AN EFFECTIVE NULLSTELLENSATZ IN TERMS OF RESIDUE CURRENTS

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ABSTRACT. We prove an effective Nullstellensatz for complex polynomials where the polynomial degree of the solution is related to the vanishing of a certain residue current in \( \mathbb{P}^n \). We also provide explicit integral representations of the solutions.

1. INTRODUCTION

If \( P_1, \ldots, P_m \) are given polynomials in \( \mathbb{C}^n \) with no common zeros, then by Hilbert’s Nullstellensatz there are polynomials \( Q_j \) such that

\[
\sum_{j=1}^{m} P_j Q_j = 1.
\]

An analytic proof can be obtained in the following way. Since the homogenizations of \( P_j \) are holomorphic even across the hyperplane at infinity in \( \mathbb{P}^n \) it follows that

\[
\sum_{j=1}^{m} \frac{|P_j(z')|^2}{(1 + |z'|^2)^{d_j}} \geq c \frac{1}{(1 + |z'|^2)^M},
\]

for some integer \( M \) (here \( z' = (z_1, \ldots, z_n) \) and \( d_j = \deg P_j \)). One can then use an explicit integral formula, see [6], [5], to obtain a solution \( Q = (Q_1, \ldots, Q_m) \) to (1.1). One can also use the Koszul complex to reduce to a sequence of \( \bar{\partial} \)-equations in \( \mathbb{C}^n \) which are to be solved by polynomial growth, or one can apply Skoda’s \( L^2 \)-estimate for vector bundle homomorphisms, [15].

It is sometimes of interest to get some bound of the degree of the resulting solution. The breakthrough was in the paper [8] where Brownawell proved that (1.2) holds with \( M = (n - 1)d_{\min(m,n)} - 1 + d \) (assuming \( \deg P_j \leq d \)) and by applying theorem he obtained a solution \( Q \) with \( \deg Q_j \leq n \min(m, n) d_{\min(m,n)} + \min(m, n) d \). Soon after that Kollár [13] obtained by purely algebraic methods the optimal bound

\[
\deg P_j Q_j \leq N(d_1 \cdots d_m),
\]

where \( N(d_1 \cdots d_m) = d_1 = \cdots d_m \) if \( m \leq n \); for the case when \( m > n \), see [13]. More generally he proved

\[
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\]

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Theorem 1.1 (Kollár). Let $P_1, \ldots, P_n$ and $\Phi$ be polynomials in $\mathbb{C}^n$ of degrees $d_j$, and $r$, respectively, and assume that $\Phi$ vanishes on the common zero set of $P_j$. Then (if $d_j \neq 2$), one can find polynomials $Q_j$ and a natural number $s$ such that $\sum P_j Q_j = \Phi^s$, and such that $s \leq N(d_1 \cdots d_m)$ and $\deg(P_j Q_j) \leq (1 + r) N(d_1 \cdots d_m)$.

The bounds of the degrees here are optimal. From this theorem he deduces the best possible constant $M$ in (1.2) which is $M = N(d_1 \cdots d_m)$. On the other hand, if we start with this estimate, the analytic tools only give back $\deg P_j Q_j \leq \min(m, n) N(d_1 \cdots d_m)$ cf. Corollary 1.4 below. This is thus weaker than the optimal result due to Kollár. The factor $\min(m, n)$ is related to the Briançon-Skoda theorem, [7]; see [5] for a further discussion. Thus, as long as we only consider the degrees of the polynomials $P_j$ the problem is completely solved by Kollár. However, in more specialized situations one can obtain sharper results.

In this paper we formulate a sufficient condition, in terms of a global residue current, to have a solution $Q$ satisfying $\deg(P_j Q_j) \leq r$.

Remark 1. In this paper we only consider polynomials over $\mathbb{C}$. However, Kollár’s theorem holds for an arbitrary field. Berenstein and Yger, [4], have obtained variants of Brownawell’s result for subfields of $\mathbb{C}$, by means of explicit integral formulas; see also [5] for a thorough discussion and more references.

In [9] Brownawell has given a prime power version of the Nullstellensatz which shed more geometric light on Kollár’s theorem, and there is a generalization to smooth algebraic manifolds in [11].

Let $P_1, \ldots, P_m$ be polynomials in $\mathbb{C}^n$. If $p_j$ denote homogenizations of $P_j$, i.e., $p_j(z) = z_0^{d_j} P_j(z'/z_0)$, where $d_j \geq \deg P_j$, then each $p_j$ defines a global holomorphic section to the line bundle $L^{d_j} \to \mathbb{P}^n$, and hence $p = p_1 + \cdots + p_m$ is a section to the rank $m$ bundle $E^* = L^{d_1} \oplus \cdots \oplus L^{d_m}$ over $\mathbb{P}^n$. If $E^*$ is equipped with the natural hermitian structure, then

$$
||p(z)||^2 = \sum_1^m \frac{|p_j(z)|^2}{|z|^{2d_j}}.
$$

Following [2] we can define the residue current $R^p$ to the section $p$, which is an element in $\bigoplus_t \mathcal{D}_{0,t}(\mathbb{P}^n, \Lambda^t E)$ and with support on the zero set

$$
Z^p = \{[z] \in \mathbb{P}^n; p(z) = 0\}.
$$

If we assume that the polynomials $P_j$ have no common zeros in $\mathbb{C}^n$, then of course $Z^p$ is a subset of the hyperplane at infinity. If $\text{codim} Z^p = m$, i.e., $p$ is a complete intersection, then $R^p$ is a $(0, m)$-current with values in $\text{det } E = L^{-\Sigma d_j}$; more precisely a Coleff-Herrera current which formally can be written

$$
R^p = \left[ \overline{\partial} \frac{1}{p_1} \wedge \ldots \wedge \overline{\partial} \frac{1}{p_m} \right],
$$

see Section 2.
Theorem 1.2. Let $P_1, \ldots, P_m$ be polynomials in $\mathbb{C}^n$, $\deg P_j \leq d_j$, let $p = p_1 + \cdots + p_m$ be the corresponding section to $E^* = L^1 \oplus \cdots \oplus L^m$ over $\mathbb{P}^n$, and let $R^p$ be its residue current. Moreover, assume that

\begin{equation}
\tag{1.4}
m \leq n \quad \text{or} \quad r \geq \sum_{j=1}^{m} d_j - n.
\end{equation}

Let $\Phi$ be a polynomial, $\deg \Phi \leq r$, and let $\phi \in \mathcal{O}(\mathbb{P}^n, L')$ denote its $r$-homogenization. If

\begin{equation}
\tag{1.5}
\phi R^p = 0,
\end{equation}

then there are polynomials $Q_j$ such that

$$
\Phi = \sum_{j=1}^{m} P_j Q_j
$$

and $\deg(P_j Q_j) \leq r$. If $p$ is a complete intersection (then the condition (1.4) is fulfilled) and there are such polynomials $Q_j$, then the condition (1.5) holds.

It is clear that the conclusion about $\deg P_j Q_j$ cannot be improved. If $\Phi = 1$ the condition (1.5) means that $P_j$ have no common zeros in $\mathbb{C}^n$ and that $z_0^r$ annihilates the residue $R^p$ at infinity. If $Z^p$ is empty and $m = n + 1$ we get a solution to (1.1) such that $\deg P_j Q_j \leq \sum d_j - n$; this is a classical theorem of Macauley, [14].

If

\begin{equation}
\tag{1.6}
\|\phi\| \leq C\|p\|,
\end{equation}

then, see [2], $\phi \min\{m, n\} R^p = 0$, and hence we have

Corollary 1.3. Let $P_j$ and $\Phi$ be as in Theorem 1.2 and assume that

$$
m \leq n \quad \text{or} \quad r \min\{m, n\} \geq \sum_{j=1}^{m} d_j - n.
$$

If (1.6) holds, then there are polynomials $Q_j$ such that

$$
\sum_{j=1}^{m} P_j Q_j = \Phi \min\{m, n\}
$$

and $\deg(P_j Q_j) \leq r \min\{m, n\}$.

Since there are examples where $p$ is a complete intersection and the full power of $\min\{m, n\}$ of $\phi$ is needed to kill $R^p$, this result is then sharp.

In particular, (1.2) means that $P_j$ have no common zeros in $\mathbb{C}^n$ and that

$$
\|z_0\|^M \leq C\|p\|.
$$

Thus $z_0^M \min\{m, n\} R^p = 0$ so we have

Corollary 1.4. Let $P_1, \ldots, P_m$ be polynomials in $\mathbb{C}^n$ of degrees $d_j$ such that (1.2) holds for some number $M$, and assume that

$$
m \leq n \quad \text{or} \quad M \min\{m, n\} \geq \sum_{j=1}^{m} d_j - n.
$$

Then there is a solution to $\sum P_j Q_j = 1$ with $\deg(P_j Q_j) \leq \min(m, n) M$. 

Example 1. Let $M$ be a given positive integer. In $\mathbb{C}^n$ we take $P_j = \zeta_j^{Mm}$ for $j = 1, \ldots, m$ and let $\Phi = (\zeta_1 + \cdots + \zeta_m)^{Mm}$. Then $Z^p$ is just the origin, thus $p$ is a complete intersection, and $\|\Phi\| \leq C\|P\|$. It is easily checked that $\Phi^n R^p = 0$ but $\phi^{m-1} R^p \neq 0$. One can just as well see that $\sum Q_j P_j = \Phi^n$ is solvable whereas $\sum Q_j P_j = \Phi^{m-1}$ is not. Thus the statement in Corollary 1.3 is optimal.

Taking $z_0 = \zeta_1 + \cdots + \zeta_m$, $z_j = -\zeta_j$ for $j < m$, $p_j = z_j^{Mm}$ for $j < m$ and $p_m = (z_0 + z_1 + \cdots + z_{m-1})^{Mm}$, we have that $\|z_0\|^M \leq C\|p\|$. We need the power $Mm - m + 1$ of $z_0$ to kill $R^p$, which is close to $Mm = M\min(m, n)$ if $M$ is large, and thus Corollary 1.4 is almost optimal.

Remark 2. The condition (1.6) means that $\phi$ locally on $\mathbb{P}^n$ belongs to the integral closure of $p$. In [12], Hickel proves that if $\Phi$ is in the integral closure of $P$ in $\mathbb{C}^n$, then one can solve (assuming $m \leq n$ for simplicity) $\Phi^n = \sum P_j Q_j$ with $\deg(P_j Q_j) \leq m\deg \Phi + md_1 \cdots d_m$. This result would follow from Theorem 1.2 if one could prove that the current $z_0^{m(d_1 \cdots d_m)} \phi^m R^p$ vanishes ($\phi$ is the deg $\Phi$ homogenization of $\Phi$). In $\mathbb{C}^n$ it vanishes since $|\Phi| \leq C|P|$ locally. If the zero set is contained in $\{z_0 = 0\}$ the current vanishes there by Kollár’s theorem. We do not know how one can see the general case.

Theorem 1.2 is a special case of the following more general result, for which we formulate only the homogeneous version. Let $\delta_p$ denote the mapping $\mathcal{E}(\mathbb{P}^n, \Lambda^{\ell+1} E \otimes L') \to \mathcal{E}(\mathbb{P}^n, \Lambda^p E \otimes L')$ defined as interior multiplication with the section $p$ to $E^*$. Thus for instance, if $q = q_1 + \cdots + q_m$ is a section to $E \otimes L'$, then $\delta_p q$ is equal to the section $\sum_j p_j q_j$ to $L'$.

Theorem 1.5. Let $p$ be holomorphic section to $E^* = L^{d_1} \oplus \cdots \oplus L^{d_m}$ and assume $\ell \geq 0$ is given and that

$$m - \ell \leq n \quad \text{or} \quad r \geq \sum_{i=1}^m d_i - n.$$

If $\phi \in \mathcal{O}(\mathbb{P}^n, \Lambda^{\ell+1} E \otimes L')$, then $\phi = \delta_p \psi$ for some $\psi \in \mathcal{O}(\mathbb{P}^n, \Lambda^\ell E \otimes L')$ if and only if

$$\nabla_p (w \wedge R^p) = \phi \wedge R^p$$

for some smooth $w$ defined in a neighborhood of $Z^p$.

If $\ell > m - p$ then (1.7) is void; if $\ell = m - p$, then (1.7) just means that $\phi \wedge R^\ell = 0$. If $p$ is a complete intersection, then $m \leq n$ and therefore we have

Corollary 1.6. Let $p$ be a holomorphic section to $E^* = L^{d_1} \oplus \cdots \oplus L^{d_m}$ that is a complete intersection, and assume that $r \geq 0$. If $\phi \in \mathcal{O}(\mathbb{P}^n, L')$, then $\phi \cdot q$ is solvable with $q \in \mathcal{O}(\mathbb{P}^n, E \otimes L')$ if and only if $\phi R^p = 0$. 
AN EFFECTIVE NULLSTELLENSATZ IN TERMS OF RESIDUE CURRENTS 5

Proof of Theorem 1.2. If the hypotheses in Theorem 1.2 are fulfilled, then Theorem 1.5 provides a section \( q = q_1 + \ldots + q_m \) to \( E \otimes L^r \) such that \( \sum p_j q_j = \delta_p q = \phi \); here \( q_j \) are sections to \( L^{-a_j} \). After dehomogenization this means that \( Q_j \) are polynomials such that \( \deg P_j Q_j \leq r \). □

In Section 2 we recall the necessary background from [2] about the residue currents, and present a general result about the image of a holomorphic morphism \( f \). Combined with well-known vanishing results for the line bundles \( L^r \to \mathbb{P}^n \) it leads to a proof of Theorem 1.5.

In the last section we construct explicit integral representations of the solutions in Theorem 1.2. They give essentially the same results except for a small loss of precision. The construction is based on the ideas in [2] and [3].

2. THE RESIDUE CURRENT OF A HOLOMORPHIC SECTION

Let \( E \to X \) be a holomorphic hermitean vector bundle and let \( f \) be a holomorphic section to the dual bundle \( E^* \), or in other words, a holomorphic morphism \( f : E \to X \times \mathbb{C} \). Let

\[
\mathcal{L}^r = \bigoplus_{\ell} \mathcal{D}_{0,\ell+1}(X, \Lambda^\ell E);
\]

we consider \( \mathcal{L}^r \) as a subbundle to \( \Lambda(T_0^* \oplus E) \), so that \( \delta f \) (i.e., interior multiplication with \( f \)) and \( \bar{\partial} \) anticommutes. Then \( \nabla_f = \delta_f - \bar{\partial} \) induces the complex \( \to \mathcal{L}^{r-1} \to \mathcal{L}^r \to \cdots \). Let \( s \) be the dual section to \( E \) of \( f \) so that in particular \( \delta_f s = |f|^2 \). In [2] we defined the current

\[
R^f = \bar{\partial}|f|^{2\lambda} \wedge s \frac{s}{\bar{s}}|_{\lambda-0};
\]

for large \( \Re \lambda \) the right hand side is integrable and therefore a well-defined current, and by a nontrivial argument based on Hironaka’s theorem one can make an analytic continuation to \( \lambda = 0 \). The resulting current is an element in \( \mathcal{L}^0 \) with support on \( Z^f = \{ z; f(z) = 0 \} \) and it satisfies the basic equality

\[
\nabla_f U^f = 1 - R^f,
\]

where \( U \in \mathcal{L}^{-1} \) is defined as

\[
U_f = |f|^{2\lambda} \frac{s}{\bar{s}}|_{\lambda-0}.
\]

Moreover,

\[
R^f = R^f_{c,c} + \ldots + R^f_{m,m},
\]

where \( c = \text{codim} Z^f \); here lower index \( p,q \) means a \( \Lambda^p E \)-valued \( (0,q) \)-form. It was also proved in [2] that \( h^{\min(m,n)} R^f = 0 \) if \( h \) is holomorphic.
and $|h| \leq C|f|$. Let $L \to X$ be a holomorphic line bundle and let $\phi$ be a holomorphic section to $\Lambda^k E \otimes L$.

**Theorem 2.1.** Let $\ell \geq 0$ and suppose that $H^{0,s}(X, \Lambda^{s+\ell+1} E \otimes L) = 0$ for all $1 \leq s \leq m - \ell - 1$. Moreover, let $\phi \in \mathcal{O}(X, \Lambda^\ell E \otimes L)$. Then $\delta_f \psi = \phi$ has a solution $\psi \in \mathcal{O}(X, \Lambda^{\ell+1} E \otimes L)$ if and only if there is a smooth solution $w$, defined in a neighborhood of $Z^I$, to

$$\nabla_f (w \wedge R^I) = \phi \wedge R^I. \tag{2.3}$$

In view of (2.2), the condition (2.3) is void if $\ell > m - c$, since $w = w_{\ell+1,0} + w_{\ell+2,1} + \cdots$ the condition means precisely that $\phi \wedge R^I = 0$ if $\ell = m - c$.

**Proof.** Suppose that the holomorphic solution $\psi$ exists. Then $\nabla_f \psi = \phi$ and hence $\nabla_f (\psi \wedge R^I) = \phi \wedge R^I$ since $\nabla_f R^I = 0$. Conversely, if (2.3) holds, then $\nabla_f v = \phi$, where

$$v = (-1)^\ell \phi \wedge U^I + w \wedge R^I;$$

this means that

$$\delta v_{m,m-\ell-1} = 0 \quad \text{and} \quad \delta_f v_{k+1,k-\ell} = \delta v_{k,k-\ell-1}. \tag{2.4}$$

By the assumption on the Dolbeault cohomology, we can successively solve the equations

$$\delta v_{m,m-\ell-2} = v_{m,m-\ell-1}, \quad \delta f v_{k,k-\ell-2} = v_{k,k-\ell-1} + \delta f v_{k+1,k-\ell-1}, \quad k \geq \ell,$n

and then finally $\psi = v_{\ell,0} + \delta_f v_{k+1,0}$ is the desired holomorphic solution.

**Example 2.** Suppose that $X$ is a compact and $L$ is a strictly positive line bundle. Then there is an $r_0 > 0$ such that $H^{0,k}(X, \Lambda^k E^* \otimes L^*) = 0$ for all $k \geq 1$ if $r \geq r_0$. If $f$ is a holomorphic section to $E^*$, then a holomorphic section $\phi \in \mathcal{O}(\Lambda^\ell E \otimes L^*)$, $r \geq r_0$, is in the image of the morphism

$$\mathcal{O}(X, \Lambda^{\ell+1} E \otimes L^*) \to \mathcal{O}(X, \Lambda^\ell E \otimes L^*) \tag{2.4}$$

if $\phi \wedge R^I = 0$. If $\ell = m - c$ the condition is necessary. □

We shall now focus on the case when $X = \mathbb{P}^n$ and $E$ is the hermitean vector bundle from Section 1. Let $E_1, \ldots, E_m$ be trivial line bundles over $\mathbb{P}^n$ with basis elements $e_1, \ldots, e_m$, and let $E_j^*$ be the dual bundles, with bases $e_j^*$. Then we have that

$$E^* = L^{d_1} \otimes E_1^* \oplus \cdots \oplus L^{d_m} \otimes E_m^*, \quad E = L^{-d_1} \otimes E_1 \oplus \cdots \oplus L^{-d_m} \otimes E_m,$$

and for instance our section $p$ can be written

$$p = \sum_{j=1}^m p_j e_j^*.$$
Its dual section $s$ is then, cf., (1.3),
\[
s = \sum_j \frac{p_j(z)}{|z|^{2d_j}} e_j,
\]
so
\[
R^p = \partial ||p||^{2\lambda} \wedge \sum_{\ell+1}^{m} s \wedge (\partial s)^{\ell-1}_{|p|^{2\ell}} |_{\lambda=0}^{-}.
\]
In $\mathbb{C}^n = \{z_0 \neq 0\} \subset \mathbb{P}^n$ we have the coordinates $z'$ and the natural holomorphic frame $e_j = z_0^{-d_j} e_j$ and its dual $e_j^* = z_0^{d_j} e_j^*$. If $p_j'(z') = p_j(1,z')$ then
\[
p = \sum_1^m p_j' e_j^*
\]
and
\[
s = \sum_1^m \frac{p_j(z')}{(1 + |z'|)^{d_j}} e_j.
\]
When codim $Z^p = m$, the residue current $R^p$ is independent of the metric, it just contains the top degree term $R^p_{m,m}$, and in fact, see [2],
\[
R^p = [\partial^{-1}_{p'_m} \wedge \ldots \wedge \partial^{-1}_{p'_1}] \wedge e_1 \wedge \ldots \wedge e_m,
\]
where the expression in brackets is the Coleff-Herrera residue current which we can rewrite formally as
\[
(2.5) \quad [\partial^{-1}_{p'_m} \wedge \ldots \wedge \partial^{-1}_{p'_1}] \wedge e_1 \wedge \ldots \wedge e_m.
\]

**Proof of Theorem 1.5.** It is wellknown, see, e.g., [10], that $H^{0,k}(\mathbb{P}^n, L^\nu) = 0$ for all $\nu$ if $1 \leq k \leq n - 1$ and that $H^{0,n}(\mathbb{P}^n, L^\nu) = 0$ if $\nu \geq -n$. Since $E = L^{-d_1} \oplus \ldots \oplus L^{-d_m}$ we have that
\[
\Lambda^r E \oplus L^r = \bigoplus_{|J| - \nu} L^{-d_{j_1}} \otimes \ldots \otimes L^{-d_{j_\nu}} \otimes L^r = \bigoplus_{|J| - \nu} L^{-d_{j_1}, \ldots, d_{j_\nu}}.
\]
Thus $H^{0,s}(\mathbb{P}^n, \Lambda^{s+\ell+1} E \otimes L^r) = 0$ for $1 \leq s \leq m - \ell - 1$ if either $m - \ell - 1 \leq n - 1$ or $r - \sum d_j \geq -n$. Now Theorem 1.5 follows from Theorem 2.1 with $f = p$. 
\[\square\]
3. Integral Representation

The aim of this section is to present an explicit integral representation of the solution $Q_j$ to the division problem in Theorem 1.2. We have

**Theorem 3.1.** Let $P_1, \ldots, P_m, \Phi$ be polynomials in $\mathbb{C}^n$, let $p$ and $R^p$ be as before, and let $\phi$ be the $r$-homogenization of $\Phi$ (deg $\Phi \leq r$). Then there is an explicit decomposition

$$
\Phi(z') = \sum_{1}^{m} P_j(z') \int_{\mathbb{P}^n} T^j(\zeta, z') \phi(\zeta) + \int_{\mathbb{P}^n} S(\zeta, z') \wedge R^p(\zeta) \phi(\zeta),
$$

where $T^j(\zeta, z'), S(\zeta, z')$ are smooth forms (in $|\zeta|$) on $\mathbb{P}^n$ and holomorphic polynomials in $z'$, such that

$$
deg_{z'}(P_j(z') T^j(\zeta, z')) \leq d_1 + d_2 + \cdots + d_{\mu+1} + r,
$$

if $\mu = \min(n, m - 1)$ and $d_1 \geq d_2 \geq \cdots \geq d_m$.

Thus, if $\phi R^p = 0$ we get back the conclusion of Theorem 1.2 but with the extra term $d_1 + \cdots + d_{\mu+1}$ in the estimate of the degree.

For fixed $z \in \mathbb{C}^n$,

$$
\eta = 2\pi i \sum_{0}^{n} \frac{z_j \partial}{\partial \zeta_j}
$$

is an $L_z \otimes L^{-1}_{\zeta}$-valued $(1,0)$-form on $\mathbb{P}^n$, and if $\delta_{\eta}$ denotes interior multiplication with $\eta$, then

$$
\delta_{\eta} : \mathcal{D}'_{\ell+1,0}(\mathbb{P}^n, L_{r+1}) \to \mathcal{D}'_{\ell,0}(\mathbb{P}^n, L^r).
$$

**Remark 3.** When we say that $\eta$ is a section to $L_z \otimes L^{-1}_{\zeta}$ rather than $L^{-1} = L^{-1}_{\zeta}$, we just indicate that it is 1-homogeneous in $z$; it would be more correct, but less convenient, to consider $\eta$ as a section to the bundle $L_z \otimes L^{-1}_{\zeta} \otimes (T^*_\zeta)_{0,1}$ over $\mathbb{P}_z \times \mathbb{P}^n$.

Let $\nabla_{\eta} = \delta_{\eta} - \bar{\partial}$. Notice that if

$$
\alpha = \alpha_0 + \alpha_1 = \frac{z \cdot \bar{\zeta}}{|\zeta|^2} - \frac{\bar{\partial} \zeta \cdot d\zeta}{2\pi i |\zeta|^2},
$$

then the first term, $\alpha_0$, is a section to $L_z \otimes L^{-1}_{\zeta}$ and the second term, $\alpha_1$, is a projective form (since $\delta_\zeta \alpha_1 = 0$); moreover

$$
\nabla_{\eta} \alpha = 0.
$$

We have the following basic integral representation of global holomorphic sections to $L^r$.

**Proposition 3.2.** Assume that $r \geq 0$ and that $\phi \in \mathcal{O}(\mathbb{P}^n, L^r)$. Then

$$
\phi(z) = \int_{\mathbb{P}^n} \alpha_{n+r} \phi.
$$
For degree reasons, actually
\[
\phi(z) = \frac{(n+r)!}{n!r!} \int_{\mathbb{P}^n} \alpha_0^r \wedge \alpha_1^n;
\]
this formula has appeared at several places, and expressed in affine coordinates it is a well-known representation formula for polynomials in \( \mathbb{C}^n \). However, we prefer to supply a direct proof on \( \mathbb{P}^n \), following the ideas in [1].

**Proof.** Let \( \sigma \) be the \( L_z^{-1} \otimes L_\zeta \otimes T^*_1(\mathbb{P}^n) \) valued \((1,0)\)-form on \( \mathbb{P}^n \) that is dual, with respect to the natural metric, to \( \eta \). Then, since \( \eta \) has a first order zero at \([z]\) (and no others), it follows (see, e.g., [1]) that
\[
\nabla_{\eta} \frac{\sigma}{\nabla_{\eta} \sigma} = 1 - [[z]].
\]
The rightmost term is the \( L_z^{n} \otimes L_\zeta^r \)-valued \((n,n)\)-current point evaluation at \([z]\) for sections to \( L_z^{-n} \). If \( \phi \) is a global holomorphic section to \( L' \) it follows by (3.2) that
\[
\nabla_{\eta} \left( \frac{\sigma}{\nabla_{\eta} \sigma} \wedge \alpha_1^{n+r} \phi \right) = \phi \alpha_1^{n+r} - \phi[[z]],
\]
where this time the last term is \( \phi \) times the \( L_z' \otimes L_\zeta'^r \)-valued current point evaluation at \([z]\). If we integrate this equality over \( \mathbb{P}^n \) we get the desired representation formula. \( \square \)

Let \( E_1, \ldots, E_m \) be the trivial line bundles over \( \mathbb{P}^n \) with basis elements \( \epsilon_1, \ldots, \epsilon_m \), so that \( E = L^{-d_1} \otimes E_1 \oplus \cdots \oplus L^{-d_m} \otimes E_m \) as in Section 2. We also introduce disjoint copies \( \bar{E}_j \) of \( E_j \) with bases \( \bar{\epsilon}_j \) and the bundle
\[
\bar{E} = L^{-d_1} \otimes \bar{E}_1 \oplus \cdots \oplus L^{-d_m} \otimes \bar{E}_m.
\]
Let \( \Lambda \) be the exterior algebra bundle over the direct sum of all the bundles \( E, \bar{E}, E^*, \) and \( T^*(\mathbb{P}^n) \). Any form \( \gamma \) with values in \( \Lambda \) can be written uniquely as \( \gamma = \gamma' \wedge \left( \sum \epsilon_j^* \wedge \epsilon_j \right)^m / m! + \gamma'' \) where \( \gamma'' \) denotes terms that do not contain a factor \( \left( \sum \epsilon_j^* \wedge \epsilon_j \right)^m / m! \), and we define
\[
\int_\epsilon \gamma = \gamma'.
\]
We have a globally defined form
\[
\tau = \sum_j \epsilon_j^* \wedge (\epsilon_j - \bar{\epsilon}_j).
\]

From now on we consider \([z]\) as a fixed arbitrary point in \( \mathbb{C}^n \subset \mathbb{P}^n \), and let \( z = (1, \bar{z}') \). We also introduce the section
\[
p_z = \sum_j \zeta^d_j p_j(1, z) \epsilon_j^* = \sum \zeta^d_0 P_j(z') \epsilon_j^*
\]
to \( E^* \) and let \( \bar{p}_z \) be the corresponding section to \( \bar{E}^* \).
Lemma 3.3. There is a holomorphic section $H = \sum H_j \wedge \epsilon_j$ to $E^* \otimes L \otimes T^*_{1,0}$, thus $H_j$ are sections to $L^{d_j} \otimes L \otimes T^*_{1,0}$, such that

$$\delta_\eta H = p - p_z,$$

and such that the coefficients in $H_j$ are polynomials in $z'/z_0$ of degrees (at most) $d_j - 1$.

Proof. For each $P_j(z')$ we can find Hefer functions $h_j^k(\zeta', z')$, polynomials of degree $d_j - 1$ in $(\zeta', z')$, such that

$$\sum_{k=1}^n h_j^k(\zeta', z')(\zeta_k - z_k) = P_j(\zeta') - P_j(z').$$

If we then take

$$H_j = \frac{\zeta_0^{d_j+1}}{2\pi i} \sum_{k=1}^n h_j^k(\zeta'/\zeta_0, z')d(\zeta_k/\zeta_0),$$

then clearly $H_j$ is a projective $(1,0)$-form, and moreover,

$$\delta_\eta H_j = p_j(\zeta) - \zeta_0^{d_j} P_j(z')$$

as wanted.

Let $\delta_F$ denote interior multiplication with the section $F = p + p_z$ to $E^* \oplus \tilde{E}^*$. Then $\delta_F\tau = p - p_z = -\delta_\eta H$. If

$$\nabla = \delta_F + \delta_\eta - \bar{\partial},$$

thus

$$\nabla(\tau + H) = 0.\quad(3.3)$$

We are now ready to define the explicit division formula.

Proof of Theorem 3.1. From (3.3) it follows that

$$(\nabla_\eta + \delta_F)(e^{\tau + H} \wedge U^p) = e^{\tau + H} \wedge (1 - R^p).\quad(3.4)$$

We can rewrite this as

$$\delta_F(e^{\tau + H} \wedge U^p) + e^{\tau + H} \wedge R^p = e^{\tau + H} - \nabla_\eta(e^{\tau + H} \wedge U^p).\quad(3.5)$$

We claim that the component of full bidegree $(n, n)$ of

$$\int_\epsilon [e^{\tau + H} - \nabla_\eta(e^{\tau + H} \wedge U^p)] \wedge \alpha^\tau \phi$$

is equal to

$$\frac{(n + r)!}{n!r!} \alpha_0^r \alpha_0^r \phi + \check{\delta}(\cdots)$$

where $(\cdots)$ is a scalar valued $(n, n - 1)$-form. In fact, since $\alpha^{n+r}$ has bidegree $(\ast, \ast)$ the factor $U_{k, k-1}$ must be combined with $H_k$, and then it follows that $\tau$ can be replaced by $\omega = \sum_{j} \epsilon_j^* \wedge \epsilon_j$. Observe that the component of $U_{k, k-1}$ with basis element $\epsilon_{j_1} \wedge \ldots \wedge \epsilon_{j_k}$ takes values in $L^{-\left(d_{j_1} + \cdots + d_{j_k}\right)}$, whereas the component of $H_k$ with basis element $\epsilon_{j_1} ^* \wedge$
\[ \mathcal{S}(\zeta, z') \cap \mathbb{R}^p(\zeta) = \int e^{\omega + H} \cap \mathbb{R}^p \cap \alpha^{n+r} = \sum_{k=0}^{m-1} e^{\omega_{m-k-1} \cap H_k \cap U_{k+1, k} \cap \alpha_1^{n-k} \alpha_0^{k+r}} \]

and

\[ T^j(\zeta, z') = \int e^{\omega + H} \cap \mathbb{R}^p(\zeta) = \sum_{k=0}^{m-1} e^{\omega_{m-k-1} \cap H_k \cap U_{k+1, k} \cap \alpha_1^{n-k} \alpha_0^{k+r}} \]

Both \( \alpha \) and \( H \) are polynomials in \( z' \) so it just remains to check the degrees of \( T^j \). The worst case occur when \( k \) is as large as possible which is \( k = \mu = \min(m-1, n) \). Then the factor \( \alpha_0^{k+r} \) has degree \( k+r \). Recall that \( H = \sum H_j \cap e^\ast_j \) and that \( \deg H_j = d_j - 1 \). The term \( H_j \) cannot occur, because of the presence of \( e^\ast_j \), and thus we get that \( d_j + \deg Q_j \) is at most \( d_1 - 1 + d_2 - 1 + \cdots + d_{\mu+1} - 1 + 1 + \mu + r = d_1 + \cdots + d_{\mu+1} + r \). \( \square \)
The main novelty with these formulas is that they contains the global residues $R^p$ and $U^p$. However, even if we suppress the residues by assuming $\phi$ is vanishing enough, and consider them as formulas in $\mathbb{C}^n$, they differ from the formulas from [6] and admit sharper estimates of the degrees, see the discussion in [3].

REFERENCES


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