Volume fraction of some models for non-intersecting grains, in particular dead leaves models

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Abstract

We consider the volume fraction of two random models of non-intersecting grains: the intact grains of Matheron's "dead leaves" model, and a generalisation of one of Matérn's hard-core processes. In both models the grains are supposed to have a fixed, convex shape, while the sizes and orientations may be random. The focus is on how the shape of the grains affects the volume fraction. In particular, we show that for grains of a fixed shape and orientation, centrally symmetric sets give the highest volume fraction, while simplices give the lowest. If the grains are randomly rotated, then the volume fraction achieves its highest value only for spheres, and the lower bound of the volume fraction is zero.

Keywords: Dead leaves model, hard-core process, non-intersecting grains, Poisson process, volume fraction.

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1 Introduction

Models for random patterns of non-intersecting grains are of importance in many areas, such as material sciences, forestry, physics and chemistry. The grains can, for instance, represent inhomogeneities in a material, or trees in a forest. Spherical grains are obviously an important special case.

One kind of random models for non-intersecting spheres are the so-called hard-core point processes. In these models the constituent points are not allowed to lie closer than some minimum distance D. By regarding the points as centres of spheres with the diameter D, we can interpret hard-core point processes as models for random patterns of non-intersecting spheres with fixed radii.

In a forestry context, Matérn (1960) introduced two hard-core point processes. These processes are obtained in two steps. First a stationary Poisson point process is generated. Then the point pattern is thinned so that no points are closer than the minimal distance D. To obtain Matérn's first model every point with its nearest neighbour closer than D is excluded. In the second model, referred to as Matérn's hard-core model in the sequel, weights are independently assigned to the points according to a uniform distribution over (0,1). A point is then kept if there is no other point with a lower weight within the distance D.

In Stoyan and Stoyan (1985), Matérn's hard-core model is generalised by letting random, independent radii be associated with the points of the Poisson process. A point in x with the radius r_x is then retained if there is no other point with lower weight within distance r_x. If the radii are fixed, this process coincides with Matérn's hard-core model, in which the associated spheres are non-intersecting. However, if the radii are random, then spheres may intersect.

In Månsson and Rudemo (2002), the sphere-process associated with Matérn's hard-core model is generalised from fixed-sized spheres to convex grains of possibly varying sizes and orientations. Furthermore, the weights are allowed to have a more general distribution, and possibly depend on the sizes of the corresponding grains. In the case of spherical grains of a fixed radius and continuous weight distribution, this model also coincides with Matérn's hard-core model. In contrast to the generalisation by Stoyan and Stoyan (1985), the resulting process now consists of non-intersecting grains, also when the sizes are random.

Another model closely related to Matérn's hard-core model is the dead leaves model introduced by Matheron (1968). The dead leaves model is mostly described in two dimensions; one can think of random leaves falling to the ground from time $-\infty$ to 0. The intact leaves are then the leaves which are intact at time zero, when you look from above. A proper description of this model is given in Section 2.

In a material science context a model for non-intersecting spheres was proposed by Stienen (1982). In his model, each point of a stationary Poisson process is the centre of a sphere with diameter equal to the distance to its closest neighbour. In a related model, the dynamic lily-pond model (see Häggström and Meester (1996)), let spheres grow radially at the points of a stationary Poisson process at unit speed until they meet another sphere. The union of the spheres of the Stienen model is a subset of the spheres of the lily-pond model, if one starts with the same Poisson process. Note that in neither of these process one can control the radius distribution of the spheres; it is determined by the underlying Poisson process.

Another type of model is the random sequential adsorption model, RSA, or simple sequential inhibition model, SSI. Grains are placed sequentially and randomly in a bounded region. Only grains which do not intersect any of the previous ones are retained. There are also versions of this model for unbounded regions. See the review papers by Evans (1993) and Talbot, Tarjus, Van Tassel and Viot (2000). Yet another process for non-intersecting spheres, of possibly random sizes, is the Poisson hard-core model, which is a kind of Gibbs process, see Mase, Møller, Stoyan, Waagepetersen and Döge (2001).

In any model where random, closed grains are placed at the points of some d-dimensional point process, the union of all grains can be considered a random closed subset of $\mathbb{R}^d$. The *volume fraction* of this union set Ξ, is in the stationary case defined by $\rho = \mathbf{E}(l_d(\Xi \cap [0,1]^d)) = \Pr(o \in \Xi)$, where o denotes the origin and l_d denotes the d-dimensional Lebesgue measure. If the grains are non-intersecting, the volume fraction can be written as

$$\rho = \lambda \overline{V}, \tag{1.1}$$

where λ is the intensity of the points and $\overline{V}$ denotes the mean volume of a typical grain. The *size* of a convex grain $K \subset \mathbb{R}^d$ we measure by $D(K)/2$, where

$$D(K) = \sup_{x,y \in K} |x - y|.$$

is the *diameter* of A. Note that for a sphere, the size equals the radius.

The main aim of this paper is to present results for the volume fraction of one of the models introduced in Månsson and Rudemo (2002), referred to as Model I here, and of the intact grains of the dead leaves model. The results are first derived for Model I. It turns out that, asymptotically, Model I coincides with the intact grains of the dead leaves model, so that results for the intact grains model follow directly from results for Model I. The grains all have the same convex shape, but are of possibly varying sizes and orientations. The focus is on how the volume fraction depends on the shape of the grains. Furthermore, we derive the distribution of the sizes of the grains, given the original, so-called proposal, distribution. Many ideas and some results in the present paper come from Månsson and Rudemo (2002).

The plan of the paper is as follows. In Section 2, the two models of concern here are presented, and it is shown that Model I equals a finite version of the dead leaves model. The technique which is used to derive the volume fraction is illustrated by two examples in Section 3, and the notation is introduced in Section 4. After that the volume fraction and size distribution are derived for varying-sized convex grains of fixed or random orientations. In Section 5, Model I is dealt with, while Section 6 discusses the dead leaves model. In Section 6 we also show which shapes are extreme, in the sense of producing models with the lowest and highest possible volume fraction among the convex grains, and we look more carefully at some examples when the sizes are fixed. The paper ends with some concluding remarks.

2 The models

2.1 The dead leaves model

The original dead leaves model was introduced by Matheron (1968) and is a random tessellation of the space, as well as a model for non-intersecting sets. It can be defined as follows. Consider a stationary Poisson process $\{(x_i, t_i)\}$ with unit intensity in $\mathbb{R}^d \times (-\infty, 0]$. Interpret t_i as the arrival time of the point $x_i \in \mathbb{R}^d$. Let d-dimensional possibly random, compact grains be implanted at the points x_i sequentially in time, so that a new grain deletes portions of the "older" ones. At time $t = 0$ the space $\mathbb{R}^d$ is completely occupied, and the grains which are not completely deleted constitute a tessellation of $\mathbb{R}^d$ which is called the dead leaves model.

The grains which are intact, that is not intersected by any later ones, constitute a model of non-intersecting grains. This model we refer to as the intact grains of the dead leaves model or, for short, the intact grains model (IGM). Note that the union of the intact grains constitute a random closed set in $\mathbb{R}^d$.

A finite version of the dead leaves model is constructed as above, but with a finite time interval. Let the corresponding intact grains model be referred to as the finite intact grains model (FIGM), and denote the grains of a FIGM with the time interval restricted to $[-T, 0]$ by Ψ_T. Letting Ψ_i, $i = 1, 2, \ldots$, be based on the same Poisson process, it follows that $\Psi_i \subset \Psi_{i+1}$, and the limiting random set $\cup_{i=1}^{\infty} \Psi_i$ equals the intact grains of the IGM.

The dead leaves model and generalisations of it, for instance the colour dead leaves, are studied in a number of papers by Jeulin, see e.g. Jeulin (1997). Results on the intensity and size distribution of the intact grains may be found in Jeulin (1989). In Stoyan and Schlater (2000) the dead leaves model is constructed in an alternative way, which is useful when comparing with Matérn's hard-core model, and with the RSA model.

2.2 Model I

This model was already introduced in Section 1 as a generalisation of Matérn's hard-core model by Månsson and Rudemo (2002). It is constructed in two steps as follows:

1. In the first step we generate a Poisson process with constant intensity in $\mathbb{R}^d$. At the points of this process independent grains of a given shape, but possibly of random size and orientation, are implanted. Furthermore, to each grain a weight is given, according to a uniform distribution on $(0, 1)$. The weights are independent of each other and of everything else.

2. In the second step we thin the process by letting all pairs of grains which intersect 'compete'. A grain is kept if it has the higher weight in all pairwise comparisons.

The resulting process consists of non-intersecting grains, and we refer to it as Model I. Obviously the intensity after thinning is lower than the intensity before thinning. Furthermore, the size distribution of the grains has changed and the sizes and orientations are no longer independent. We will come back to these issues later.

The original Poisson process together with the grains can be seen as a *proposal* model. In accordance with this, we let λ_{pr} and F_{pr} denote the *proposal* intensity and *proposal* distribution function of the sizes of the grains, respectively, before thinning. The intensity measure and distribution function of the sizes in Model I, will be denoted by λ_{th} and F_{th}, respectively.

2.3 The relation between Model I and the FIGM

In the constructions of Model I and of the FIGM, the following processes are used:

Model I: A Poisson process with intensity λ_{pr} in $\mathbb{R}^d$, with independent and identically distributed d-dimensional grains of a given shape, but possibly of random size and orientation, implanted at the points. Associated to the points are independent weights, uniformly distributed on $(0, 1)$.

FIGM: A Poisson process in $\mathbb{R}^d \times [-T, 0]$, where the last coordinates are considered time instances, and with d-dimensional grains distributed as in Model I, implanted at the points in $\mathbb{R}^d$.

First note that the weight distribution in the construction of Model I need not be uniform on $(0, 1)$; it can be replaced by any other continuous distribution, since it is only the order of the weights which is of importance, not their actual values. Now let $\lambda_{pr} = T$, and let the weights be uniformly distributed on $[-T, 0]$ in Model I. Then the points of the original Poisson process together with the weights can be regarded as a Poisson process in $\mathbb{R}^d \times [-T, 0]$ with unit intensity, which coincides with the Poisson process used to construct the FIGM. Since high weights in Model I correspond to late arrivals in FIGM, the grains which are retained in the thinning step in Model I equal the intact ones in the FIGM. This means that results concerning Model I with $\lambda_{pr} = T$ also hold for the FIGM with the time interval $[-T, 0]$.

Results for Model I, and thus also for the FIGM, can now be used to achieve results for the infinite IGM. For instance, the volume fraction of the IGM is achieved by letting $\lambda_{pr} = T \to \infty$ in the volume fraction of Model I (see Theorems 5.4 and 5.5). This can be described as follows: Let E_i denote the event that the origin belongs to Ψ_i. Then the volume fraction of Ψ_i is $\rho = P(o \in \Psi_i) = P(E_i)$. If the origin belongs to Ψ_i, then it also belongs to Ψ_j, $j > i$, so that E_i, $i = 1, 2, \ldots$ is an increasing sequence with $\cup_{i=1}^{\infty} E_i = \{$the origin belongs to the IGM$\}$. It follows that $P(E_i)$ tends to the volume fraction of the IGM.

3 Two illustrating examples

Consider a stationary point process Φ with intensity λ_{pr}. As is common in the literature, we will here use the term *typical point* of Φ. The probability of an event involving a typical point is calculated by means of the Palm distribution P_o (see Stoyan, Kendall and Mecke (1995)). By a *typical grain* we mean a grain at a typical point. Assume that Φ is thinned, and that p_{ret} is the probability that a typical point is retained. Then the intensity of the thinned process Φ_{th} is

$$\lambda_{th} \quad = \quad \lambda_{pr} p_{ret}. \tag{3.2}$$

In Model I

$$p_{ret} = \int_0^1 \int_0^\infty p_{ret}(r, w) dF_{pr}(r)\, dw,$$

where $p_{ret}(r, w)$ is the probability that a typical point with weight w and associated grain of size r is retained in the thinning. Recall that a point and its associated grain are retained if they "win", that is have the higher weight, over all intersecting grains.

To illustrate how the intensity and volume fraction can be derived for Model I and the intact grains in the dead leaves model, we start by two examples.

Example 3.1. Discs in $\mathbb{R}^2$ with fixed radius r_0.

1. Consider a typical point at the origin with weight w, which we denote by $[o, w]$. The points which win over $[o, w]$ are those which lie in the disc with the radius $2r_0$, centred at the origin, and have a weight which is higher than w. The number of such points is Poisson distributed with the parameter $\lambda_{pr}(1-w)4\pi r_0^2$. If no points beat $[o, w]$, it is retained, which thus happens with the probability $p_{ret}(r_0, w) = \exp\{-\lambda_{pr}(1-w)4\pi r_0^2\}$.

2. By integrating over the weight, the retaining probability of a typical point follows:
$$p_{ret} = \int_0^1 p_{ret}(r_0, w) = \frac{1 - \exp\{-\lambda_{pr} 4\pi r_0^2\}}{\lambda_{pr} 4\pi r_0^2}.$$
By (3.2), the intensity of the point process after thinning, Model I, is
$$\lambda_{th} = \frac{1 - \exp\{-\lambda_{pr} 4\pi r_0^2\}}{4\pi r_0^2}.$$
Since all discs have the radius r_0, the mean area is $\overline{V} = \pi r_0^2$, and, by (1.1), the volume fraction of Model I is
$$\rho = \lambda_{th}\pi r_0^2 = \frac{1 - \exp\{-\lambda_{pr} 4\pi r_0^2\}}{4}.$$

3. By letting $\lambda_{pr} \to \infty$, it follows that the intensity of the intact discs in the dead leaves model is $\lambda_{ig} = (4\pi r_0^2)^{-1}$, and that the volume fraction is $\rho = \lambda_{ig}\pi r_0^2 = 1/4$.

$\square$

Example 3.2. Let $K \subset \mathbb{R}^2$ be a triangle of fixed size and orientation. As in the above example we consider a typical point $[o, w]$. The points which win over the typical point are those which have a weight which is higher than w, and lie in the set $\{x \in \mathbb{R}^2 : K \cap (K + \{x\}) \neq \emptyset\}$. This set is a polygon with six edges which has area $6l_2(K)$. This is easily seen by drawing a picture, or by using (4.3) and (4.5) below. Continuing as in the above example we get
$$\lambda_{th} = \frac{1 - \exp\{-\lambda_{pr} 6l_2(K)\}}{6l_2(K)} \text{ and } \rho = \frac{1 - \exp\{-\lambda_{pr} 6l_2(K)\}}{6},$$
for Model I. It follows that in the IGM the intensity is $\lambda_{ig} = (6l_2(K))^{-1}$ and that the volume fraction is $\rho = 1/6$. $\square$

Note that the intensities of the IGMs, when the grains are discs and triangles with fixed size and orientation, are $(4l_2(K))^{-1}$ and $(6l_2(K))^{-1}$, respectively, and the volume fractions are $1/4$ and $1/6$, respectively. Hence it is the shape and the volume of the grains which determine the intensity, while it is the shape which determines the volume fraction. As we will see later in (6.22), the above examples are extremal in the sense that the intensity and volume fraction for all other shapes of convex grains lie between those for discs and triangles.

The procedure in the above examples will be generalised to arbitrary convex grains, higher dimensions, random sizes, and random orientations in Sections 5 and 6. Before that we need some more notation.

4 Notation and convex set theory

We will now introduce further notation and some necessary theory of convex sets. Most of the concepts and results presented here can be found in Schneider (1993).

As already introduced, l_d denotes the d-dimensional Lebesgue measure and o denotes the origin. Furthermore, let $B_d(z,r) = \{x \in \mathbb{R}^d : |z-x| \leq r\}$ denote the d-dimensional ball centred at z with radius r, and let $\kappa_d = l_d(B_d(o,1))$ be the volume of the unit ball in $\mathbb{R}^d$. Then $r^d \kappa_d$ is the volume of $B_d(z,r)$. The surface area of $B_d(o,1)$ is denoted by ω_d.

For $K, L \subset \mathbb{R}^d$ and $c \in \mathbb{R}$ the *Minkowski sum* and *scalar multiple* are defined as

$$K \oplus L = \{x+y : x \in K, y \in L\} \quad \text{and} \quad cK = \{cx : x \in K\},$$

respectively. If $c = -1$ we get $\check{K} = \{-x : x \in K\}$, which we call the *reflected set* of K. For $x \in \mathbb{R}^d$, $K + \{x\}$ is the *translate* of K by x. If $K = \check{K} + \{x\}$ for some $x \in \mathbb{R}^d$, K is said to be *centrally symmetric*. An alternative way of writing the Minkowski sum is

$$K \oplus L = \{x \in \mathbb{R}^d : K \cap (\check{L} + \{x\}) \neq \emptyset\}. \tag{4.3}$$

For the set $xK \oplus yL$, where $x, y \in \mathbb{R}^+$ and $K, L \subset \mathbb{R}^d$ are non-empty convex sets, the volume can be written as:

$$l_d(xK \oplus yL) = \sum_{i=0}^{d} \binom{d}{i} x^i y^{d-i} V_{i,d-i}(K,L), \tag{4.4}$$

where $V_{i,d-i}(K,L) = V(\overbrace{K,\ldots,K}^{i},\overbrace{L,\ldots,L}^{d-i})$ are the *mixed volumes* (*areas* in $\mathbb{R}^2$) of K and L. Note the special cases $V_{d,0}(K,L) = l_d(K)$ and $V_{0,d}(K,L) = l_d(L)$.

A set which is of special interest in the present paper is $K \oplus \check{K} = \{x \in \mathbb{R}^d : K \cap (K + \{x\}) \neq \emptyset\}$, which is called the *difference body* of $K \subset \mathbb{R}^d$. If K is convex, then

$$2^d l_d(K) \leq l_d(K \oplus \check{K}) \leq \binom{2d}{d} l_d(K), \tag{4.5}$$

where the lower bound is attained if and only if K is centrally symmetric, and the upper bound is attained if and only if K is a simplex.

Furthermore, if K is a convex set, then

$$l_d(K) \leq V_{i,d-i}(K,\check{K}) \leq d^{\min\{i,d-i\}} l_d(K), \tag{4.6}$$

with equality on the left-hand side iff K is centrally symmetric or $i(d-i)=0$. On the right-hand side, there is equality in dimension 2 and 3 if and only if K is a triangle and a tetrahedron, respectively, or $i(d-i)=0$. It is furthermore conjectured by Godbersen (1938) and Makai jr. (1974) that

$$V_{i,d-i}(K,\check{K}) \leq \binom{d}{i} l_d(K), \tag{4.7}$$

with equality if and only if K is a simplex (see Schneider (1993) p.412). For arbitrary polygons in $\mathbb{R}^2$ and $\mathbb{R}^3$, a simple formula for the mixed volumes can be found in Eggleston (1963, p. 85). For polytopes in $\mathbb{R}^d$, $d \geq 2$, a formula is given in Betke (1992).

Let $S_{d-1}(K)$ denote the $(d-1)$-dimensional surface area and $\bar{b}(K)$ the so-called mean width of the convex set $K \subset \mathbb{R}^d$. The mean width can be defined as follows: For each line g through the origin, let $g(K)$ denote the smallest distance between two parallel hyperplanes perpendicular to g such that K is in between them. Then $\bar{b}(K)$ is the average value of $g(K)$ over all lines g. A formula for the mean width of a general convex set can be found in Schneider (1993) p. 42, while a formula for the mean width of a convex polytope K in $\mathbb{R}^3$ is given in Santaló (1976) p. 226:

$$\bar{b}(K) = \frac{1}{4\pi}\sum_{i=1}^{m_K}(\pi-\alpha_i)l_i, \tag{4.8}$$

where m_K is the number of edges of K, l_i the lengths of the edges, and α_i the corresponding dihedral angles (i.e. the angles between adjacent sides).

Next, we introduce the so-called *intrinsic volumes* $V_i(K)$, $i=1,\ldots,d$, for compact, convex $K \subset \mathbb{R}^d$,

$$V_i(K) = \frac{\binom{d}{i}}{\kappa_{d-i}} V_{i,d-i}(K, B_d(o,1)). \tag{4.9}$$

Note that $V_0 = 1$ and that V_d is the volume. Furthermore $2V_{d-1} =$ the surface area and $2\kappa_{d-1}V_1/\omega_d =$ the mean width.

Let $\mathcal{C}^d$ denote the family of compact, convex sets K with interior points in $\mathbb{R}^d$, such that $o \in K$ and with the size $D(K)/2 = 1$. Furthermore, let $K(z,r)$ denote a set with the same shape as K, translated by z and with size $r > 0$, that is $K(z,r) = \{ry + z : y \in K\}$, and $l_d(K(z,r)) = r^d l_d(K)$. Note that if $K \in \mathcal{C}^d$, then $K(o,1) = K$.

5 Volume fraction and size distribution of Model I

In Examples 3.1 and 3.2 we derived the volume fraction when the grains were discs or triangles of fixed size and orientation. We will now generalise the method used there to convex grains of the same shape as $K \in \mathcal{C}^d$, of possibly random size and orientation. By (1.1) and (3.2) the volume fraction of Model I is

$$\rho = \lambda_{th}\overline{V}_{th} = \lambda_{pr}p_{ret}\overline{V}_{th}, \tag{5.10}$$

where λ_{pr} is the proposal intensity, p_{ret} is the probability that a typical point of the original point pattern and its associated grain is retained in the thinning step, and $\overline{V}_{th}$ denotes the mean volume of a typical grain after thinning. Letting F_{th} denote the distribution function of the sizes of the grains after thinning, we get

$$\overline{V}_{th} = l_d(K) \int_0^\infty r^d F_{th}(dr), \tag{5.11}$$

since $l_d(K(o,r)) = r^d l_d(K(o,1)) = r^d l_d(K)$ for any $r > 0$, if $K \in \mathcal{C}^d$. To calculate the mean volume we thus need to derive F_{th}. The size of a grain after thinning equals its size before thinning, given that it is retained. Hence the distribution function is given by

$$F_{th}(r) \quad = \quad 1 - \frac{1}{p_{ret}} \int_r^\infty p_{ret}(s) F_{pr}(ds), \tag{5.12}$$

where $p_{ret}(s)$ is the probability that a typical grain of size s is retained, and

$$p_{ret} \quad = \quad \int_0^\infty p_{ret}(r)\, F_{pr}(dr). \tag{5.13}$$

By (5.10) – (5.13) we note that we need p_{ret} and $p_{ret}(s)$ to calculate the volume fraction and the size distribution after thinning. The next step is thus to derive these probabilities. This is done separately for the two different cases of fixed and random orientations of the grains in the next two sections. In Section 5.3 we return to the volume fraction and the size distribution after thinning.

5.1 Retaining probability when the orientation of the grains is fixed

As seen in Examples 3.1 and 3.2, the points which are possible "winners" over a typical point at the origin lie in the set $\{x \in \mathbb{R}^d : K \cap (K + \{x\}) \neq \emptyset\}$, if size and orientation of the grains are fixed. By (4.3) this set can be written as $K \oplus \check{K}$. In order to derive the retaining probability p_{ret} when the size is random rather than fixed, we need to find the corresponding set for possible winners over a typical grain of a given size r. Let R_{pr} be a random variable with distribution F_{pr}. We introduce the notation $\Lambda_{fix}(K, r)$ for the expected volume of $\{x \in \mathbb{R}^d : K(o,r) \cap (K(o, R_{pr}) + \{x\}) \neq \emptyset\} = K(o,r) \oplus \check{K}(o, R_{pr})$, that is

$$\begin{aligned} \Lambda_{fix}(K,r) \quad &= \quad \int_0^\infty l_d(K(o,r) \oplus \check{K}(o,y)) F_{pr}(dy) \\ &= \quad \sum_{i=0}^{d} \binom{d}{i} r^i V_{i,d-i}(K, \check{K}) \mathbf{E}[R_{pr}^{d-i}], \end{aligned} \tag{5.14}$$

where the equality follows by (4.4). If all grains have the size r_0, then

$$\Lambda_{fix}(K, r_0) = r_0^d l_d(K \oplus \check{K}) = r_0^d \sum_{i=0}^{d} \binom{d}{i} V_{i,d-i}(K, \check{K}). \tag{5.15}$$

Lemma 5.1 *Using the notation introduced above, the retaining probability for a typical grain of size r, $K(o,r)$ where $K \in \mathcal{C}^d$, in the case where the orientation of the grains is fixed, is given by*

$$p_{ret}(r) \quad = \quad \frac{1-\exp\{-\lambda_{pr}\Lambda_{fix}(K,r)\}}{\lambda_{pr}\Lambda_{fix}(K,r)}.$$

Proof. View the homogeneous Poisson process of points in $\mathbb{R}^d$ before thinning, together with the sizes and the weights of the points, as an inhomogeneous Poisson process in $\mathbb{R}^d \times \mathbb{R}^+ \times [0,1]$, with the intensity measure $\lambda_{pr}\, dx\, F_{pr}(dy)\, dz$.

Let a typical point at the origin with associated grain of size r and weight w be denoted by $[o,r,w]$. Recall that a point "wins" over $[o,r,w]$ if its weight is higher than w, and if its associated grain intersects $K(o,r)$, that is if it lies in the set

$$\{x \in \mathbb{R}^d : K(o,r) \cap K(x,y) \neq \emptyset\} = K(o,r) \oplus \check{K}(o,y).$$

Since the points that win over $[o,r,w]$ is the result of an independent thinning of the original Poisson process, they constitute an inhomogeneous Poisson process in $\mathbb{R}^d \times \mathbb{R}^+ \times [0,1]$, whose intensity measure is

$$\lambda_{pr} 1\{w \leq z\} 1\{x \in K(o,r) \oplus \check{K}(o,y)\}\, dx\, F_{pr}(dy) dz.$$

The total number of points of this process is Poisson distributed with the expectation

$$\lambda_{pr} \int_0^\infty \int_{K(o,r)\oplus\check{K}(o,y)} \int_w^1 dz\, dx\, F_{pr}(dy) \quad = \quad \lambda_{pr}(1-w)\Lambda_{fix}(K,r).$$

Since $[o,r,w]$ is retained if no points beat it,

$$p_{ret}(r,w) = \exp\{-\lambda_{pr}(1-w)\Lambda_{fix}(K,r)\},$$

and by integrating over the weight, we get the retaining probability of a typical grain of size r at the origin:

$$\begin{aligned} p_{ret}(r) \quad &= \quad \int_0^1 p_{ret}(r,w)dw \\ &= \quad \frac{1-\exp\{-\lambda_{pr}\Lambda_{fix}(K,r)\}}{\lambda_{pr}\Lambda_{fix}(K,r)}. \end{aligned}$$

■

5.2 Retaining probability when the grains are randomly rotated

A rotation about the origin is a map $m : \mathbb{R}^d \to \mathbb{R}^d$, which can be represented in the form $mx = \mathbf{A}x$, $x \in \mathbb{R}^d$, where $\mathbf{A}$ is an orthogonal matrix with det$\mathbf{A}$=1. Let $SO(d)$ denote the group of rotations about the origin. By a uniformly distributed rotation we mean an element from $SO(d)$, chosen according to the

Haar measure ν, with $\nu(SO(d)) = 1$ (see e.g. Schneider and Wieacker (1993) for details).

In this section we let the grains be rotated according to the uniform distribution described above, and furthermore, the rotations are independent of each other, the sizes of the grains and the positions. The expected volume of the set where the possible winners over a typical point $[o, r, w]$ can lie, when the orientation and size is random, is

$$\begin{aligned}
&\int_0^\infty \int_{SO(d)} l_d(\{x \in \mathbb{R}^d : K(o,r) \cap (\vartheta K(x,y)) \neq \emptyset\})\nu(d\vartheta)F_{pr}(dy) \\
&= \int_0^\infty \int_{SO(d)} l_d(K(o,r) \oplus \vartheta \check{K}(o,y))\nu(d\vartheta)F_{pr}(dy) \\
&= \int_0^\infty \int_{SO(d)} l_d(K(o,r) \oplus \vartheta K(o,y))\nu(d\vartheta)F_{pr}(dy). \qquad (5.16)
\end{aligned}$$

Since the distribution ν is rotation invariant we can apply the generalised Steiner formula (see eg. Weil and Wieacker (1993), p. 1407), from which it immediately follows that

$$\int_{SO(d)} l_d(K(o,r) \oplus \vartheta K(o,y))\nu(d\vartheta) = \frac{1}{\kappa_d} \sum_{k=0}^{d} \frac{\kappa_k \kappa_{d-k}}{\binom{d}{k}} r^k V_k(K) y^{d-k} V_{d-k}(K). \qquad (5.17)$$

Recall from Section 4 that κ_d is the volume of the d-dimensional unit ball, and that V_i is the ith intrinsic volume, defined in (4.9). Letting $\Lambda_{rot}(K, r)$ denote this expected volume, we get by (5.16) and (5.17)

$$\begin{aligned}
\Lambda_{rot}(K,r) &= \int_0^\infty \int_{SO(d)} l_d(K(o,r) \oplus \vartheta K(o,y))\nu(d\vartheta)F_{pr}(dy) \\
&= \frac{1}{\kappa_d} \sum_{k=0}^{d} \frac{\kappa_k \kappa_{d-k}}{\binom{d}{k}} r^k V_k(K) \mathbf{E}[R_{pr}^{d-k}] V_{d-k}(K). \qquad (5.18)
\end{aligned}$$

If all grains have the size r_0, then

$$\begin{aligned}
\Lambda_{rot}(K,r_0) &= r_0^d \int_{SO(d)} l_d(K \oplus \vartheta K)\nu(d\vartheta) \\
&= \frac{r_0^d}{\kappa_d} \sum_{k=0}^{d} \frac{\kappa_k \kappa_{d-k}}{\binom{d}{k}} V_k(K) V_{d-k}(K). \qquad (5.19)
\end{aligned}$$

Note that $\Lambda_{rot}(K,r) = \Lambda_{fix}(K,r)$ for spheres.

Lemma 5.2 *Using the notation introduced above, the retaining probability for a typical grain of size r, $K(o,r)$ where $K \in \mathcal{C}^d$, in the case where the grains are independently and uniformly rotated, is given by*

$$p_{ret}(r) = \frac{1 - \exp\{-\lambda_{pr}\Lambda_{rot}(K,r)\}}{\lambda_{pr}\Lambda_{rot}(K,r)}.$$

Proof. The proof of this lemma is identical to the proof of Lemma 5.1, where the orientation was fixed, except for an adjustment which handles the random rotations. Now the Poisson process of points which beat a typical point $[o, r, w]$ is defined on $\mathbb{R}^d \times \mathbb{R}^+ \times [0,1] \times SO(d)$, and has the intensity measure

$$\lambda_{pr} 1\{w \leq z\} 1\{x \in K(o,r) \oplus \vartheta K(o,y)\}\, dx\, F_{pr}(dy)\, dz\, \nu(d\vartheta),$$

with expected total number of points

$$\begin{aligned}
&\lambda_{pr} \int_0^\infty \int_{SO(d)} \int_{K(o,r)\oplus\vartheta K(o,y)} \int_w^1 dz\, dx\, \nu(d\vartheta)\, F_{pr}(dy) \\
&= \lambda_{pr}(1-w) \int_0^\infty \int_{SO(d)} l_d(K(o,r) \oplus \vartheta K(o,y))\, \nu(d\vartheta)\, F_{pr}(dy) \\
&= \lambda_{pr}(1-w)\Lambda_{rot}(K,r).
\end{aligned}$$

Since the number of points that beat the typical one is Poisson distributed also when the orientation is random, the proof proceeds as for Lemma 5.1.

■

5.3 Size distribution and volume fraction of Model I

Now we are able to state the main results for Model I, by combining results from previous sections. We start to present the size distribution.

Theorem 5.3 *Assume that the grains have the same shape as $K \in \mathcal{C}^d$, and let $\Lambda(K,r) = \Lambda_{fix}(K,r)$ if the orientation is fixed, and $\Lambda(K,r) = \Lambda_{rot}(K,r)$ otherwise, where $\Lambda_{fix}(K,r)$ and $\Lambda_{rot}(K,r)$ are given by (5.14) and (5.18), respectively. Then the size distribution of the grains in Model I is given by*

$$F_{th}(r) = 1 - k \int_r^\infty \frac{1-\exp\{-\lambda_{pr}\Lambda(K,s)\}}{\Lambda(K,s)} F_{pr}(ds), \qquad (5.20)$$

where

$$k^{-1} = \int_0^\infty \frac{1-\exp\{-\lambda_{pr}\Lambda(K,x)\}}{\Lambda(K,x)} F_{pr}(dx).$$

If the original size distribution is continuous with density f_{pr}, then the distribution of the sizes in Model I is also continuous with density

$$f_{th}(r) = \frac{(1-\exp\{-\lambda_{pr}\Lambda(K,r)\})\Lambda(K,r)^{-1} f_{pr}(r)}{\int_0^\infty (1-\exp\{-\lambda_{pr}\Lambda(K,x)\})\Lambda(K,x)^{-1} f_{pr}(x) dx}.$$

Proof. This result follows from (5.12), (5.13), Lemmas 5.1 and 5.2. ■

The above theorem gives the size distribution of the grains in Model I given the proposal size distribution. However, when trying to fit this model to data, it is natural to go the opposite way: for a desired size distribution of the final process, one needs to find the proposal distribution. In general this is a much more difficult task, and it seems hard to find an explicit expression for F_{pr} given F_{th}. Instead we have to use some iterative method for computing F_{pr} from F_{th}. One such method is described in Månsson and Rudemo (2002).

In the next two theorems we give the volume fractions for the cases of fixed and random orientations.

Theorem 5.4 *(i) Using the notation introduced above, the volume fraction of Model I when the grains have the same shape and orientation as $K \in \mathcal{C}^d$, is*

$$\rho = \overline{V}_{th} \int_0^\infty \frac{1 - \exp\{-\lambda_{pr}\Lambda_{fix}(K,r)\}}{\Lambda_{fix}(K,r)} F_{pr}(dr),$$

where $\Lambda_{fix}(K,r)$ is given by (5.14) and $\overline{V}_{th}$ by (5.11).
(ii) If both the orientation and size of the grains are fixed, then

$$\rho = \frac{l_d(K)}{l_d(K \oplus \check{K})}(1 - \exp\{-\lambda_{pr}\Lambda_{fix}(K,r_0)\}),$$

where r_0 is the size and Λ_{fix} is given by (5.15).

Proof. (i) The results follows directly by (5.10), (5.13) and Lemma 5.1.
(ii) For fixed size r_0 the mean volume is $\overline{V}_{th} = r_0^d l_d(K)$, which combined with (5.15) and (i) gives the result. ■

Theorem 5.5 *(i) Using the notation introduced above, the volume fraction of Model I with uniformly and independently rotated grains of the same shape as $K \in \mathcal{C}^d$, is*

$$\rho = \overline{V}_{th} \int_0^\infty \frac{1 - \exp\{-\lambda_{pr}\Lambda_{rot}(K,r)\}}{\Lambda_{rot}(K,r)} F_{pr}(dr),$$

where $\Lambda_{rot}(K,r)$ is given by (5.18) and $\overline{V}_{th}$ is given by (5.11).
(ii) In particular, for fixed size r_0,

$$\begin{aligned} \rho &= \frac{l_d(K)}{\int_{SO(d)} l_d(K \oplus \vartheta K)\nu(d\vartheta)}(1 - \exp\{-\lambda_{pr}\Lambda_{rot}(K,r_0)\}) \\ &= l_d(K)\left(\frac{1}{\kappa_d}\sum_{k=0}^{d} \frac{\kappa_k\kappa_{d-k}}{\binom{d}{k}} V_k(K)V_{d-k}(K)\right)^{-1} (1 - \exp\{-\lambda_{pr}\Lambda_{rot}(K,r_0)\}), \end{aligned}$$

where Λ_{rot} is given by (5.19).

Proof. (i) Combining Lemma 5.2 with (5.10) and (5.13) gives the result.
(ii) Since $\overline{V}_{th} = r_0^d l_d(K)$, the result follows by (i) and (5.19). ■

Remark 5.6. Since

$$\rho = \lambda_{th}\overline{V}_{th},$$

it is clear that by the above theorems we also get λ_{th}, the intensity of Model I. □

6 Intact grains of the dead leaves model

In this section we consider results for the IGM. However, we start by considering the FIGM. Recall that in Section 2.3 we concluded that the intact grains in a FIGM, based on a Poisson process restricted in time to the interval $[-T, 0]$, Ψ_T, coincide with the grains of Model I with $\lambda_{pr} = T$. Hence we immediately have results on the size distribution and volume fraction of the grains of the FIGMs.

Lemma 6.1 *For a FIGM restricted to the time interval $[-T, 0]$, Ψ_T, Theorems 5.3, 5.4 and 5.5 hold if λ_{pr} is replaced by T.*

6.1 Size distribution of the intact grains

Recall from Section 2.1 that the limiting set $\cup_{T=1}^{\infty}\Psi_T$ equals the intact grains of the dead leaves model. By Lemma 6.1 and Theorem 5.3 we thus get the size distribution for the grains of the IGM directly, by letting $\lambda_{pr} = T \to \infty$.

Theorem 6.2 *Assume that the grains have the same shape as $K \in \mathcal{C}^d$, and let F_{ig} and F_{pr} denote the distribution function of the sizes of the grains in the IGM and the original size distribution, respectively. Then*

$$F_{ig}(r) \quad = \quad 1 - \frac{\int_r^{\infty} \Lambda(K,s)^{-1} F_{pr}(ds)}{\int_0^{\infty} \Lambda(K,x)^{-1} F_{pr}(dx)},$$

where $\Lambda(K,r) = \Lambda_{fix}(K,r)$ if the orientation is fixed, and $\Lambda(K,r) = \Lambda_{rot}(K,r)$ otherwise, where $\Lambda_{fix}(K,r)$ and $\Lambda_{rot}(K,r)$ are given by (5.14) and (5.18), respectively.

If the original size distribution is continuous with density f_{pr}, then the distribution of the sizes in the IGM is also continuous with density

$$f_{ig}(r) = \frac{f_{pr}(r)}{\Lambda(K,r)\int_0^{\infty} \Lambda(K,x)^{-1} f_{pr}(x)dx}.$$

As is easy to believe, the probability for small grains to stay intact is larger than that for large grains. This is illustrated by the following example.

Example 6.3. Consider a mixture model of discs with two different sizes, r_1 and r_2, where $r_2 = 2r_1$, and with the probability 1/2 for each size in the original process. By Theorem 6.2 we get $p_{ig}(r_1) = 25/38$ and $p_{ig}(r_2) = 13/38$. This example was considered in Månsson and Rudemo (2002), for an arbitrary relation between the sizes of d-dimensional spheres. □

By (5.11) it follows that the mean volume for a typical grain of the IGM is

$$\overline{V}_{ig} = l_d(K)\int_0^{\infty} r^d F_{ig}(dr), \tag{6.21}$$

where F_{ig} is given by Theorem 6.2. Furthermore, if the original size distribution is continuous with density f_{pr}, then

$$\overline{V}_{ig} \quad = \quad l_d(K)\frac{\int_0^{\infty} r^d \Lambda(K,r)^{-1} f_{pr}(r)dr}{\int_0^{\infty} \Lambda(K,x)^{-1} f_{pr}(x)dx}.$$

6.2 Volume fraction

The following theorem follows immediately from Theorems 5.4 and 5.5 by letting $\lambda_{pr} \to \infty$.

Theorem 6.3 *(i) Using the notation introduced above, the volume fraction in the IGM with grains of the same shape and orientation as $K \in \mathcal{C}^d$, is*

$$\rho = \overline{V}_{ig} \int_0^\infty \left(\Lambda_{fix}(K,r)\right)^{-1} F_{pr}(dr),$$

where $\overline{V}_{ig}$ is given by (6.21) and Λ_{fix} is given by (5.14). In particular, if both size and orientation are fixed, then

$$\rho = \frac{l_d(K)}{l_d(K \oplus \check{K})} = \frac{l_d(K)}{\sum_{i=0}^d \binom{d}{i} V_{i,d-i}(K,\check{K})}.$$

(ii) The volume fraction in the IGM with independently and uniformly rotated grains of the same shape as $K \in \mathcal{C}^d$, is

$$\rho = \overline{V}_{ig} \int_0^\infty \left(\Lambda_{rot}(K,r)\right)^{-1} F_{pr}(dr),$$

where Λ_{rot} is given by (5.18). In particular, for fixed size and random orientations

$$\begin{aligned} \rho &= l_d(K) \left(\int_{SO(d)} l_d(K \oplus \vartheta K) \nu(d\vartheta) \right)^{-1} \\ &= l_d(K) \left(\frac{1}{\kappa_d} \sum_{k=0}^d \frac{\kappa_k \kappa_{d-k}}{\binom{d}{k}} V_k(K) V_{d-k}(K) \right)^{-1}. \end{aligned}$$

Example 6.4. In Example 6.3 a mixture model with discs of two sizes was introduced. By Theorem 6.3 (i) we get the volume fraction $\rho = 75053/255035 \approx 0.294$. Note that this is higher than 1/4, which is the volume fraction in the case of a fixed size of the discs (Example 3.1). □

6.3 Bounds on the volume fraction

As was remarked after Example 3.2, the intensity and volume fraction depend on the shape of the grains; for instance the volume fraction is higher for discs than for triangles of a fixed size and orientation. In the formulas in Theorem 6.3 it can be seen that the volume fraction depends on the shape through the mixed volumes if the orientation is fixed. If the orientations are random, it is the intrinsic volumes rather than the mixed volumes which involve the shape.

Fixed orientation

We will now use the inequalities concerning mixed volumes in Section 4 to give upper and lower bounds on the volume fraction.

Corollary 6.4 *(i) Let the grains have the same shape and orientation as $K \in \mathcal{C}^d$, and let R_{ig} and R_{pr} be random variables with distribution functions F_{ig}*

and F_{pr}, respectively. Then the volume fraction in the IGM has the following bounds:

$$\rho \;\geq\; \mathbf{E}[R_{ig}^d] \int_0^\infty \left(\sum_{i=0}^d \binom{d}{i} r^i d^{\min\{i,d-i\}} \mathbf{E}[R_{pr}^{d-i}] \right)^{-1} F_{pr}(dr),$$
$$\rho \;\leq\; \mathbf{E}[R_{ig}^d] \int_0^\infty \left(\sum_{i=0}^d \binom{d}{i} r^i \mathbf{E}[R_{pr}^{d-i}] \right)^{-1} F_{pr}(dr),$$

where the upper bound is attained if and only if K is centrally symmetric. In $d = 2$ and $d = 3$ the lower bound is attained if and only if K is a triangle and a tetrahedron, respectively. Furthermore, if the conjectured inequality (4.7) is true, then

$$\rho \;\geq\; \mathbf{E}[R_{ig}^d] \int_0^\infty \left(\sum_{i=0}^d \binom{d}{i}^2 r^i \mathbf{E}[R_{pr}^{d-i}] \right)^{-1} F_{pr}(dr),$$

with equality if and only if K is a simplex.
(ii) If both size and orientation are fixed, then the volume fraction in the IGM has the following bounds:

$$\frac{1}{\binom{2d}{d}} \leq \rho \leq \frac{1}{2^d}, \tag{6.22}$$

where the upper bound is attained if and only if K is centrally symmetric, and the lower bound is attained if and only if K is a simplex.

Proof. (i) Follows from Theorem 6.3, (4.6) and (4.7).
(ii) Follows from Theorem 6.3 and (4.5). ■

Random orientations

Now we present upper bounds on the volume fraction when the orientations of the grains are random. The lower bound is zero, which can be motivated as follows. If the size is fixed, the volume fraction in two dimensions is, by Theorem 6.3,

$$\rho \;=\; \frac{l_2(K)}{l_2(K)2 + S_1(K)^2/(2\pi)}, \tag{6.23}$$

where $S_1(K)$ denotes the perimeter of K. Think of a triangle of a fixed area. The more the triangle is stretched out in one direction, while keeping the area fixed, the larger is the perimeter, and hence, by (6.23), the closer to 0 is the volume fraction.

Corollary 6.5 *(i) Using the notation introduced above, the volume fraction for the IGM with independently and uniformly rotated grains of the same shape as $K \in \mathcal{C}^d$, is*

$$\rho \;\leq\; \mathbf{E}[R_{ig}^d] \int_0^\infty \left(\sum_{i=0}^d \binom{d}{i} r^i \mathbf{E}[R_{pr}^{d-i}] \right)^{-1} F_{pr}(dr),$$

with equality if and only if K is a sphere.
(ii) If the size is fixed and the orientations are random, then

$$\rho \leq 1/2^d,$$

with equality if and only if K is a sphere.

Proof. (i) From the Brunn-Minkowski theorem it follows that

$$l_d(\{x : K(o,r) \cap \vartheta K(x,y) \neq \emptyset\}) \geq (r+y)^d l_d(K),$$

with equality if and only if K and $\vartheta(\check{K})$ are translates. Hence

$$\begin{aligned}
&\int_0^\infty \int_{SO(d)} l_d(\{x : K(o,r) \cap \vartheta(K)(x,y) \neq \emptyset\})\nu(d\vartheta)F_{pr}(dy) \\
&\geq \int_0^\infty (r+y)^d l_d(K) F_{pr}(dy) \\
&= l_d(K) \sum_{i=0}^{d} \binom{d}{i} r^i \mathbf{E}[R_{pr}^{d-i}],
\end{aligned}$$

with equality if and only if K is a sphere, since K and $\vartheta(\check{K})$ are translates for all $\vartheta \in SO(d)$ if and only if K is a sphere. By Theorem 6.3 and (5.18), the result in (i) follows.
(ii) Follows immediately from (i). ■

Contrary to the case of fixed orientations, the centrally symmetric sets do not all behave in the same way now – here spheres are the only sets for which the upper bound of the volume fraction is attained. Furthermore, all triangles and tetrahedra do not give the same volume fraction when the orientations are random.

6.4 Discs, triangles and other extremal sets

Finally, we calculate the volume fraction for the IGM with fixed-sized grains of the following shapes: discs, squares and equilateral triangles in two dimensions, and spheres, cubes and regular tetrahedra in three dimensions. The results are presented in Table 1.

For fixed orientations we have according to (6.22)

$$\begin{aligned}
\frac{1}{6} \leq \rho \leq \frac{1}{4}, &\qquad \text{if } d = 2, \\
\frac{1}{20} \leq \rho \leq \frac{1}{8}, &\qquad \text{if } d = 3,
\end{aligned}$$

with equalities on the right-hand sides if and only if K is centrally symmetric (for instance for discs, spheres, squares and cubes), and equalities on the left-hand sides if and only if K is a triangle and a tetrahedron, respectively.

Dimension 2		
	Fixed orientation	Random orientation
Disc	1/4	1/4
Square	1/4	$(2+8/\pi)^{-1} \approx 0.22$
Equilateral triangle	1/6	$\sqrt{3}(2\sqrt{3}+18/\pi)^{-1} \approx 0.19$
Dimension 3		
	Fixed orientation	Random orientation
Sphere	1/8	1/8
Cube	1/8	1/11
Regular tetrahedron	1/20	$(2+18\sqrt{1.5}(1-\arccos 3^{-1}/\pi))^{-1} \approx 0.068$

Table 1: The volume fraction in some special cases.

In the case of random orientations, some more effort is needed. First note that from Theorem 6.3 it follows that

$$\rho \quad = \quad \begin{cases} \dfrac{l_2(K)}{l_2(K)2 + S_1(K)^2/(2\pi)}, & \text{if } d = 2, \\ \dfrac{l_3(K)}{l_3(K)2 + S_2(K)\bar{b}(K)}, & \text{if } d = 3, \end{cases}$$

where S_1 is the perimeter, S_2 is the surface area and $\bar{b}$ is the mean width. Note that the volume fraction does not depend on the size of the grains, and hence one can choose a size for which is it easy to determine these quantities. For the examples we have chosen in two dimensions, the quantities are straightforward to calculate. In three dimensions, there are no problems with the sphere, and for a cube of side-length 1 we get the mean width 3/2 by (4.8). For the more complicated tetrahedron, the following values are given in Månsson and Rudemo (2002) if the size is 1: $l_3(K) = 2\sqrt{2}/3$, $S_2(K) = 4\sqrt{3}$ and $\bar{b}(K) = \frac{3}{\pi}(\pi - \arccos 3^{-1})$.

Note that for squares and cubes the volume fraction is lower if the grains are randomly rotated than if they have a fixed orientation, while for triangles and tetrahedra it is the other way around.

7 Conclusions and future work

In this paper we have showed how the shape of the grains affects the volume fraction of two random models for non-intersecting grains. For instance, if the grains have a fixed shape and orientation, the volume fraction assumes its highest value for all centrally symmetric sets and its lowest value for all simplices. To be more precise, it is the mixed volumes that determine the volume fraction if the orientation is fixed, while it is the intrinsic volumes if the orientations are random. Is it the same quantities that determine the volume fraction also for other models of non-intersecting grains? In, for instance, the RSA model the answer to this question is no. In two dimensions, simulation studies proposes that squares and discs do not give the same volume fraction, see Evans (1993) p. 1293 and 1312. A natural question to pose is then which geometrical properties of the grains determine the volume fraction in other models.

We have considered grains of either a fixed orientation or uniformly rotated in the original process. Which distribution of the orientations of Model I and of the IGM does a uniform proposal distribution give, and which distribution should we start with to have uniformly rotated grains at the end? Furthermore, it would be interesting to study which size and orientation distributions are beneficial from a volume fractions perspective. For instance, in Example 6.4 we saw that the volume fraction increased when we had two sizes of the spheres – which relation between the radii gives the highest volume fraction? And if the grains are triangles, it might be a good idea to let them have only two possible directions, which are opposite to each other and have equal probability.

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