COLOURED PERMUTATIONS CONTAINING AND AVOIDING CERTAIN PATTERNS

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Abstract

Following [M2], let $S_n^{(r)}$ be the set of all coloured permutations on the symbols $1,2,\dots,n$ with colours $1,2,\dots,r$, which is the analogous of the symmetric group when $r=1$, and the hyperoctahedral group when $r=2$. Let $I\subseteq\{1,2,\dots,r\}$ be subset of d colours; we define $T_{k,r}^m(I)$ be the set of all coloured permutations $\phi\in S_k^{(r)}$ such that $\phi_1=m^{(c)}$ where $c\in I$. We prove that, the number $T_{k,r}^m(I)$-avoiding coloured permutations in $S_n^{(r)}$ equals $(k-1)!r^{k-1}\prod_{j=k}^n h_j$ for $n\geq k$ where $h_j=(r-d)j+(k-1)d$. We then prove that for any $\phi\in T_{k,r}^1(I)$ (or any $\phi\in T_{k,r}^k(I)$), the number of coloured permutations in $S_n^{(r)}$ which avoid all patterns in $T_{k,r}^1(I)$ (or in $T_{k,r}^k(I)$) except for ϕ and contain ϕ exactly once equals $\prod_{j=k}^n h_j\cdot\sum_{j=k}^n \frac{1}{h_j}$ for $n\geq k$. Finally, for any $\phi\in T_{k,r}^m(I)$, $2\leq m\leq k-1$, this number equals $\prod_{j=k+1}^n h_j$ for $n\geq k+1$. These results generalize recent results due to Mansour [M1], Mansour and West [MW], and due to Simion [S].

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1. Introduction

The main goal of this note is to give analogies of enumerative results on certain classes of permutations characterized by pattern-avoidance in the symmetric group (see [M1]), and in the hyperoctahedral group (see [S, MW]). In $S_n^{(r)}$ (see [M2]), the natural analogue of the symmetric group and of the hyperoctahedral group, we identify classes of restricted coloured permutations with enumerative properties analogous to results in the symmetric group and hyperoctahedral group. In the remainder of this section we present a brief account of earlier work which motivated out investigation, summarize the main results, and present the basic definitions used throughout the note.

Pattern avoidance in the symmetric group proved to be a useful language in a variety of seemingly unrelated problems, from stack sorting [K, T, W] to the theory of Kazhdan-Lusztig polynomials [Br], singularities of Schubert varieties [LS, Bi], Chebyshev polynomials [CW, MV1, Kr, MV2, MV3], and rook polynomials [MV4]. Signed pattern avoidance in the hyperoctahedral group proved to be a useful language in combinatorial statistics defined in type-B noncrossing partitions, enumerative combinatorics [S, BS], algebraic combinatorics [FK, BK, Be, Mo, R].

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Let $\pi \in S_n$ and $\tau \in S_k$ be two permutations. An *occurrence* of τ in π is a subsequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $(\pi_{i_1}, \ldots, \pi_{i_k})$ is order-isomorphic to τ; in such a context τ is usually called a *pattern*. We say that π ***avoids*** τ, or is τ-***avoiding***, if there is no occurrence of τ in π. The set of all τ-avoiding permutations in S_n is denoted by $S_n(\tau)$. For an arbitrary finite collection of patterns T, we say that π avoids T if π avoids any $\tau \in T$; the corresponding subset of S_n is denoted by $S_n(T)$. The first case examined was the case of permutations avoiding one pattern of length 3. Knuth [K] found that $|S_n(\tau)| = C_n$ for all $\tau \in S_3$, where C_n is the nth Catalan number. Later, Simion and Schmidt [SS] found the cardinalities of $|S_n(T)|$ for all $T \subset S_3$.

The hyperoctahedral group B_n is an analog of the symmetric group S_n. Let us view the elements of B_n as signed permutation $b = b_1 b_2 \ldots b_n$ in which each of the symbols $1, 2, \ldots, n$ appears once, possibly barred. Thus, the cardinality of B_n is $n!2^n$. Simion [S] was looking for the analogs of Knuth's results for B_n; she discovered that for every 2-letter signed pattern τ; the number of τ-avoiding signed permutations in B_n is $\sum_{j=0}^{n} \binom{n}{j}^2 j!$. Besides, Simion [S] found the number of all signed permutations in B_n avoiding double 2-letter signed patterns in B_2. Recently, Mansour and West [MW] studied the number of signed permutations in B_n avoiding a set of 2-letter signed patterns in B_2. This invites us to define a further generalizations for avoiding a pattern in S_n and avoiding a signed pattern in B_n.

Following [M2] (see also [St]), the group $S_n^{(r)} = S_n \wr C_r$ where C_r is the cyclic group of order r, is an analog of the symmetric group (S_n) and of the hyperoctahedral group (B_n). We will view the elements of the set $S_n^{(r)}$ as coloured permutations $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ in which each of the symbols $1, 2, \ldots, n$ appears once, coloured by one of the colours $1, 2, \ldots, r$ (more generally, we denote by $S_{\{a_1,\ldots,a_n\}}^{\{s_1,\ldots,s_r\}}$ the set of all permutations of the symbols $a_1, \ldots, a_n$ where each symbol appears once and is coloured by one of the colours $s_1, \ldots, s_r$). Thus, $S_n^{(1)} = S_n$, $S_n^{(2)} = B_n$, and the cardinality of $S_n^{(r)}$ is $n!r^n$. The absolute value notation means $|\phi|$ is the permutation $(|\phi_1|, \ldots, |\phi_n|)$ where $|\phi_j|$ is the symbol which appear in ϕ at the position j. An example $\phi = (1^{(1)}, 3^{(2)}, 2^{(1)})$ is a coloured permutation in $S_3^{(2)}$, and $|\phi| = (1, 3, 2)$.

Let $\phi = (\tau_1^{(s_1)}, \ldots, \tau_k^{(s_k)}) \in S_k^{(r)}$, and $\psi = (\alpha_1^{(v_1)}, \ldots, \alpha_n^{(v_n)}) \in S_n^{(r)}$; we say that ψ ***contains*** ϕ (or is ϕ-containing) if there is a sequence of k indices, $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that the following two conditions hold: (i) $(\alpha_{i_1}, \ldots, \alpha_{i_k})$ is order-isomorphic to $|\phi|$, and (ii) $v_{i_j} = s_j$ for all $j = 1, 2, \ldots, k$. Otherwise, we say that ψ ***avoids*** ϕ (or is ϕ-avoiding). The set of all ϕ-avoiding coloured permutations in $S_n^{(r)}$ is denoted by $S_n^{(r)}(\phi)$, and in this context ϕ is called a *coloured pattern*. For an arbitrary finite collection of coloured patterns T, we say that ψ avoids T if ψ avoids any $\phi \in T$; the corresponding subset of $S_n^{(r)}$ is denoted by $S_n^{(r)}(T)$. As an example, $\psi = (1^{(1)}, 2^{(2)}, 3^{(2)}) \in S_3^{(2)}$ avoids $(1^{(1)}, 2^{(1)})$; that is, $\psi \in S_3^{(2)}((1^{(1)}, 2^{(1)}))$.

In this note, we extend Mansour's (see [M1]) results for avoiding and containing certain patterns in S_n, and Simion's (see [S, Section 3]) results for avoiding signed patterns in B_n (also, see [MW]).

2. Coloured permutations avoiding $T_k^m(I_d)$

Let I_d be any subset of $\{1, 2, \ldots, r\}$ of d elements, and let us define $T_{k,r}^m(I_d)$ be the set of all coloured permutations $\phi \in S_k^{(r)}$ such that $\phi_1 = m^{(c)}$ where $c \in I_d$; that is,

$$T_{k,r}^m(I_d) = \bigcup_{c \in I_d} \{\phi \in S_k^{(r)} | \phi_1 = m^{(c)}\}.$$

Theorem 2.1. *Let $k, r \geq 1$, $n \geq k$, and $k \geq m \geq 1$. Then*

$$|S_n^{(r)}(T_{k,r}^m(I_d))| = (k-1)!r^{k-1}\prod_{j=k}^{n}((k-1)d + (r-d)j).$$

Proof. Let $G_n = S_n^{(r)}(T_{k,r}^m(I_d))$, and define the functions $f_{h,c} : S_n^{(r)} \to S_{n+1}^{(r)}$ by:

$$[f_{h,c}(\phi)]_i = \begin{cases} h^{(c)}, & \text{when } i = 1 \\ \phi_{i-1}, & \text{when } |\phi_{i-1}| < h \\ (|\phi_{i-1}|+1)^{(a_{i-1})}, & \text{when } |\phi_{i-1}| \geq h \end{cases}$$

for every $i = 1, ..., n+1$, $\phi \in S_n^{(r)}$, $1 \leq c \leq r$ and $h = 1, ..., n+1$, where a_i is the colour of the symbol $|\phi_i|$ in ϕ. From this we see that if $\phi \in G_n$ then

$$f_{n+1,c_j}(\phi), f_{n,c_j}(\phi), ..., f_{n+m-k+2,c_j}(\phi), f_{1,c_j}(\phi), ..., f_{m-1,c_j}(\phi) \in G_{n+1}$$

for all $j = 1, 2, \ldots, d$, and $f_{h,c}(\phi) \in G_{n+1}$ for all $c \notin I_d$ and $h = 1, 2, \ldots, n+1$. So we have that $((k-1)d + (r-d)(n+1))|G_n| \leq |G_{n+1}|$ where $n \geq k$.

Assume that $((k-1)d + (r-d)(n+1))|G_n| < |G_{n+1}|$. Then there exists a coloured permutation $\psi \in G_{n+1}$ such that $m \leq |\psi_1| \leq n+m-k+1$ and the symbol $|\psi_1|$ coloured by $c \in I_d$, so there exist $k-1$ positions $1 < i_1 < \cdots < i_{k-1} \leq n+1$ such that the subsequence $\phi_1, \phi_{i_1}, \ldots, \phi_{i_{k-1}}$ contains $\eta \in T_{k,r}^m(I_d)$, which contradicts the definition of G_{n+1}. So we have that $|G_{n+1}| = ((k-1)d + (r-d)(n+1))|G_n|$ for all $n \geq k$. Besides $|G_k| = (rk-d)(k-1)!r^{k-1}$ (from the definitions), hence the theorem holds. □

Example 2.2. (see [M1, Theorem 1]) *Let $T_k^m = T_{k,1}^m(1)$ and $k \geq m \geq 1$. Theorem 2.1 yields for all $n \geq k$ that $|S_n(T_k^m)| = (k-1)!(k-1)^{n-k+1}$.*

Example 2.3. *Theorem 2.1 yields for all $n \geq k \geq m \geq 1$ that $|S_n^{(2)}(T_{k,2}^m(1))| = \frac{(n+k-1)!}{\prod_{i=1}^{k-1}(2i-1)}$. For $k = 2$ see* [S, Equation 47] *and* [MW, Equation 2.2].

Example 2.4. *Theorem 2.1 for $k = 2$, $m = 1$, and $I_1 = \{1\}$ gives $|S_n^{(r)}(T_{2,r}^1(1))| = \prod_{j=0}^{n}(1+j(r-1))$. For $r = 2$ see* [MW, Equation 4.2].

Corollary 2.5. *Let $k, r \geq 1$, and $k \geq b \geq a \geq 1$. For all $n \geq k$,*

$$|S_n^{(r)}(\cup_{m=a}^{b} T_{k,r}^m(I_d))| = (k-1)!r^{k-1}\prod_{j=k}^{n}(d(k+a-b-1) + j(r-d)).$$

Proof. Let $G_n = S_n^{(r)}(T_{k,r}^m(I_d))$. From Theorem 2.1 we get that $\phi \in G_n$ if and only if either

$$f_{1,c_j}(\phi), \ldots, f_{a-1,c_j}(\phi), f_{n+b-(k-2),c_j}(\phi), \ldots, f_{n+1,c_j}(\phi) \in G_{n+1}$$

where $j = 1, 2, \ldots, d$, or $f_{h,c}(\phi) \in G_{n+1}$ where $h = 1, 2, \ldots, n+1$ and $c \notin I_d$. So we have that $|G_{n+1}| = (d(k+a-b-1) + (r-d)(n+1))|G_n|$. Besides $|G_k| = (rk + d(a-b-1))(k-1)!r^{k-1}$, hence the theorem holds. □

Example 2.6. *Corollary 2.5 gives $|S_n^{(2)}(T_{k,2}^1(1) \cup T_{k,2}^2(1))| = \frac{2^{k-1}(k-1)!}{(2k-3)!}(n+k-2)!$ where $n \geq k$. For $k = 2$ see* [MW, Equation 4.5].

This example invite us to generalize. By using Corollary 2.5 we get the following result.

Corollary 2.7. *Let $k, r \geq 1$ and $n \geq k$. Then $|S_n^{(r)}(\cup_{m=1}^{k} T_{k,r}^m(I_d))| = r^{k-1}(r-d)^{n+1-k}n!$.*

3. Avoiding $T^1_{k,r}(I_d)\backslash\{\phi\}$ and containing ϕ exactly once

Let $M^m_{k,r}(\phi, I_d) = T^m_{k,r}(I_d)\backslash\{\phi\}$, for $\phi \in T^m_{k,r}(I_d)$. We denote by $S^{(r)}_n(T^m_{k,r}(I_d);\tau)$ the set of all permutations in $S^{(r)}_n$ that avoid $M^m_{k,r}(\phi, I_d)$ and contain ϕ exactly once.

Theorem 3.1. *Let $k, r \geq 1$, and $n \geq k$. Then*

$$|S^{(r)}_n(T^1_{k,r}(I_d);\phi)| = \prod_{j=k}^{n}((k-1)d + (r-d)j)\sum_{j=k}^{n}\frac{1}{(k-1)d+(r-d)j},$$

for all $\phi \in T^1_{k,r}(I_d)$.

Proof. Let $\psi \in S^{(r)}_n(T^1_{k,r}(I_d);\phi)$, and let us consider the possible values of ψ_1:

(1) $|\psi_1|$ is coloured by colour $c \notin I_d$. Evidently $\psi \in S^{(r)}_n(T^1_{k,r}(I_d);\phi)$ if and only if $(\psi_2,\ldots,\psi_n)$ avoids $M^m_{k,r}(\phi, I_d)$ and contains ϕ exactly once.

(2) $|\psi_1| \geq n-k+2$ is coloured by $c \in I_d$. Evidently $\psi \in S^{(r)}_n(T^1_{k,r}(I_d);\phi)$ if and only if $(\psi_2,\ldots,\psi_n)$ avoids $M^m_{k,r}(\phi, I_d)$ and contains ϕ exactly once.

(3) $|\psi_1| \leq n-k$ is coloured by $c \in I_d$. Then there exist $1 < i_1 < \cdots < i_k \leq n$ such that $(\psi_1, \psi_{i_1}, \ldots, \psi_{i_k})$ is a coloured permutation of the symbols $n, \ldots, n-k+1, \psi_1$ coloured by any colours such that ψ_1 coloured by $c \in I_d$. For any choice of $k-1$ positions out of $i_1, \ldots, i_k$, the corresponding coloured permutations proceeded by $|\psi_1|$, is containing some coloured pattern in $T^1_{k,r}(I_d)$. Since ψ avoids $M^m_{k,r}(\phi, I_d)$, it is, in fact, order-isomorphic to ϕ. We thus get at least k occurrences of ϕ in ψ, a contradiction.

(4) $|\psi_1| = n-k+1$. Then there exist $1 < i_1 < \cdots < i_{k-1} \leq n$ such that $\eta = (\psi_1, \psi_{i_1}, \ldots, \psi_{i_{k-1}})$ is a coloured permutation of the symbols $n, \ldots, n-k+1$. As above, we get that η is order-isomorphic to ϕ. Let A_n be the set of all coloured permutations $\beta \in S^{(r)}_n(T^1_{k,r}(I_d);\phi)$ such that the symbol $|\beta_1|$ is $n-k+1$ and coloured by $c \in I_d$. Define the functions $f_{h,c} : A_n \to S^{(r)}_{n+1}$ by:

$$[f_{h,c}(\beta)]_i = \begin{cases} 1^{(c)}, & \text{when } i = h \\ (|\beta_i|+1)^{(a_i)}, & \text{when } i < h \\ (|\beta_{i-1}|+1)^{(a_i)}, & \text{when } i > h, \end{cases}$$

for every $i = 1, ..., n+1$, $\beta \in A_n$, $h = 1, ..., n+1$ and $c = 1, \ldots, r$, where the symbol $|\beta_i|$ coloured by the colour a_i. It is easy to see that for all $\beta \in A_n$,

$$f_{n+1,c_j}(\beta), \ldots, f_{n-k+3,c_j}(\beta) \in A_{n+1},$$

and $f_{h,c}(\beta) \in A_{n+1}$ for $c \notin I_d$ and $h = 2, \ldots, n$ (for $h = 1$ we added in the first case), hence $((k-1)d + (r-d)n)|A_n| \leq |A_{n+1}|$. Now we define another function $g : A_{n+1} \to S^{(r)}_n$ by:

$$[g(\beta)]_i = \begin{cases} (|\beta_i|-1)^{(a_i)}, & \text{when } i < h \\ (|\beta_{i+1}|-1)^{(a_i)}, & \text{when } i+1 > h, \end{cases}$$

where $|\beta_h| = 1$, $i = 1, ..., n$, $\beta \in A_{n+1}$.

Observe that $h \geq n-k+3$ such that $|\beta_h|$ coloured by $c \in I_d$, since otherwise already $(\beta_h, \beta_{h+1}, \ldots, \beta_{n+1})$ contains a pattern from $T^1_k(I_d)$, a contradiction. Therefore $g(\beta) \in A_n$ for all $\beta \in A_{n+1}$, so $|A_{n+1}| \leq ((k-1)d + (r-d)n)|A_n|$. Hence, using the fact that $|A_k| = 1$ we get that $|A_{n+1}| = (d(k-1) + n(r-d))|A_n| = \prod_{j=k}^{n}((k-1)d + (r-d)j)$.

Since the above cases 1-4 are disjoint we obtain

$$|S_n^{(r)}(T_{k,r}^1(I_d);\phi)| = ((k-1)d+(r-d)n)|S_{n-1}^{(r)}(T_{k,r}^1(I_d);\phi)| + \prod_{j=k}^{n-1}((k-1)d+(r-d)j).$$

Besides $|S_k^{(r)}(T_{k,r}^1(I_d);\phi)| = 1$, hence the theorem holds. □

Example 3.2. *If $\phi \in T_{k,2}^1(1)$ then Theorem 3.1 for $n \geq k$ gives*

$$|S_n^{(2)}(T_{k,2}^1(1);\phi)| = \frac{(n+k-1)!}{(2k-2)!}\sum_{j=k}^{n}\frac{1}{k-1+j}.$$

Using the natural bijection between the set $S_n^{(r)}(T_{k,r}^1(I_d);\phi)$ and the set $S_n^{(r)}(T_{k,r}^k(I_d);\phi')$ for all $\phi \in T_{k,r}^1(I_d)$, where $\phi'_i = (k+1-|\phi_i|)^{(a_i)}$ and a_i is a colour of the symbol $|\phi_i|$ in ϕ, together with Theorem 3.1 we get as follows.

Corollary 3.3. *Let $k, r \geq 1$ and $n \geq k$. Then*

$$|S_n^{(r)}(T_{k,r}^k(I_d);\phi)| = \prod_{j=k}^{n}((k-1)d+(r-d)j)\sum_{j=k}^{n}\frac{1}{(k-1)d+(r-d)j},$$

for all $\phi \in T_{k,r}^k(I_d)$.

4. Avoiding $T_{k,r}^m(I_d)\backslash\{\phi\}$ and containing ϕ exactly once, $2 \leq m \leq k-1$

Now we find the cardinalities of the sets $S_n^{(r)}(T_{k,r}^m(I_d);\phi)$ where $2 \leq m \leq k-1$, $\phi \in T_{k,r}^m(I_d)$.

Theorem 4.1. *Let $r \geq 1$, $k \geq 3$, and $n \geq k+1$. Then*

$$|S_n^{(r)}(T_{k,r}^m(I_d);\phi)| = \prod_{j=k+1}^{n}((k-1)d+(r-d)j),$$

for all $2 \leq m \leq k-1$ and $\phi \in T_{k,r}^m(I_d)$.

Proof. Let $G_n = S_n^{(r)}(T_{k,r}^m(I_d);\phi)$, $\psi \in G_n$, and let us consider the possible values of ψ_1:

(1) $|\psi_1| \leq m-1$ and coloured by $c \in I_d$. Evidently $\psi \in G_n$ if and only if $(\psi_2,\ldots,\psi_n)$ avoids $M_{k,r}^m(\phi, I_d)$ and contains ϕ exactly once.
(2) $|\psi_1| \geq n-k+m+1$ and coloured by $c \in I_d$. Evidently $\psi \in G_n$ if and only if $(\psi_2,\ldots,\psi_n)$ avoids $M_{k,r}^m(\phi, I_d)$ and contains ϕ exactly once.
(3) $m \leq |\psi_1| \leq n-k+m$ and coloured by one of the colours $c_1,\ldots,c_d$. By definition we have that $|G_k| = 1$, so let $n \geq k+1$. If $m+1 \leq |\psi_1|$ then ψ contains at least $m \geq 2$ occurrences of a pattern from $T_{k,r}^m(I_d)$, and If $|\psi_1| \leq n-k+m-1$ then ψ contains at least $k-m+1 \geq 2$ occurrences of a pattern from $T_{k,r}^m(I_d)$, a contradiction.
(4) $|\psi_1|$ coloured by a colour $c \notin I_d$. Evidently $\psi \in G_n$ if and only if $(\psi_2,\ldots,\psi_n)$ avoids $M_{k,r}^m(\phi, I_d)$ and contains ϕ exactly once.

Since the above cases 1-4 are disjoint we obtain $|G_n| = ((k-1)d+(r-d)n)|G_{n-1}|$ for all $n \geq k+1$. Besides $|G_k| = 1$, hence the theorem holds. □

Example 4.2. *Let* $\phi \in T^m_{k,2}(1)$ *where* $2 \le m \le k-1$ *and* $k \ge 3$. *Then Theorem 4.1 gives* $|S^{(2)}_n(T^m_{k,2}(1);\phi)| = \frac{(n+k-1)!}{(2k-1)!}$ *for* $n \ge k$.

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