Characterization of the Geometric and Exponential Random Variables

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Abstract. Let the random variable X be distributed over the non-negative integers and let L_m and R_m be the quotient and the remainder in the division of X by m. It is shown that X is geometric if and only if L_m and R_m are independent for $m = 2, 3, \ldots$. In similar terms is characterized also the exponential random variable.

Keywords: geometric distribution, exponential distribution, characterization.

1 Introduction.

A random variable X is said to be geometric with parameter p if

$$P\{X = k\} = p(1-p)^k, \quad k = 0, 1, \ \ldots \ , \ 0 < p < 1,$$

and exponential with parameter λ if

$$P\{X \leq x\} = 1 - e^{x/\lambda}, \quad x > 0, \quad \lambda > 0.$$

It is well known that only the geometric and exponential distributions possess the lack of memory property

$$P\{X \leq t + x \mid X > t\} = P\{X \leq x\}, \quad x > 0, \quad t > 0,$$

which makes these distributions play a special role in stochastic modelling. In particular, any characterization property of these distributions, apart from being possibly of theoretical interest, may turn out to be useful in applications by providing another view to the lack of memory property.

There are many works dealing with characterization of the geometric and exponential distributions, with [1]–[12] being just a few examples, of which [2], [7], and [8] contain extensive bibliographies in the matter.

The idea of characterizations presented here comes from communication. Suppose the terminal produces messages according to a sequence of independent Bernoulli trials. The length X of a message is then described by a geometric random variable. For transmission, the message is broken up into packets of $m \geq 2$ bytes. Let L_m be the number of full packets and let R_m be the number of bytes left over. Then for $l = 0, 1, \ldots$ and $r = 0, 1, \ldots, m-1$ we have

$$P\{L_m = \ell,\ R_m = r\} = P\{X = m\ell + r\}$$
$$= p(1-p)^{m\ell+r} = (1-(1-p)^m)(1-p)^{m\ell}\frac{p(1-p)^r}{1-(1-p)^m}.$$

The last expression above is a product of two probability mass functions, the first one of the geometric distribution with parameter $1-(1-p)^m$ and the second one of the truncated geometric distribution on $\{0, 1, \ldots, m-1\}$. Thus the random variables L_m and R_m are independent, i.e.,

$$\begin{gathered} P\{L_m = \ell,\ R_m = r\} = P\{L_m = \ell\}P\{R_m = r\}, \\ \ell = 0, 1, \ldots \text{ and } r = 0, 1, \ldots, m-1, \end{gathered} \tag{1}$$

and have probability mass functions

$$P\{L_m = \ell\} = (1-(1-p)^m)(1-p)^{m\ell},\ \ell = 0, 1, \ldots,$$

and

$$P\{R_m = r\} = \frac{p(1-p)^r}{1-(1-p)^m}, \quad r = 0, 1, \ldots, m-1,$$

respectively. In this way, for a geometric random variable X the quotient L_m in the division of X by the integer $m \geq 2$ and the remainder R_m are independent random variables. It is now natural to ask whether there exist non-geometric random variables, distributed over the non-negative integers, with the same property. The answer to this question turns out to be negative for random variables with at least three values with positive probabilities. In fact, we will show in Theorem 1 that if the quotient and the remainder in the division by $m \geq 2$ of a non-negative integer-valued random variable with at least three values with positive probabilities are independent when m equals two or three, and satisfy the equalities (1) for all $\ell = 0, 1, \ldots$ but only for $r = 0, 1, 2$ when $m \geq 4$, then the random variable is geometric. This of course would imply, as we saw above, the independence of the quotient and the remainder for any $m \geq 2$. Thus although the conditions of Theorem 1 seem to be weaker than the independence of the quotient and the remainder for $m \geq 2$, they are actually not and then, as stated in Theorem 2, this independence property makes the geometric random variable unique in the class of all non-negative, integer-valued random variables with at least three values with positive probabilities.

One interesting question regarding the above theorems is whether they remain true without the assumption that the random variable under consideration has at least three values with positive probabilities. They do not, in fact, and this is shown by the example following Theorem 2.

Since only the geometric and the exponential distributions possess the memoryless property among the discrete and the continuous distributions, respectively, the geometric distribution is considered as a discrete analogue of the exponential one, and many studies have been made on the relationship between these distributions, see, e.g. [2] and [4–6]. One particular result from [2] is a characterization of the exponential distribution by the geometric distribution. We give this result in Theorem 3 and then use it together with the characterization from Theorem 2 of the geometric distribution to characterize in Theorem 4 also the exponential distribution.

2 Characterization of the geometric random variable

We shall show the following theorem. Recall that the floor function $\lfloor x \rfloor$ maps $x \in R$ onto the greatest integer not greater than x.

Theorem 1. *Let the non-negative integer-valued random variable X have at least three values with positive probabilities, and let L_m and R_m denote the quotient and the remainder of the division of X by m, respectively:*

$$L_m = \left\lfloor \frac{X}{m} \right\rfloor, \qquad R_m = X - mL_m.$$

If the random variables L_m and R_m are independent when $m = 2,\ 3$, and if when $m \geq 4$ the events $\{L_m = \ell\}$ and $\{R_m = r\}$ are independent for $\ell = 0,\ 1,\ \ldots$ and $r = 0,\ 1,\ 2$ then X is geometric.

Before the proof we introduce some notation and show two lemmas. Assume that the random variable X satisfies the conditions of the theorem, and introduce the probability mass functions of X and R_m

$$p_k = P\{X = k\}, \quad k = 0, 1, \dots, \tag{2}$$

and

$$\beta_m(r) = P\{R_m = r\}, \quad r = 0, 1, \dots, m-1,$$

respectively. Since

$$P\{L_m = \ell, R_m = r\} = P\{X = m\ell + r\}$$

the conditions of the theorem imply

$$\begin{aligned} &\beta_m(r) = \frac{p_{m\ell+r}}{P\{L_m = \ell\}} \quad \text{for all} \quad \ell \quad \text{such that} \quad P\{L_m = \ell\} > 0 \\ &\text{and} \quad r = 0,\,1 \quad \text{when} \quad m = 2, \quad r = 0,\,1,\,2 \quad \text{when} \quad m \geq 3. \end{aligned} \tag{3}$$

Lemma 1. *The probabilities p_k in the probability mass function (2) of the random variable X are positive for $k = 0,\,1,\,\dots$.*

Proof. Let $s \geq 0$ and $n > s$ be the smallest number and the next to the smallest, respectively, among those numbers k for which $p_k > 0$. We consider two cases, depending on the value of n.

Case 1: n=1

We get $s = 0$ in this case, thus $p_0 > 0$ and $p_1 > 0$. According to the conditions of the theorem, there is at least one more positive probability in (2). Let the number $v \geq 2$ be the smallest with $p_v > 0$. We must have $v = 2$, because otherwise $v - 1 \geq 2$ and we would then have

$$P\{L_{v-1} = 0\} = P\{0 \leq X < v - 1\} > p_0 > 0,$$

$$P\{L_{v-1} = 1\} = P\{v - 1 \leq X < 2(v-1)\} \geq p_v > 0,$$

which together with $p_{v-1} = 0$ and (3) would give the contradiction

$$\beta_{v-1}(0) = \frac{p_0}{P\{L_{v-1} = 0\}} > 0$$

and

$$\beta_{v-1}(0) = \frac{p_{v-1}}{P\{L_{v-1} = 1\}} = 0.$$

Thus the first three probabilities in (2) are positive in this case. We will show now that the rest of them are positive as well. Assume that $p_k > 0$ for $k = 0, \ldots, u$, where $u \geq 3$. From

$$P\{L_u = 0\} = P\{0 \leq X < u\} > p_1 > 0,$$

$$P\{L_u = 1\} = P\{u \leq X < 2u\} \geq p_u > 0,$$

and (3) we obtain

$$\beta_u(1) = \frac{p_1}{P\{L_u = 0\}} = \frac{p_{u+1}}{P\{L_u = 1\}}$$

which implies that also $p_{u+1} > 0$. By induction, $p_k > 0$ for $k > u$.

In this way, all probabilities p_k in (2) are positive in the present case.

Case 2: $n \geq 2$.

From

$$P\{L_n = 0\} = P\{0 \leq X < n\} = p_s > 0,$$

$$P\{L_n = 1\} = P\{n \leq X < 2n\} \geq p_n > 0,$$

we get

$$\beta_n(0) = \frac{p_0}{P\{L_n = 0\}} = \frac{p_n}{P\{L_n = 1\}}$$

and since $p_n > 0$ we must have $p_0 > 0$ as well, which implies $s = 0$. Now, the values $n \geq 3$ are impossible, since for these values $n - 1 \geq 2$ and we would then have

$$P\{L_{n-1} = 0\} = P\{0 \leq X < n - 1\} = p_0 > 0,$$

$$P\{L_{n-1} = 1\} = P\{n - 1 \leq X < 2(n - 1)\} \geq p_n > 0,$$

which by (3) would give the contradiction

$$\beta_{n-1}(1) = \frac{p_1}{P\{L_{n-1} = 0\}} = 0$$

and

$$\beta_{n-1}(1) = \frac{p_n}{P\{L_{n-1} = 1\}} > 0.$$

In this way, it is impossible in this case that $n \geq 3$, and we have to assume that $n = 2$. Now, there are at least three values of X with positive probabilities, of which 0 and 2 are the smallest ones. Let $w \geq 3$ be the next smallest value for which $p_w > 0$. Since $w - 1 \geq 2$ we have

$$P\{L_{w-1} = 0\} = P\{0 \leq X < w - 1\} \geq p_0 > 0,$$

$$P\{L_{w-1} = 1\} = P\{w - 1 \leq X < 2(w - 1)\} \geq p_w > 0,$$

which implies the contradiction

$$\beta_{w-1}(1) = \frac{p_1}{P\{L_{w-1} = 0\}} = 0$$

and

$$\beta_{w-1}(1) = \frac{p_w}{P\{L_{w-1} = 1\}} > 0.$$

Thus neither $n = 2$ is possible here, i.e., the present case is not possible at all.

In this way, only *Case* 1 ($n = 1$) takes place and then, as shown there, all p_k in (2) are positive. □

We can now write the conditions (3) as

$$\frac{p_{m\ell} + \ldots + p_{m(\ell+1)-1}}{p_{m\ell+r}} = \frac{1}{\beta_m(r)} \quad \text{for} \quad \ell = 0, 1, \ldots \tag{4}$$
$$\text{and} \quad r = 0, 1 \quad \text{when} \quad m = 2, \quad r = 0, 1, 2 \quad \text{when} \quad m \geq 3.$$

Lemma 2. *The probabilities p_k in the probability mass function (2) of the random variable X satisfy the relationship $p_k = c^k p_0$, $k = 1, 2, \ldots$.*

Proof. By setting $m = 2$ and $r = 0$ in (4) we obtain

$$\frac{p_{2k} + p_{2k+1}}{p_{2k}} = \frac{1}{\beta_2(0)}, \quad k = 0, 1, \ldots .$$

Let $c = \frac{1}{\beta_2(0)} - 1$. Thus $p_{2k+1} = cp_{2k}, \quad k \geq 0$, and the sequence of probabilities in the probability mass function (2) of X is of the form

$$p_0, \quad cp_0, \quad p_2, \quad cp_2, \quad p_4, \quad cp_4, \quad \ldots \quad p_{2k}, \quad cp_{2k}, \quad \ldots \quad . \tag{5}$$

We will now use (5) in (4). With $m = 3, \quad r = 1, \quad \ell = 0$ and then $\ell = 1$ (4) gives

$$\frac{p_0 + cp_0 + p_2}{cp_0} = \frac{cp_2 + p_4 + cp_4}{p_4}, \tag{6}$$

and with $m = 3, \quad r = 2, \quad \ell = 0$ and then $\ell = 1$ it gives

$$\frac{p_0 + cp_0 + p_2}{p_2} = \frac{cp_2 + p_4 + cp_4}{cp_4}.$$

Dividing the above two inequalities we obtain $p_2 = c^2 p_0$ and then from (6) we get $p_4 = c^2 p_2 = c^4 p_0$ and from (5) $p_3 = c^3 p_0$ and $p_5 = c^5 p_0$.

In this way, using (4) with the values $m = 2$ and $m = 3$ we have shown that $p_k = c^k p_0$ for $k = 1, \ldots, 5$.

We complete the proof of the lemma by induction. Let $m_0 \geq 3$ be fixed and assume that

$$p_k = c^k p_0, \quad k = 1, \ldots, 2m_0 - 1. \tag{7}$$

This, (4) with $m = m_0 + 1$, $r = 0$ and first with $\ell = 0$ then with $\ell = 1$, and (5) give

$$\frac{p_0 + cp_0 + \ldots + c^{m_0-2}p_0 + c^{m_0-1}p_0 + c^{m_0}p_0}{p_0}$$

$$= \frac{c^{m_0+1}p_0 + c^{m_0+2}p_0 + \ldots + c^{2m_0-1}p_0 + p_{2m_0} + cp_{2m_0}}{c^{m_0+1}p_0},$$

or

$$1 + c + \ldots + c^{m_0-2} + c^{m_0-1} + c^{m_0} = 1 + c + \ldots + c^{m_0-2} + \frac{(1+c)}{c^{m_0+1}} \frac{p_{2m_0}}{p_0},$$

which implies $p_{2m_0} = c^{2m_0} p_0$. By this and (5) again, $p_{2m_0+1} = c^{2m_0+1} p_0$. Thus the assumption (7) holds for $m_0 + 1$ as well. By induction, (7) is true for any $k \geq 1$ and the lemma is proved. □

We are now ready for the

Proof of Theorem 1

Since by Lemma 2 the probabilities p_k in the probability mass function (2) of X satisfy $p_k = c^k p_0, \quad k = 1, 2, \ldots$, the number c must be equal to $1 - p_0$ and X is then geometric with parameter p_0. □

From Theorem 1 and (1) we obtain the following characterization of the geometric random variable in terms of the quotient and the remainder of division by a positive integer.

Theorem 2. *A non-negative integer valued random variable X with at least three values with positive probabilities is geometric if and only if the quotient of the division of X by m, $L_m = \left\lfloor \frac{X}{m} \right\rfloor$, and the remainder $R_m = X - mL_m$ are independent for all $m \geq 2$.*

The following example shows that theorems 1 and 2 above do not remain true when dropping the assumption that the random variable under consideration has at least three values with positive probabilities.

Example. Consider a random variable X with two values, 0 and 2, taken with probabilities $p_0 > 0$ and $p_2 = 1 - p_0 > 0$, respectively. It is easy to see that for this random variable $P\{R_2 = 0\} = 1$ and $P\{L_m = 0\} = 1$ when $m \geq 3$. Thus the random variables L_m and R_m are independent for all $m \geq 2$, although X is not geometric.

3 Characterization of the exponential random variable

One particular result on the relationship between the geometric and exponential distributions presented in [2] is the following.

Let X be a non-negative random variable with distribution function $F(x)$ such that $0 < F(x) < 1$ for $x > 0$. For a specified number $t > 0$ consider the random variable tX and introduce its discrete analogue

$$Y_t = \lfloor tX \rfloor = kI\{k \leq tX < k+1\}, \quad k = 0, 1, \dots , \tag{8}$$

taking values $k = 0, 1, \dots$ with probabilities

$$P\{Y_t = k\} = P\{k \leq tX < k+1\}.$$

Theorem 3. *The random variable X is exponential if and only if the discrete analogues Y_t of tX are geometric for all $t > 0$.*

The proof of this result may be found in [2], p. 81. □

We now will use the above characterization and the characterization of the geometric random variable obtained in Theorem 2 to characterize in a similar way also the exponential distribution.

Let X be a non-negative random variable and Y be its discrete analogue as defined in (8):

$$Y = \lfloor X \rfloor = kI\{k \leq X < k+1\}, \quad k = 0, 1, \ldots .$$

As in the previous section, we let L_m and R_m denote the quotient and the remainder in the division of X by the integer $m \geq 2$.

Lemma 3. *Let X be a non-negative random variable with distribution function $F(x)$ such that $0 < F(x) < 1$ for $x > 0$, and let Y be the discrete analogue of X. If the random variables $L_m = \left\lfloor \frac{X}{m} \right\rfloor$ and $R_m = X - mL_m$ are independent for all $m \geq 2$, then the random variable Y is geometric.*

Proof. Denote by L'_m and R'_m the quotient and the remainder in the division of the random variable Y by m. For $m \geq 2$ and $\ell = 0, 1, \ldots, r = 0, 1, \ldots, m-1$, we have

$$P\{L'_m = \ell,\, R'_m = r\} = P\{Y = m\ell + r\}$$

$$= P\{m\ell + r \leq X < m\ell + r + 1\} = P\{L_m = \ell,\ r \leq R_m < r+1\}$$

$$= P\{L_m = \ell\} P\{r \leq R_m < r+1\},$$

where we have used the independence of the random variables L_m and R_m and where the term $P\{r \leq R_m < r+1\}$ in the last line does not involve ℓ. Consequently,

$$P\{R'_m = r\} = \sum_{\ell=0}^{\infty} P\{L'_m = \ell,\, R'_m = r\}$$

$$= P\{r \leq R_m < r+1\} \sum_{\ell=0}^{\infty} P\{L_m = \ell\} = P\{r \leq R_m < r+1\}$$

and hence

$$P\{L'_m = \ell\} = P\{L_m = \ell\}$$

as well. Thus the random variables L'_m and R'_m are independent for $m \geq 2$. Since the distribution function of X is such that $0 < F(x) < 1$ for $x > 0$, the discrete analogue of X cannot have less than three values with positive probabilities. In this way, the random variable Y satisfies the conditions of Theorem 2 and is then geometric. □

Theorem 4. *A non-negative random variable X with distribution function $F(x)$ such that $0 < F(x) < 1$ for $x > 0$ is exponential if and only if the random variables*

$$L_m^t = \left\lfloor \frac{tX}{m} \right\rfloor \quad \textit{and} \quad R_m^t = tX - mL_m^t$$

are independent for all $m \geq 2$ and $t > 0$.

Proof. Assume first that If X is exponential with parameter λ and let $t > 0$. The random variable tX is exponential with parameter $t\lambda$ and we have for all $m \geq 2$ that

$$P\{L_m^t = \ell,\ R_m^t < r\} = P\{m\ell \leq tX < m\ell + r\}$$
$$= e^{-\frac{m\ell}{t\lambda}}(1 - e^{-\frac{r}{t\lambda}}) = (1 - e^{-\frac{m}{t\lambda}})e^{-\frac{m\ell}{t\lambda}}\frac{1 - e^{-\frac{r}{t\lambda}}}{1 - e^{-\frac{m}{t\lambda}}}, \quad \ell = 0, 1, \ldots, \quad 0 \leq r < m,$$

showing that L_m^t is geometric with parameter $1 - e^{-\frac{m}{t\lambda}}$, R_m^t is truncated exponential with parameter $t\lambda$ in $[0, m)$, and that L_m^t and R_m^t are independent.

Assume now that L_m^t and R_m^t are independent for all $m \geq 2$ and $t > 0$. The random variable tX satisfies the conditions of Lemma 3 and its discrete analogue Y_t is then geometric. Then by Theorem 3 the random variable X is exponential. □

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