2 \\
\chi_2(t_1, t_2, t_3) &= \chi_1(t_1, t_2)\chi_1(t_2, t_3),
\end{aligned}$$

we have

$$\begin{aligned}
&\mathbb{E}[d(z_\epsilon(t_1), z_\epsilon(t_2))d(z_\epsilon(t_2), z_\epsilon(t_3))] \leq |t_2 - t_1||t_3 - t_2| + 4\mathbb{E}(\chi_2(t_1, t_2, t_3)) \\
&\quad +2[|t_2 - t_1|\mathbb{E}(\chi_1(t_2, t_3)) + |t_3 - t_2|\mathbb{E}(\chi_1(t_1, t_2))] \\
&\quad \leq |t_3 - t_1|^2 + 4C_1|t_3 - t_2||t_2 - t_1| + 4C_2|t_3 - t_2||t_2 - t_1| \leq C_3|t_3 - t_1|^2\,,
\end{aligned}$$

which is nothing but the estimate (57). This concludes the proof of Proposition 7. $\square$

Next we wish to prove that $\tilde{\mu}_\epsilon$ is close to μ_ϵ. This is done in [CPR] by defining a "bad" subset of $D_{[0,T]}(\mathbb{R}^2 \times S^1)$, which is small for μ_ϵ, because it is small for the measure μ, and then the result follows by proving the statement on the complement of this bad set.

Because of a technical difficulty in defining the bad subset, we here take a somewhat different path.

First we note that the two measures $\tilde{\mu}_\epsilon$ and μ_ϵ are concentrated on subsets of $D_{[0,T]}(\mathbb{R}^2 \times S^1)$ which consist of trajectories that have constant velocity, or change velocities at a finite set of points, namely the points where the trajectory meets an obstacle site; moreover, the two measures differ only on subsets where the trajectory meets with the same obstacle site more than once. When $\delta = 1$, i.e. the case considered in CPR, this happens with positive probability for both measures; though not a proof, an explanation is that there is a positive probability that a trajectory crosses itself, and fraction of the area occupied by obstacle sites is $\pi/4$ independently of ϵ; this is not a real obstruction for obtaining the desired result, as we shall see, but we begin by proving that for $0 < \delta < 1$, the probability that a trajectory loops back to the same obstacle site converges to zero with ϵ.

Consider thus a trajectory that somewhere along its path makes a loop, i.e. one that meets the same obstacle site a second time. It might have several loops, but here we always consider a fixed one. Such a trajectory can be indexed by a sequence $\xi_j \in \mathbb{Z}^2$, $0 \to \xi_1 \to \xi_2 \cdots \to \xi_n \to 0$, where the 0 in the beginning and the end indicates the starting point, and where the ξ_j denote the relative integer coordinates of the obstacle sites where the trajectory changes direction. We can assume that the absolute coordinates of the obstacles are distinct, i.e. that the loop is a "simple loop", but of course the ξ_j need not be distinct.

Let $\boldsymbol{\xi} = (\xi_1, ...\xi_n) \in (\mathbb{Z}^2)^n$ denote this sequence. Note that the real length of such a loop is approximately $\epsilon^{1/(2-\delta)} \left(| \sum_{j=1}^n \xi_j| + \sum_{j=1}^n |\xi_j| \right)$, and that this length must be less than T.

Let A_0 denote the obstacle site where the loop starts; the trajectory could have traversed the site, or it could have been reflected on ∂A_0, the boundary of A_0. In either case, the trajectory meets ∂A_0 in a unique point (x_0, v_0) that satisfies $v \cdot \omega > 0$, where ω is the outward normal to A_0. In this setting, ∂A_0 is part of the boundary of the billiard table, $\partial \Lambda_{\mathbf{c}}$, as defined in Section 2. The *billiard map* is a transformation of $\partial \Lambda_{\mathbf{c}} \times S^1_+$ to itself, defined by $(x_0, v_0) \mapsto (x_1, v_1)$, where $x_1 \in \partial \Lambda_{\mathbf{c}}$ is the next point where the trajectory hits the boundary, and where v_1) is the reflected velocity.

Let now ds denote the length measure on the $\partial \Lambda_{\mathbf{c}}$; in the present case all of the boundary consist of circular arcs with the same radius, r, and this measure can be written $r d\omega$, where ω can be identified with the outgoing normal at the point x. Let θ be the angle between v and ω. Then the *billiard measure* is defined as $\cos(\theta) r \, d\omega dv$. Below it is more convenient to parameterize $\partial A_0 \times S^1_+$ by v and ζ where ζ is the distance between the center of A_0 and the line containing the trajectory defined by (x_0, v_0). With this parameterization, the billiard measure becomes $d\zeta dv$. Now it is a fact that this measure is preserved under the billiard map (see eg. [BS1, P] for classical and more recent results concerning billiards and their asymptotic behavior).

Consider a fixed trajectory that starts at (x_0, ω_0), and then returns to A_0 after being reflected on a sequence of other obstacles A_j; consider also the corresponding sequence $\boldsymbol{\xi} = (\xi_1, ..., \xi_n)$. We define the set $\Omega_{\boldsymbol{\xi}} \subset \partial A_0 \times S^1_+$ as the set of all trajectories going out from ∂A_0 that can return to A_0 via the sequence $\boldsymbol{\xi}$. Note that this is a well defined set, that does not depend on whether a trajectory is realized or not.

Lemma 1 *There are constants C_0 and C_1 such that*

$$d\zeta dv\text{-}meas(\Omega_{\boldsymbol{\xi}}) \quad \leq \quad C_o C_1^n r^{n+2} \frac{1}{\epsilon^{1/(2-\delta)} |\sum_{j=1}^n \xi_j|} \prod_{j=1}^n \frac{1}{|\epsilon^{1/(2-\delta)} \xi_j|} \,. \qquad (58)$$

Proof: In Figure 6, we denote A_j the j-th obstacle along the path. This is always an obstacle where the trajectory changes direction. The calculation is not carried out in full detail, although it is easy to see how to make each step completely rigorous.

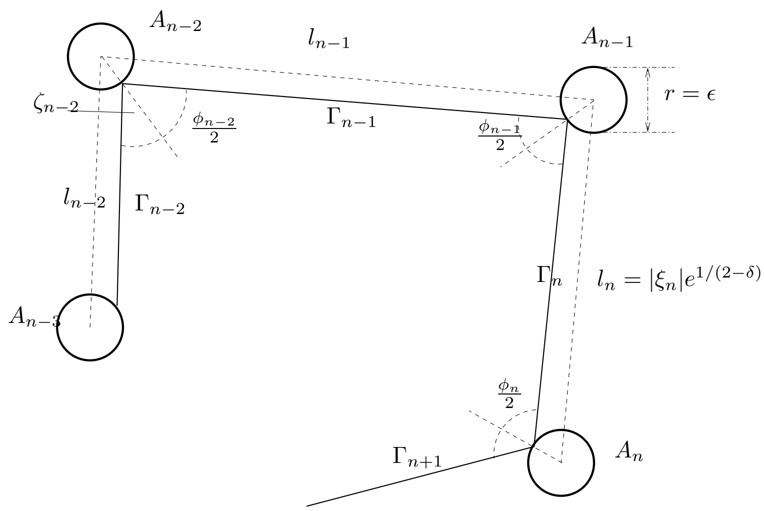

Figure 6: Part of a looping trajectory

The notation in the figure should be clear, except perhaps for ζ_{n-2}, et.c.; in general ζ_k denotes the distance between the line segment Γ_n and the center of A_n. This means that for $k = 1, ..., n$,

$$\sin(\frac{\phi_k}{2}) \quad = \quad \frac{\zeta_k}{r} \,,$$

and that if ϕ_k belongs to an interval $\Delta\phi_k$ of size $|\Delta\phi_k|$, then ζ_k belongs to an interval that satisfies $|\zeta_k| \leq \frac{r}{2}|\Delta\phi_k|$. We set $l_k = |\epsilon^{1/(2-\delta)}\xi_k$

From the figure we note that the length of the k-th lap is $|\Gamma_k| = e^{1/(2-\delta)}|\xi_k| \pm \epsilon$. Moreover, Γ_k is almost parallel to ξ_k; more precisely, if β_k denotes the angle between these two lines, then $\beta_k = o(r/l_k)$, as $\epsilon \to 0$.

Now, if the trajectory Γ_{n+1} is to join obstacle A_0, then ϕ_n must belong to a set $\Delta\phi_n$ which satisfies

$$|\Delta\phi_n| \leq \frac{r}{|\Gamma_{n+1}|}(1+o),$$

where o denotes a rest term which is small compared to the first term, and vanishing when ϵ goes to zero. But then ζ_n must belong to a set $\Delta\zeta_n$ such that

$$|\Delta\zeta_n| \leq \frac{r}{2}|\cos(\frac{\phi_n}{2})|\ |\Delta\phi_n| \leq \frac{r}{2}\frac{r}{|\Gamma_{n+1}|}(1+o)$$

Continuing backwards, this requires that ϕ_{n-1} belongs to a set $\Delta\phi_{n-1}$ that satisfies

$$|\Delta\phi_{n-1}| \leq \frac{|\Delta\zeta_n|}{|l_n|}(1+o)\,,$$

where the rest term o results from the fact that Γ_{n-1} and ξ_{n-1} are not exactly parallel, and do not have exactly the same length; the o goes to zero as ϵ goes to zero, uniformly in $|\xi|$. This gives

$$|\Delta\zeta_{n-1}| \leq r\frac{r}{2l_n}\frac{r}{2|\Gamma_{n+1}|}(1+o)^2\,,$$

and inductively,

$$|\Delta\zeta_{n-k}| \leq r\frac{r}{2|\Gamma_{n+1}|}(1+o)^{k+1}\prod_{j=n-k+1}^{n}\frac{r}{2l_j}\,.$$

In summary

$$\begin{aligned} dvd\zeta\text{-meas}(\Omega_{\boldsymbol{\xi}}) &\leq \frac{r}{|l_1|}r\frac{r}{2|\Gamma_{n+1}|}(1+o)^n\prod_{j=2}^{n}\frac{r}{2l_{n-j}} \\ &= \left(\frac{1+o}{2}\right)^n r^{n+1}\frac{r}{2|\Gamma_{n+1}|}\prod_{j=1}^{n}\frac{1}{l_j}. \end{aligned}$$

This is exactly our claim, once one has set $l_j = \epsilon^{1/(2-\delta)}|\xi_j|$, and similarly with Γ_{n+1}. □

Next we prove that $\tilde{\mu}_\epsilon$ converges weakly to μ.

Proposition 8 *For each continuous function* $F : D_{[0,T]}(\mathbb{R}^2 \times S^1)$,

$$\lim_{\epsilon\to 0}\int F(z)\tilde{\mu}_\epsilon(dz) \to \int F(z)\mu(dz)\,. \tag{59}$$

Proof: Fix $\varepsilon_0 > 0$ arbitrarily. Using Proposition 7,

$$\begin{aligned} \int F(z)\tilde{\mu}_\epsilon(dz) &= \int F(z)\mu(dz) \qquad (60)\\ &+ \left(\int F(z)\mu_\epsilon(dz) - \int F(z)\mu(dz)\right) + \left(\int F(z)\tilde{\mu}_\epsilon(dz) - \int F(z)\mu_\epsilon(dz)\right) \end{aligned}$$

where

$$\left|\int F(z)\mu_\epsilon(dz) - \int F(z)\mu(dz)\right| \;\; \leq \;\; \varepsilon_0/2\,,$$

provided that ϵ is sufficiently small. Also,

$$\begin{aligned}
&\left|\int F(z)\tilde{\mu}_\epsilon(dz) - \int F(z)\mu_\epsilon(dz)\right| \;\leq \\
&\quad\leq \;\; \left|\int_{K_\epsilon} F(z)\tilde{\mu}_\epsilon(dz) - \int_{K_\epsilon} F(z)\mu_\epsilon(dz)\right| + \left|\int_{K_\epsilon^c} F(z)\tilde{\mu}_\epsilon(dz) - \int_{K_\epsilon^c} F(z)\mu_\epsilon(dz)\right|
\end{aligned}$$

where $K_\epsilon \subset D_{[0,T]}(\mathbb{R}^2 \times S^1)$ is the set of trajectories that contains at least one loop, as defined above. On the complementary set, K_ϵ^c, the measures $\tilde{\mu}_\epsilon$ and μ_ϵ are identical, so the last term vanishes, and

$$\left|\int_{K_\epsilon} F(z)\tilde{\mu}_\epsilon(dz) - \int_{K_\epsilon} F(z)\mu_\epsilon(dz)\right| \leq \sup|F| \, \left(\tilde{\mu}_\epsilon(K_\epsilon) + \mu_\epsilon(K_\epsilon)\right). \tag{61}$$

Each trajectory in the set K_ϵ contains at least one simple loop. Let A_0 denote the obstacle site where the loop starts, and let $\boldsymbol{\xi}$ be the index sequence for the loop. Then let $\Omega_{\boldsymbol{\xi}}$ be the set of all (v,ζ) giving trajectories that have the same index sequence. The probability that a given loop is realized *given that the trajectory starts in* $\Omega_{\boldsymbol{\xi}}$ is $p^n(1-p)^{\sum_{j=1}^{n+1} s_j} \leq p^n$, where s_j is the number of obstacles sites that the trajectory crosses along the path between the $j-1$:th and the j:th reflection, and where n is the length of the sequence $\boldsymbol{\xi}$. Hence

$$\begin{aligned}
P_{\boldsymbol{\xi},n} \;\; &\equiv \;\; \text{Pr(there is a loop of type } \boldsymbol{\xi} \text{ along a randomly chosen trajectory)} \\
&\leq \;\; p^n \,\text{Pr(there is a } t \in [0,T] \text{ such that } T^t(x,v) \in \Omega_{\boldsymbol{\xi}}) \\
&\leq \;\; p^n \frac{T\; dv\, d\zeta\text{-meas}(\Omega_{\boldsymbol{\xi}})}{2\pi\epsilon^{2/(2-\delta)}}\,,
\end{aligned} \tag{62}$$

i.e. p^n times the probability that a trajectory starting at the random initial position (x,v) at some time t has evolved to A_0, the first obstacle of the loop, and leaves A_0 in the set $\Omega_{\boldsymbol{\xi}}$. The last expression can be derived as follows. Because of the periodicity, we consider a random choice of a starting point (x,v), where x is chosen i a lattice cell with area $\epsilon^{2/(2-\delta)}$; hence the denominator. Consider $(v',\zeta') \in \Omega_{\boldsymbol{\xi}}$, and the corresponding point $(x',v') \in \mathbb{R}^2 \times S^1$. We consider the history of an infinitesimal set $\Delta v \times \Delta\zeta$ around (v',ζ'); there are two possible histories, also if we assume that there are no other encounters with an obstacle before the one at the starting point (x',v'): either the trajectory continues backwards in the direction $-v'$, or it continues backward in the direction $-v''$, where the latter corresponds to a reflection (see the figure). The probability that a trajectory reaches the set $\Delta v \times \Delta\zeta$ within a time interval Δt is

$$\begin{aligned}
&p\; dxdv\text{-meas}(\{(x'-tv'',v'')\,|\zeta \in \Delta\zeta, v' \in \Delta v, t \in [0,\Delta t]\}) \\
&\qquad +(1-p)\; dxdv\text{-meas}(\{(x'-tv',v')\,|\zeta \in \Delta\zeta, v' \in \Delta v, t \in [0,\Delta t]\}) \\
&\quad = \quad dxdv\text{-meas}(\{(x'-tv',v')\;,\; \zeta \in \Delta\zeta\;,\; v' \in \Delta v\;,\; t \in [0,\Delta t]\})\;;
\end{aligned}$$

this is because the reflection leaves the measure $dvd\zeta$ invariant. And the same then holds for all possible histories, which shows the claim in the inequality (62). Obviously the full history should be mapped into one lattice cell, hence the normalization with $2\pi\epsilon^{2/(2-\delta)}$.

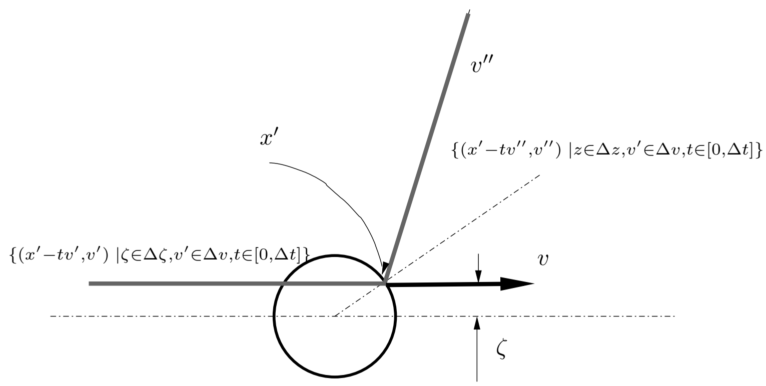

Figure 7: Different histories leading to starting points in $\Omega_{\boldsymbol{\xi}}$

But then each of the measures in the right hand side of (61) can be estimated by summing over all n and over all $\boldsymbol{\xi}$ with length n:

$$\tilde{\mu}_\epsilon(K_\epsilon) \quad \leq \quad \frac{1}{2\pi\epsilon^{2/(2-\delta)}} \sum_{n=1}^{\infty} \sum_{\boldsymbol{\xi}\in\Xi_n} TC_oC_1^n p^n r^{n+2} \frac{1}{\epsilon^{1/(2-\delta)}|\sum_{j=1}^n \xi_j|} \prod_{j=1}^n \frac{1}{|\epsilon^{1/(2-\delta)}\xi_j|} \tag{63}$$

where the set Ξ_n is defined by

$$\Xi_n = \left\{ (\xi_1, ..., \xi_n) \bigm| |\textstyle\sum_{j=1}^n \xi_j| + \displaystyle\sum_{j=1}^n |\xi_j| \leq \epsilon^{-1/(2-\delta)}\, T \ ; \ \xi_j \in \mathbb{Z}^2 \setminus \{0\} \right\} .$$

Because we consider here only loops that are simple, i.e. all the velocity jumps take place at distinct obstacles, exactly the same estimate holds for $\tilde{\mu}_\epsilon(K_\epsilon)$ and $\mu_\epsilon(K_\epsilon)$.

We can approximate the sum over Ξ_n by an integral:

$$\sum_{\Xi_n} \frac{1}{\epsilon^{1/(2-\delta)}} \prod_{j=1}^n \frac{1}{|\epsilon^{1/(2-\delta)}\xi_j|} \leq$$

$$\leq \quad C^{n+1} \int_{\mathbf{x}\in\mathbf{X}_{\epsilon,T}} \frac{1}{\epsilon^{1/(2-\delta)}|\sum_{j=1}^n x_j|} \prod_{j=1}^n \frac{1}{\epsilon^{1/(2-\delta)}|x_j|}\ dx_1\cdots dx_n\,,$$

where $\mathbf{X}_{\epsilon,T} = \Big\{ \mathbf{x} = (x_1, ..., x_n) \in (\mathbb{R}^2)^n\ ,\ x_j \in \mathbb{R}^2\ ,\ |x_j| \geq 1\ ,\ \sum_{j=1}^n |x_j| \leq \epsilon^{-1/(2-\delta)}\, T \Big\}$. With a change of variables, $y_j = \epsilon^{1/(2-\delta)} x_j$, one gets $dx_1\cdots dx_n = \epsilon^{-2n/(2-\delta)} dy_1\cdots dy_n$, and then

$$\sum_{\Xi_n} \frac{1}{\epsilon^{1/(2-\delta)}} \prod_{j=1}^{n} \frac{1}{|\epsilon^{1/(2-\delta)}\xi_j|} \leq$$
$$\epsilon^{-2n/(2-\delta)} C^{n+1} \int_{y\in\mathbf{Y}} \frac{1}{|\sum_{j=1}^n y_j|} \prod_{j=1}^{n} \frac{1}{|y_j|}\, dy_1 \cdots dy_n\,, \qquad (64)$$

for some set $\mathbf{Y} \subset (\mathbb{R}^2)^n$, which particular satisfies $\epsilon^{1/(2-\delta)} < |\sum_{j=1}^n y_j| < T$, $\epsilon^{1/(2-\delta)} < |y_j| < T$, and $\sum_{j=1}^n |y_j| < T$. We have

$$\frac{1}{|\sum_{j=1}^n y_j|} \prod_{j=1}^{n} \frac{1}{|y_j|} \quad\leq\quad \frac{1}{2}\left(\frac{1}{|\sum_{j=1}^n y_j|^2} + \frac{1}{|y_1|^2}\right) \prod_{j=2}^{n} \frac{1}{|y_j|}\,,$$

(note that this expression also holds for $n=1$), and then the integral in (64) is smaller than

$$\int_{\epsilon^{1/(2-\delta)}<|y_1|<2T} \frac{1}{|y_1|^2}\, dy_1 \quad \int_{\sum_{j=2}^n |y_j|<T} \prod_{j=2}^{n} \frac{1}{|y_j|}\, dy_2 \cdots dy_n\,,$$

which, expressed in polar coordinates for each $y_j \in \mathbb{R}^2$ is

$$\begin{aligned}(2\pi)^n &\int_{\epsilon^{1/(2-\delta)}<r_1<2T} \frac{1}{r_1}\, dr \quad \int_{(\sum_{j=2}^n r_j)<T} dr_2 \cdots dr_n\,,\\ &\leq \quad (2\pi)^n \left(\log(2T) - \log(\epsilon^{1/(2-\delta)}\right) T^{n-1}/(n-1)!\end{aligned}$$

Now we put this back into (63):

$$\begin{aligned}\tilde{\mu}_\epsilon(K_\epsilon) \quad&\leq\quad \frac{1}{2\pi\epsilon^{2/(2-\delta)}} \left(\log(2T) - \log(\epsilon^{1/(2-\delta)}\right)\\ &\qquad\qquad \times \sum_{n=1}^{\infty} T C_o C_1^n p^n r^{n+2} \epsilon^{-2n/(2-\delta)} T^{n-1}/(n-1)!\\ &=\quad C_0 T \left(\log(2T) - \log(\epsilon^{1/(2-\delta)}\right) p\, r^3 \epsilon^{-4/(2-\delta)}\\ &\qquad\qquad \times \sum_{n=1}^{\infty} \left(C_1 T p\, r\, \epsilon^{-2/(2-\delta)}\right)^{n-1} /(n-1)!\\ &=\quad C_0 T \left(\log(2T) - \log(\epsilon^{1/(2-\delta)}\right) \epsilon^{2(1-\delta)/(2-\delta)} e^{C_1 T} \qquad (65)\end{aligned}$$

Here, in the last line, C_0 and C_1 are new constants independent of T and ϵ, and the last expression follows by setting $r=\epsilon$, and $p=\epsilon^{\delta/(2-\delta)}$; then $p\, r\, \epsilon^{-2/(2-\delta)} = 1$. Clearly, when $0<\delta<1$, $\tilde{\mu}_\epsilon(K_\epsilon) \to 0$ when $\epsilon \to 0$, and because this calculation holds in the same way for μ_ϵ, we can choose ϵ so small that $\sup|F|\big(\tilde{\mu}_\epsilon(K_\epsilon)\big) + \big(\mu_\epsilon(K_\epsilon)\big) \leq \varepsilon_0/2$ so that finally the last two terms in (60) together are smaller than ε_0, and this concludes the proof of the proposition, because ε_0 was arbitrary. □

This proposition also concludes the proof of Theorem 1. Note that the proof of Proposition 8 really says something more than what is needed: that if trajectories cross itself, this will very rarely happen inside an obstacle site, and hence the trajectories don't get a chance to test the difference between the two measures $\tilde{\mu}_\epsilon$ and μ_ϵ. The trajectories actually only *do* test this difference, if there is a real collision the starting point A_0 of a loop. This gives another factor ϵ, and so the measure of this restricted set converges to zero, also when $\delta = 1$ as in [CPR], and so this calculation would give a proof also in that case. However, one would then need to make a more careful calculation when proving Lemma 1.

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