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Operator Synthesis II. Individual Synthesis and Linear Operator Equations

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Operator synthesis II. Individual synthesis and linear operator equations

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Abstract

The second part of our work on operator synthesis deals with individual operator synthesis of elements in some tensor products, in particular in Varopoulos algebras, and its connection with linear operator equations. Using a developed technique of "approximate inverse intertwining" we obtain some generalizations of the Fuglede and the Fuglede-Weiss theorems and solve some problems posed in [O, W2, W3]. Additionally, we give some applications to spectral synthesis in Varopoulos algebras and to partial differential equations.

1 Introduction

This work is a sequel of [ShT] where the problems of operator synthesis were treated "globally" for lattices of subspaces, bilattices, or, in coordinate setting, for subsets of direct products of measure spaces. Here we consider operator-synthetic properties of elements of some tensor products, first of all of the Varopoulos algebras $V(X,Y) = C(X)\hat{\otimes}C(Y)$. The topic is deeply connected to the theory of linear operator equations and, more generally, to the spectral theory of multiplication operators in the space of bounded operators and in symmetrically normed ideals of operators. We obtain some extensions of the Fuglede and Fuglede-Weiss theorems, answer several questions posed in [O, W2, W3], give applications to spectral synthesis in Varopoulos algebras and (somewhat unexpectedly) to partial differential equations.

Let us describe the results of the paper in more detail. In Section 2 we consider some pseudo-topologies and functional spaces on direct products of measure spaces. Basic definitions and results from [A] and [ShT] related to operator synthesis for subsets in a direct product $X \times Y$ are recalled. It is proved that a subset with a scattered family of X-sections is equivalent to a countable union of rectangles. A consequence which is used later on is that the set of all solutions (x, y) of an equation of the form $\sum_{i=1}^{n} a_i(x)b_i(y) = 0$ is a union of countable family of rectangles and a set of measure null. A special case of this result was established in [W2, Proposition 12].

Section 3 deals with a kind of spectral synthesis in commutative Banach algebras – the synthesis with respect to Banach modules. The real distinction of this theory from the

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classical one is that given a module we get a special class of ideals, the annihilators of subsets in the module, and work with them only. Here our aim is to compare the conditions for an element to be synthetic with respect to a module and to admit spectral synthesis in the algebra. We also relate these conditions to spectra and spectral subspaces of the corresponding multiplication operators.

In Section 4 the approach is reduced to the case of operator modules over Varopoulos algebras. Let μ, ν be regular measures on compacts X, Y and H_1, H_2 the corresponding L_2-spaces. Then the space $B(H_1, H_2)$ of all bounded operators from H_1 to H_2 becomes a $V(X, Y)$-module with respect to the action $(f \otimes g) \cdot T = M_g T M_f$, where M_f, M_g are the multiplication operators. It is proved that $F \in V(X, Y)$ admits spectral synthesis iff it is synthetic with respect to all modules of this kind. This allows to obtain results on spectral synthesis in an operator-theoretical way. The following auxiliary statement (Corollary 4.8) appears to be useful: a function $F \in V(X, Y)$ is synthetic with respect to $B(H_1, H_2)$ iff the space of all solutions of the equation $F \cdot X = 0$ is reflexive (in the sense of [LSh]) and iff the 0-spectral subspace of the operator of multiplication by F coincides with its kernel.

The topics of Section 5 are linear operator equations of general type and modules over weak*-Haagerup tensor products of L_∞-algebras. Some estimates for the action of a linear multiplication operator with normal coefficients on its 0-spectral subspace are obtained. The extension of the approach allows to relate the topic with global operator synthesis.

In Section 6 we develop a general technique which relates solutions of the "same" linear equations in different linear topological spaces. More strictly speaking we are given two operators, S and T, acting in spaces $\mathfrak{X}$, $\mathfrak{Y}$, and a linear injection, $\Phi : \mathfrak{X} \to \mathfrak{Y}$, that intertwines them: $T\Phi = \Phi S$. Our main tool then is the "approximate inverse intertwining" (AII), that is a net $\{F_\alpha\}$ of maps from $\mathfrak{Y}$ to $\mathfrak{X}$ satisfying the conditions that $F_\alpha \Phi \to 1_{\mathfrak{X}}$, $\Phi F_\alpha \to 1_{\mathfrak{Y}}$ and $F_\alpha T - S F_\alpha \to 0_{\mathfrak{X}}$ in the topology of simple convergence. It appears to be possible to obtain some non-completely trivial results on inclusions of images or norm-inequalities in such a general abstract scheme.

In applications of the AII-technique to linear operator equations, the spaces $\mathfrak{X}$, $\mathfrak{Y}$ are symmetrically normed ideals of the algebra $B(H)$ (actually the case $\mathfrak{Y} = B(H)$ is the most important) and S, T are the restrictions of a multiplication operator Δ to $\mathfrak{X}$, $\mathfrak{Y}$. The conditions under which an AII exists are considered in Section 7. They are close in spirit to Voiculescu's conditions of quasidiagonality modulo a symmetrically normed ideal, but formally are more weak: instead of the condition $|||[A, P_n]||| \to 0$ we need only the boundedness of the norms $|||[A, P_n]|||$ (semidiagonality). Note that for the usual operator norm the semidiagonality holds automatically while quasidiagonality is an intriguing property which was explored in a great number of publications. We discuss examples of $\mathfrak{S}_p$-quasidiagonal families; the most simple ones are families of weighted shifts ($p = 1$) and families of commuting normal operators with thin joint spectra.

In Section 8 the applications of AII's to the problem of triviality of the trace of a commutator and to some related problems are gathered. In [W1] Weiss proved that if a commutator $[A, X]$ of a normal operator A and a Hilbert-Schmidt operator X belongs to $\mathfrak{S}_1$ then $\mathrm{tr}([A, X]) = 0$. In Proposition 8.1 we extend this result as follows: if a family, $\{A_k\}_{k=1}^n$ of operators is $\mathfrak{S}_{p/(p-1)}$-semidiagonal and if a sum $\sum_{k=1}^n [A_k, X_k]$ belongs to $\mathfrak{S}_1$ then $\mathrm{tr}(\sum_{k=1}^n [A_k, X_k]) = 0$. We answer also some questions posed in [W1] and [O] which are formulated in purely function-theoretical terms but in their essence are about the trace of (sums of) commutators.

The famous Fuglede Theorem can be formulated as the equality $\ker\Delta = \ker\tilde{\Delta}$, where $\Delta(X) = AX - XA$, $\tilde{\Delta}(X) = A^*X - XA^*$ and A is a normal operator. Weiss [W1] strengthen the result to $||\Delta(X)||_2 = ||\tilde{\Delta}(X)||_2$. Weiss also proposed to consider the case when Δ is a more general multiplication operator $\Delta(X) = \sum_{k\in K} B_k X A_k$ with commuting normal coefficients and $\tilde{\Delta}(X) = \sum_{k\in K} B_k^* X A_k^*$. He proved the equality $||\Delta(X)||_2 = ||\tilde{\Delta}(X)||_2$ in the case when K is finite and both parts of the equality are finite (that is $\Delta(X)$ and $\tilde{\Delta}(X)$ belong to $\mathfrak{S}_2$). In general, these restrictions can not be dropped ([Sh1]). We show that if the Hausdorff dimension of the joint spectrum of the family $\{A_k\}$ does not exceed 2 the equality holds without the restrictions. We discuss also a "non-commutative version" of the Fuglede Theorem: $\ker\tilde{\Delta}\Delta = \ker\Delta$, where Δ is arbitrary (the coefficients are not supposed to be normal or commuting). It is proved in Theorem 9.1 that the equality holds if $\{A_k\}_{k\in K}$ is 1-semidiagonal.

In Theorem 9.3 we show the inequality $||\Delta(X)||_2 \geq ||\tilde{\Delta}(X)||_2$ for $\Delta(X) = AX - XB$, provided that A and B^* are hyponormal operators of finite multiplicity.

Section 10 is devoted to multiplication operators with normal finite families of coefficients. It is proved that the ascent of such an operator does not exceed $d/2$ where d is the Hausdorff dimension of the joint spectrum of the left coefficient family. This result is applied in Section 11 to the evaluation of the number on which the chain of closed ideals generated by the powers of an element F of a Varopoulos algebra is stabilized. In particular, it is proved that if $F(x,y) = \sum_{k=1}^n f_k(x)g_k(y) \in V(X,Y)$, $\dim X \leq 2$ and the functions f_k are Lipschitsian then F admits spectral synthesis.

Our last application is to partial differential equations with constant coefficients. Corollary 11.4 states that the space of all bounded solutions of the equation $p(i\frac{\partial}{\partial x_1}, i\frac{\partial}{\partial x_2})u = 0$ depends only on the variety of zeros of the polynomial p.

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2 Pseudo-topologies and functional spaces on direct products of measure spaces

Let (X,μ), (Y,ν) be standard measure spaces with finite measures, $m = \mu\times\nu$ the product measure on $X\times Y$. In this section we recall some definitions and results from [A, EKS, ShT] and obtain a few others auxiliary results.

A rectangle in $X\times Y$ is a measurable subset of the form $A\times B$, where $A\subset X$, $B\subset Y$.

A subset $E \subset X \times Y$ is called *marginally null* (with respect to $\mu \times \nu$) if $E \subset (X_1 \times Y) \cup (X \times Y_1)$ and $\mu(X_1) = \nu(Y_1) = 0$. Two subsets E_1, E_2 are marginally equivalent ($E_1 \sim^M E_2$ or simply $E_1 \cong E_2$) if their symmetric difference is marginally null. Furthermore, $E_1 \subset^M E_2$ means that $E_1 \setminus E_2$ is marginally null, a property holds marginally almost everywhere if it holds everywhere apart of a marginally null set, and so on.

A subset E is called *pseudo-open* (more strictly, ω-pseudo-open) if it is marginally equivalent to a countable union of measurable rectangles. The complements of pseudo-open sets are *pseudo-closed* sets.

It is easy to see that the family of all pseudo-open sets defines a pseudo-topology on $X \times Y$: it is stable under finite intersections and countable unions. This pseudo-topology is denoted by ω.

A complex-valued function f on $X \times Y$ is pseudo-continuous if f-preimages of open sets are pseudo-open. It is known ([EKS]) that pseudo-continuous functions form a functional algebra on $X \times Y$. In particular, all functions of finite length $f(x,y) = \sum_{i=1}^n a_i(x)b_i(y)$ (with measurable a_i, b_i) are pseudo-continuous.

Set $\Gamma(X,Y) = L_2(X,\mu)\hat{\otimes}L_2(Y,\nu)$, where $\hat{\otimes}$ denotes the projective tensor product. Clearly, every $\Psi \in \Gamma(X,Y)$ can be identified with a function $\Psi : X \times Y \to \mathbb{C}$ which admits a representation

$$\Psi(x,y) = \sum_{n=1}^{\infty} f_n(x)g_n(y) \tag{1}$$

where $f_n \in L_2(X,\mu)$, $g_n \in L_2(Y,\nu)$ and $\sum_{n=1}^{\infty} ||f_n||_{L_2} \cdot ||g_n||_{L_2} < \infty$. Such a representation defines a function marginally almost everywhere (m.a.e.), so two functions in $\Gamma(X,Y)$ which coincides m.a.e. are identified. $L_2(X,\mu)\hat{\otimes}L_2(Y,\nu)$-norm of Ψ is

$$||\Psi||_\Gamma = \inf \sum_{n=1}^{\infty} ||f_n||_{L_2} \cdot ||g_n||_{L_2},$$

where the infimum is taken over all sequences f_n, g_n for which (1) holds m.a.e.

We consider also the space $V^\infty(X,Y)$ of all (marginal equivalence classes of) functions $\Psi(x,y)$ that can be written in the form (1) with $f_n \in L^\infty(X,\mu)$, $g_n \in L^\infty(Y,\nu)$ and

$$\sum_{n=1}^{\infty} |f_n(x)|^2 \le C, \quad x \in X, \quad \sum_{n=1}^{\infty} |g_n(y)|^2 \le C, \quad y \in Y.$$

The least possible C here is the norm of Ψ in $V^\infty(X,Y)$. In tensor notations $V^\infty(X,Y) = L^\infty(X,\mu)\hat{\otimes}^{w^*h}L^\infty(Y,\nu)$, the weak*-Haagerup tensor product ([BSm]). Since measures μ, ν are finite, $V^\infty(X,Y) \subset \Gamma(X,Y)$.

Lemma 2.1. *[EKS] All functions $\Psi \in \Gamma(X,Y)$ are ω-pseudo-continuous.*

Now we discuss the null sets of (families of) functions in $\Gamma(X,Y)$. We say that $F \in \Gamma(X,Y)$ vanishes on $E \subset X \times Y$ if $F\chi_E = 0$ m.a.e., χ_E being the characteristic function of E. For $\mathcal{F} \subset \Gamma(X,Y)$, the *null set*, null $\mathcal{F}$, is defined to be the largest, up to marginal equivalence, pseudo-closed set such that each function $F \in \mathcal{F}$ vanishes on it. If E is a pseudo-closed subset of $X \times Y$, let

$$\Phi(E) = \{F \in \Gamma(X,Y) \mid F \text{ vanishes on } E\},$$

$$\Phi_0(E) = \overline{\{F \in \Gamma(X,Y) \mid F \text{ vanishes on a nbhd of } E\}},$$

where by a neighborhood we mean a pseudo-open set containing E and the closure is taken in $\Gamma(X,Y)$. By [ShT][Theorem 2.1], $\Phi_0(E)$ and $\Phi(E)$ are the smallest and the largest invariant (with respect to the multiplication by functions $f \in L_\infty(X,\mu)$ and $g \in L_2(Y,\nu)$) closed subspaces of $\Gamma(X,Y)$ whose null set is E.

We will also need another pseudo-topology on $X \times Y$. Let us say that a subset $E \subset X \times Y$ is τ-pseudo-open if it is a union of an m-null set and a countable family of rectangles. It is not difficult to check that the class of all such sets is stable under finite intersections and countable unions. Clearly, the pseudo-topology τ is stronger than ω. In particular, all functions of finite length and functions in $\Gamma(X,Y)$ are τ-pseudo-continuous.

Our next aim is to obtain some sufficient condition for a set to be τ-pseudo-open.

For U, $V \subset X$ let us write $U \subset^\mu V$ if $\mu(U \setminus V) = 0$. If $U \subset^\mu V$ and $V \subset^\mu U$ we say that U and V are μ-equivalent and write $U \sim^\mu V$.

A family $\mathcal{F}$ of measurable subsets of X is called μ-scattered if *any* decreasing sequence $U_1 \supset U_2 \supset \ldots$ of its members μ-stabilizes (that is $U_n \sim^\mu U_{n+1} \sim^\mu U_{n+2} \sim \ldots$ for some n).

Let now E be a subset of $X \times Y$. An X-section of E is a subset of the form

$$E^y = \{x \in X \mid (x,y) \in E\}.$$

E is called *X-scattered* if the family of all finite intersections of its X-sections is μ-scattered. The following result gives us important examples of X-scattered sets.

Proposition 2.2. *Let $h(x,y)$ be a complex-valued function on $X \times Y$ that has a finite length, that is $h(x,y) = \sum_{i=1}^N a_i(x)b_i(y)$, where a_i and b_i are measurable. Then*

$$E = \{(x,y) \mid h(x,y) = 0\}$$

is an X-scattered set.

Proof. Let $\vec{a} : X \to \mathbb{C}^N$, $\vec{b} : X \to \mathbb{C}^N$ be defined by $\vec{a}(x) = \{a_i(x)\}_{i=1}^N$, $\vec{b}(y) = \{b_i(y)\}_{i=1}^N$. For any $y \in Y$, the X-section E^y is the preimage with respect to $\vec{a}$ of the hyperplane

$$\{\vec{z} \in \mathbb{C}^N \mid \sum_{i=1}^N z_i b_i(y) = 0\}.$$

Since intersections of hyperplanes must stabilize, the family of their preimages is scattered. □

It is easy to see that any set, which is m-equivalent to a finite union of rectangles, is X-scattered.

Theorem 2.3. *Any X-scattered set is m-equivalent to a countable union of rectangles.*

Proof. Let $E \subset X \times Y$ be an X-scattered set. Denote by $\mathcal{U}$ the set of all countable unions of rectangles and set

$$\underline{m}(E) = \sup\{m(U) \mid U \in \mathcal{U}, U \subset^m E\}.$$

Choosing $U_n \in \mathcal{U}$ with $U_n \subset^m E$ and $m(U_n) > \underline{m}(E) - \frac{1}{n}$, we set $U = \cup_{n=1}^{\infty} U_n$. Then $m(U) = \underline{m}(E)$, $U \in \mathcal{U}$, $U \subset^m E$. Hence the set $S = E \setminus U$ has the property that $m(S \cap \Pi) = 0$ for any rectangle $\Pi \subset^m E$. It remains to show that $m(S) = 0$.

We define a measure ν on X by $\nu(A) = m(S \cap (A \times Y))$. Clearly, $\nu \ll \mu$ whence by the Jordan Theorem $X = X_0 \cup X_1$, $X_0 \cap X_1 = \emptyset$, $\nu(X_0) = 0$ and $\nu \sim \mu$ on X_1. If $\mu(X_1) = 0$ then $m(S) = 0$ and we are done.

Assume $\mu(X_1) \neq 0$ and let $E_1 = E \cap (X_1 \times Y)$. Clearly E_1 is X-scattered. Let $A_1 \supset A_2 \supset \ldots \supset A_n$ be a maximal chain in the family of (the equivalence classes of) finite intersections of X-sections of E_1, and let A be its final non-zero element ($A = A_n$ if $\mu(A_n) \neq 0$, $A = A_{n-1}$ otherwise). Then any X-section of E_1 either μ-contains A or is μ-disjoint with A. Set

$$K = \{y \in Y \mid A \subset^{\mu} E_1^y\}.$$

Then all X-sections of the set $(A \times (Y \setminus K)) \cap E_1$ are μ-null. Therefore $m((A \times (Y \setminus K)) \cap E_1) = 0$, $m((A \times (Y \setminus K)) \cap S) = 0$ and $m((A \times K) \cap S) = m((A \times Y) \cap S) = \nu(A) \neq 0$ (because $\mu(A) \neq 0$).

On the other hand, $(A \times K) \setminus E_1$ has μ-null sections whence $m((A \times K) \setminus E_1) = 0$. Thus the rectangle $\Pi = A \times K$ is m-contained in E_1 and has non-trivial intersection with S, a contradiction. □

Corollary 2.4. *A pseudo-closed X-scattered set is τ-pseudo-open.*

Proof. Let E be a pseudo-closed X-scattered set. If $\Pi \subset^m E$, where Π is a rectangle, then $\Pi \subset^M E$. Indeed, $m(\Pi \cap \Pi') = 0$ for each rectangle $\Pi' \subset E^c$ so $\Pi \cap \Pi' \sim^M 0$.

It follows that Π can be changed by a sub-rectangle $\tilde{\Pi}$ of the same measure such that $\tilde{\Pi} \subset E$. This clearly implies our statement: since $E \sim^m \cup_{j=1}^{\infty} \Pi_j$ we have $E \sim^m \cup_{j=1}^{\infty} \tilde{\Pi}_j \subset E$. □

Corollary 2.5. *The set of zeros of a function of finite length is τ-pseudo-open.*

Now it is easy to deduce a more general result.

Corollary 2.6. *Let h_j, $1 \leq j \leq n$, be real-valued functions of finite length on $X \times Y$. The set*

$$E = \{(x, y) \mid h_j(x, y) \leq 0, 1 \leq j \leq n\}$$

is τ-pseudo-open.

Proof. Clearly $E = \cap_{j=1}^{n} E_j$, where $E_j = \{(x, y) \mid h_j(x, y) \leq 0\}$, and $E_j = E_j' \cup E_j''$, where

$$E_j' = \{(x, y) \mid h_j(x, y) = 0\}, \quad E_j'' = \{(x, y) \mid h_j(x, y) < 0\}.$$

All E_j' are τ-pseudo-open by Corollary 2.5. All E_j'' are ω-pseudo-open by Lemma 2.1 and hence τ-pseudo-open. □

For us the space $\Gamma(X,Y)$ is important because it is predual to the space of bounded operators, $B(H_1,H_2)$, from $H_1 = L_2(X,\mu)$ to $H_2 = L_2(Y,\nu)$. The duality is given by

$$\langle T, \Psi\rangle = \sum_{n=1}^{\infty}(Tf_n, \bar{g}_n),$$

with $T \in B(H_1,H_2)$ and $\Psi(x,y) = \sum_{n=1}^{\infty} f_n(x)g_n(y)$.

Let P_U and Q_V denote the multiplication operators by the characteristic functions of $U \subset X$ and $V \subset Y$. We say that $T \in B(H_1,H_2)$ is *supported* in $E \subset X \times Y$ (or E supports T) if $Q_V T P_U = 0$ for each Borel sets $U \subset X$, $V \subset Y$ such that $(U \times V) \cap E = \emptyset$. Then there exists the smallest (up to a marginally null set) pseudo-closed set, supp T, which supports T. More generally, for any subset $\mathfrak{M} \subset B(H_1,H_2)$ there is the smallest pseudo-closed set supp $\mathfrak{M}$, which supports all operators in $\mathfrak{M}$. In the seminal paper [A] Arveson defined a support in a similar way but using closed sets instead of pseudo-closed (in his setting X, Y are topological spaces). This closed support, $\mathrm{supp}_A T$ can be strictly larger than supp T.

For any pseudo-closed set $E \subset X \times Y$ the set, $\mathfrak{M}_{max}(E)$, of all operators T, supported in E, has support E and is the largest set with this property. It is easy to check that $\mathfrak{M}_{max}(E)$ is a $\mathcal{D}_1 \times \mathcal{D}_2$-bimodule, where $\mathcal{D}_1$, $\mathcal{D}_2$ are the algebra of multiplications by functions in $L_\infty(X,\mu)$ and $L_\infty(Y,\nu)$ respectively. There is also the smallest bimodule $\mathfrak{M}_{min}(E) \subset B(H_1,H_2)$ with support equal to E and, moreover,

$$\mathfrak{M}_{max}(E) = \Phi_0(E)^{\perp}, \quad \mathfrak{M}_{min}(E) = \Phi(E)^{\perp}.$$

(see [ShT]). We say that a pseudo-closed set $E \subset X \times Y$ is *operator synthetic* (or $\mu \times \nu$-synthetic) if the following equivalent conditions hold:

- $\Phi(E) = \Phi_0(E)$.
- $\mathfrak{M}_{max}(E) = \mathfrak{M}_{min}(E)$.
- $\langle T, F\rangle = 0$ for any $T \in B(H_1,H_2)$ and $F \in \Gamma(X,Y)$ with supp $T \subset E \subset$ null F.

In further sections we will use also the following characterization of $\mathfrak{M}_{min}(E)$ by pairs of projections. Identifying projections $P \in B(l_2)\bar{\otimes}\mathcal{D}_1$ and $Q \in B(l_2)\bar{\otimes}\mathcal{D}_2$ with projection-valued functions $P(x) : X \to B(l_2)$ and $Q(y) : Y \to B(l_2)$ we say that a pair (P,Q) is an E-pair if $P(x)Q(y)$ vanishes on E m.a.e. Then by [ShT][Corollary 4.4]

$$\mathfrak{M}_{min}(E) = \{T \in B(H_1,H_2) \mid Q(1 \otimes T)P = 0 \text{ for any } E\text{-pair } (P,Q)\}.$$

3 Synthesis with respect modules over Banach algebras

Let A be a semisimple, regular, commutative Banach algebra with unit and let X_A be its spectrum. For any $a \in A$ we shall denote by $\hat{a}$ its Gelfand transform and set

$$null(a) = \{\chi \in X_A \mid \hat{a}(\chi) = 0\}.$$

More generally, for any subset $B \subset A$, we define $null(B)$ as $\cap_{a \in B} null(a)$.

To any closed subset $E \subset X_A$ there correspond ideals

$$I(E) = \{r \in A \mid \hat{r}^{-1}(0) \text{ contains } E\},$$
$$J_0(E) = \{r \in A \mid \hat{r}^{-1}(0) \text{ contains a nbhd of } E\} \text{ and } J(E) = \overline{J_0(E)}.$$

It is known that $null(J(E)) = null(I(E)) = E$ and $J(E) \subset K \subset I(E)$, for any closed ideal K with $null(K) = E$.

For $a \in A$ we define

$$I_a = I(null(a)),$$
$$J_a^0 = J_0(null(a)), \text{ and } J_a = J(null(a)).$$

One says that $a \in A$ *admits spectral synthesis* if $a \in J_a$.

Let M be a Banach A-module. For any $x \in M$ set

$$ann(x) = \{a \in A \mid a \cdot x = 0\},$$
$$Supp(x) = null(ann(x)).$$

Then $ann(x)$ is a closed ideal and $Supp(x)$ is a closed subset in X_A.

In a similar way one defines $ann(N)$ and $Supp(N)$ for arbitrary subset $N \subset M$.

Definition 3.1. We say that an element $a \in A$ *admits synthesis* (or *IS synthetic*) with respect to an A-module M if $a \cdot x = 0$ for any $x \in M$ such that $Supp(x) \subset null(a)$.

Example 3.2. A is a module over itself with the action defined by $a \cdot x = ax$ for any $a, x \in A$. Each $a \in A$ admits synthesis with respect to A as A-module. In fact, assuming $ax \neq 0$ for some $a, x \in A$ with $Supp(x) \subset null(a)$, one can find $\chi \in X_A$ such that $\widehat{ax}(\chi) = \hat{a}(\chi)\hat{x}(\chi) \neq 0$ and hence $\hat{a}(\chi) \neq 0$ and $\hat{x}(\chi) \neq 0$. Since $Supp(x) \subset null(a)$, $\hat{a}(\chi) \neq 0$ implies $\hat{b}(\chi) \neq 0$ for some $b \in ann(x)$. However, $\widehat{bx}(\chi) = \hat{b}(\chi)\hat{x}(\chi) \neq 0$, a contradiction.

Let A' denote the Banach space dual to A. Setting $a \cdot x(\cdot) = x(a\cdot)$ for any $a \in A$, $x \in A'$, we have that A' is an A-module. This example is especially important because of the following result that connects the notions of spectral synthesis and synthesis with respect to A-modules.

Theorem 3.3. *For $a \in A$ the following conditions are equivalent.*

1) *a admits spectral synthesis;*

2) *a admits synthesis with respect to A';*

3) *a admits synthesis with respect to any A-module.*

Proof. 1) $\Rightarrow$ 3). Let M be an A-module and $x \in M$, $Supp(x) \subset null(a)$. Let $J = J(Supp(x))$, then since $null(ann(x)) = Supp(x) \subset null(a)$ and a admits spectral synthesis in A, we have $a \in J_a \subset J \subset ann(x)$ and hence $a \cdot x = 0$.

3) $\Rightarrow$ 2) is obvious.

2) $\Rightarrow$ 1). Assume that $a \notin J_a$. Then there is $x \in A'$ such that $x(J_a) = 0$ and $x(a) \neq 0$. Since a admits synthesis with respect to A', $Supp(x)$ is not a subset of $null(a)$. On the other hand, $J_a \subset ann(x)$ and $Supp(x) = null(ann(x)) \subset null(J_a) = null(a)$, a contradiction. $\square$

Lemma 3.4. *Let M be a Banach A-module. For $x \in M$, $a \cdot x = 0$ for any $a \in J(E)$ if and only if $Supp(x) \subset E$.*

Proof. Assume that $Supp(x) \subset E$ and that $\hat{a} = 0$ on a nbhd of E. Then there exists an open set F such that $null(a) \supset F \supset Supp(x)$. Clearly, $J = ann(x)$ is an ideal in A with $null(J) = Supp(x)$. Since $F^c \cap Supp(x) = \emptyset$, there exists $b \in J$ such that $\hat{b} = 1$ on F^c and therefore $\widehat{ab} = \hat{a}$ implying $a \cdot x = ab \cdot x = 0$.

If $a \cdot x = 0$ for any $a \in J(E)$ then $J(E) \subset ann(x)$ and $E = null(J(E)) \supset null(ann(x)) = Supp(x)$. $\square$

As usually by L_a we denote the operators of the "left" multiplication by $a \in A$, acting in an A-module.

Lemma 3.5. *Let M be a Banach A-module and let N be the closed submodule generated by $x \in M$. Then $\sigma(L_a|_N) = \hat{a}(Supp(x))$.*

Proof. Let $\tilde{J} = J(Supp(x))$. Since, by Lemma 3.4, $b \cdot x = 0$ for any $b \in \tilde{J}$, N is also an $A/\tilde{J}$-module with the action $(c + \tilde{J}) \cdot y = c \cdot y$. $X_{A/\tilde{J}}$ can be identified with $\{\chi \in X_A : \chi(\tilde{J}) = 0\} = Supp(x)$ and therefore $\sigma(a + \tilde{J}) = \hat{a}(Supp(x))$. If $\lambda \in \hat{a}(Supp(x))^c$ then $a - \lambda e + \tilde{J}$ is invertible in $A/\tilde{J}$ and so is the operator $(L_a - \lambda I)|_N$. This shows that $\sigma(L_a|_N) \subset \hat{a}(Supp(x))$.

Assume now that $(L_a - \hat{a}(\chi)I)|_N$, for some $\chi \in Supp(x)$, is an invertible operator. Then there exists $c \in A$ such that $(a - \hat{a}(\chi))cb \cdot x = b \cdot x$ for any $b \in A$ and therefore $((a - \hat{a}(\chi))c - 1)A \subset ann(x)$. Since $\chi \in Supp(x) = null(ann(x))$, we have $\chi(A) = 0$. A contradiction. $\square$

Theorem 3.6. *Let M be a Banach A-module. For $x \in M$ and $a \in A$, $Supp(x) \subset null(a)$ if and only if $||a^n x||^{1/n} \to 0$ as $n \to \infty$.*

Proof. Consider the Banach algebra $V = A/J_a$. The element $a + J_a$ is quasi-nilpotent in V. In fact, if χ is a character of A/J_a then $\rho(\chi)$, defined by $\rho(\chi)(b) := \chi(b + J_a)$, $b \in A$, is a character of A. Therefore, there exists $\tau \in X_A$ such that $\rho(\chi)(b) = \tau(b)$. This gives $\tau(b) = 0$ for any $b \in J_a$ and hence $\tau \in null(J_a) = null(a)$ and $\chi(a + J_a) = \tau(a) = 0$.

By Lemma 3.4, $J_a \cdot x = 0$ whence $J_a \cdot N = 0$, N being the closed submodule generated by x. N becomes a Banach V-module by setting $(b + J_a) \cdot y := b \cdot y$, for $y \in N$ and $b \in A$. Moreover,

$$||a^n \cdot x||^{1/n} = ||(a + J_a)^n \cdot x||^{1/n} \leq ||(a + J_a)^n||^{1/n} ||x||^{1/n} \to 0, n \to \infty.$$

The converse follows immediately from Lemma 3.5. $\square$

Let us also mention the module version of the "global" synthesis. Let M be a Banach A-module and E be a closed subset of X_A. E is called a *set of synthesis over* M (*synthetic over* M) if $a \cdot x = 0$ for any $x \in M$ and $a \in A$ such that $Supp(x) \subset E \subset null(a)$. Clearly, if, for $a \in A$, $null(a)$ is synthetic over M then a admits synthesis with respect to M.

4 Modules over tensor algebras and linear operator equations

Let X and Y be compact Hausdorff spaces and consider the projective tensor product $V(X,Y) = C(X)\hat{\otimes}C(Y)$. Recall that $V(X,Y)$ (the Varopoulos algebra) consists of all functions $F \in C(X \times Y)$ which admit a representation

$$F(x,y) = \sum_{i=1}^{\infty} f_i(x) g_i(y), \tag{2}$$

where $f_i \in C(X)$, $g_i \in C(Y)$ and

$$\sum_{i=1}^{\infty} ||f_i||_{C(X)} ||g_i||_{C(Y)} < \infty.$$

$V(X,Y)$ is a Banach algebra with the norm

$$||F||_V = \inf \sum_{i=1}^{\infty} ||f_i||_{C(X)} ||g_i||_{C(Y)},$$

where inf is taken over all representations of F in the above form (see [V]). We note that $V(X,Y)$ is a semi-simple regular Banach algebra with spectrum $X \times Y$.

Any element of $V(X,Y)'$ can be identified with a bounded bilinear form $B(f,g) = \langle B, f\otimes g\rangle$ on $C(X) \times C(Y)$ (a bimeasure, in short).

Let $M(X)$ denote the space of finite Borel measures on X. For $\mu \in M(X)$, $\nu \in M(Y)$, set $H_1 = L_2(X,\mu)$, $H_2 = L_2(Y,\nu)$. Then $V(X,Y) \subset V^{\infty}(X,Y) \subset \Gamma(X,Y)$. Note that, for $\mathcal{F} \subset V(X,Y)$, null $\mathcal{F}$ coincides with $\cap_{F\in\mathcal{F}} F^{-1}(0)$.

By M_f, M_g we denote the multiplication operators in H_1, H_2 by functions $f(x)$, $g(y)$ respectively.

Setting, for $F(x,y) = \sum_{n=1}^{\infty} f_n(x) g_n(y) \in V(X,Y)$ and $T \in B(H_1,H_2)$,

$$F \cdot T = \sum_{n=1}^{\infty} M_{g_n} T M_{f_n} \tag{3}$$

we obtain a $V(X,Y)$-module structure on $B(H_1,H_2)$. So, for $T \in B(H_1,H_2)$, we have $Supp(T) = \cap\, null(F)$, the intersection being taken over all functions $F \in V(X,Y)$ such that $F \cdot T = 0$. Now we compare $Supp(T)$ with the "inner" definitions of a support introduced in Section 2.

Lemma 4.1. *For $F \in V(X,Y)$ and $T \in B(H_1,H_2)$, if $F \cdot T = 0$ then T is supported in $null(F)$.*

Proof. Take Borel sets $U \subset X$, $V \subset Y$ such that $(U \times V) \cap null(F) = \emptyset$ and consider $Q_V T P_U \in B(L_2(U,\mu), L_2(V,\nu))$. Then $\chi_U(x)\chi_V(y)F(x,y) \neq 0$ on $U \times V$ and if Ψ denote the set $\{FG \mid G \in \Gamma(U,V)\} \subset \Gamma(U,V)$, we have null $\Psi \cong \emptyset$. By [ShT, Corollary 4.3], Ψ is dense in $\Gamma(U,V)$. As

$$0 = \langle F \cdot Q_V T P_U, G\rangle = \langle Q_V T P_U, FG\rangle, \quad G \in \Gamma(U,V),$$

we obtain $Q_V T P_U = 0$ and therefore $null(F)$ supports T. □

Proposition 4.2. *$Supp(T)$ is the smallest closed set which supports the operator T.*

Proof. Set $E = Supp(T)$. We show first that E supports T. By Lemma 3.4, $F \cdot T = 0$ for any $F \in V(X,Y)$ vanishing on a nbhd of E. Therefore, by Lemma 4.1, $null(F)$ supports T for each $F \in J(E)$. Since $E = null(J(E))$, E supports T.

Let $W \subset X \times Y$ be a closed set supporting T. By [ShT, Theorem 4.3], given $F \in J(W)$, $\langle T, FG\rangle = 0$ for any $G \in \Gamma(X,Y)$ and hence $F \cdot T = 0$. Applying now Lemma 3.4, we obtain $Supp(T) \subset W$, showing that $Supp(T)$ is the smallest closed set supporting T. □

By the proposition we have therefore supp $T \subset^M Supp(T) = \text{supp}_A T$.

Let K_F denote the set of all operators T satisfying the condition $F \cdot T = 0$, that is $K_F = ann(F)$ in $B(H_1,H_2)$. It coincides with the space of solutions of the linear operator equation

$$\sum_{n=1}^{\infty} M_{g_n} T M_{f_n} = 0, \tag{4}$$

where $F(x,y) = \sum_{n=1}^{\infty} f_n(x) g_n(y)$.

It follows from Lemma 4.1 and Proposition 4.2 that $Supp(K_F) \subset null(F)$.

Proposition 4.3.

$$\text{supp } K_F = Supp(K_F) = null(F).$$

Proof. It suffices to show that $null(F) \subset$ supp K_F. Let $P = P(null(F))$ be the set of all pseudo-integral operators T_σ with $supp(\sigma) \subset null(F)$ (see [A, Section 1.5]). It follows from [A, Section 2.2] that supp $P = null(F)$. On the other hand it is easy to see that

$$(F \cdot T_\sigma u, v) = \iint F(x,y)u(x)v(y)d\sigma(x,y) = 0,$$

for any $u \in H_1$, $v \in H_2$. Hence, $P \subset K_F$ and $null(F) = \text{supp } P \subset \text{supp } K_F$. □

Remark 4.4. The result can be proved without the use of pseudo-integral operators (see the proof of a more general result, Proposition 5.3, below).

We say that $F \in V(X,Y)$ is *operator synthetic* with respect to (μ,ν) (we also write (μ,ν)-synthetic or operator synthetic if μ, ν are fixed) if it is synthetic with respect to the $V(X,Y)$-module $B(H_1,H_2)$.

The following proposition can be considered as a local version of Theorem 6.1 from [ShT]. Unlike the latter it is "two-sided".

Proposition 4.5. *$F \in V(X,Y)$ admits spectral synthesis if and only if it is operator synthetic for any choice of finite measures on X, Y.*

Proof. The necessity follows from Theorem 3.3. To prove the sufficiency it is enough to show that F admits synthesis in $V(X,Y)'$. Assume that synthesis fails for F. Then we can find a bimeasure $B \in V(X,Y)'$ such that $Supp(B) \subset null(F)$ and $F \cdot B \neq 0$. By the Grothendieck theorem [G] there exist measures $\mu \in M(X)$, $\nu \in M(Y)$ such that

$$|\langle B, f \otimes g\rangle| \leq C||f||_{L_2(X,\mu)}||g||_{L_2(Y,\nu)}$$

for any $f \in C(X)$ and $g \in C(Y)$. Thus there exists an operator $T \in B(L_2(X,\mu), L_2(Y,\nu))$ such that

$$\langle T, \Psi\rangle = \langle B, \Psi\rangle, \quad \Psi \in V(X,Y),$$

where in the left hand side we used the inclusion $V(X,Y) \subset \Gamma(X,Y)$. Therefore, $F \cdot T \neq 0$. We will get a contradiction if prove that $Supp(T) \subset null(F)$. For $G \in J_F$, $v \in C(X)$ and $w \in C(Y)$, we have

$$(G \cdot Tv, w) = \langle G \cdot B, v \otimes w\rangle = 0,$$

implying $G \cdot T = 0$ and $Supp(T) \subset null(G)$. Since $null(J_F) = null(F)$, $Supp(T) \subset null(F)$. □

Theorem 4.6. *$F \in V(X,Y)$ is operator synthetic if and only if $K_F = \mathfrak{M}_{max}(null(F))$.*

Proof. Assume $F \in V(X,Y)$ is operator synthetic with respect to (μ,ν). We have $F \cdot T = 0$ for each T such that $Supp(T) \subset null(F)$. As $Supp(T)$ is the smallest closed set which supports T, we have that each $T \in \mathfrak{M}_{max}(null(F))$ is a solution of the equation $F \cdot T = 0$, i.e., $T \in K_F$. Conversely, if $T \in K_F$ then, by Lemma 4.1, T is supported in $null(F)$.

Assume now that $\mathfrak{M}_{max}(null(F)) = K_F$, but F is not synthetic with respect to $B(H_1,H_2)$. Then there exists T with $Supp(T) \subset null(F)$ and therefore $T \in \mathfrak{M}_{max}(null(F))$, such that $F \cdot T \neq 0$. A contradiction. □

For $F \in V(X,Y)$, let Δ_F denote the multiplication operator $X \mapsto F \cdot X$ on $B(H_1,H_2)$. Then $K_F = \ker \Delta_F$. Let $\mathcal{E}_{\Delta_F}(0)$ be the 0-spectral subspace of Δ_F:

$$\mathcal{E}_{\Delta_F}(0) = \{T \in B(H_1,H_2) \mid ||\Delta_F^n(T)||^{1/n} \to 0, n \to \infty\}.$$

Recall that if $\mathcal{L}$ is a subspace in $B(H_1,H_2)$ then its *reflexive hull*, Ref $\mathcal{L}$, is the set of all operators A such that $Ax \in \overline{\mathcal{L}x}$ for any $x \in H_1$. $\mathcal{L}$ is said to be *reflexive* if Ref $\mathcal{L} = \mathcal{L}$.

Proposition 4.7.

$$Ref\ K_F = \mathfrak{M}_{max}(null(F)) = \mathcal{E}_{\Delta_F}(0).$$

Proof. It is a standard fact (see [A] or [ShT]) that for arbitrary $L^\infty(X,\mu)\times L^\infty(Y,\nu)$-bimodule G the space Ref G consists of operators supported by supp (G). Hence the first equality follows from Proposition 4.3. By Theorem 3.6, $\mathcal{E}_{\Delta_F}(0)$ consists of all operators T such that $Supp(T)\subset null(F)$. But, by Proposition 4.2, this condition is equivalent to supp $T\subset null(F)$. Hence $\mathcal{E}_{\Delta_F}(0)=\mathfrak{M}_{max}(null(F))$. ☐

Corollary 4.8. *The following are equivalent:*

(a) $F\in V(X,Y)$ *is operator synthetic;*

(b) the solution space of the equation $F\cdot T=0$ *is reflexive;*

(c) $\ker\Delta_F=\mathcal{E}_{\Delta_F}(0)$.

Proof. (a)⇒ (b). It is easily seen that $\mathfrak{M}_{max}(null(F))$ is reflexive. The implication follows now from Theorem 4.6.

(b)⇒ (c). Follows from the equality Ref $\ker\Delta_F=\mathcal{E}_{\Delta_F}(0)$ which is due to Proposition 4.7.

(c)⇒(a). Let $T\in B(H_1,H_2)$ be an operator supported in $null(F)$. By Proposition 4.2, $Supp(T)\subset null(F)$ and, by Theorem 3.6, T is in $\mathcal{E}_{\Delta_F}(0)$ and therefore in $\ker\Delta_F$. Hence, $\mathfrak{M}_{max}(null(F))\subset\ker\Delta_F$. The reverse inclusion follows from Lemma 4.1. Thus $\mathfrak{M}_{max}(null(F))=\ker\Delta_F$. The statement now follows from Theorem 4.6. ☐

Proposition 4.5 and Corollary 4.8 reduce the problem of verification of individual synthesis to a purely operator problem. The comparison of 0-spectral subspace and kernels for multiplication operators will be one of the main topics in the further sections.

5 Equations of more general form. Relations to "global" operator synthesis

The study of the equations of the form (4) is a part of the general theory of linear operator equations

$$\Delta(X)=\sum_{k\in K}B_kXA_k=0 \tag{5}$$

where $\{A_k\}_{k\in K}=\mathbb{A}$, $\{B_k\}_{k\in K}=\mathbb{B}$ are finite or countable families of operators.

If $\mathbb{A}$, $\mathbb{B}$ are commutative families of normal operators one says that (5) is a linear operator equation with normal coefficients. Among them equations (4) correspond to those whose coefficients satisfy the restriction

$$\sum_{k\in K}||A_k||^2<\infty,\quad \sum_{k\in K}||B_k||^2<\infty. \tag{6}$$

Indeed, realizing all A_k and B_k as multiplication operators by continuous functions f_k, g_k on $L_2(X,\mu)$, $L_2(Y,\nu)$, (5) can be rewritten in a form (4); clearly $F(x,y)=\sum_{k\in K}f_k(x)g_k(y)\in V(X,Y)$.

It is more convenient sometimes to choose "spectral" realization of coefficient families. Let $\sigma(\mathbb{A})$, $\sigma(\mathbb{B})$ be the maximal ideal spaces of the unital C^*-algebras generated by the families $\mathbb{A}$ and $\mathbb{B}$ respectively. To any $t \in \sigma(\mathbb{A})$ we associate a sequence $\lambda(t) = (t(A_1), t(A_2), \ldots) \in l_2$; the map $t \mapsto \lambda(t)$ is continuous and identifies $\sigma(\mathbb{A})$ with a compact subset of l_2. Thus $C^*(\mathbb{A})$ can be considered as $C(\sigma(\mathbb{A}))$ and the operators A_i correspond to the coordinate functions on l_2 (restricted to $\sigma(\mathbb{A})$). In a similar way we realize $C^*(\mathbb{B})$. The space $B(H_1, H_2)$ becomes a $V(\sigma(\mathbb{A}), \sigma(\mathbb{B}))$-module with respect to the operation

$$(f \otimes g) \cdot T = f(\mathbb{B}) T g(\mathbb{A}).$$

In particular, $\Delta(T) = F \cdot T$, where $F(\lambda, \mu) = \sum_{k \in K} \lambda_k \mu_k$.

Let $E_{\mathbb{A}}(\cdot)$, $E_{\mathbb{B}}(\cdot)$ be the spectral measures of $\mathbb{A}$ and $\mathbb{B}$ (on $\sigma(\mathbb{A})$, $\sigma(\mathbb{B})$). We say that an operator T is *supported in* $U \subset \sigma(\mathbb{A}) \times \sigma(\mathbb{B})$ if

$$E_{\mathbb{B}}(\beta) T E_{\mathbb{A}}(\alpha) = 0$$

for any Borel sets $\alpha \subset \sigma(\mathbb{A})$, $\beta \subset \sigma(\mathbb{B})$ such that $(\alpha \times \beta) \cap U = \emptyset$. The following result directly follows from Proposition 4.7 if the families $\mathbb{A}$, $\mathbb{B}$ have cyclic vectors (one only needs to realize H_1, H_2 as the L_2-spaces for scalar spectral measures of $\mathbb{A}$, $\mathbb{B}$). In the general case the proof is similar to the proof of Proposition 4.7.

Proposition 5.1. *Let $S = \{(\lambda, \mu) \in \sigma(\mathbb{A}) \times \sigma(\mathbb{B}) \mid \sum_{k \in K} \lambda_k \mu_k = 0\}$. Then $\mathcal{E}_\Delta(0)$ consists of all operators supported in S.*

In our further study of spectral behavior of multiplication operators Δ the following estimate will be useful.

Lemma 5.2. *If $X \in \mathcal{E}_\Delta(0)$ then*

$$||\Delta(X)|| \leq 2(\sum_{k \in K} ||B_k||^2)^{1/2} ||X|| \text{diam } \sigma(\mathbb{A}).$$

Proof. Let $S = \{(\lambda, \mu) \in \sigma(\mathbb{A}) \times \sigma(\mathbb{B}) \mid \sum_{k \in K} \lambda_k \mu_k = 0\}$. By Proposition 5.1, if $X \in \mathcal{E}_\Delta(0)$ then X is supported in S and therefore $X = E_{\mathbb{B}}(\hat{\beta}) X$ where $\hat{\beta} = (\cup \beta)^c$ with the union taken over all relatively open $\beta \subset \sigma(\mathbb{B})$ such that $(\sigma(\mathbb{A}) \times \beta) \cap S = \emptyset$. We can certainly assume $\hat{\beta} = \sigma(\mathbb{B})$, for if not we set $B'_k = B_k|_{E_{\mathbb{B}}(\hat{\beta}) H_2}$ and replace Δ by the operator Δ' with coefficients $\{A_k\}$, $\{B'_k\}$ (acting on $B(H_1, E_{\mathbb{B}}(\hat{\beta}) H_2)$).

Let $\{\lambda_k\}_{k \in K} \in \sigma(\mathbb{A})$. Then

$$\Delta(X) = \sum_{k \in K} B_k X \lambda_k + \sum_{k \in K} B_k X (A_k - \lambda_k I).$$

For each $f \in H$, we have

$$\begin{aligned} ||(\sum_{k \in K} B_k X (A_k - \lambda_k I) f||^2 &\leq (\sum_{k \in K} ||B_k|| \cdot ||X|| \cdot ||(A_k - \lambda_k I) f||)^2 \leq \\ &\leq ||X||^2 \sum_{k \in K} ||B_k||^2 \sum_{k \in K} ||(A_k - \lambda_k I) f||^2 \leq \\ &\leq ||X||^2 \sum_{k \in K} ||B_k||^2 (\sum_{k \in K} (A_k - \lambda_k I)^* (A_k - \lambda_k I) f, f) \leq \\ &\leq ||X||^2 \sum_{k \in K} ||B_k||^2 (\text{diam } \sigma(\mathbb{A}))^2 ||f||^2, \end{aligned}$$

and therefore

$$||\sum_{k\in K} B_k X(A_k - \lambda_k I)|| \leq ||X||(\sum_{k\in K} ||B_k||^2)^{1/2}(\text{diam } \sigma(\mathbb{A})).$$

To estimate the norm of the first summand, note that the spectrum $\sigma(\sum_{k\in K} B_k\lambda_k)$ is equal to $\{\sum_{k\in K}\mu_k\lambda_k \mid (\mu_k)_{k\in K} \in \sigma(\mathbb{B})\}$. By our assumption, for given $\mu = (\mu_k)_{k\in K} \in \sigma(\mathbb{B})$, there exists $\lambda(\mu) = (\lambda_k(\mu))_{k\in K} \in \sigma(\mathbb{A}))$ such that $(\lambda(\mu), \mu) \in S$. Therefore,

$$|\sum_{k\in K}\mu_k\lambda_k| = |\sum_{k\in K}\mu_k(\lambda_k - \lambda_k(\mu))| \leq (\sum_{k\in K}|\mu_k|^2)^{1/2}(\sum_{k\in K}|\lambda_k - \lambda_k(\mu)|^2)^{1/2} \leq$$
$$\leq (\sum_{k\in K}||B_k||^2)^{1/2}(\text{diam } \sigma(\mathbb{A}).$$

Thus

$$||\sum_{k\in K} B_k\lambda_k|| \leq (\sum_{k\in K}||B_k||^2)^{1/2}(\text{diam } \sigma(\mathbb{A})),$$

implying $||\sum_{k\in K} B_k X\lambda_k|| \leq (\sum_{k\in K}||B_k||^2)^{1/2}(\text{diam } \sigma(\mathbb{A}))||X||$, and

$$||\Delta(X)|| \leq 2(\sum_{k\in K}||B_k||^2)^{1/2}||X||\text{diam } \sigma(\mathbb{A}).$$

□

The condition (6) is not necessary for a linear operator equation (5) to have sense. In many situations one may only suppose that

$$\sum_{k\in K} B_k B_k^* < \infty, \quad \sum_{k\in K} A_k^* A_k < \infty, \tag{7}$$

which means that the norms of partial sums are bounded and the series strongly converge. For equations with normal coefficients one can realize $\mathbb{A}$ and $\mathbb{B}$ as families of multiplication operators on the spaces $H_1 = L_2(X,\mu)$, $H_2 = L_2(Y,\nu)$:

$$A_k u(x) = f_k(x)u(x), \quad B_k v(y) = g_k(y)v(y).$$

By (7),

$$\sum_{k\in K}|f_k(x)|^2 < \infty, \quad \sum_{k\in K}|g_k(y)|^2 < \infty.$$

In other words we may write (5) as $F\cdot X = 0$ with $F \in V^\infty(X,Y)$ if $V^\infty(X,Y)$-module structure in $B(H_1,H_2)$ is defined by (3) (now the series converge strongly). It is not difficult to see that $F(x,y)\Psi(x,y) \in \Gamma(X,Y)$, for each $\Psi(x,y) \in \Gamma(X,Y)$, and $\langle F\cdot T, \Psi\rangle = \langle T, F\Psi\rangle$, showing that the action is well-defined.

Our next aim is to show that in terms of $V^\infty(X,Y)$-module structure one can describe $\mathfrak{M}_{min}(E)$ for arbitrary pseudo-closed set $E \subset X\times Y$.

Proposition 5.3.

$$\mathfrak{M}_{min}(E) = \{T \in B(H_1,H_2) \mid F\cdot T = 0, \textit{ for any } F \in V^\infty(X,Y) \textit{ that vanishes on } E\}.$$

Proof. Let us denote the right hand side of the equality by $\mathfrak{M}_e(E)$. Recall that $\mathfrak{M}_{min}(E)$ can be characterized by E-pairs of projections (P,Q) (see Section 2). Let (P,Q) be such a pair and $P = P(x) = (P_{ij}(x))$, $Q = Q(y) = (Q_{ij}(y))$, $P_{ij}(x) \in L^\infty(X,\mu)$, $Q_{ij}(y) \in L^\infty(Y,\nu)$ be the matrix representations with respect to a fixed basis in l_2. Since P, Q are projections,

$$\sum_j P_{ij}(x)P_{ij}^*(x) = \sum_j P_{ij}(x)P_{ji}(x) = P_{ii}(x) \in L^\infty(X,\mu)$$

and

$$\sum_j Q_{jk}^*(y)Q_{jk}(y) = \sum_j Q_{kj}(y)Q_{jk}(y) = Q_{kk}(y) \in L^\infty(Y,\nu)$$

so that $F_{ik}(x,y) = \sum_j P_{ij}(x)Q_{jk}(y) \in V^\infty(X,Y)$. Moreover, each F_{ik} vanishes on E. Therefore, assuming $T \in \mathfrak{M}_e(E)$ we obtain $F_{ik} \cdot T = 0$, for any i, k, implying $Q(1 \otimes T)P = 0$ and hence $T \in \mathfrak{M}_{min}(E)$.

Conversely, if $T \in \mathfrak{M}_{min}(E)$ then $\langle T, \Psi \rangle = 0$ for any $\Psi \in \Phi(E)$. Therefore for any $F \in V^\infty(X,Y)$ such that F vanishes on E and any $\Psi \in \Gamma(X,Y)$ the following holds

$$\langle F \cdot T, \Psi \rangle = \langle T, F\Psi \rangle = 0$$

which implies $F \cdot T = 0$, i.e. $T \in \mathfrak{M}_e(E)$. □

Corollary 5.4. *If a pseudo-closed set $E \subset X \times Y$ is a set of operator synthesis then any operator supported in E satisfies each operator equation $F \cdot T = 0$ with $F \in V^\infty(X,Y)$ vanishing on E.*

Corollary 5.5. *If a pseudo-closed set $E \subset X \times Y$ is a set of synthesis and $F_1, F_2 \in V^\infty(X,Y)$ such that* null $F_i \cong E$ *then the corresponding linear operator equations $F_1 \cdot T = 0$ and $F_2 \cdot T = 0$ are equivalent.*

Proof. Let T be a solution of the equation $F_1 \cdot T = 0$. In order to show that $F_2 \cdot T = 0$ it is sufficient, by Proposition 5.3, to show that $T \in \mathfrak{M}_{max}(E)$ ($= \mathfrak{M}_{min}(E)$ in our case).

Take $U \subset X$, $V \subset Y$ such that $(U \times V) \cap E \cong \emptyset$ and consider the operator $Q_V T P_U$ in $B(L_2(U,\mu), L_2(V,\nu))$. We have $\chi_U(x)\chi_V(y)F_1(x,y) \neq 0$ m.a.e. on $U \times V$. Let Ψ denote the set

$$\{F_1 \cdot F \mid F \in \Gamma(U,V)\} \subset \Gamma(U,V).$$

Then null $\Psi \cong \emptyset$ and, by [ShT, Corollary 4.3], Ψ is dense in $\Gamma(U,V)$. As

$$0 = \langle F_1 \cdot Q_V T P_U, F \rangle = \langle Q_V T P_U, F_1 \cdot F \rangle, \quad F \in \Gamma(U,V),$$

we obtain $Q_V T P_U = 0$. □

Remark 5.6. The result extends to systems of equations $F_1^i \cdot T = 0$, $1 \le i \le n$. In this case we have that supp $T \subset$ null F_1^i, for any i, and therefore supp $T \subset E$ for $E = \cap_{i=1}^n$null F_1^i.

Corollary 5.7. *Let f_i, g_i, $1 \le i \le n$, be Borel functions on standard Borel spaces (X,μ), (Y,ν). If $T \in B(L_2(X,\mu), L_2(Y,\nu))$ satisfies operator equations*

$$M_{f_i} T = T M_{g_i}, \quad 1 \le i \le n,$$

then $F \cdot T = 0$ for any $F \in V^\infty(X,Y)$ vanishing on $\{(x,y) \mid f_i(x) = g_i(y), 1 \le i \le n\}$.

Proof. By [ShT, Theorem 4.8] the set $\{(x,y) \mid f_i(x) = g_i(y), 1 \le i \le n\}$ is synthetic. As it was noticed in Remark 5.6, T is supported in $\{(x,y) \mid f_i(x) = g_i(y), 1 \le i \le n\}$ and the statement now follows from Corollary 5.4. □

The result implies, in particular, the Fuglede-Putnam Theorem, a useful tool in the operator theory, which states the equivalence of the relations $AT = TB$ and $A^*T = TB^*$, where A, B are normal bounded operators on a Hilbert space and T is just a bounded one acting on the same space.

It is natural to ask if the Fuglede-Putnam Theorem extends to the equations of the form $\sum_{i=1}^n B_i T A_i = 0$ and $\sum_{i=1}^n B_i^* T A_i^* = 0$, where $\{A_i\}_{1\le i\le n}$ and $\{B_i\}_{1\le i\le n}$ are commutative families of normal operators. This question of Gary Weiss [W2] has been answered negatively in [Sh2]. The proof in [Sh2] (see also [SShT] where it is written more transparently) was based exactly on the connection between the individual and global operator synthesis and related to the Schwartz example of a non-synthetic set in the operator version due to Arveson [A]).

In what follows we will find various conditions providing the equivalence for the equations of this kind and for more general linear operator equations. One of the main tools will be reducing (under some assumptions) the problem to equations in the space of Hilbert-Schmidt operators. So we begin with a general approach that relates the spaces of the solutions of linear equations in different topological vector spaces.

6 Approximate inverse intertwinings

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be topological vector spaces, $\Phi : \mathfrak{X} \to \mathfrak{Y}$ a continuous imbedding with dense range, and S and T operators acting in $\mathfrak{X}$ and $\mathfrak{Y}$, respectively, intertwined by the mapping Φ: $T\Phi = \Phi S$. We write in this case that we are given an intertwining triple (or just an intertwining) (Φ, S, T).

A net of linear mappings $F_\alpha : \mathfrak{Y} \to \mathfrak{X}$ is called an *approximate inverse intertwining* (AII) for the intertwining (Φ, S, T) if $F_\alpha \Phi \to 1_{\mathfrak{X}}$, $\Phi F_\alpha \to 1_{\mathfrak{Y}}$ and $F_\alpha T - S F_\alpha \to 0_{\mathfrak{X}}$ in the topology of simple convergence.

Denote by Φ^{-1} the full inverse image under the mapping Φ: $\Phi^{-1}(M) = \{x \in \mathfrak{X} \mid \Phi(x) \in M\}$ for any $M \subset \mathfrak{Y}$ (non-necessarily $M \subset \Phi(\mathfrak{X})$). As usually the image of a map X is denoted by Im X.

Theorem 6.1. *If the intertwinings (Φ, S_i, T_i), $1 \le i \le n$, have a common AII, then*

$$\Phi^{-1}\left(\sum_i \mathit{Im}\ T_i\right) \subset \overline{\sum_i \mathit{Im}\ S_i}.$$

Proof. If $\Phi x = \sum_i T_i y_i$, then

$$x = \lim_\alpha F_\alpha \Phi x = \lim_\alpha F_\alpha \sum_i T_i y_i =$$

$$= \lim_\alpha (\sum_i (F_\alpha T_i - S_i F_\alpha) y_i + \sum_i S_i F_\alpha y_i) = \lim_\alpha \sum_i S_i F_\alpha y_i \in \overline{\sum_i \text{Im } S_i}.$$

□

Let $\mathcal{H}$ be a Hilbert space equipped with the weak operator topology.

Corollary 6.2. *If $\mathfrak{X} = \mathcal{H}$ and (Φ, S, T) has an AII, then*

$$\Phi(\ker S^*) \cap \mathit{Im}\ T = \{0\}.$$

Proof.

$$\Phi(\ker S^*) \cap \text{Im } T = \Phi(\ker S^* \cap \Phi^{-1}(\text{Im } T)) \subset \Phi(\ker S^* \cap \overline{\text{Im } S}) = \{0\}.$$

□

In applications $\mathfrak{X}$, $\mathfrak{Y}$ will be Banach spaces (of operators) supplied with weak or weak* topology. Nevertheless AII's can be used to obtain some norm inequalities. Such a possibility is provided by the following result:

Proposition 6.3. *Suppose that $\mathfrak{X}$ is a Banach space with the weak topology. If $G_\lambda : \mathfrak{Y} \to \mathfrak{X}$, $\lambda \in \Lambda$, is an AII for an intertwining (Φ, S, T) then there is another AII, $\{F_\alpha\}_{\alpha \in \mathfrak{A}}$, satisfying the conditions*

$$||F_\alpha \Phi x - x|| \to 0, \text{ for any } x \in \mathfrak{X}, \tag{8}$$

and

$$||(F_\alpha T - S F_\alpha) y|| \to 0, \text{ for any } y \in \mathfrak{Y}. \tag{9}$$

Proof. Let $\mathfrak{A}$ be the set of all triples $\alpha = (E, \lambda, \varepsilon)$, where E is a finite subset of $\mathfrak{X}$, $\lambda \in \Lambda$, $\varepsilon > 0$. Setting $\alpha_1 < \alpha_2$ if $E_1 \subset E_2$, $\lambda_1 < \lambda_2$, $\varepsilon_1 < \varepsilon_2$ we invert $\mathfrak{A}$ into a directed set.

Fix $\alpha = (E, \lambda, \varepsilon) \in \mathfrak{A}$. We claim that there is a convex combination, $F = F_\alpha$, of operators G_μ with $\mu > \lambda$ such that $||F\Phi x - x|| < \varepsilon$ for any $x \in E$.

Indeed, let $x_\mu = G_\mu \Phi x - x$. By our assumption $x_\mu \to 0$ (weakly in $\mathfrak{X}$), for any $x \in \mathfrak{X}$. Let $N = \text{card } E$ and let $\mathfrak{X}^N$ be the direct sum of N copies of $\mathfrak{X}$. Then the net $e_\mu = \oplus_{x \in E} x_\mu$ tends to 0 weakly in $\mathfrak{X}^N$, whence there is a convex combination $e = \sum_{i=1}^n c_i e_{\mu_i}$ with $\mu_i > \lambda$ such that $||e|| < \varepsilon$. Now setting $F = \sum_{i=1}^n c_i G_{\mu_i}$ we prove the claim.

It is clear that the net $\{F_\alpha\}_{\alpha \in \mathfrak{A}}$ is an AII for (Φ, S, T) and that (8) is satisfied. To obtain (9) one should repeat the trick (clearly the property (8) will be preserved). □

Theorem 6.4. *Let Φ intertwine pairs S_i, T_i $(i = 1, 2)$. Suppose that $\mathfrak{X}$ is a Banach space equipped with a weak topology and $||S_2x|| \leq ||S_1x||$ for any $x \in \mathfrak{X}$. If (Φ, S_1, T_1) has AII then*

$$T_1^{-1}(Im\ \Phi) \subset T_2^{-1}(Im\ \Phi)$$

and

$$||\Phi^{-1}T_2y|| \leq ||\Phi^{-1}T_1y|| \tag{10}$$

for any $y \in T_1^{-1}(Im\ \Phi)$.

Proof. By Proposition 6.3 we can assume that (8) and (9) hold. Let $y \in T_1^{-1}(\text{Im } \Phi)$. Thus, $T_1y = \Phi x_1$ for some $x_1 \in \mathfrak{X}$. Hence,

$$x_1 = \lim_\alpha F_\alpha \Phi x_1 = \lim_\alpha F_\alpha T_1 y = \lim_\alpha((F_\alpha T_1 - S_1 F_\alpha)y + S_1 F_\alpha y) = \lim_\alpha S_1 F_\alpha y.$$

Since $\{S_1F_\alpha y\}$ is a Cauchy net and $||S_2F_\alpha y - S_2F_\beta y|| \leq ||S_1F_\alpha y - S_1F_\beta y||$, we have that $\{S_2F_\alpha y\}$ is Cauchy. Let $x_2 = \lim_\alpha S_2F_\alpha y$. Then $||x_2|| \leq ||x_1||$ and

$$\Phi x_2 = \lim_\alpha \Phi S_2 F_\alpha y = \lim_\alpha T_2 \Phi F_\alpha y = T_2 y,$$

the convergence being in the weak topology. This imply $y \in T_2^{-1}(\text{Im } \Phi)$ and

$$||\Phi^{-1}T_2y|| = ||x_2|| \leq ||x_1|| = ||\Phi^{-1}T_1y||.$$

□

The following result has some similarity to Theorem 6.4 but it does not use *AII's*.

Proposition 6.5. *Let $\Phi : \mathcal{H} \to \mathfrak{Y}$ intertwine a normal operator S with T_1 and its adjoint S^* with T_2. Suppose that* $\ker S \cap \overline{\Phi^{-1}(T_2\mathfrak{Y})} = \{0\}$*. Then (10) holds for any $y \in \mathfrak{Y}$ such that $T_iy \in \Phi\mathcal{H}$, $i = 1, 2$.*

Proof. Note first that T_1 and T_2 commute. Indeed, $(T_1T_2 - T_2T_1)\Phi y = \Phi(SS^* - S^*S)y = 0$; since $\Phi\mathcal{H}$ is dense in $\mathfrak{Y}$ the claim follows.

Let U be a partially isometric operator such that $SU = S^*$, $U\mathcal{H} = \overline{S\mathcal{H}} = \overline{S^*\mathcal{H}}$. Let $T_1y = \Phi x_1$, $T_2y = \Phi x_2$. We have to prove that $||x_2|| \leq ||x_1||$. For $x = x_2 - Ux_1$, one has

$$\Phi Sx = \Phi Sx_2 - \Phi S^* x_1 = T_1 \Phi x_2 - T_2 \Phi x_1 = T_1T_2y - T_2T_1y = 0,$$

and hence $Sx = 0$, $x \in \ker S$.

On the other hand, $\Phi x_2 = T_2y \in T_2\mathfrak{Y}$, $x_2 \in \Phi^{-1}(T_2\mathfrak{Y})$, $Ux_1 \in \overline{S\mathcal{H}} \subset \overline{\Phi^{-1}(T_2\mathfrak{Y})}$, so that $x \in \overline{\Phi^{-1}(T_2\mathfrak{Y})} \cap \ker S = \{0\}$. Hence $x_2 = Ux_1$, $||x_2|| \leq ||x_1||$. □

We return to AII's. The following result is an immediate consequence of Theorem 6.4.

Corollary 6.6. *Suppose that S is a normal operator on $\mathcal{H}$ and (Φ, S, T_1), (Φ, S^*, T_2) have approximate inverse interwtinings (non-necessarily coinciding). Then*

$$||\Phi^{-1}T_2y|| = ||\Phi^{-1}T_1y||$$

for any $y \in T_1^{-1}(Im\ \Phi) = T_2^{-1}(Im\ \Phi)$ and, in particular, $\ker T_1 = \ker T_2$.

In many cases the verification that a net $\{F_\alpha\}$ is an AII can be considerably simplified by using the following result.

Proposition 6.7. *Let* (Φ, S, T) *be an intertwining triple and* $\{F_\alpha\}$ *a net of operators from* $\mathfrak{Y}$ *to* $\mathfrak{X}$.

(i) If $\mathfrak{X}$ *is a dual Banach space with the weak-* topology and* $\{F_\alpha\}$ *satisfies the conditions*

(a) $\Phi F_\alpha y \to y$, *for any* $y \in \mathfrak{Y}$;

(b) $\{F_\alpha \Phi x\}$ *is bounded for any* $x \in \mathfrak{X}$;

(c) $\{(F_\alpha T - SF_\alpha)y\}$ *is bounded for any* $y \in \mathfrak{Y}$,

then $\{F_\alpha\}$ *is an AII.*

(ii) If $\mathfrak{Y}$ *is a Banach space with the weak topology and* $\{F_\alpha\}$ *satisfies the conditions*

(d) $F_\alpha \Phi x \to x$, *for any* $x \in \mathfrak{X}$;

(e) $\sup_\alpha ||\Phi F_\alpha|| < \infty$;

(f) for any neighbourhood U *of* 0 *in* $\mathfrak{X}$ *there is* $\delta > 0$ *such that* $(F_\alpha T - SF_\alpha)y \in U$ *for all* α, *when* $||y|| < \delta$,

then $\{F_\alpha\}$ *is an AII.*

Proof. (i) Let $x \in \mathfrak{X}$. We have to prove that $F_\alpha \Phi x \to x$. Since the net $\{F_\alpha \Phi x\}$ is bounded it is precompact (in the chosen topology of $\mathfrak{X}$) so it suffices to show that x is its only limit point. But if x_1 is a limit point of $\{F_\alpha \Phi x\}$ then Φx_1 is a limit point of $\{\Phi F_\alpha \Phi x\}$ which tends to Φx. So $\Phi x_1 = \Phi x$, $x_1 = x$.

The proof of the condition $(F_\alpha T - SF_\alpha)y \to 0$ is similar.

(ii) The uniform boundedness permits us to prove the limit condition $\Phi F_\alpha y \to y$ and $(F_\alpha T - SF_\alpha)y \to 0$ on a dense subset. But for $y \in \Phi\mathfrak{X}$ they evidently follow from (d). □

In general a net $\{F_\alpha\}$ satisfying the conditions (a), (b), (c) of Proposition 6.7 is called an *approximate inverse semi-intertwining* (AIS).

Denote by $\mathfrak{X}^*$ the space of continuous antilinear functionals on $\mathfrak{X}$, endowed with the weak-* topology (in particular, $\mathcal{H}^* = \mathcal{H}$). The adjoint operators (on $\mathfrak{X}^*$ or between $\mathfrak{X}^*$ and $\mathfrak{Y}^*$) are defined in the usual way. In particular, the adjoint of an operator on $\mathcal{H}$ has the usual meaning.

It is not difficult to see that if $\{F_\alpha\}$ is an AII for (Φ, S, T) then $\{F_\alpha^*\}$ is an AII for (Φ^*, T^*, S^*).

Let $\Phi : \mathcal{H} \to \mathfrak{Y}$ intertwine operators S, S^* with T_1, T_2. Let $\{F_\alpha\} : \mathfrak{Y} \to \mathcal{H}$ be an AII for the intertwining (Φ, S, T_1). It is called a $*$-*approximate inverse intertwining* ($*$-AII) for the ordered pair $((\Phi, S, T_1), (\Phi, S^*, T_2))$ if $\{F_\alpha^* F_\alpha\}$ is an AII for $(\Phi\Phi^*, T_1^*, T_2)$.

A $*$-*approximate semi-intertwining* ($*$-AIS) is defined in a similar way: it is an AIS $\{F_\alpha\}$ such that $\{F_\alpha^* F_\alpha\}$ is an AIS for $(\Phi\Phi^*, T_1^*, T_2)$. Since $\mathcal{H}$ is reflexive, $\{F_\alpha\}$ is in fact an AII by Proposition 6.7.

Corollary 6.8. *If $\mathfrak{Y}$ is a Banach space with the weak topology then any $*$-AIS is a $*$-AII.*

Proof. Follows from Proposition 6.7(i) (with $\mathfrak{Y}^*$ as the space $\mathfrak{X}$). $\square$

Theorem 6.9. *(i) If the pair $((\Phi, S, T_1), (\Phi, S^*, T_2))$ has $*$-AIS, then*

$$(Im\ T_1) \cap T_2^{-1}(\Phi\Phi^*(\mathfrak{Y}^*)) \subset \Phi(\mathcal{H}).$$

(ii) If $((\Phi, S, T_1), (\Phi, S^*, T_2))$ has $*$-AII then

$$||\Phi^{-1}(T_1 y)||^2 = \langle (\Phi\Phi^*)^{-1}(T_2 T_1 y), y\rangle$$

for any $y \in (T_2 T_1)^{-1}(\Phi\Phi^*(\mathfrak{Y}^*))$.

Proof. Let $\hat{y} \in (\text{Im}\ T_1) \cap T_2^{-1}(\Phi\Phi^*(\mathfrak{Y}^*))$. Then $\hat{y} = T_1 y$ and $T_2\hat{y} = \Phi\Phi^* z$, for some $z \in \mathfrak{Y}^*$ and $y \in \mathfrak{Y}$. Let $\{F_\alpha\}$ be a $*$-AIS for the pair $((\Phi, S, T_1), (\Phi, S^*, T_2))$. Then $\{F_\alpha^* F_\alpha T_2 \hat{y} - T_1^* F_\alpha^* F_\alpha \hat{y}\}$ is bounded. Because $F_\alpha^* F_\alpha T_2 \hat{y} = F_\alpha^* F_\alpha \Phi\Phi^* z \to z$, we obtain also boundedness of the net $\{||F_\alpha \hat{y}||^2\}$:

$$||F_\alpha \hat{y}||^2 = (F_\alpha^* F_\alpha \hat{y}, T_1 y) = (T_1^* F_\alpha^* F_\alpha \hat{y}, y).$$

Thus there exists a subnet $\{F_{n(\alpha)}\hat{y}\}$ converging weakly to a vector $h \in \mathcal{H}$. This gives us

$$\hat{y} = \lim_\alpha \Phi F_{n(\alpha)} \hat{y} = \Phi h \in \Phi(\mathcal{H}),$$

and the first statement of the theorem.

Let now $\{F_\alpha\}$ be a $*$-AII. Then the net $\{F_\alpha^* F_\alpha T_2 T_1 y - T_1^* F_\alpha^* F_\alpha T_1 y\}$ converges to zero in the weak*-topology in $\mathfrak{Y}^*$. By the previous arguments we have that $F_\alpha^* F_\alpha T_2 T_1 y \to z$, where $T_2 T_1 y = \Phi\Phi^* z$, implying $\{T_1^* F_\alpha^* F_\alpha T_1 y\} \to z$ and

$$\langle T_1^* F_\alpha^* F_\alpha T_1 y, y\rangle \to \langle z, y\rangle.$$

On the other hand, by the first statement, $T_1 y = \Phi h$ for some $h \in \mathcal{H}$ and

$$\langle T_1^* F_\alpha^* F_\alpha T_1 y, y\rangle = \langle F_\alpha^* F_\alpha T_1 y, T_1 y\rangle = \langle F_\alpha^* F_\alpha \Phi h, \Phi h\rangle = (\Phi^* F_\alpha^* F_\alpha \Phi h, h) \to ||h||^2.$$

One can see the convergence here in the following way. Since $\{F_\alpha \Phi h\}$, $h \in \mathcal{H}$, is bounded, the net $\{\Phi^* F_\alpha^* F_\alpha \Phi h\}$ is bounded and therefore precompact in the weak topology of $\mathcal{H}$. Moreover, as $\Phi\Phi^* F_\alpha^* F_\alpha \Phi h \to \Phi h$ and Φ is injective, we have that the only limit point of $\{\Phi^* F_\alpha^* F_\alpha \Phi h\}$ is h.

Finally we obtain

$$||\Phi^{-1}(T_1 y)||^2 = ||h||^2 = \langle z, y\rangle = \langle (\Phi\Phi^*)^{-1}(T_2 T_1 y), y\rangle.$$

$\square$

Corollary 6.10. *If $((\Phi, S, T_1), (\Phi, S^*, T_2))$ has $*$-AIS then $Im\ T_1 \cap \ker T_2 = \{0\}$.*

Proof. Clearly, $\ker T_2 \subset T_2^{-1}(\Phi\Phi^*(\mathfrak{Y}^*))$. Therefore, by Theorem 6.9, $\text{Im}\ T_1 \cap \ker T_2 \subset \Phi(\mathcal{H})$ and, by Corollary 6.2,

$$\text{Im}\ T_1 \cap \ker T_2 = \text{Im}\ T_1 \cap \ker T_2 \cap \Phi(\mathcal{H}) = \text{Im}\ T_1 \cap \Phi(\ker S^*) = 0.$$

$\square$

7 AII for inclusions of symmetrically normed ideals and multiplication operators

Here we apply the results of the previous chapter to multiplication operators on symmetrically normed ideals of operators on Hilbert spaces.

Let H be a Hilbert space, $B(H)$ be the space of bounded linear operators on H. For a symmetrically normed ideal J we denote by $||\cdot||_J$ the associated norm. If $J = \mathfrak{S}_p$, $1 \le p < \infty$, a Shatten-von Neumann ideal, we simply write $||\cdot||_p$ instead of $||\cdot||_{\mathfrak{S}_p}$.

Given J, we set

$$\begin{aligned} J^* &= \{X \in B(H) \mid XY \in \mathfrak{S}_1, \text{ for any } Y \in J\}, \\ J' &= \{X \in B(H) \mid XY \in \mathfrak{S}_2, \text{ for any } Y \in J\}, \\ \tilde{J} &= \{X \in B(H) \mid XY \in J^*, \text{ for any } Y \in J\}. \end{aligned}$$

It is clear that in this way one obtains ideals of $B(H)$ which become symmetrically normed if the norm of an operator X in J^* (J' and $\tilde{J}$) is defined as the norm of the mapping $Y \mapsto XY$ from J to $\mathfrak{S}_1$ (from J to $\mathfrak{S}_2$ and from J to J^* respectively).

One can easily see that

$$\begin{aligned} B(H)^* &= \mathfrak{S}_1, \quad \mathfrak{S}_1^* = B(H), \quad \mathfrak{S}_\infty^* = \mathfrak{S}_1, \\ B(H)' &= \mathfrak{S}_2, \quad \mathfrak{S}_1' = B(H), \quad \mathfrak{S}_\infty' = \mathfrak{S}_2, \\ \tilde{B}(H) &= \mathfrak{S}_1, \quad \tilde{\mathfrak{S}}_1 = B(H), \quad \tilde{\mathfrak{S}}_\infty = \mathfrak{S}_1, \end{aligned}$$

and if $J = \mathfrak{S}_p$, $1 < p \le \infty$, then

$$J^* = \mathfrak{S}_{p/(p-1)}, \quad J' = \begin{cases} \mathfrak{S}_{2p/(p-1)}, & p > 2, \\ B(H), & p \le 2, \end{cases} \quad \tilde{J} = \begin{cases} \mathfrak{S}_{p/(p-2)}, & p > 2, \\ B(H), & p \le 2 \end{cases}$$

(the equality for symmetrically normed ideals assumes the equality of the norms).

Let $\{A_k\}_{k\in K}$ and $\{B_k\}_{k\in K}$ be arbitrary families of operators (not necessarily commuting or normal) acting on H such that

$$\sum_{k\in K} ||A_k|| \cdot ||B_k|| < \infty.$$

Note that multiplying by constants $A_k \mapsto \lambda_k A_k$, $B_k \mapsto \lambda_k^{-1} B_k$ we may (and will) assume that

$$\sum_{k\in K} ||A_k||^2 < \infty, \qquad \sum_{k\in K} ||B_k||^2 < \infty. \tag{11}$$

It is easy to check that in this case the multiplication operator $\Delta : X \mapsto \sum_{k\in K} B_k X A_k$ is continuous on $B(H)$ and preserves all symmetrically normed ideals. We shall also denote by $\tilde{\Delta}$ the formal adjoint to Δ: $\tilde{\Delta}(X) = \sum_{k\in K} B_k^* X A_k^*$. Note that $\tilde{\Delta}|_{\mathfrak{S}_2} = (\Delta|_{\mathfrak{S}_2})^*$ and $\tilde{\Delta}|_{\mathfrak{S}_1} = \Delta^*$. For simplicity of notation, we write Δ_J and $\tilde{\Delta}_J$ instead of $\Delta|_J$ and $\tilde{\Delta}|_J$ for a symmetrically normed ideal J of $B(H)$.

If $J_1 \subset J_2$ then the natural inclusion $J_1 \hookrightarrow J_2$ will be denoted by Φ_{J_1,J_2}. Clearly Φ_{J_1,J_2} intertwines Δ_{J_1} with Δ_{J_2}. For brevity we will denote this intertwining triple by $(\Phi, \Delta_{J_1}, \Delta_{J_2})$.

Now we should look for the approximate inverse intertwinings. They will be constructed by means of increasing sequences $\{P_n\}$ of finite-dimensional projections and will have the form $F_n(X) = XP_n$. But for this the coefficient families of a multiplication operator must satisfy some restrictions.

For any family $\{X_k\}_{k\in K}$ of operators and a finite-dimensional projection P we set

$$\varphi_P^J(\{X_k\}_{k\in K}) = (\sum_{k\in K} ||[X_k, P]||_J^2)^{1/2},$$

where $||\cdot||_J$ is the norm in the ideal J.

A family $\{X_k\}_{k\in K}$ is said to be *J-semidiagonal* if there exists a sequence of projections P_n of finite rank such that $P_n \to^s 1$ and $\sup_n \varphi_{P_n}^J(\{X_k\}_{k\in K}) < \infty$. If $J = \mathfrak{S}_p$, $1 \le p \le \infty$, we write simply *p-semidiagonal.* It is clear that if $p_1 < p_2$ then each p_1-semidiagonal family is p_2-semidiagonal. In particular, 1-semidiagonality is the strongest of these conditions. Clearly, any finite family is $\mathfrak{S}_\infty$-semidiagonal.

Examples of semidiagonal families will be discussed later on.

In the following theorem J stands for either a separable symmetrically normed ideal with the weak topology or for a symmetrically normed ideal, dual to a separable one, with the weak*-topology. In both cases J^* is identified with the space of continuous antilinear functionals by means of the map $f_Y(X) = tr(X^*Y)$. Under this correspondence $\tilde{\Delta}_{J^*} = \Delta_J^*$.

Theorem 7.1. *Assume that $\mathfrak{S}_2 \subset J$.*

(i) If $\{A_k\}_{k\in K}$ is J'-semidiagonal then there is an AII for $(\Phi, \Delta_{\mathfrak{S}_2}, \Delta_J)$.

(ii) If $\{A_k\}_{k\in K}$ is $\tilde{J}$-semidiagonal then there is a $*$-AIS for $((\Phi, \Delta_{\mathfrak{S}_2}, \Delta_J),\ (\Phi, \tilde{\Delta}_{\mathfrak{S}_2}, \tilde{\Delta}_J))$, and, moreover, a $*$-AII if J is separable with the weak topology.

Assume that $\mathfrak{S}_1 \subset J$.

(iii) If $\{A_k\}_{k\in K}$ is J^*-semidiagonal then there exists an AIS for $(\Phi, \Delta_{\mathfrak{S}_1}, \Delta_J)$ in general and an AII if J is separable with the weak topology.

Proof. (i) We define $F_n : J \to \mathfrak{S}_2$ by $F_n(X) = XP_n$, $X \in J$, where $\{P_n\}$ is a sequence of finite rank projections such that $\sup_n \varphi_{P_n}^J(\{A_k\}_{k\in K}) < \infty$ and $P_n \to^s 1$, $n \to \infty$. Clearly, $F_n\Phi \to 1$, and $\Phi F_n \to 1$. For $X \in J$ one can easily check the equality

$$(\Delta_{\mathfrak{S}_2} F_n - F_n \Delta_J)(X) = \sum_{k\in K} B_k X [A_k, P_n]$$

and

$$||(\Delta_{\mathfrak{S}_2} F_n - F_n \Delta_J)(X)||_2 \le \sum_{k\in K} ||B_k X||_J ||[A_k, P_n]||_{J'} \le$$
$$\le (\sum_{k\in K} ||B_k X||_J^2)^{1/2} \varphi_{P_n}^{J'}(\{A_k\}_{k\in K}) \le ||X||_J (\sum_{k\in K} ||B_k||^2)^{1/2} \varphi_{P_n}^{J'}(\{A_k\}_{k\in K}),$$

showing that $\{F_n\}$ is an AIS. Since $\mathfrak{S}_2$ is reflexive, $\{F_n\}$ is an AII by Proposition 6.7(i).

(ii) Define $F_n : J \to \mathfrak{S}_2$ as before: $F_n(X) = XP_n$, $X \in J$, where $\sup_n \varphi^{\tilde{J}}_{P_n}(\{X_k\}_{k\in K}) < \infty$ and $P_n \to^s 1$, $n \to \infty$.

Similar arguments shows that

$$||\Delta^*_J F^*_n F_n(X) - F^*_n F_n \tilde{\Delta}_J(X)||_{J^*} \leq \sum_{k\in K} ||B^*_k X[A^*_k, P_n]||_{J^*} \leq$$
$$\leq \sum_{k\in K} ||B^*_k X||_J ||[A^*_k, P_n]||_{\tilde{J}} \leq ||X||_J (\sum_{k\in K} ||B_k||^2)^{1/2} \varphi^{\tilde{J}}_{P_n}(\{A_k\}_{k\in K}).$$

giving $||\Delta^*_J F^*_n F_n - F^*_n F_n \tilde{\Delta}_J|| < \infty$. Thus $\{F^*_n F_n\}$ is an AIS for $(\Phi\Phi^*, \Delta^*_J, \tilde{\Delta}_J)$ and if J is supplied with the weak topology it is even an AII by Proposition 6.7.

The sequence $\{F_n\}$ is an AIS for $(\Phi, \Delta_{\mathfrak{S}_2}, \Delta_J)$ and, by reflexivity of $\mathfrak{S}_2$, it is also an AII. In fact,

$$||(\Delta_{\mathfrak{S}_2} F_n - F_n \Delta_J)(X)||_2 \leq \sum_{k\in K} ||B_k X[A_k, P_n]||_2 \leq$$
$$\leq C \sum_{k\in K} ||B_k X[A_k, P_n]||_{J^*} \leq \sum_{k\in K} ||B_k X||_J ||[A_k, P_n]||_{\tilde{J}} \leq$$
$$\leq (\sum_{k\in K} ||B_k X||^2_J)^{1/2} \varphi^{\tilde{J}}_{P_n}(\{A_k\}_{k\in K}) \leq ||X||_J (\sum_{k\in K} ||B_k||^2)^{1/2} \varphi^{\tilde{J}}_{P_n}(\{A_k\}_{k\in K}).$$

(we used the fact that $\mathfrak{S}_2 \subset J$ and therefore $J^* \subset \mathfrak{S}_2$ so that $||\cdot||_2 \leq C||\cdot||_{J^*}$ for some constant C).

(iii) In a similar way one shows that $F_n : J \to \mathfrak{S}_1$, $F_n(X) = XP_n$, $X \in J$, is an AIS for the intertwining $(\Phi, \Delta_{\mathfrak{S}_1}, \Delta_J)$. Moreover,

$$||\Delta_{\mathfrak{S}_1} F_n - F_n \Delta_J|| \leq \sum_{k\in K} (||B_k||^2)^{1/2} \varphi^{J^*}_{P_n}(\{A_k\}_{k\in K}).$$

Therefore, to prove that $\{F_n\}$ is an AII for separable J (endowed with the weak topology) it is sufficient to prove that $B_k X[A_k, P_n] \to^w 0$, as $n \to \infty$, for any $X \in J$.

Given $Z \in B(H)$,

$$|\mathrm{tr}(ZB_k X[A_k, P_n])| = |\mathrm{tr}(ZB_k X(1-P_n)A_k P_n) - \mathrm{tr}(ZB_k X P_n A_k(1-P_n))| \leq$$
$$\leq |\mathrm{tr}(ZB_k X(1-P_n)A_k P_n)| + |\mathrm{tr}((1-P_n)ZB_k X P_n A_k(1-P_n))| \leq$$
$$\leq ||(ZB_k X)(1-P_n)||_J \cdot ||(1-P_n)A_k P_n||_{J^*} + ||(1-P_n)(ZB_k X)||_J \cdot ||(P_n A_k(1-P_n)||_{J^*}.$$

Since $\sup_n ||(P_n A_k(1-P_n)||_{J^*} < \infty$, and $||ZB_k X(1-P_n)||_J$, $||(1-P_n)ZB_k X||_J \to 0$ as $n \to \infty$ if J is separable, we have the statement. □

Remark 7.2. As in the proof of (i) ((iii) respectively) one can show that having a finite number of multiplication operators $\Delta_i : B(H) \to B(H)$, $\Delta_i(X) = \sum_{k\in K} B^i_k X A^i_k$ such that the family $\{A^i_k\}^n_{i=1,k\in K}$ is J'-semidiagonal ($\tilde{J}$-semidiagonal) there exists a common AII for the intertwinings $(\Phi, (\Delta_i)_{\mathfrak{S}_2}, (\Delta_i)_J)$, $1 \leq i \leq n$ $((\Phi, (\Delta_i)_{\mathfrak{S}_1}, (\Delta_i)_J)$, $1 \leq i \leq n)$.

Corollary 7.3. *(i) If $\{A_k\}_{k\in K}$ is 1-semidiagonal, then there exist an AIS for $(\Phi, \Delta_{\mathfrak{S}_1}, \Delta)$, an AII for $(\Phi, \Delta_{\mathfrak{S}_2}, \Delta_\infty)$, a $*$-AIS for $((\Phi, \Delta_{\mathfrak{S}_2}, \Delta),\ (\Phi, (\Delta_{\mathfrak{S}_2})^*, \tilde{\Delta}))$ and a $*$-AII for the pair $((\Phi, \Delta_{\mathfrak{S}_2}, \Delta_\infty),\ (\Phi, (\Delta_{\mathfrak{S}_2})^*, \tilde{\Delta}_\infty))$.*

(ii) If $\{A_k\}_{k\in K}$ is 2-semidiagonal, then there exists an AII for $(\Phi, \Delta_{\mathfrak{S}_2}, \Delta)$.

(iii) If $\{A_k\}_{k\in K}$ is $p/(p-1)$-semidiagonal, then there exists an AII for $(\Phi, \Delta_{\mathfrak{S}_1}, \Delta_{\mathfrak{S}_p})$.

(iv) If $\{A_k\}_{k\in K}$ is $2p/(p-2)$-semidiagonal, then there exists an AII for $(\Phi, \Delta_{\mathfrak{S}_2}, \Delta_{\mathfrak{S}_p})$.

(v) If $\{A_k\}_{k\in K}$ is $p/(p-2)$-semidiagonal, then there exists a $*$-AII for $((\Phi, \Delta_{\mathfrak{S}_2}, \Delta_{\mathfrak{S}_p}),$ $(\Phi, (\Delta_{\mathfrak{S}_2})^*, \tilde{\Delta}_{\mathfrak{S}_p}))$.

Now we list some examples of semidiagonal families.

Proposition 7.4. *If in some basis the matrices of all the operators A_k have all their nonzero entries on a finite number of diagonals (and $\sum_{k\in K} ||A_k||^2 < \infty$), then the family $\{A_k\}_{k\in K}$ is 1-semidiagonal.*

Proof. Let $\{e_k\}$ be a basis satisfying the assumptions. Then $(A_k e_i, e_j) = 0$ for $|i-j| > n$, where n is a positive integer. Let P_m be the projection onto the subspace generated by $e_1, \ldots, e_k$. One can easily see that for each m and k the rank of the operator $[A_k, P_m]$ does not exceed $2n+1$ and therefore

$$||[A_k, P_m]||_1 \le (2n+1)||[A_k, P_m]|| \le 2(2n+1)||A_k||,$$

$$\sup_m \varphi_{P_m}^{\mathfrak{S}_1}(\{A_k\}_{k\in K}) \le 2(2n+1)(\sum_{k\in K} ||A_k||^2)^{1/2} < \infty.$$

□

The simplest class of such examples consists of finite families of weighted shifts.

More generally one can consider operators with matrices whose entries a_{ij} sufficiently quickly decrease with $|i-j| \to \infty$. Let $|A|_k = \sup_{|i-j|=k} |a_{ij}|$ and $|A|_{diag} = \sum_k k|A|_k$. Then $||[A, P_m]||_1 < |A|_{diag}$ for each m. We call A *diagonally bounded* if $|A|_{diag} < \infty$. We have

Proposition 7.5. *Any finite family of diagonally bounded operators is 1-semidiagonal.*

Corollary 7.6. *Let $\mathcal{A}$ be the algebra of operators on $L_2(\mathbb{T})$ generated by shifts $u(t) \mapsto u(t-\theta)$ and multiplication operators M_f, $f \in C^2(\mathbb{T})$. Then any finite family of elements of $\mathcal{A}$ is 1-semidiagonal.*

Proof. It suffices to show that any shift operator and any multiplication operator M_f, $f \in C^2(\mathbb{T})$ are diagonally bounded for the standard basis $e_n = e^{int}$, $n \in \mathbb{N}$. Shifts are diagonally bounded because their matrices are diagonal. If $f = \sum_n a_n e^{int} \in C^2(\mathbb{T})$ then $\sum_n n|a_n| < \infty$ and $|M_f|_k = max\{|a_k|, |a_{-k}|\}$. Hence $|M_f|_{diag} < \infty$. □

In particular, all Bishop's operators $u(t) \mapsto e^{it}u(t-\theta)$ are 1-semidiagonal. This was established by Voiculescu in [Vo2].

For a family, $\mathbb{A} = \{A_k\}_{k\in K}$, of normal operators with $\sum_{k\in K}||A_i||^2 < \infty$, the Hausdorff dimension, dim of its spectrum $\sigma(\mathbb{A}) \subset l_2(K)$ is appeared to be important in our study. We say that the ("essential") dimension, ess-dim, of $\mathbb{A}$ does not exceed $r > 0$ if there is a subset D of $\sigma(\mathbb{A})$ such that $E_{\mathbb{A}}(\sigma(\mathbb{A}) \setminus D) = 0$ and $\dim(D) \le r$ (meaning that there exists $C > 0$ such that for $\epsilon > 0$ there is a covering $\mathcal{B} = \{\beta_j\}$ of D by pairwise disjoint Borel sets with $\mathrm{diam}\beta_j < \epsilon$ and $|\mathcal{B}|_r := (\sum_j (\mathrm{diam}\beta_j)^r)^{1/r} \le C$). In particular, if K is finite and all A_k are Lipschitz functions of one Hermitian (normal) operator then ess-dim$(\mathbb{A}) \le 1$ (respectively 2). If $\mathbb{A}$ is diagonal then ess-dim$(\mathbb{A}) = 0$.

Proposition 7.7. *If $\mathbb{A} = \{A_k\}_{k\in K}$ is a commutative family of normal operators of finite multiplicity such that ess-dim$(\mathbb{A}) \le 2$, then $\mathbb{A}$ is 2-semidiagonal. If it is a finite family of commuting normal operators of finite multiplicity such that ess-dim$(\mathbb{A}) \le p$, $p < 2$, then $\mathbb{A}$ is p-semidiagonal.*

Proof. Suppose first that $\mathbb{A}$ has a cyclic vector. Then all A_k can be realized on $L_2(\sigma(\mathbb{A}), \mu)$ as multiplication operators by the coordinate functions. Without loss of generality we may assume that $\dim(\sigma(\mathbb{A})) \le p$. Given a family $\mathcal{B} = \{\beta_j\}_{j=1}^N$ of pairwise disjoint Borel subsets of $\sigma(\mathbb{A})$ we denote by $P_{\mathcal{B}}$ the projection onto the subspace generated by the characteristic functions χ_j of the subsets β_j. Then

$$\sum_{k\in K} ||[P_{\mathcal{B}}, A_k]||_p^2 \le D|\mathcal{B}|_p^2,$$

where D is a constant. In fact, let $e_j = \chi_j/||\chi_j||$ and $\lambda \in \beta_j$. In what follows we assume K to be finite if $p \ne 2$.

$$\begin{aligned}
\sum_{k\in K} ||(1-P_{\mathcal{B}})A_kP_{\mathcal{B}}e_j||^p & \sum_{k\in K} ||(1-P_{\mathcal{B}})(A_k-\lambda_k)P_{\mathcal{B}}e_j||^p \le \\
\le \sum_{k\in K} ||(A_k-\lambda_k)P_{\mathcal{B}}e_j||^p & \le C(\sum_{k\in K} ||(A_k-\lambda_k)P_{\mathcal{B}}e_j||^2)^{p/2} \le \\
& \le C(\sum_{k\in K} |((A_k-\lambda_k)^*(A_k-\lambda_k)E_{\mathbb{A}}(\beta_j)e_j, e_j)|)^{p/2} \le \\
& \le C(\sum_{k\in K} (A_k-\lambda_k)^*(A_k-\lambda_k)E_{\mathbb{A}}(\beta_j)e_j, e_j))^{p/2} \le \\
& \le C||\sum_{k\in K} (A_k-\lambda_k)^*(A_k-\lambda_k)E_{\mathbb{A}}(\beta_j)||^{p/2} \le \\
& \le C(\sup_{\lambda'\in\beta_j} \sum_{k\in K} |\lambda'_k-\lambda_k|^2)^{p/2} \le C(\mathrm{diam}\beta_j)^p.
\end{aligned}$$

Here we use the fact that the norms $||\alpha||_p = (\sum_{i=1}^n |\alpha_i|^p)^{1/p}$, $\alpha = (\alpha_i) \in \mathbb{C}^n$, $1 \le p < \infty$ are equivalent on $\mathbb{C}^n$, C is a corresponding constant. Since $\{e_j\}_{j=1}^N$ is a basis of the subspace $P_{\mathcal{B}}H$ and $p \le 2$ we have

$$\sum_{k\in K} ||(1-P_{\mathcal{B}})A_kP_{\mathcal{B}}||_p^2 \le \sum_{k\in K} (\sum_{j=1}^N ||(1-P_{\mathcal{B}})A_kP_{\mathcal{B}}e_j||^p)^{2/p} \le$$

$$\le M(\sum_{k\in K}\sum_{j=1}^N ||(1-P_{\mathcal{B}})A_kP_{\mathcal{B}}e_j||^p)^{2/p} \le MC^{2/p}(\sum_{j=1}^N (\mathrm{diam}\beta_j)^p)^{2/p} = MC^{2/p}|\mathcal{B}|_p^2$$

for some constant M coming from the equivalence of the norms on $\mathbb{C}^n$.

Similarly, $\sum_{k\in K} ||(1-P_{\mathcal{B}})A_k^*P_{\mathcal{B}}||_p^2 \leq L|\mathcal{B}|_p^2$ ($L = MC^{2/p}$), and therefore $\sum_{k\in K} ||P_{\mathcal{B}}A_k(1-P_{\mathcal{B}})||_2^2 \leq L|\mathcal{B}|_p^2$ so that

$$\sum_{k\in K} ||[P_{\mathcal{B}}, A_k]||_p^2 = \sum_{k\in K} ||(1-P_{\mathcal{B}})A_kP_{\mathcal{B}}||_p^2 + \sum_{k\in K} ||P_{\mathcal{B}}A_k(1-P_{\mathcal{B}})||_2^2 \leq 2L|\mathcal{B}|_p^2.$$

Let now $\mathcal{B}^{(i)} = \{\beta_j^{(i)}\}_{j=1}^{\infty}$ be a sequence of the partitions of $\sigma(\mathbb{A})$ such that $\sup_j \operatorname{diam}\beta_j^{(i)} \to 0$ and $|\mathcal{B}^{(i)}|_p \to m_p(\sigma(\mathbb{A}))^{1/p}$, where m_p is the Hausdorff measure on $l_2(K)$. Then $P_{\mathcal{B}^{(i)}} \to I$ strongly. Since $P_{\mathcal{B}^{(i)}} = s.\lim P_{\mathcal{B}_n^{(i)}}$, where $\mathcal{B}_n^{(i)} = \{\beta_j^{(i)}\}_{j=1}^{n}$, one can find a subsequence of (finite-dimensional) projections $P_{\mathcal{B}_{n(i)}^{(i)}}$ which converges strongly to I. Finally,

$$\underline{\lim} \sum_{k\in K} ||[P_{\mathcal{B}_{n(i)}^{(i)}}, A_k]||_p^2 \leq 2L \cdot \underline{\lim}|\mathcal{B}_{n(i)}^{(i)}|_p^2 \leq 2L \cdot \lim |\mathcal{B}^{(i)}|_p^2 = 2L(m_p(\sigma(\mathbb{A})))^{2/p} < \infty,$$

implying p-semidiagonality of $\mathbb{A}$.

Generally, we decompose the Hilbert space H into a direct sum of subspaces $H = \oplus_j H_j$, where each H_j is invariant with respect to $\mathbb{A}$ and $\mathbb{A}|_{H_j}$ has a cyclic vector. If $\mathbb{A}$ has a finite multiplicity, we have a finite number of subspaces H_j and the statement easily follows from what we have already proved. □

Recall that an operator A is almost normal if $[A^*, A] \in \mathfrak{S}_1$. The following result was established by Voiculescu [Vo1, Corollary 2].

Proposition 7.8. *Any almost normal operator of finite multiplicity is 2-semidiagonal.*

In what follows we apply the obtained results to various problems on multiplication operators.

8 Application related to the traces of commutators

In [W1] Weiss proved that if A is a normal operator, $X \in \mathfrak{S}_2$ and $[A, X] \in \mathfrak{S}_1$, then $\operatorname{tr}([A, X]) = 0$. The following proposition extends this in several directions.

Proposition 8.1. *Let $p \in (1, \infty]$. If $\{A_k\}_{k=1}^n$ is $p/(p-1)$-semidiagonal, $X_k \in \mathfrak{S}_p$ and $\sum_{k=1}^n [A_k, X_k] \in \mathfrak{S}_1$ then*

$$tr(\sum_{k=1}^{n} [A_k, X_k]) = 0.$$

Proof. Let $T_k : \mathfrak{S}_p \to \mathfrak{S}_p$, $T_k(X) = [A_k, X]$, and $S_k = T_k|_{\mathfrak{S}_1}$. By Proposition 7.1(iii) and Remark 7.2, there exists a common AII for the intertwinings (Φ, S_k, T_k). By assumption, $\sum_{k\in K}[A_k, X_k] = R = \Phi(R)$, for some $R \in \mathfrak{S}_1$ and therefore $R \in \Phi^{-1}(\sum_{k=1}^n \operatorname{Im} T_k)$. Then Theorem 6.1 gives $R \in \overline{(\sum_{k=1}^n \operatorname{Im} S_k)}$ and, since $\operatorname{Im} S_k \subset \ker(\operatorname{tr})$, we obtain $\operatorname{tr}(R) = 0$. □

Remark 8.2. The result of Proposition 8.1 extends to infinite family of operators $\{A_k\}_{k\in K}$, $\{X_k\}_{k\in K}$ provided that that $\sum_{k\in K}||X_k||_p^2 < \infty$.

Corollary 8.3. *Let $\{A_k\}_{k=1}^n$ and $\{B_k\}_{k=1}^n$ be families of operators satisfying*

$$\sum_{k=1}^{n} B_k A_k = 0.$$

If $\{A_k\}_{k=1}^n$ is $p/(p-1)$-semidiagonal, $X \in \mathfrak{S}_p$ and $\Delta(X) = \sum_{k=1}^n A_k X B_k \in \mathfrak{S}_1$ then

$$tr(\Delta(X)) = 0.$$

Proof. We have

$$\Delta(X) = \sum_{k=1}^{n} A_k X B_k - X \sum B_k A_k = \sum_{k=1}^{n} [A_k, XB_k].$$

Apply now Proposition 8.1 with $X_k = XB_k$. □

Corollary 8.4. *If $f_k \in Lip_{1/2}([0,1])$, $1 \le k \le n$, then no functions $F_k \in L_2([0,1]^2)$ satisfying the condition*

$$\sum_{k=1}^{n} (f_k(x) - f_k(y)) F_k(x,y) = 1. \tag{12}$$

Proof. Let X_k be the integral operator on $L_2([0,1])$ with the kernel $F_k(x,y)$. Clearly, $X_k \in \mathfrak{S}_2$. Now (12) can be rewritten in the form $\sum\limits_k [M_{f_k}, X_k] = Q$, where Q is the rank-one operator with kernel $F(t,s) = 1$. By Proposition 7.7, the family $\{M_{f_k}\}_{k=1}^n$ is 2-semidiagonal. It remains to apply Proposition 8.1 to X_k, M_{f_k}, $p = 2$, we obtain tr $Q = 0$. A contradiction. □

One can consider more general classes of functions F_k imposing more restrictive conditions on f_k (and applying Proposition 8.1 with $p > 2$). For example, for $f_k \in Lip_1[0,1]$, (12) can not hold with arbitrary function F_k which are the integral kernels of compact operators.

We do not know if the constant $p/(p-1)$ in Proposition 8.1 is strict for all p, but for $p = 2$ it is. This follows from

Example 8.5. Let $\mathbb{D}$ be the unit disk and dA the area measure on $\mathbb{D}$. In terms of polar coordinates, we have $dA(z) = rdrd\theta$, $z = re^{i\theta}$. Let $H = L^2(\mathbb{D}, dA(z))$ and let

$$Au(z) = zu(z), \quad Xu(z) = \int\int (\xi - z)^{-1} u(\xi) dA(\xi).$$

Then $[A, X] = Q$, where $Qu(z) = \int\int u(\xi)dA(\xi)$. Clearly, rank $Q = 1$, tr $Q > 0$, A is normal and cyclic and hence 2-semidiagonal. We next claim that $X \in \cap_{\epsilon>0}\mathfrak{S}_{2+\epsilon}$. In order to prove this we decompose X into the sum $X = X_1 + X_2$, where X_i are defined by similar integrals but in X_1 we integrate by the disk $|\xi| \le |z|$, in X_2 by the annulus $|z| \le |\xi| \le 1$.

Actually, both X_i are represented as operator-weighted bilateral shifts. Indeed, let us denote, for any $k \in \mathbb{Z}$, by H_k the subspace of $L_2(\mathbb{D})$, consisting of functions $u(r,\theta) =$

$f(r)e^{ik\theta}$, where (r,θ) are the polar coordinates. The map $u \mapsto f$ identifies H_k with the space $L_2([0,1], dm)$, where $dm = 2\pi x dx$. For $u \in H_k$ one has

$$X_1 u(z) = \sum_{n=0}^{\infty} \int \int_{r \leq |z|} f(r) e^{ik\theta} r^n \frac{e^{in\theta}}{z^{n+1}} r dr d\theta = \begin{cases} 2\pi \int_{r\leq |z|} f(r) r^{-k} z^{k-1} r dr, & k \leq 0, \\ 0, & k > 0. \end{cases}$$

Writing $z = re^{i\theta}$ we have therefore

$$X_1 u(r,\theta) = h(r) e^{i(k-1)\theta},$$

where $h(r) = A_k f(r)$, A_k is the integral operator on $L_2([0,1], dm)$ with kernel $K(r,t) = r^{k-1} t^{-k} \chi_{t<r}(r,t)$. One can easily compute that $||A_k||_2^2 = \pi^2/(|k|+1)$ so that for $p > 2$

$$||X_1||_p^p = \sum_{k=0}^{\infty} ||A_k||_p^p \leq \sum_{k=0}^{\infty} ||A_k||_2^p \leq \sum_{k=0}^{\infty} \pi^p/(|k|+1)^{p/2} < \infty,$$

i.e., $K_1 \in \mathfrak{S}_p$ for any $p > 2$. Similar arguments shows that $K_2 \in \mathfrak{S}_p$, $p > 2$, verifying the statement.

The above construction answers a question of Weiss [W3] ("does a nuclear commutator of a compact and a normal operators have zero trace?"). It was first published in [Sh1] with the reference to [BirS] for the proof of the inclusion $X \in \cap_{\epsilon>0} \mathfrak{S}_{2+\epsilon}$. We included the proof because the reference was a mistake - [BirS] does not contain this fact. Using the arguments of [W1] we deduce from the above example an answer to Question 2 in [W3]:

Corollary 8.6. *There is a normal operator A and a compact operator X such that $[A,X] \in \mathfrak{S}_1$, $[A*, X] \notin \mathfrak{S}_1$.*

Weiss [O] asks also if (12) can be satisfied with $n = 2$, and $F_k \in L_2([0,1]^2)$ if f_k are only supposed to be continuous. The answer is positive as the following result shows.

Proposition 8.7. *There are functions f_1, $f_2 \in Lip_{1/3}[0,1]$ and F_1, $F_2 \in L^2([0,1]^2)$ such that*

$$(f_1(x) - f_2(y))F_1(x,y) + (f_2(x) - f_2(y))F_2(x,y) = 1 \tag{13}$$

Proof. Let $\Pi = [0,1]^3 \subset \mathbb{R}^3$, π_i be the coordinate functions on Π $(1 \leq i \leq 3)$. Set

$$\varphi_1(\lambda) = \pi_1(\lambda), \quad \varphi_2(\lambda) = \pi_2(\lambda) + i\pi_3(\lambda)$$
$$\Phi_1(\lambda,\mu) = \frac{\pi_1(\lambda) - \pi_1(\mu)}{|\lambda - \mu|^2}, \quad \Phi_2(\lambda,\mu) = \frac{\pi_2(\lambda) - i\pi_3(\lambda) - \pi_2(\mu) + i\pi_3(\mu)}{|\lambda-\mu|^2}.$$

Then

$$\sum_{i=1}^{2} (\varphi_i(\lambda) - \varphi_i(\mu))\Phi_i(\lambda,\mu) = 1.$$

An easy calculation shows that $\Phi_i \in L^2([0,1]^6)$.

Let γ be the standard Peano curve $[0,1] \to \Pi$. It can be checked that $\gamma \in Lip_{1/3}([0,1], \mathbb{R}^3)$ and γ preserves the Lebesgue measure: $m_1(\gamma^{-1}(E)) = m_3(E)$, where m_n is the Lebesgue measure on $\mathbb{R}^n$. So, if we set $f_i(x) = \varphi_i(\gamma(x))$, $F_i(x,y) = \Phi_i(\gamma(x), \gamma(y))$ then $F_i \in L_2([0,1]^2, m_2)$, $f_i \in Lip_{1/3}[0,1]$ and (13) holds. ☐

Note that if f_j in (13) are supposed to be real-valued then (13) fails. Indeed, in this case M_{f_1}, M_{f_2} is a pair of commuting self-adjoint operators, hence is 2-semidiagonal by Proposition 7.7. So it suffices to apply Proposition 8.1.

9 Non-commutative version of Fuglede theorem; extensions of Fuglede-Weiss theorem

The well-known problem of the existence of an operator A for which the image of the derivation $X \mapsto [A, X]$ has non-trivial intersection with the commutant of the operator A^* can be formulated for general multiplication operators as the problem of the validity of the equality

$$\ker \tilde{\Delta} \cap \mathrm{Im}\Delta = 0 \tag{14}$$

or of the equivalent equality

$$\ker \tilde{\Delta}\Delta = \ker \Delta. \tag{15}$$

It should be noted that (15) seems to be the "right" form of the "Fuglede Theorem for non-normal operators". Indeed, while the Fuglede theorem can be considered as the analogs for normal derivations of the fact that $\ker A = \ker A^*$, for a normal operator A, the equality (15) is an analog of the equality $\ker A^*A = \ker A$, for arbitrary operators. Clearly if (15) holds and Δ commutes with $\tilde{\Delta}$ then (15) immediately gives $\ker \Delta = \ker \tilde{\Delta}\Delta = \ker \tilde{\Delta}$. Since as we know there are multiplication operators with commuting normal coefficients for which $\ker \Delta \neq \ker \tilde{\Delta}$, the equality (15) fails in general. Nevertheless the following result shows that it holds for a broad class of multiplication operators.

Theorem 9.1. *If $\{A_k\}_{k\in K}$ is 1-semidiagonal then (15) is valid.*

Proof. By Corollary 7.3 there exists a $*$-AIS for $(\Phi_2, \Delta_{\mathfrak{S}_2}, \Delta)$ and $(\Phi, \tilde{\Delta}_{\mathfrak{S}_2}, \tilde{\Delta})$. The statement now follows from Corollary 6.10 applied to $\mathcal{H} = \mathfrak{S}_2$, $\mathfrak{Y} = B(H)$, $T_1 = \Delta$, $T_2 = \tilde{\Delta}$, $S = \Delta_{\mathfrak{S}_2}$. ☐

Proposition 9.2. *If $\{A_k\}_{k\in K}$ is 1-semidiagonal and $\tilde{\Delta}\Delta(X) \in \mathfrak{S}_1$, then $\Delta(X) \in \mathfrak{S}_2$.*

Proof. By Corollary 7.3, there exists a $*$-AIS for the pair $((\Phi_2, \Delta_{\mathfrak{S}_2}, \Delta),\ (\Phi_2, \tilde{\Delta}_{\mathfrak{S}_2}, \tilde{\Delta}))$. Moreover,

$$\Delta(X) \in \tilde{\Delta}^{-1}(\mathfrak{S}_1) \cap \mathrm{Im}\Delta = \tilde{\Delta}^{-1}(\Phi_2\Phi_2^*((B(H)^*)) \cap \mathrm{Im}\Delta.$$

Applying now Theorem 6.9, we obtain $\Delta(X) \in \Phi_2(\mathfrak{S}_2) = \mathfrak{S}_2$. ☐

Let us write $||X||_2 = \infty$ if $X \notin \mathfrak{S}_2$. In this notation the famous Fuglede-Weiss theorem [W3] states that

$$||AX - XB||_2 = ||A^*X - XB^*||_2$$

for any normal operators A, B and any operator X. Our next task is to extend this result to hyponormal operators.

Recall that an operator $A \in B(H)$ is said to be *hyponormal* if $[A^*, A]$ is positive.

Theorem 9.3. *Let $A \in B(H)$ be a hyponormal operator of finite multiplicity and let $B \in B(H)$ be such that B^* is hyponormal. Then for each $X \in B(H)$*

$$||AX - XB||_2 \geq ||A^*X - XB^*||_2.$$

Proof. Let first $X \in \mathfrak{S}_2$. Then

$$\begin{aligned}||AX - XB||_2^2 &= \operatorname{tr}((X^*A^* - B^*X^*)(AX - XB)) = \\ &= \operatorname{tr}(X^*A^*AX - AXB^*X^* - XBX^*A^* + XBB^*X^*) \geq \\ &\geq \operatorname{tr}(X^*AA^*X - AXB^*X^* - XBX^*A^* + XB^*BX^*) = ||A^*X - XB^*||_2^2.\end{aligned}$$

The inequality for arbitrary X follows now from Theorem 6.4 applied to $\mathfrak{X} = \mathfrak{S}_2$, $\mathfrak{Y} = B(H)$, $T_1 : X \mapsto AX - XB$, $T_2 : X \mapsto A^*X - XB^*$ and $S_i = T_i|_{\mathfrak{S}_2}$. The only thing we need to show is the existence of AII for (Φ_2, S_1, T_1) and, by Proposition 7.1, it would be sufficient to prove that A is 2-semidiagonal. But beacuse A is hyponormal and has finite multiplicity, A is almost normal, i.e. $[A, A^*] \in \mathfrak{S}_1$ (see, e.g. [C]), and therefore 2-semidiagonal by Proposition 7.8. □

Now we consider an extensions of the Fuglede-Weiss theorem to general multiplication operators with normal coefficients.

Let $\mathbb{A} = \{A_k\}_{k\in K}$ and $\mathbb{B} = \{B_k\}_{k\in K}$, be two separately commutating families of normal operators satisfying (11).

Proposition 9.4. *Suppose that for each $j \in J$ we are given bounded functions f_j, u_j on $\sigma(\mathbb{A})$ and g_j, v_j on $\sigma(\mathbb{B})$ such that*

$$|\sum_{j\in J} f_j(x)g_j(y)| \leq |\sum_{j\in J} u_j(x)v_j(y)|.$$

If ess-dim $(\mathbb{A}) \leq 2$, all $f_j \in Lip_1(\sigma(\mathbb{A}))$ and $\sum_{j\in J} ||f_j||_{Lip_1}^2 < \infty$, then

$$||\sum_{j\in J} g_j(\mathbb{B})Xf_j(\mathbb{A})||_2 \leq ||\sum_{j\in J} v_j(\mathbb{B})Xu_j(\mathbb{A})||_2 \tag{16}$$

for each $X \in B(H)$.

Proof. Let $\Delta_1(X) = \sum_{j\in J} g_j(\mathbb{B})Xf_j(\mathbb{A})$ and $\Delta_2(X) = \sum_{j\in J} v_j(\mathbb{B})Xu_j(\mathbb{A})$. The assumption on $\sigma(\mathbb{A})$ and the functions f_j, $j \in J$, implies ess-dim$(\{f_j(\mathbb{A})\}_{j\in J}) \leq 2$. By Proposition 7.7, $\{f_j(\mathbb{A})\}_{j\in J}$ is 2-semidiagonal and hence there exists an AII for $(\Phi_2, (\Delta_1)_{\mathfrak{S}_2}, \Delta_1)$. Using the same arguments as in the proof of Proposition 9.3, we see that it is enough to show the inequality (16) for $X \in \mathfrak{S}_2$. For this we consider the families $\mathbb{A}$, $\mathbb{B}$ concretely represented as $(A_kf)(x) = a_k(x)f(x)$ on $L_2(T,\mu)$ and $(B_kg)(x) = b_k(x)g(x)$ on $L_2(S,\nu)$ ($a_k \in L_\infty(T,\mu)$, $b_k \in L_\infty(S,\nu)$). Then $X \in \mathfrak{S}_2(L_2(T,\mu), L_2(S,\nu))$ is an integral operator with a kernel $K(x,y) \in L_2(T \times S, \mu \times \nu)$ and so are the operators $\Delta_1(X)$ and $\Delta_2(X)$ with kernels

$$K_1(x,y) = K(x,y)\sum_{j\in J} f_j(a(x))g_j(b(y)) \text{ and } K_2(x,y) = K(x,y)\sum_{j\in J} u_j(a(x))v_j(b(y))$$

respectively, where $a(x) = (a_k(x))_{k\in K}$, $b(x) = (b_k(x))_{k\in K}$. Since $a(x) \in \sigma(\mathbb{A})$, $b(y) \in \sigma(\mathbb{B})$ for almost all $(x, y) \in T \times S$, we have

$$|K_1(x,y)| \le |K_2(x,y)| \text{ a.e.},$$

whence

$$||\Delta_1(X)||_2^2 \le ||\Delta_2(X)||_2^2.$$

□

We mention two special cases of this result; they extend to "long" multiplication operators the Fuglede-Weiss and Fuglede theorems respectively.

Corollary 9.5. *If ess-dim* $(\mathbb{A}) \le 2$*, then*

$$||\sum_{k\in K} B_k X A_k||_2 = ||\sum_{k\in K} B_k^* X A_k^*||_2 \tag{17}$$

for any $X \in B(H)$.

Corollary 9.6. *Let A be a normal operator, $f_k \in Lip_1\sigma(A)$, such that $\sum_{k\in K} ||f_k||^2_{Lip_1} < \infty$, and let $A_k = f_k(A)$. Then the equations $\sum_{k\in K} A_k X B_k = 0$ and $\sum_{k\in K} A_k^* X B_k^* = 0$ are equivalent in $B(H)$.*

Proof. By the assumptions ess-dim $\{A_k\}_{k\in K} \le 2$. The statement now trivially follows from Corollary 9.5. □

It would be desirable to have a "qualitative" version of non-commutative Fuglede Theorem which would imply simultaneously the Fuglede-Weiss theorem for a sufficiently general class of multiplication operators. The following result is a first step in this direction.

Proposition 9.7. *If the coefficient family $\{A_k\}_{k\in K}$ of Δ is 1-semidiagonal then*

$$||\Delta(X)||_2^2 = tr(X^*\tilde{\Delta}\Delta(X)) \tag{18}$$

for any compact operator X such that $\tilde{\Delta}\Delta(X) \in \mathfrak{S}_1$.

Proof. Follows from Theorem 6.9(ii) and Corollary 7.3(i). □

Clearly the proposition shows that on $\mathfrak{S}_\infty$ $\ker\tilde{\Delta}\Delta = \ker\Delta$ (a restrictive form of Theorem 9.1). On the other hand if Δ has normal coefficients and, for some $X \in \mathfrak{S}_\infty$, $\tilde{\Delta}\Delta(X) \in \mathfrak{S}_1$ then $\Delta\tilde{\Delta}(X) \in \mathfrak{S}_1$ and from (18) we get $||\tilde{\Delta}(X)||_2^2 = ||\Delta(X)||_2^2$ - a special case of Corollary 9.5.

Clearly, the "Fuglede theorem" for arbitrary Δ holds in $\mathfrak{S}_2$ and therefore in $\mathfrak{S}_p$, $p \le 2$. Now we will show that for its validity in $\mathfrak{S}_p$, $p > 2$, some restrictions on the coefficient families are necessary.

Proposition 9.8. *For any $p > 2$ there is a multiplication operator Δ with commuting normal coefficients satisfying (10) such that the equations $\Delta(X) = 0$ and $\tilde{\Delta}(X) = 0$ are non-equivalent in $\mathfrak{S}_p$.*

Proof. Using the arguments in [R, 7.8.3-7.8.6] one could show that there are sequences $c \in l^p(\mathbb{Z})$ and $d \in l^1(\mathbb{Z})$ such that $c * d = 0$ and $c * \hat{d} \neq 0$ where $\hat{d}_n = \bar{d}_{-n}$. Let $d_n = a_n b_n$ with $|a_n| = |b_n|$, for each $n \in \mathbb{Z}$. We denote by U the bilateral shift acting on the space $l^2(\mathbb{Z})$ ($Ue_n = e_{n+1}$, where (e_n) is the standard basis) and set $A_n = a_n U^n$, $B_n = b_n U^{-n}$, $X = diag(c)$ which means that $Xe_n = c_n e_n$. Clearly $\{A_n\}_{n\in\mathbb{Z}}$, $\{B_n\}_{n\in\mathbb{Z}}$ are commuting families of normal operators satisfying (10) and $X \in \mathfrak{S}_p$. An easy calculation shows that

$$\sum_{n\in\mathbb{Z}} A_n X B_n = diag(d * c) = 0, \quad \sum_{n\in\mathbb{Z}} A_n^* X B_n^* = (diag(d * \hat{c}))^* \neq 0.$$

□

A very interesting result of Weiss [W2] states that if the coefficient families $\mathbb{A}$, $\mathbb{B}$ are (normal and) finite then equality (17) is valid for any X such that $\Delta(X)$ and $\tilde{\Delta}(X)$ belong to $\mathfrak{S}_2$. We will finish this section by showing how this result may be obtained by using the technique of intertwinings. The intermediate steps of the proof are of their own interest and will be used in the next section.

Lemma 9.9. *If card $K < \infty$ then $\mathfrak{S}_1 \cap \ker \Delta_{\mathfrak{S}_2}$ is $\mathfrak{S}_2$-dense in $\ker \Delta_{\mathfrak{S}_2}$.*

Proof. Using the spectral theorem we represent our operators A_k, B_k as $(A_k f)(x) = a_k(x)f(x)$ on $H_1 = L_2(X, \mu)$ and $(B_k g)(y) = b_k(y)g(y)$ on $H_2 = L_2(Y, \nu)$. Set

$$\Phi(x, y) = \sum_{k\in K} f_k(x)g_k(y) \text{ and } E = \{(x, y) \in X \times Y \mid \Phi(x, y) = 0\}.$$

The space $\mathfrak{S}_2(H_1, H_2)$ is naturally identified with $L_2(X\times Y, m)$, where $m = \mu\times\nu$. Moreover, one can easily see that an operator T belongs to $\ker \Delta_{\mathfrak{S}_2}$ iff the corresponding kernel $K(x, y) \in L_2(X \times Y, m)$ satisfies

$$K(x, y)\Phi(x, y) = 0 \quad m\text{-almost everywhere.}$$

This means that the space $\ker \Delta_{\mathfrak{S}_2}$ is isomorphic to the space of all functions $K \in L_2(X \times Y, m)$ vanishing outside E m-almost everywhere, i.e. to the space $L_2(E, m)$. Since card K $< \infty$, we have, by Corollary 2.5, that E is τ-pseudo-open, i.e. it is a union of a countable number of rectangles $A_i \times B_i$, $A_i \subset X$, $B_i \subset Y$ and a m-null set. Therefore

$$L_2(E, m) \simeq \oplus_{i=1}^{\infty} L_2(A_i \times B_i, m).$$

It remains to see that for each rectangle Π the space $L_2(\Pi, m)$ is generated by functions of type $f(x)g(y)$ corresponding to operators of rank one, the proof is complete. □

Let $\overline{\Delta(B(H))}^{w^*}$ denote the weak*-closure of $\Delta(B(H))$.

Corollary 9.10. *If card $K < \infty$ then*

$$\overline{\Delta(B(H))}^{w^*} \cap \mathfrak{S}_2 \cap \ker \Delta = \{0\}. \quad (19)$$

Proof. Let $X \in \overline{\Delta(B(H))}^{w^*} : X = \lim_t \Delta(Z_t)$. For any $Y \in \mathfrak{S}_1 \cap \ker \Delta$ we have

$$\mathrm{tr}(Y^* X) = \lim_t \mathrm{tr}(Z_t(\tilde{\Delta}(Y))^*).$$

Hence if $X \in \mathfrak{S}_2$ then by the previous lemma $\mathrm{tr}(Y^*X) = 0$ for any $Y \in \mathfrak{S}_2 \cap \ker \Delta$. So if also $X \in \ker \Delta$ then $\mathrm{tr}(X^*X) = 0$ and $X = 0$. □

Corollary 9.11. *[W2]. If card $K < \infty$, $\Delta(X), \tilde{\Delta}(X) \in \mathfrak{S}_2$ then (17) holds.*

Proof. The equality (19) implies immediately that the $\mathfrak{S}_2$-closures of $(\mathrm{Im}\ \Delta) \cap \mathfrak{S}_2$ and the $\mathfrak{S}_2$-closure of $(\mathrm{Im}\ \tilde{\Delta}) \cap \mathfrak{S}_2$ have trivial intersections with $\ker \Delta$. Now the result follows directly from Proposition 6.5. □

10 Linear operator equations with normal coefficients.

The purpose of this section is the study of the thin spectral structure of the multiplication operators with commuting normal coefficients satisfying (11). The results will be applied in the next section to the individual synthesis in Varopoulos algebras, convolution equations and partial differential equations with constant coefficients.

Lemma 10.1. *Let (Ω, μ) be a space with finite measure. Assume that Ω is metrizable such that $m_p(\Omega) < \infty$, where m_p is the Hausdorff measure corresponding to the measure function $h(r) = r^p$. If $T : L_2(\Omega, \mu) \to H$ is such that*

$$||TP(\alpha)||^2 \leq C(diam\ \alpha)^p$$

for any $\alpha \subset \Omega$, where $P(\alpha)$ is the multiplication operator by the characteristic function of α, then $T \in \mathfrak{S}_2(L_2(\Omega, \mu), H)$ and

$$||T||_2^2 \leq Cm_p(\Omega).$$

Proof. For any covering $\mathcal{E} = (\alpha_1, \ldots, \alpha_n)$ of Ω, set $|\mathcal{E}| = \sum_{k=1}^n (\mathrm{diam}\ \alpha_k)^p$. Let $e_k = \chi_k/||\chi_k||$, where χ_k is the characteristic function of α_k, and let $Q_{\mathcal{E}}$ be the projection onto the linear span of $\{e_k\}_{k=1}^n$ in $L_2(\Omega, \mu)$. Then

$$||TQ_{\mathcal{E}}||_{\mathfrak{S}_2} = \sum_{k=1}^n ||Te_k||^2 \leq C|\mathcal{E}|.$$

Taking a sequence of coverings $\{\mathcal{E}_j\}$ such that $\mathcal{E}_{j+1}$ is a refinement of $\mathcal{E}_j$ and $|\mathcal{E}_j| \to m_p(\Omega)$, we obtain $Q_{\mathcal{E}_j} \to^s 1$ and

$$\overline{\lim_j} ||TQ_{\mathcal{E}_j}||_{\mathfrak{S}_2}^2 \leq Cm_p(\Omega).$$

Therefore, $T \in \mathfrak{S}_2$ and $||T||_{\mathfrak{S}_2}^2 \leq Cm_p(\Omega)$. □

Let $\Delta(X) = \sum_{k\in K} B_k X A_k$, where $\mathbb{A} = \{A_k\}_{k\in K}$, $\mathbb{B} = \{B_k\}_{k\in K}$ are families of commuting normal operators acting on Hilbert spaces H_1 and H_2 respectively and satisfying (6). Recall that by $\mathcal{E}_\Delta(0)$ we denote the space

$$\{T \in B(H_1, H_2) \mid ||\Delta^n(T)||^{1/n} \to 0, n \to \infty\}.$$

Lemma 10.2. *Assume that ess-dim $\mathbb{A} \leq 2n$ and $\mathbb{A}$ has a cyclic vector. Then*

$$\Delta^n(X) \in \mathfrak{S}_2$$

for any $X \in \mathcal{E}_\Delta(0)$.

Proof. Without loss of generality we can assume that dim $\sigma(\mathbb{A}) \leq 2n$. Let α be a closed subset of $\sigma(\mathbb{A})$, $A_k' = A|_{E_{\mathbb{A}}(\alpha)H_1}$, Δ' is the multiplication operator with coefficients $\{A_k'\}_{k\in K}$, $\{B_k\}$ and $X' = X|_{E_{\mathbb{A}}(\alpha)}H_1$. Then it is easy to check that $X' \in \mathcal{E}_{\Delta'}(0)$. Applying Lemma 5.2 to Δ', X' we obtain the equality

$$||\Delta(X)E_{\mathbb{A}}(\alpha)|| \leq 2(\sum_{k\in K} ||B_k||^2)^{1/2}||X||(\text{diam } \alpha).$$

Changing repeatedly X by $\Delta(X)E_{\mathbb{A}}(\alpha)$ we obtain

$$||\Delta^n(X)E_{\mathbb{A}}(\alpha)|| \leq C(\text{diam } \alpha)^{2n},$$

where $C = (2(\sum_{k\in K} ||B_k||^2)^{1/2}||X||)^{2n}$.

It remains to note that since dim $\sigma(\mathbb{A}) \leq 2n$, we have $m_{2n}(\sigma(\mathbb{A})) < \infty$, and, since $\mathbb{A}$ has a cyclic vector, $E_{\mathbb{A}}(\alpha)$ the multiplication by the characteristic functions of α on $L_2(\sigma(\mathbb{A}), \mu)$, where μ is the scalar spectral measure of $\mathbb{A}$. The statement now follows directly from Lemma 10.1. ☐

Theorem 10.3. *If ess-dim $(\mathbb{A}) \leq 2$, then $\mathcal{E}_\Delta(0) = \ker \Delta$.*

Proof. Assume first that $\mathbb{A}$ has a cyclic vector. If $X \in \mathcal{E}_\Delta(0)$, then, by Lemma 10.2, $\Delta(X) \in \Phi_2(\mathfrak{S}_2)$. We have

$$\Delta(X) \in \mathcal{E}_\Delta(0) \cap \mathfrak{S}_2 = \mathcal{E}_{\Delta_{\mathfrak{S}_2}}(0) = \ker \Delta_{\mathfrak{S}_2} = \ker(\Delta_{\mathfrak{S}_2})^*.$$

The last equality holds because $\Delta_{\mathfrak{S}_2}$ is normal. Therefore, by Corollary 6.2, $\Delta(X) \in \Phi_2(\ker \Delta_{\mathfrak{S}_2}^*) \cap \text{Im } \Delta = \{0\}$.

Generally, decompose H into a direct sum of subspaces $H = \oplus_{j=1}^\infty H_j$, where each H_j is invariant with respect to $\{A_k\}_{k\in K}$ and $\{A_k\}_{k\in K}|_{H_j}$ has a cyclic vector. Then each $X \in B(H)$ can be written as a row operator $X = (X_1, X_2, \ldots)$, where $X_j = X|_{H_j}$, and $\Delta(X) = (\Delta_1(X_1), \Delta_2(X_2), \ldots)$, where Δ_j is the restriction of Δ to $B(H_j, H)$. Now if $X \in \mathcal{E}_\Delta(0)$, $X_j \in \mathcal{E}_{\Delta_j}(0) \ker \Delta_j$ and hence $X \in \ker \Delta$. ☐

Corollary 10.4. *If card $K < \infty$ and ess-dim $(\mathbb{A}) \leq 2n$ then $\mathcal{E}_\Delta(0) = \ker \Delta^n$*

Proof. We can assume that $\{A_k\}_{k\in K}$ has a cyclic vector (the general case reduces to this one as above). By Lemma 10.2, $\Delta^n(X) \in \mathfrak{S}_2$ for any $X \in \mathcal{E}_\Delta(0)$. The arguments similar to one in the proof of Theorem 10.3, gives

$$\Delta^n(X) \in \text{Im } \Delta^n \cap \ker \Delta^*_{\mathfrak{S}_2} \subset \text{Im } \Delta \cap \ker \Delta^*_{\mathfrak{S}_2}.$$

Therefore it is enough to show that Im $\Delta \cap \ker \Delta^*_{\mathfrak{S}_2} = \{0\}$. But this follows immediately from Corollary 9.10. □

Let $\mathfrak{X}$ be a Banach space and let T be a linear mapping on $\mathfrak{X}$. The ascent, asc T, is defined as the least positive integer n such that $\ker T^n = \ker T^{n+1}$. If no such integers exist we put asc $T = \infty$. Since $\ker \Delta^k \subset \mathcal{E}_\Delta(0)$, we obtain from Corollary 10.4 an estimate of the ascent

$$\text{ess-dim } (\mathbb{A}) \leq 2n \Rightarrow \text{asc } \Delta \leq n \tag{20}$$

or setting $(x] = -[-x]$, the smallest integer $\geq x$,

$$\text{asc } \Delta \leq (\frac{1}{2}\text{ess-dim } (\mathbb{A})] \tag{21}$$

This implies a more simple and rough result: asc $\Delta \leq k =$ card K. Somewhat more precise (but of course also rough) estimate is given in the following result:

Proposition 10.5. *asc* $\Delta \leq k - 1$.

Proof. Clearly, if Δ has length k then, by (21), asc $(\Delta - zI) \leq k$ for any constant z.

Assume first that the operators A_1, B_1 are invertible and denote by R_{A_1}, L_{B_1} the right and the left multiplication by A_1 and B_1 respectively. Then

$$\Delta = R_{A_1} L_{B_1}(\Delta' + 1),$$

where Δ' is a multiplication operator of length $\leq k-1$. Since clearly asc $\Delta =$ asc $(\Delta'+1)$, we obtain asc $\Delta \leq k - 1$.

To prove the statement in general case consider the hyperplane $S = \{(z_i) \in \mathbb{C}^k \mid z_1 = 0\}$ and the set M of all closed subsets $K \subset \mathbb{C}^k$ such that either $K \cap S = \emptyset$ or $K = S$. Let Q be the family of the projections $R_{E_{\mathbb{A}}(K_1)} L_{E_{\mathbb{B}}(K_2)}$, K_1, $K_2 \in M$. One can easily see that Q is complete, meaning that, for any $X \in B(H_1, H_2)$, the closed subspace generated by $P(X)$, $P \in Q$, contains X.

Next we note that

$$\text{asc } \Delta = \sup\{\text{asc}(\Delta P) \mid P \in Q\}$$

for any complete family Q of projections commuting with Δ and that for any such projection P either asc$(\Delta P) = 1$ or asc(ΔP)asc$(\Delta P|_{P\mathfrak{X}})$, $\mathfrak{X} = B(H_1, H_2)$. Hence it is enough to show that asc $(\Delta P|_{P\mathfrak{X}}) \leq n - 1$ for any $P \in Q$. But if some of K_1 and K_2 equals S then for $P = R_{E_{\mathbb{A}}(K_1)} L_{E_{\mathbb{B}}(K_2)}$, $R_{A_1} L_{B_1} P = 0$ and $\Delta P|_{P\mathfrak{X}}$ has length $\leq n - 1$ implying asc $(\Delta P|_{P\mathfrak{X}}) \leq n - 1$. Otherwise, the restrictions of the operators R_{A_1}, L_{B_1} to $P\mathfrak{X}$ are invertible reducing the problem to the case treated in the beginning. □

Remark 10.6. Proposition 10.5, being much less general than (21) has its advantages. For example, it implies immediately the result of Weiss [W3] on multiplication operators of the length 2.

Corollary 10.7. *If ess-dim* $(\mathbb{A}) \leq 2$*, then the solution space of the equation*

$$\sum_{k \in K} B_k X A_k = 0$$

is reflexive.

Proof. It follows from Theorem 10.3, Proposition 5.1 and Corollary 4.8. □

In the next section it will be shown that this statement can be regarded as an operator version of the Beurling-Pollard theorem on synthesis of $Lip_{1/2}$-functions on the circle.

11 Individual synthesis in Varopoulos algebras; some applications

We can now return to our initial topic and apply Proposition 4.5 and Theorem 10.3 to obtain a criterium for synthesis of functions in the Varopoulos algebra $V(X,Y)$.

Theorem 11.1. *Let* $F = \sum_{i=1}^{\infty} f_i \otimes g_i \in V(X,Y)$*,* $f(x) = (f_1(x), f_2(x), \ldots) \in l_2$*. If dim* $f(X) \leq 2$*, then* F *admits spectral synthesis.*

Proof. Take arbitrary borel measures μ, ν on X, Y and let Δ_F be the multiplication operator as defined in Section 4. Then its left coefficient family $\mathbb{A} = \{A_i\}_{i=1}^{\infty}$ coincides with $\{M_{f_i}\}_{i=1}^{\infty}$, whence $\sigma(\mathbb{A}) \subset f(X)$, dim $\sigma(\mathbb{A}) \leq$ dim $f(X) \leq 2$. The statement now follows from Theorem 10.3 and Corollary 4.8. □

Note that if dim $X = 1$ (or 2) and $f_i \in Lip_{1/2}(X)$ (respectively $f_i \in Lip(X)$) with the Lipschitz constants C_i such that $\sum C_i < \infty$ the theorem says that $F(x,y) = \sum_i f_i(x) g_i(y)$ admits spectral synthesis in $V(X,Y)$. This shows that Theorem 11.1 can be considered as a tensor algebra version of the Beurling-Pollard theorem and Corollary 10.7 as its operator version.

Theorem 11.2. *Let* $F(x,y) = \sum_{i=1}^{k} f_i(x) g_i(y) \in V(X,Y)$*. Let* $m = (\text{dim } f(X)/2]$*, where* $f : X \to \mathbb{C}^k$ *is the map* $x \mapsto (f_1(x), \ldots, f_k(x))$*. Then the sequence of closed ideals* $J_j = \overline{F^j V(X,Y)}$ *of* $V(X,Y)$ *stabilizes on a number* $n \leq m$*. Moreover, for any Banach module* M *over* $V(X,Y)$ *the sequence of submodules* $\overline{F^j M}$ *stabilizes on a number* $n \leq m$*.*

Proof. Clearly the annihilator $J_j^{\perp}$ of J_j in $V(X,Y)'$ coincides with the space

$$W_j = \{B \in V(X,Y)' \mid F^j B = 0\}.$$

It suffices to prove that $W_{m+1} = W_m$. Let $B \in W_{m+1}$. As in the proof of Proposition 4.5 there are measures μ, ν on X, Y and an operator $T : L_2(X, \mu) \to L_2(Y, \nu)$ such that

$$\langle u \otimes v, B \rangle = (Tu, v)$$

for any $u \in C(X)$, $v \in C(Y)$.

If Δ_F is the multiplication operator corresponding to F then for its left coefficient family $\mathbb{A} = \{M_{f_i}\}_{i=1}^k$, we have $\sigma(\mathbb{A}) \subset f(X)$, and dim $\sigma(\mathbb{A}) \leq$ dim $f(X)$. The bimeasure FB corresponds to the operator $\Delta_F(T)$, $F^j B$ to $\Delta_F^j(T)$. Thus $T \in \ker \Delta_{F^{m+1}} = \ker \Delta_{F^m}$ by (21), and $B \in W_m$.

We actually established that $F^m = \lim_j F^{m+1} G_j$ for some sequence $\{G_j\}$ in $V(X, Y)$. This implies immediately the equality $\overline{F^m M} = \overline{F^{m+1} M}$ for any Banach module M. □

Next results relate the problematic with harmonic analysis and ordinary differential equations. Let $\mathcal{F}$ denote the Fourier transform (acting in any of the spaces of ordinary or generalized functions considered below), D be the space of compactly supported infinitely differentiable functions on $\mathbb{R}^n$, D' its dual space, $\mathcal{F}L_1(\mathbb{R}^n)$ the Fourier algebra, $PM(\mathbb{R}^n)$ the space dual to $\mathcal{F}L_1(\mathbb{R}^n)$ (the space of pseudomeasures), $\varphi * \psi$ the convolution of two functions in D. The imbedding $PM \subset D'$ permits one to consider the distribution $p\Phi \in D'$ for any polynomial p in n variables and any pseudomeasure Φ.

Corollary 11.3. *Let p be a polynomial in two variables then for $\Phi \in PM(\mathbb{R}^2)$ the inclusion supp $\Phi \subset p^{-1}(0)$ is equivalent to the condition $p\Phi = 0$.*

Proof. Let supp $\Phi \subset p^{-1}(0)$. Clearly, there exist polynomials s_i, r_i, $1 \leq i \leq N$, such that $p(x - y) = \sum_{i=1}^N s_i(x) r_i(y)$, x, $y \in \mathbb{R}^2$. For u, $v \in D$ set $a_i(x) = u(x)s_i(x)$ and $b_i(y) = v(y) r_i(y)$. Since the Fourier transform $\mathcal{F}\Phi$ of a pseudomeasure Φ belongs to $L_\infty(\mathbb{R}^2)$ we have a well-defined bounded operator $T = \mathcal{F}^{-1} M_{\mathcal{F}\Phi} \mathcal{F}$ on $L_2(\mathbb{R})$, here $M_{\mathcal{F}\Phi}$ is the multiplication operator by the function $\mathcal{F}\Phi$. T is supported in

$$E = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid x - y \in \text{supp } \Phi\}.$$

In fact, for f, $g \in L_2(\mathbb{R}^2)$, one can easily see that

$$(Tf, g) = \langle \Phi, f * \tilde{\bar{g}} \rangle,$$

where $\tilde{\phi}(x) = \phi(-x)$. Let α, β be closed sets such that $(\alpha \times \beta) \cap E = \emptyset$. Since E is closed, there exist open sets $\alpha_0 \supset \alpha$, $\beta_0 \supset \beta$ such that $\bar{\alpha}_0 \times \bar{\beta}_0$ does not intersect E. For every pair of functions f, $g \in L_2(\mathbb{R}^2) \cap C(\mathbb{R}^2)$, which vanish outside the sets α_0 and β_0 respectively, we have supp $(f * \tilde{\bar{g}}) \subset$ supp $(f) -$ supp $(g) \subset \bar{\alpha}_0 - \bar{\beta}_0$, and since $(\bar{\alpha}_0 - \bar{\beta}_0) \cap$ supp $\Phi = \emptyset$, we have $(Tf, g) = \langle \Phi, f * \tilde{\bar{g}} \rangle = 0$, implying $M_{\chi_{\beta_0}} T M_{\chi_{\alpha_0}} = 0$ and $M_{\chi_\beta} T M_{\chi_\alpha} = 0$. By the regularity of the Lebesgue measure we have that this is true for any Borel sets α, β. Clearly

$$E \subset \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid p(x - y) = 0\} \subset \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \sum_{i=1}^N a_i(x) b_i(y) = 0\}.$$

Since a_i are smooth functions on $\mathbb{R}^2$, we have that ess-dim $(\{M_{a_i}\}) \leq 2$. Applying now Proposition 4.7 and Theorem 10.3 we conclude that $\sum M_{b_i} T M_{a_i} = 0$. A direct computation shows that for $\varphi, \psi \in D$

$$(\sum_{i=1}^{N} M_{b_i} T M_{a_i} \varphi, \psi) = \langle p\Phi, u\varphi * \widetilde{\bar{v}\psi} \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the pairing of the spaces D and D'. Therefore, $\langle p\Phi, u\varphi * \widetilde{\bar{v}\psi} \rangle = 0$, for any u, φ, v, $\psi \in D$; this shows that $p\Phi = 0$.

The reverse implication is obvious. □

Corollary 11.4. *The space of all bounded solutions of the equation*

$$p(i\frac{\partial}{\partial x_1}, i\frac{\partial}{\partial x_2})u = 0 \tag{22}$$

in $\mathbb{R}^2$ *is completely determined by the variety of zeros of the polynomial* p *in* $\mathbb{R}^2$.

Proof. The equation (22) is equivalent to $p\Phi = 0$, where $\Phi = \mathcal{F}^{-1}u$ is a pseudomeasure. By Corollary 11.3, the space of its solutions is the set of all pseudomeasures such that supp $\Phi \subset p^{-1}(0)$. □

Remark 11.5. (i) The result of Corollary 11.4 clearly extends to a wide class of infinite order equations (that is (22) with a smooth functions p instead of polynomial).

(ii) For polynomials in $n > 2$ variables the result obtains the following form: if p, q are polynomials with $p^{-1}(0) \subset q^{-1}(0)$ then any bounded solution of the equation

$$p(i\frac{\partial}{\partial x_1}, \ldots, i\frac{\partial}{\partial x_n})u = 0$$

satisfies the equation

$$q(i\frac{\partial}{\partial x_1}, \ldots, i\frac{\partial}{\partial x_n})^m u = 0,$$

where $m = (n/2]$. For the proof one should only use Corollary 10.4 instead of Theorem 10.3.

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