AN EFFECTIVE NULLSTELLENSATZ IN TERMS OF RESIDUE CURRENTS

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Abstract. We prove an effective Nullstellensatz for complex polynomials where the polynomial degree of the solution is related to the vanishing of a certain residue current in $\mathbb{P}^n$. We also provide explicit integral representations of the solutions.

1. Introduction

If $P_1, \ldots, P_m$ are given polynomials in $\mathbb{C}^n$ with no common zeros, then by Hilbert's Nullstellensatz there are polynomials Q_j such that

$$\sum_1^m P_j Q_j = 1. \tag{1.1}$$

An analytic proof can be obtained in the following way. Since the homogenizations of P_j are holomorphic even across the hyperplane at infinity in $\mathbb{P}^n$ it follows that

$$\sum_1^m \frac{|P_j(z')|^2}{(1+|z'|^2)^{d_j}} \geq c \frac{1}{(1+|z'|^2)^M}, \tag{1.2}$$

for some integer M (here $z' = (z_1, \ldots, z_n)$ and $d_j = \deg P_j$). One can then use an explicit integral formula, see [6], [5], to obtain a solution $Q = (Q_1, \ldots, Q_m)$ to (1.1). One can also use the Koszul complex to reduce to a sequence of $\bar\partial$-equations in $\mathbb{C}^n$ which are to be solved by polynomial growth, or one can apply Skoda's L^2-estimate for vector bundle homomorphisms, [15].

It is sometimes of interest to get some bound of the degree of the resulting solution. The breakthrough was in the paper [8] where Brownawell proved that (1.2) holds with $M = (n-1)d^{\min(m,n)} - 1 + d$ (assuming $\deg P_j \leq d$) and by applying theorem he obtained a solution Q with $\deg Q_j \leq n \min(m,n) d^{\min(m,n)} + \min(m,n)d$. Soon after that Kollár [13] obtained by purely algebraic methods the optimal bound

$$\deg P_j Q_j \leq N(d_1 \cdots d_m),$$

where $N(d_1 \cdots d_m) = d_1 = \cdots d_m$ if $m \leq n$; for the case when $m > n$, see [13]. More generally he proved

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Theorem 1.1 (Kollár). *Let $P_1, \ldots, P_n$ and Φ be polynomials in $\mathbb{C}^n$ of degrees d_j, and r, respectively, and assume that Φ vanishes on the common zero set of P_j. Then (if $d_j \neq 2$), one can find polynomials Q_j and a natural number s such that $\sum P_j Q_j = \Phi^s$, and such that $s \leq N(d_1 \cdots d_m)$ and deg $(P_j Q_j) \leq (1+r)N(d_1 \cdots d_m)$.*

The bounds of the degrees here are optimal. From this theorem he deduces the best possible constant M in (1.2) which is $M = N(d_1 \cdots d_m)$. On the other hand, if we start with this estimate, the analytic tools only give back $\deg P_j Q_j \leq \min(m,n) N(d_1 \cdots d_m)$ cf. Corollary 1.4 below. This is thus weaker than the optimal result due to Kollár. The factor $\min(m,n)$ is related to the Briançon-Skoda theorem, [7]; see [5] for a further discussion. Thus, as long as we only consider the degrees of the polynomials P_j the problem is completely solved by Kollár. However, in more specialized situations one can obtain sharper results. In this paper we formulate a sufficient condition, in terms of a global residue current, to have a solution Q satisfying $\deg(P_j Q_j) \leq r$.

Remark 1. In this paper we only consider polynomials over $\mathbb{C}$. However, Kollár's theorem holds for an arbitrary field. Berenstein and Yger, [4], have obtained variants of Brownawell's result for subfields of $\mathbb{C}$, by means of explicit integral formulas; see also [5] for a thorough discussion and more references.

In [9] Brownawell has given a prime power version of the Nullstellensatz which shed more geometric light on Kollár's theorem, and there is a generalization to smooth algebraic manifolds in [11]. □

Let $P_1, \ldots, P_m$ be polynomials in $\mathbb{C}^n$. If p_j denote homogenizations of P_j, i.e., $p_j(z) = z_0^{d_j} P_j(z'/z_0)$, where $d_j \geq \deg P_j$, then each p_j defines a global holomorphic section to the line bundle $L^{d_j} \to \mathbb{P}^n$, and hence $p = p_1 + \cdots + p_m$ is a section to the rank m bundle $E^* = L^{d_1} \oplus \cdots \oplus L^{d_m}$ over $\mathbb{P}^n$. If E^* is equipped with the natural hermitean structure, then

$$\|p(z)\|^2 = \sum_1^m \frac{|p_j(z)|^2}{|z|^{2d_j}}. \tag{1.3}$$

Following [2] we can define the residue current R^p to the section p, which is an element in $\oplus_\ell \mathcal{D}'_{0,\ell}(\mathbb{P}^n, \Lambda^\ell E)$ and with support on the zero set

$$Z^p = \{[z] \in \mathbb{P}^n;\ p(z) = 0\}.$$

If we assume that the polynomials P_j have no common zeros in $\mathbb{C}^n$, then of course Z^p is a subset of the hyperplane at infinity. If $\operatorname{codim} Z^p = m$, i.e., p is a complete intersection, then R^p is a $(0,m)$-current with values in $\det E = L^{-\sum d_j}$; more precisely a Coleff-Herrera current which formally can be written

$$R^p = \Big[\bar\partial \frac{1}{p_1} \wedge \ldots \wedge \bar\partial \frac{1}{p_m}\Big],$$

see Section 2.

Theorem 1.2. *Let $P_1, \dots, P_m$ be polynomials in $\mathbb{C}^n$, $\deg P_j \le d_j$, let $p = p_1 + \cdots + p_m$ be the corresponding section to $E^* = L^{d_1} \oplus \cdots \oplus L^{d_m}$ over $\mathbb{P}^n$, and let R^p be its residue current. Moreover, assume that*

$$m \le n \quad \text{or} \quad r \ge \sum_1^m d_j - n. \tag{1.4}$$

Let Φ be a polynomial, $\deg \Phi \le r$, and let $\phi \in \mathcal{O}(\mathbb{P}^n, L^r)$ denote its r-homogenization. If

$$\phi R^p = 0, \tag{1.5}$$

then there are polynomials Q_j such that

$$\Phi = \sum_1^m P_j Q_j$$

and $\deg(P_jQ_j) \le r$. If p is a complete intersection (then the condition (1.4) is fulfilled) and there are such polynomials Q_j, then the condition (1.5) holds.

It is clear that the conclusion about $\deg P_jQ_j$ cannot be improved. If $\Phi = 1$ the condition (1.5) means that P_j have no common zeros in $\mathbb{C}^n$ and that z_0^r annihilates the residue R^p at infinity. If Z^p is empty and $m = n+1$ we get a solution to (1.1) such that $\deg(P_jQ_j) \le \sum d_j - n$; this is a classical theorem of Macauley, [14].

If

$$\|\phi\| \le C\|p\|, \tag{1.6}$$

then, see [2], $\phi^{\min(m,n)} R^p = 0$, and hence we have

Corollary 1.3. *Let P_j and Φ be as in Theorem 1.2 and assume that*

$$m \le n \quad \text{or} \quad r \min(m,n) \ge \sum d_j - n.$$

If (1.6) holds, then there are polynomials Q_j such that

$$\sum P_j Q_j = \Phi^{\min(m,n)}$$

and $\deg(P_jQ_j) \le r \min(m,n)$.

Since there are examples where p is a complete intersection and the full power $\min(m,n)$ of ϕ is needed to kill R^p, this result is then sharp.

In particular, (1.2) means that P_j have no common zeros in $\mathbb{C}^n$ and that

$$\|z_0\|^M \le C\|p\|.$$

Thus $z_0^{M \min(m,n)} R^p = 0$ so we have

Corollary 1.4. *Let $P_1, \dots, P_m$ be polynomials in $\mathbb{C}^n$ of degrees d_j such that (1.2) holds for some number M, and assume that*

$$m \le n \quad \text{or} \quad M \min(m,n) \ge \sum d_j - n.$$

Then there is a solution to $\sum P_j Q_j = 1$ with $\deg(P_jQ_j) \le \min(m,n) M$.

Example 1. Let M be a given positive integer. In $\mathbb{C}^m$ we take $P_j = \zeta_j^{Mm}$ for $j = 1, \ldots, m$ and let $\Phi = (\zeta_1 + \cdots + \zeta_m)^{Mm}$. Then Z^p is just the origin, thus p is a complete intersection, and $\|\Phi\| \leq C\|P\|$. It is easily checked that $\Phi^m R^P = 0$ but $\phi^{m-1} R^p \neq 0$. One can just as well see directly that $\sum Q_j P_j = \Phi^m$ is solvable whereas $\sum Q_j P_j = \Phi^{m-1}$ is not. Thus the statement in Corollary 1.3 is optimal.

Taking $z_0 = \zeta_1 + \cdots \zeta_m$, $z_j = -\zeta_j$ for $j < m$, $p_j = z_j^{Mm}$ for $j < m$ and $p_m = (z_0 + z_1 + \cdots + z_{m-1})^{Mm}$, we have that $\|z_0\|^M \leq C\|p\|$. We need the power $Mm - m + 1$ of z_0 to kill R^p, which is close to $Mm = M\min(m, n)$ if M is large, and thus Corollary 1.4 is almost optimal. □

Remark 2. The condition (1.6) means that ϕ locally on $\mathbb{P}^n$ belongs to the integral closure of p. In [12], Hickel proves that if Φ is in the integral closure of P in $\mathbb{C}^n$, then one can solve (assuming $m \leq n$ for simplicity) $\Phi^m = \sum P_j Q_j$ with $\deg(P_j Q_j) \leq m\deg\Phi + md_1 \cdots d_m$. This result would follow from Theorem 1.2 if one could prove that the current $z_0^{md_1 \cdots d_m} \phi^m R^p$ vanishes (ϕ is the $\deg\Phi$ homogenization of Φ). In $\mathbb{C}^n$ it vanishes since $|\Phi| \leq C|P|$ locally. If the zero set is contained in $\{z_0 = 0\}$ the current vanishes there by Kollár's theorem. We do not know how one can see the general case. □

Theorem 1.2 is a special case of the following more general result, for which we formulate only the homogeneous version. Let δ_p denote the mapping $\mathcal{E}(\mathbb{P}^n, \Lambda^{\nu+1} E \otimes L^r) \to \mathcal{E}(\mathbb{P}^n, \Lambda^\nu E \otimes L^r)$ defined as interior multiplication with the section p to E^*. Thus for instance, if $q = q_1 + \ldots + q_m$ is a section to $E \otimes L^r$, then $\delta_p q$ is equal to the section $\sum_j p_j q_j$ to L^r.

Theorem 1.5. *Let p be holomorphic section to $E^* = L^{d_1} \oplus \cdots \oplus L^{d_m}$ and assume $\ell \geq 0$ is given and that*

$$m - \ell \leq n \quad \text{or} \quad r \geq \sum_1^m d_j - n.$$

If $\phi \in \mathcal{O}(\mathbb{P}^n, \Lambda^\ell E \otimes L^r)$, then $\phi = \delta_p \psi$ for some $\psi \in \mathcal{O}(\mathbb{P}^n, \Lambda^{\ell+1} E \otimes L^r)$ if and only if

$$\nabla_p (w \wedge R^p) = \phi \wedge R^p \tag{1.7}$$

for some smooth w defined in a neighborhood of Z^p.

If $\ell > m - p$ then (1.7) is void; if $\ell = m - p$, then (1.7) just means that $\phi \wedge R^f = 0$. If p is a complete intersection, then $m \leq n$ and therefore we have

Corollary 1.6. *Let p be a holomorphic section to $E^* = L^{d_1} \oplus \cdots \oplus L^{d_m}$ that is a complete intersection, and assume that $r \geq 0$. If $\phi \in \mathcal{O}(\mathbb{P}^n, L^r)$, then $\phi = p \cdot q$ is solvable with $q \in \mathcal{O}(\mathbb{P}^n, E \otimes L^r)$ if and only if $\phi R^p = 0$.*

Proof of Theorem 1.2. If the hypotheses in Theorem 1.2 are fulfilled, then Theorem 1.5 provides a section $q = q_1 + \ldots + q_m$ to $E \otimes L^r$ such that $\sum p_j q_j = \delta_p q = \phi$; here q_j are sections to L^{-d_j+r}. After dehomogenization this means that Q_j are polynomials such that $\deg P_j Q_j \le r$. □

In Section 2 we recall the necessary background from [2] about the residue currents, and present a general result about the image of a holomorphic morphism f. Combined with wellknown vanishing results for the line bundles $L^r \to \mathbb{P}^n$ it leads to a proof of Theorem 1.5.

In the last section we construct explicit integral representations of the solutions in Theorem 1.2. They give essentially the same results except for a small loss of precision. The construction is based on the ideas in [2] and [3].

2. The residue current of a holomorphic section

Let $E \to X$ be a holomorphic hermitean vector bundle and let f be a holomorphic section to the dual bundle E^*, or in other words, a holomorphic morphism $f\colon E \to X \times \mathbb{C}$. Let

$$\mathcal{L}^r = \bigoplus_\ell \mathcal{D}'_{0,l+r}(X, \Lambda^\ell E);$$

we consider $\mathcal{L}^r$ as a subbundle to $\Lambda(T^*_{0,1} \oplus E)$, so that δ_f (i.e., interior multiplication with f) and $\bar\partial$ anticommutes. Then $\nabla_f = \delta_f - \bar\partial$ induces the complex $\to \mathcal{L}^{r-1} \to \mathcal{L}^r \to$. Let s be the dual section to E of f so that in particular $\delta_f s = |f|^2$. In [2] we defined the current

$$R^f = \bar\partial |f|^{2\lambda} \wedge \frac{s}{\nabla_f s}\Big|_{\lambda=0};$$

for large $\operatorname{Re}\lambda$ the right hand side is integrable and therefore a welldefined current, and by a nontrivial argument based on Hironaka's theorem one can make an analytic continuation to $\lambda = 0$. The resulting current is an element in $\mathcal{L}^0$ with support on $Z^f = \{z;\ f(z) = 0\}$ and it satisfies the basic equality

$$\nabla_f U^f = 1 - R^f, \tag{2.1}$$

where $U \in \mathcal{L}^{-1}$ is defined as

$$U^f = |f|^{2\lambda} \frac{s}{\nabla_f s}\Big|_{\lambda=0}.$$

Moreover,

$$R^f = R^f_{c,c} + \ldots + R^f_{m,m}, \tag{2.2}$$

where $c = \operatorname{codim} Z^f$; here lower index p, q means a $\Lambda^p E$-valued $(0,q)$-form. It was also proved in [2] that $h^{\min(m,n)} R^f = 0$ if h is holomorphic and $|h| \leq C|f|$. Let $L \to X$ be a holomorphic line bundle and let ϕ be a holomorphic section to $\Lambda^k E \otimes L$.

Theorem 2.1. *Let $\ell \geq 0$ and suppose that $H^{0,s}(X, \Lambda^{s+\ell+1} E \otimes L) = 0$ for all $1 \leq s \leq m - \ell - 1$. Moreover, let $\phi \in \mathcal{O}(X, \Lambda^\ell E \otimes L)$. Then $\delta_f \psi = \phi$ has a solution $\psi \in \mathcal{O}((X, \Lambda^{\ell+1} E \otimes L)$ if and only if there is a smooth solution w, defined in a neighborhood of Z^f, to*

$$\nabla_f (w \wedge R^f) = \phi \wedge R^f. \tag{2.3}$$

In view of (2.2), the condition (2.3) is void if $\ell > m - c$; since $w = w_{\ell+1,0} + w_{\ell+2,1} + \cdots$ the condition means precisely that $\phi \wedge R^f = 0$ if $\ell = m - c$.

Proof. Suppose that the holomorphic solution ψ exists. Then $\nabla_f \psi = \phi$ and hence $\nabla_f (\psi \wedge R^f) = \phi \wedge R^f$ since $\nabla_f R^f = 0$. Conversely, if (2.3) holds, then $\nabla_f v = \phi$, where

$$v = (-1)^\ell \phi \wedge U^f + w \wedge R^f;$$

this means that

$$\bar\partial v_{m,m-\ell-1} = 0 \quad \text{and} \quad \delta_f v_{k+1,k-\ell} = \bar\partial v_{k,k-\ell-1}.$$

By the assumption on the Dolbeault cohomology, we can successively solve the equations

$$\bar\partial \eta_{m,m-\ell-2} = v_{m,m-\ell-1}, \quad \bar\partial \eta_{k,k-\ell-2} = v_{k,k-\ell-1} + \delta_f \eta_{k+1,k-\ell-1}, \quad k \geq \ell,$$

and then finally $\psi = v_{\ell,0} + \delta_f \eta_{\ell+1,0}$ is the desired holomorphic solution. □

Example 2. Suppose that X is a compact and L is a strictly positive line bundle. Then there is an $r_0 >$ such that $H^{0,k}(X, \Lambda E^\bullet \otimes L^r) = 0$ for all $k \geq 1$ if $r \geq r_0$. If f is a holomorphic section to E^*, then a holomorphic section $\phi \in \mathcal{O}(\Lambda^\ell E \otimes L^r)$, $r \geq r_0$, is in the image of the morphism

$$\mathcal{O}(X, \Lambda^{\ell+1} E \otimes L^r) \to \mathcal{O}(X, \Lambda^\ell E \otimes L^r) \tag{2.4}$$

if $\phi \wedge R^f = 0$. If $\ell = m - c$ the condition is necessary. □

We shall now focus on the case when $X = \mathbb{P}^n$ and E is the hermitean vector bundle from Section 1. Let $E_1, \ldots, E_m$ be trivial line bundles over $\mathbb{P}^n$ with basis elements $\epsilon_1, \ldots, \epsilon_m$, and let E_j^* be the dual bundles, with bases ϵ_j^*. Then we have that

$$E^* = L^{d_1} \otimes E_1^* \oplus \cdots \oplus L^{d_m} \otimes E_m^*,$$
$$E = L^{-d_1} \otimes E_1 \oplus \cdots \oplus L^{-d_m} \otimes E_m,$$

and for instance our section p can be written

$$p = \sum_1^m p_j \epsilon_j^*.$$

Its dual section s is then, cf., (1.3),

$$s = \sum_j \frac{\overline{p_j(z)}}{|z|^{2d_j}} \epsilon_j,$$

so

$$R^p = \bar\partial \|p\|^{2\lambda} \wedge \sum_{\ell+1}^{m} \frac{s \wedge (\bar\partial s)^{\ell-1}}{\|p\|^{2\ell}}\Big|_{\lambda=0}.$$

In $\mathbb{C}^n = \{z_0 \neq 0\} \subset \mathbb{P}^n$ we have the coordinates z' and the natural holomorphic frame $e_j = z_0^{-d_j} \epsilon_j$ and its dual $e_j^* = z_0^{d_j} \epsilon_j^*$. If $p_j'(z') = p_j(1, z')$ then

$$p = \sum_1^m p_j' e_j^*$$

and

$$s = \sum_1^m \frac{\overline{p_j'(z')}}{(1 + |z'|)^{d_j}} e_j.$$

When $\operatorname{codim} Z^p = m$, the residue current R^p is independent of the metric, it just contains the top degree term $R^p_{m,m}$, and in fact, see [2],

$$R^p = \big[\bar\partial \frac{1}{p_m'} \wedge \ldots \wedge \bar\partial \frac{1}{p_1'}\big] \wedge e_1 \wedge \ldots \wedge e_m,$$

where the expression in brackets is the Coleff-Herrera residue current which we can rewrite formally as

$$\big[\bar\partial \frac{1}{p_m} \wedge \ldots \wedge \bar\partial \frac{1}{p_m}\big] \wedge \epsilon_1 \wedge \ldots \wedge \epsilon_m. \tag{2.5}$$

Proof of Theorem 1.5. It is wellknown, see, e.g., [10], that $H^{0,k}(\mathbb{P}^n, L^\nu) = 0$ for all ν if $1 \le k \le n-1$ and that $H^{0,n}(\mathbb{P}^n, L^\nu) = 0$ if $\nu \ge -n$. Since $E = L^{-d_1} \oplus \cdots \oplus L^{-d_m}$ we have that

$$\Lambda^\nu E \oplus L^r = \bigoplus_{|J|=\nu}' L^{-d_{J_1}} \otimes \cdots \otimes L^{-d_{J_\nu}} \otimes L^r = \bigoplus_{|J|=\nu}' L^{r-d_{J_1} \cdots - d_{J_\nu}}.$$

Thus $H^{0,s}(\mathbb{P}^n, \Lambda^{s+\ell+1} E \otimes L^r) = 0$ for $1 \le s \le m - \ell - 1$ if either $m - \ell - 1 \le n - 1$ or $r - \sum d_j \ge -n$. Now Theorem 1.5 follows from Theorem 2.1 with $f = p$. □

3. Integral representation

The aim of this section is to present an explicit integral representation of the solution Q_j to the division problem in Theorem 1.2. We have

Theorem 3.1. *Let $P_1, \dots, P_m, \Phi$ be polynomials in $\mathbb{C}^n$, let p and R^p be as before, and let ϕ be the r-homogenization of Φ ($deg\,\Phi \le r$). Then there is an explicit decomposition*

$$\Phi(z') = \sum_1^m P_j(z') \int_{\mathbb{P}^n} T^j(\zeta, z')\phi(\zeta) + \int_{\mathbb{P}^n} S(\zeta, z') \wedge R^p(\zeta)\phi(\zeta), \tag{3.1}$$

where $T^j(\zeta, z'), S(\zeta, z')$ are smooth forms (in $[\zeta]$) on $\mathbb{P}^n$ and holomorphic polynomials in z', such that

$$deg_{z'}\big(P_j(z')T^j(\zeta, z')\big) \le d_1 + d_2 + \cdots + d_{\mu+1} + r,$$

if $\mu = \min(n, m-1)$ and $d_1 \ge d_2 \ge \cdots \ge d_m$.

Thus, if $\phi R^p = 0$ we get back the conclusion of Theorem 1.2 but with the extra term $d_1 + \cdots + d_{\mu+1}$ in the estimate of the degree.

For fixed $z \in \mathbb{C}^n$,

$$\eta = 2\pi i \sum_0^n z_j \frac{\partial}{\partial \zeta_j}$$

is an $L_z \otimes L_\zeta^{-1}$-valued $(1,0)$-form on $\mathbb{P}^n$, and if δ_η denotes interior multiplication with η, then

$$\delta_\eta \colon \mathcal{D}'_{\ell+1,0}(\mathbb{P}^n, L^{r+1}) \to \mathcal{D}'_{\ell,0}(\mathbb{P}^n, L^r).$$

Remark 3. When we say that η is a section to $L_z \otimes L_\zeta^{-1}$ rather than $L^{-1} = L_\zeta^{-1}$, we just indicate that it is 1-homogeneous in z; it would be more correct, but less convenient, to consider η as a section to the bundle $L_z \otimes L^{-1}\zeta \otimes (T^*_\zeta)_{0,1}$ over $\mathbb{P}^n_z \times \mathbb{P}^n_\zeta$. □

Let $\nabla_\eta = \delta_\eta - \bar\partial$. Notice that if

$$\alpha = \alpha_0 + \alpha_1 = \frac{z \cdot \bar\zeta}{|\zeta|^2} - \bar\partial \frac{\bar\zeta \cdot d\zeta}{2\pi i |\zeta|^2},$$

then the first term, α_0, is a section to $L_z \otimes L_\zeta^{-1}$ and the second term, α_1, is a projective form (since $\delta_\zeta \alpha_1 = 0$); moreover

$$\nabla_\eta \alpha = 0. \tag{3.2}$$

We have the following basic integral representation of global holomorphic sections to L^r.

Proposition 3.2. *Assume that $r \ge 0$ and that $\phi \in \mathcal{O}(\mathbb{P}^n, L^r)$. Then*

$$\phi(z) = \int_{\mathbb{P}^n} \alpha^{n+r} \phi.$$

For degree reasons, actually

$$\phi(z) = \frac{(n+r)!}{n!r!} \int_{\mathbb{P}^n} \alpha_0^r \wedge \alpha_1^n;$$

this formula has appeared at several places, and expressed in affine coordinates it is a well-known representation formula for polynomials in $\mathbb{C}^n$. However, we prefer to supply a direct proof on $\mathbb{P}^n$, following the ideas in [1].

Proof. Let σ be the $L_z^{-1} \otimes L_\zeta \otimes T^*_{1,0}(\mathbb{P}^n_\zeta)$ valued $(1,0)$-form on $\mathbb{P}^n$ that is dual, with respect to the natural metric, to η. Then, since η has a first order zero at $[z]$ (and no others), it follows (see, e.g., [1]) that

$$\nabla_\eta \frac{\sigma}{\nabla_\eta \sigma} = 1 - \big[[z]\big].$$

The rightmost term is the $L_z^{-n} \otimes L_\zeta^n$-valued (n,n)-current point evaluation at $[z]$ for sections to L^{-n}. If ϕ is a global holomorphic section to L^r it follows by (3.2) that

$$\nabla_\eta \big(\frac{\sigma}{\nabla_\eta \sigma} \wedge \alpha^{n+r} \phi \big) = \phi \alpha^{n+r} - \phi\big[[z]\big],$$

where this time the last term is ϕ times the $L_z^r \otimes L_\zeta^{-r}$-valued current point evaluation at $[z]$. If we integrate this equality over $\mathbb{P}^n$ we get the desired representation formula. □

Let $E_1, \ldots, E_m$ be the trivial line bundles over $\mathbb{P}^n$ with basis elements $\epsilon_1, \ldots, \epsilon_m$, so that $E = L^{-d_1} \otimes E_1 \oplus \cdots \oplus L^{-d_m} \otimes E_m$ as in Section 2. We also introduce disjoint copies $\widetilde{E}_j$ of E_j with bases $\tilde{\epsilon}_j$ and the bundle

$$\widetilde{E} = L^{-d_1} \otimes \widetilde{E}_1 \oplus \cdots \oplus L^{-d_m} \otimes \widetilde{E}_m.$$

Let Λ be the exterior algebra bundle over the direct sum of all the bundles E, $\widetilde{E}$, E^*, and $T^*(\mathbb{P}^n)$. Any form γ with values in Λ can be written uniquely as $\gamma = \gamma' \wedge (\sum \epsilon_j^* \wedge \epsilon_j)^m/m! + \gamma''$ where γ'' denotes terms that do not contain a factor $(\sum \epsilon_j^* \wedge \epsilon_j)^m/m!$, and we define

$$\int_\epsilon \gamma = \gamma'.$$

We have a globally defined form

$$\tau = \sum_1^m \epsilon_j^* \wedge (\epsilon_j - \tilde{\epsilon}_j).$$

From now on we consider $[z]$ as a fixed arbitrary point in $\mathbb{C}^n \subset \mathbb{P}^n$, and let $z = (1, z')$. We also introduce the section

$$p_z = \sum_j \zeta_0^{d_j} p_j(1, z) \epsilon_j^* = \sum \zeta_0^{d_j} P_j(z') \epsilon_j^*$$

to E^* and let $\tilde{p}_z$ be the corresponding section to $\tilde{E}^*$.

Lemma 3.3. *There is a holomorphic section* $H = \sum H_j \wedge \epsilon_j$ *to* $E^* \otimes L \otimes T^*_{1,0}$, *thus* H_j *are sections to* $L^{d_j} \otimes L \otimes T^*_{1,0}$, *such that*

$$\delta_\eta H = p - p_z,$$

and such that the coefficients in H_j *are polynomials in* z'/z_0 *of degrees (at most)* $d_j - 1$.

Proof. For each $P_j(z')$ we can find Hefer functions $h^k_j(\zeta', z')$, polynomials of degree $d_j - 1$ in (ζ', z'), such that

$$\sum_{k=1}^{n} h^k_j(\zeta', z')(\zeta_k - z_k) = P_j(\zeta') - P_j(z').$$

If we then take

$$H_j = \frac{\zeta_0^{d_j+1}}{2\pi i} \sum_1^n h^k_j(\zeta'/\zeta_0, z') d(\zeta_k/\zeta_0),$$

then clearly H_j is a projective $(1,0)$-form, and moreover,

$$\delta_\eta H_j = p_j(\zeta) - \zeta_0^{d_j} P_j(z')$$

as wanted. □

Let δ_F denote interior multiplication with the section $F = p + \tilde{p}_z$ to $E^* \oplus \tilde{E}^*$. Then $\delta_F \tau = p - p_z = -\delta_\eta H$. If

$$\nabla = \delta_F + \delta_\eta - \bar\partial,$$

thus

$$\nabla(\tau + H) = 0. \tag{3.3}$$

We are now ready to define the explicit division formula.

Proof of Theorem 3.1. From (3.3) it follows that

$$(\nabla_\eta + \delta_F)(e^{\tau+H} \wedge U^p) = e^{\tau+H} \wedge (1 - R^p). \tag{3.4}$$

We can rewrite this as

$$\delta_F(e^{\tau+H} \wedge U^p) + e^{\tau+H} \wedge R^p = e^{\tau+H} - \nabla_\eta(e^{\tau+H} \wedge U^p). \tag{3.5}$$

We claim that the component of full bidegree (n, n) of

$$\int_\epsilon \left[e^{\tau+H} - \nabla_\eta(e^{\tau+H} \wedge U^p)\right] \wedge \alpha^{n+r} \phi \tag{3.6}$$

is equal to

$$\frac{(n+r)!}{n!r!} \alpha_1^n \alpha_0^r \phi + \bar\partial(\cdots)$$

where $(\cdots)$ is a scalarvalued $(n, n-1)$-form. In fact, since α^{n+r} has bidegree $(*,*)$ the factor $U_{\ell,\ell-1}$ must be combined with H_ℓ, and then it follows that τ can be replaced by $\omega = \sum_j \epsilon^*_j \wedge \epsilon_j$. Observe that the component of $U_{\ell,\ell-1}$ with basis element $\epsilon_{J_1} \wedge \ldots \wedge \epsilon_{J_\ell}$ takes values in $L^{-(d_{J_1} + \cdots + d_{J_\ell})}$, whereas the component of H_ℓ with basis element $\epsilon^*_{J_1} \wedge$

$\ldots \wedge \epsilon_{J_\ell}^*$ takes values in $L^{d_{J_1}+\cdots+d_{J_\ell}} \otimes L^\ell$. The product of these two factors must be combined with $\alpha_1^{n-\ell}\alpha_0^{\ell+r}\phi$ which gives a scalar valued (n,n)-form as claimed. Thus we can integrate (3.6) over $\mathbb{P}^n$, and by Proposition 3.2 and Stokes' theorem it is equal to $\phi(z)$.

We now consider the left hand side of (3.5) multiplied with $\alpha^{n+r}\phi$. To begin with,

$$\int_{\mathbb{P}^n}\int_\epsilon e^{\tau+H} \wedge R^p \wedge \alpha^{n+r}\phi$$

is welldefined with the same argument as above, and again one can replace τ by ω. Moreover, since $\alpha^{n+r}\phi$ contains no ϵ_j,

$$\int_\epsilon \delta_p(e^{\tau+H} \wedge U^p) \wedge \alpha^{n+r}\phi = \int_\epsilon \delta_p(e^{\tau+H} \wedge U^p \wedge \alpha^{n+r}\phi) = 0.$$

Since

$$\delta_{\tilde{p}_z} \sum_j \tilde{\epsilon}_j \wedge \epsilon_j^* = \sum_j P(z')\zeta_0^{d_j}\epsilon_j^* = p_z,$$

another computation shows that the component of bidegree (n,n) of

$$\int_\epsilon \delta_{\tilde{p}_z}(e^{\tau+H} \wedge U^p) \wedge \alpha^{n+r}\phi$$

is equal to

$$\int_\epsilon p_z \wedge \sum_{k=0}^{m-1} \omega_{m-k-1} \wedge H_k \wedge U_{k+1,k} \wedge \alpha_1^{n-k}\alpha_0^{k+r}\phi.$$

Again one can check that this form is scalar valued. Summing up we have the desired decomposition (3.1) with

$$S(\zeta,z') \wedge R^p(\zeta) = \int_\epsilon e^{\omega+H} \wedge R^p \wedge \alpha^{n+r} =$$
$$\sum_{k=\operatorname{codim} Z^p}^{m} \int_\epsilon \frac{(n+r)!}{(n-k)!(k+r)!}\omega_{m-k} \wedge H_k \wedge R^p_{k,k}\alpha_1^{n-k}\alpha_0^{k+r},$$

and

$$T^j(\zeta,z') =$$
$$\int_\epsilon \epsilon_j^*\zeta_0^{d_j} \wedge \sum_{k=1}^{m-1} \frac{(n+r)!}{(n-k)!(k+r)!}\tilde{I}_{n-k-1} \wedge H_k \wedge U_{k+1,k} \wedge \alpha_1^{n-k}\alpha_0^{k+r}\phi,$$

Both α and H are polynomials in z' so it just remains to check the degrees of T^j. The worst case occur when k is as large as possible which is $k=\mu=\min(m-1,n)$. Then the factor α_0^{k+r} has degree $k+r$. Recall that $H=\sum H_\ell \wedge \epsilon_\ell^*$ and that $\deg H_\ell = d_\ell - 1$. The term H_j cannot occur, because of the presence of ϵ_j^*, and thus we get that $d_j + \deg Q_j$ is at most $d_1 - 1 + d_2 - 1 + \cdots d_{\mu+1} - 1 + 1 + \mu + r = d_1 + \cdots + d_{\mu+1} + r$. □

The main novelty with these formulas is that they contains the global residues R^p and U^p. However, even if we suppress the residues by assuming ϕ is vanishing enough, and consider them as formulas in $\mathbb{C}^n$, they differ from the formulas from [6] and admit sharper estimates of the degrees, see the discussion in [3].

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