Abstract

Let $dm(z)$ be the Lebesgue measure on the unit ball $B \subset \mathbb{C}^d$. For $d < \nu < \infty$, let $d\iota_\nu(z)$ be the measure $c_\nu\,(1 - |z|^2)^{\nu-d-1}dm(z)$. Denote by $L^2_a(d\iota_\nu)$ the weighted Bergman space of all square integrable holomorphic functions on B.

The space of Hilbert-Schmidt bilinear forms on $L^2_a(d\iota_\nu)$ is decomposed under the Möbius group into a sum of irreducible subspaces, each giving rise to some Hankel forms of certain weight. The Hankel forms of weight zero correspond to the small Hankel operators, whose Schatten-von Neumann properties have been studied extensively. In this thesis we study the Schatten-von Neumann properties of bilinear Hankel forms of higher weights defined by some holomorphic vector-valued symbol functions.

We characterize bounded, compact and Hilbert-Schmidt Hankel forms in terms of the membership of the symbols in certain Besov spaces. By interpolation it follows that the symbol is in a certain Besov space if the Hankel form is of Schatten-von Neumann class $\mathcal{S}_p$, $2 < p < \infty$.

Keywords: Hankel forms, Schatten-von Neumann classes, Bergman spaces, Besov spaces, Transvectants, Unitary representations, Möbius group.

AMS 2000 Subject Classification: 32A25, 32A36, 32A37, 47B32, 47B35.

Acknowledgment

Being a student belonging to two universities, there are many persons I would like to thank.

First of all I would like to thank Örebro University for giving me the opportunity to initiate my study in mathematics and for the financial support with which I can continue my study at Göteborg University and Chalmers University of Technology.

I would like to thank Göteborg University and Chalmers University of Technology for giving me the opportunity to study mathematics in an excellent academic environment.

In addition, I would like to thank my supervisor Genkai Zhang, at Chalmers, for giving me inspiration and for always taking time to answer my questions. I would like to thank my supervisor Yang Liu, at Örebro, for all his help and support and for being the one who introduced me to operator theory.

At last but not least, I would like to thank everyone at the Department of Mathematical Sciences at Göteborg University and Chalmers University of Technology, the group of Mathematics at Örebro University and my family for kind support.

Contents

1 Summary and introduction

1.1 Introduction

Hankel operators on the unit disc have been studied extensively, see [Pe1], [Zh] and [JPR]. One of the main problems is to study their Schatten class properties. Consider the Hardy space $H^2(T) \subset L^2(T)$ of holomorphic functions, where $T = \{z \in \mathbb{C} : |z| = 1\}$. Let $P : L^2(T) \to H^2(T)$ be the Szegö projection and denote by $\tilde{H}_f$ the Hankel operator on $H^2(T)$: $\tilde{H}_f g = (I - P)(\bar{f} g)$, $g \in H^2(T)$. It can also be viewed (up to a rank one operator) as a bilinear form H_f on $H^2(T)$,

$$H_f(g_1, g_2) = \int_{\partial D} \overline{f(z)} g_1(z) g_2(z)\, d\sigma(z)\,.$$

Their Schatten properties were studied by Peller in [Pe2]. It is proved that H_f is of Schatten class if and only if f is in a certain Besov space. The corresponding problem for Hankel forms on a Bergman space has been studied in [JPR] and [R2]. It was realized later that the Hilbert-Schmidt Hankel forms on a weighted Bergman space can be viewed as the first irreducible component in the irreducible decomposition of the tensor product of two copies of the Bergman spaces, and subsequently Janson and Peetre [JP] introduced the Hankel forms of higher weights on Bergman spaces on the unit disc; see also [Ro] where multilinear Hankel forms are studied.

A natural problem is to consider Hankel forms on the unit ball in $\mathbb{C}^d$. In [P1] Peetre introduced Hankel forms on the unit ball. The spaces of Hankel forms of higher weights are explicit characterization of irreducible components in the tensor product of Bergman spaces under the Möbius group, see [JP], [P1] and [PZ]. However their Schatten-von Neumann properties have not been studied so far. In this licentiate thesis we will address this problem.

1.2 Notation

Let H_1 and H_2 be Hilbert spaces and let $T : H_1 \to H_2$ be a linear operator. Define the singular numbers $s_n(T) = \inf\{||T - K|| : \operatorname{rank}(K) \leq n\}$, $n \geq 0$. If T is compact, these singular numbers are equal to the eigenvalues of $|T| = (T^*T)^{1/2}$. We denote by $\mathcal{S}_p$ the ideal of operators for which $\{s_n(T)\}_{n \geq 0} \in l^p$, $0 < p \leq \infty$. We remark that $\mathcal{S}_\infty$ is the class of bounded operators. (The compact operators correspond to c_0, not to l^∞.)

Let dm denote the Lebesgue measure on the unit ball $B \subset \mathbb{C}^d$ and let $d\iota(z)$ be the measure $(1 - |z|^2)^{-d-1} dm(z)$. For $d < \nu < \infty$ let $d\iota_\nu(z)$ be the measure $c_\nu (1 - |z|^2)^\nu \, d\iota(z)$, where c_ν is chosen such that

$$\int_B d\iota_\nu(z) = 1\,,$$

i.e., $c_\nu = \Gamma(\nu)/(\pi^d \Gamma(\nu - d))$. The closed subspace of all holomorphic functions in $L^2(d\iota_\nu)$ is denoted by $L_a^2(d\iota_\nu)$ and is called a weighted Bergman space. Note that the space $L^2_a(d\iota_\nu)$ has a reproducing kernel $K_z(w) = (1 - \langle w, z\rangle)^{-\nu}$, that is,

$$f(z) = \langle f, K_z\rangle_\nu = \int_B f(w)\overline{K_z(w)}\, d\iota_\nu(w)\,, \quad f \in L^2_a(d\iota_\nu)\,, \quad z \in B\,. \tag{1}$$

Denote by $B(z,w)$ the Bergman operator on $V = \mathbb{C}^d$ as in [L], namely

$$B(z,w) = (1 - \langle z, w\rangle)(I - z \otimes w^*), \tag{2}$$

where $z \otimes w^*$ stands for the rank one operator given by $(z \otimes w^*)(v) = \langle v, w\rangle z$. Viewed as a matrix acting on column vectors it is

$$B(z,w) = (1 - \langle z, w\rangle)(I - z\bar{w}^t), \tag{3}$$

where w^t is the transpose of w. $B(z,w)$ is holomorphic in z and antiholomorphic in w.

The Bergman metric at $z \in B$, when we identify the tangent space with V, is $\left\langle B(z,z)^{-1}u, v\right\rangle$ for $u, v \in V$. We note that

$$B(z,w)^{-1} = (1 - \langle z, w\rangle)^{-2}\big((1 - \langle z, w\rangle)I + z \otimes w^*\big)\,. \tag{4}$$

Let $B^t(z,w)$ denote the dual of $B(z,w)$ acting on the dual space V' of V. When acting on a vector $v' \in V'$ it is

$$B^t(z,w)v' = (1 - \langle z, w\rangle)v'(I - z\bar{w}^t)\,. \tag{5}$$

Actually we may identify $B^t(z,w)$ with $(1 - \langle z, w\rangle)(I - \bar{w}z^t)$.

For a non-negative integer s, let $\otimes^s V'$ be the tensor product of s factors V' and let $\otimes^0 V' = \mathbb{C}$. The space $\otimes^s V'$ is equipped with a natural Hermitian inner product induced by that of V', so that

$$\langle v_1 \otimes \cdots \otimes v_s, w_1 \otimes \cdots \otimes w_s\rangle = \prod_{j=1}^{s} \langle v_j, w_j\rangle$$

where $v_j, w_j \in V'$, $j = 1, \ldots, s$.

Let $\{u_1, \ldots, u_d\} \subset V'$. Denote by $u_1^{i_1} \odot u_2^{i_2} \odot \cdots \odot u_d^{i_d}$ the sum

$$\frac{i_1! \cdots i_d!}{s!} \sum_{\pi \in S} \pi\,(u_1 \otimes \cdots \otimes u_1 \otimes \cdots \otimes u_d \otimes \cdots \otimes u_d)$$

where $i_1 + \ldots + i_d = s$, $S = S_s/(S_{i_1} \times \cdots \times S_{i_d})$, S_s is the permutation group acting on the tensor by permutating the factors in the tensor and $S_{i_1}, \ldots, S_{i_d}$ are the subgroups permutating the first i_1, the second i_2, ... , the last i_d elements respectively.

Let $\{e_1, \ldots, e_d\}$ be a basis for V'. Denote by $\odot^s V'$ the subspace of symmetric tensors of length s

$$\left\{ \sum_{i_1 + \cdots + i_d = s} v_i\, e_1^{i_1} \odot e_2^{i_2} \odot \cdots \odot e_d^{i_d} \;:\quad i = (i_1, \ldots, i_d) \in \mathbb{N}^d\,, \quad v_i \in \mathbb{C} \right\}.$$

Also, denote by $\otimes^s B^t(z,z)$ the operator on $\otimes^s V'$ induced by the action of $B^t(z,z)$ on V', where $\otimes^0 B^t(z,z) = I$.

1.3 Hankel forms and main results

The Transvectant $\mathcal{T}_s$ on $L^2_a(d\iota_\nu) \otimes L^2_a(d\iota_\nu)$ (introduced in [P1], see also [P2] and [PZ]) is defined by

$$\mathcal{T}_s(f,g)(z) = \sum_{k=0}^{s} \binom{s}{k} (-1)^k \frac{\partial^k f(z) \odot \partial^{s-k} g(z)}{(\nu)_k (\nu)_{s-k}} \tag{6}$$

where

$$\partial^s f(z) = \sum_{j_1 \dots j_s = 1}^{d} \partial_{j_1} \cdots \partial_{j_s} f(z)\, dz_{j_1}(z) \otimes \cdots \otimes dz_{j_s}(z) \in \odot^s V'$$

and $(\nu)_k = \nu(\nu+1)\cdots(\nu+k-1)$, $(\nu)_0 = 1$, is the Pochammer symbol.

The Hankel bilinear form H^s_F on $L^2_a(d\iota_\nu) \otimes L^2_a(d\iota_\nu)$ is defined by

$$H^s_F(f,g) = \int_B \left\langle \otimes^s B^t(z,z) \mathcal{T}_s(f,g)(z), F(z) \right\rangle \, d\iota_{2\nu}(z) \tag{7}$$

where $F : B \to \odot^s V'$ is holomorphic. We call F the symbol of the corresponding Hankel form. We remark that

$$H^0_F(f,g) = \int_B f(z) g(z) \overline{F(z)} \, d\iota_{2\nu}(z) \, .$$

This is the classical Hankel form studied in [JPR].

With the form H^s_F one can associate the operator A^s_F defined by

$$H^s_F(f,g) = \langle f, A^s_F g \rangle_\nu$$

as in [JPR]. Notice that A^s_F is an anti-linear operator on $L^2_a(d\iota_\nu)$. To get a linear operator one combines A^s_F with a conjugation, i.e., one instead considers the operator $\overline{A}^s_F : g \to \overline{A^s_F g}$. We say that H^s_F is of Schatten-von Neumann class $\mathcal{S}_p$, for $0 < p < \infty$, if and only if $\overline{A}^s_F : L^2_a(d\iota_\nu) \to \overline{L^2(d\iota_\nu)}$ is of class $\mathcal{S}_p$.

Finally we present the main results, inspired by Theorem 4.3 in [Ro], in the form of three theorems where we let s be a non-negative integer.

Theorem 1. *Let $F : B \to \odot^s V'$ be a holomorphic function.*

(a) H^s_F is bounded if and only if

$$\sup_{z \in B} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) F(z), F(z) \right\rangle < +\infty \, ,$$

(b) H^s_F is compact if and only if

$$\left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) F(z), F(z) \right\rangle \to 0 \quad \textit{as} \quad |z| \nearrow 1 \, .$$

Theorem 2. *H_F^s is of Hilbert-Schmidt class $\mathcal{S}_2$ if and only if*

$$\int_B \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z)F(z), F(z)\right\rangle \, d\iota(z) < +\infty \, .$$

Theorem 3. *If H_F^s is of class $\mathcal{S}_p$, for $2 < p < \infty$, then*

$$\int_B \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z)F(z), F(z)\right\rangle^{p/2} \, d\iota(z) < +\infty \, .$$

2 Preliminaries

2.1 $G = Aut(B)$: The automorphisms of B

We shall need some results on the group $G = Aut(B)$. We compute the differential of the Möbius transformations, which gives some refinement of the results in [Ru].

Let P_z be the orthogonal projection of $\mathbb{C}^d$ into $\mathbb{C}z$ and let $Q_z = I - P_z$. Put $s_z = (1-|z|^2)^{1/2}$ and define a linear fractional mapping ϕ_z on B by

$$\phi_z(w) = \frac{z - P_z w - s_z Q_z w}{1 - \langle w, z\rangle} \, . \tag{8}$$

If $g \in G$ and $g(z) = 0$ then there is a unique unitary operator $U : \mathbb{C}^d \to \mathbb{C}^d$ such that

$$g = U\phi_z \, .$$

Sometimes $g(z)$ will be written as gz. Define the complex Jacobian J_g by $J_g(w) = \det(g'(w))$. Then we have $J_g(w) = \det U \cdot J_{\phi_z}(w)$. This motivates us to calculate $J_{\phi_z}(w)$ (see proof of Proposition 1). To do this we need the following lemma.

Lemma 1. *Let ϕ_z be the linear fractional mapping (8) on B. Then*

$$\phi_z'(w) = \frac{-s_z^2 P_z - s_z Q_z + s_z(\langle w, z\rangle - w \otimes z^*)}{(1-\langle w, z\rangle)^2} \, .$$

The proof of this lemma is at the end of this subsection. The proposition below is a refinement of Theorem 2.2.6 in [Ru]. Actually, Theorem 2.2.6 in [Ru] is the same as Corollary 1.

Proposition 1. *Let ϕ_z be the linear fractional mapping (8) on B. Then*

$$J_{\phi_z}(w) = (-1)^d \left(\frac{s_z}{1 - \langle w, z\rangle} \right)^{d+1} \, .$$

Proof of Proposition 1. The case $d = 1$ is trivial. We will treat the cases $d = 2$ and $d \geq 3$ separately. First assume $d = 2$. Since $Q_z = I - P_z$ then $-s_z^2 P_z h - s_z Q_z h = -s_z\left((s_z - 1)P_z + I\right)h$. Then by Lemma 1 we have

$$\phi_z'(w) = \frac{-s_z}{(1-\langle w, z\rangle)^2}\Big((s_z - 1)P_z + (1 - \langle w, z\rangle)I + A \Big)$$

where $A = w\bar{z}^t$. Let

$$\begin{aligned} B &= (s_z - 1)P_z + (1 - \langle w, z\rangle)I + A \\ &= \begin{bmatrix} \frac{s_z-1}{|z|^2}|z_1|^2 + 1 - \langle w, z\rangle + \bar{z}_1 w_1 & \frac{s_z-1}{|z|^2}\bar{z}_2 z_1 + \bar{z}_2 w_1 \\ \frac{s_z-1}{|z|^2}\bar{z}_1 z_2 + \bar{z}_2 w_2 & \frac{s_z-1}{|z|^2}|z_2|^2 + 1 - \langle w, z\rangle + \bar{z}_2 w_2 \end{bmatrix}. \end{aligned}$$

Then

$$\det B = (1 - \langle w, z\rangle)s_z$$

so that

$$\det \phi_z'(w) = \frac{s_z^2}{(1 - \langle w, z\rangle)^4} \cdot \det B = \frac{s_z^3}{(1 - \langle w, z\rangle)^3}.$$

Now we consider the case when $d \geq 3$. We may assume that $w \notin \mathbb{C}z$ and that $\langle w, z\rangle \neq 0$. Then the vectors z and w span a two dimensional subspace V_0 in $\mathbb{C}^d$ and we may write $\mathbb{C}^d = V_0 \oplus V_1$ where $V_1 = V_0^{\perp}$. As in the case $d = 2$ we shall find a matrix form of $B = (s_z - 1)P_z + (1 - \langle w, z\rangle)I + w \otimes z^*$. Let $v \in \mathbb{C}^d$. Then we can write $v = v_0 + v_1$ where $v_0 = \alpha z + \beta w \in V_0$ and $v_1 \in V_1$. On one hand we have

$$Bv_1 = (1 - \langle w, z\rangle)v_1\,.$$

On the other hand

$$\begin{aligned} Bv_0 &= (s_z - 1)\frac{\langle \alpha z + \beta w, z\rangle}{|z|^2} z + (1 - \langle w, z\rangle)v_0 + \langle \alpha z + \beta w, z\rangle w \\ &= \left((s_z - 1)\left(\alpha + \frac{\beta\langle w, z\rangle}{|z|^2}\right) + \alpha(1 - \langle w, z\rangle)\right) z + \\ &+ \left(\beta(1 - \langle w, z\rangle) + \alpha|z|^2 + \beta\langle w, z\rangle\right) w\,. \end{aligned}$$

The vectors z and w are chosen as basis vectors for V_0. Thus under the decomposition $\mathbb{C}^d = V_0 \oplus V_1$, B has a block-matrix form

$$B = \left[\begin{array}{c|c} \tilde{B} & 0 \\ \hline 0 & (1 - \langle w, z\rangle)I_{d-2} \end{array}\right]$$

with

$$\tilde{B} = \begin{bmatrix} s_z - \langle w, z\rangle & (s_z - 1)\frac{\langle w,z\rangle}{|z|^2} \\ |z|^2 & 1 \end{bmatrix}.$$

This yields

$$\det B = \det \left[\begin{array}{c|c} \tilde{B} & 0 \\ \hline 0 & (1 - \langle w, z\rangle)I_{d-2} \end{array}\right] = (1 - \langle w, z\rangle)^{d-2} \det \tilde{B}$$

and $\det \tilde{B} = s_z(1 - \langle w, z\rangle)$ so that

$$\det \phi_z'(w) = \left(\frac{-s_z}{(1 - \langle w, z\rangle)^2}\right)^d \det B = (-1)^d \left(\frac{s_z}{1 - \langle w, z\rangle}\right)^{d+1}.$$

This proves the proposition. ∎

Corollary 1. *Let $g \in G$. Then the real Jacobian $J_{\mathbb{R},g}$ of g is*

$$J_{\mathbb{R},g}(w) = |J_g(w)|^2 = \left(\frac{1-|z|^2}{|1-\langle w,z\rangle|^2} \right)^{d+1} .$$

Proof of Lemma 1. We shall calculate $\phi_z(w+h)$ where $w \in B$ and $|h|$ is sufficiently small. We have

$$\phi_z(w+h) = (1-\langle w,z\rangle - \langle h,z\rangle)^{-1}\,(z - P_z w - s_z Q_z w - (P_z + s_z Q_z)h)$$

and

$$(1 - \langle w,z\rangle - \langle h,z\rangle)^{-1} = (1-\langle w,z\rangle)^{-1} + (1-\langle w,z\rangle)^{-2}\langle h,z\rangle + \mathcal{O}(|h|^2).$$

We get

$$\begin{aligned}\phi_z(w+h) = \phi_z(w) &+ (1-\langle w,z\rangle)^{-1}(-P_z - s_z Q_z)h \\ &+ (1-\langle w,z\rangle)^{-2}\langle h,z\rangle(z - P_z w - s_z Q_z w) + \mathcal{O}(|h|^2)\end{aligned}$$

so that

$$\begin{aligned}\phi_z'(w)h = (1-\langle w,z\rangle)^{-2}\Big(&(1-\langle w,z\rangle)(-P_z h - s_z Q_z h) \\ &+ \langle h,z\rangle(z - P_z w - s_z Q_z w)\Big).\end{aligned}$$

Thus

$$\phi_z'(w)h = \frac{-s_z P_z h - s_z Q_z h + s_z(\langle w,z\rangle h - \langle h,z\rangle w)}{(1-\langle w,z\rangle)^2} .$$

This completes the proof of lemma 1. □

2.2 Some elementary properties of the Bergman operator

Let $g \in G$. Combining Proposition IX.1.1 with Proposition IX.2.6 in [FK] we get

$$B(z,w)^{-1} = (dg(z))^*\, B(gz,gw)^{-1} dg(w) .$$

This yields

$$B^t(gz,gw) = \left(dg(z)^t\right)^* B^t(z,w) dg(w)^t . \tag{9}$$

Now we consider another property of the Bergman operator. It holds that

$$\begin{aligned} B^t(z,z) &= (1-|z|^2)(P_{\bar z} + Q_{\bar z} - |z|^2 P_{\bar z}) \\ &= (1-|z|^2)Q_{\bar z} + (1-|z|^2)^2 P_{\bar z} .\end{aligned}$$

Thus

$$(1-|z|^2)^2 I \leq B^t(z,z) \leq (1-|z|^2) I ; \tag{10}$$

in particular $B^t(z,z)$ is a positive operator. Actually $\otimes^s B^t(z,z)$ is positive on $\otimes^s V'$. To prove this we need a lemma.

Lemma 2. *Let H_1 and H_2 be Hilbert spaces with inner products $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ respectively. Let A and B be positive operators on H_1 and H_2 respectively. Then the operator $A\otimes B$ is positive on the induced Hilbert space $H_1\otimes H_2$ with the inner product $\langle\cdot,\cdot\rangle$.*

Proof. We compute the inner product $\langle (A\otimes B)x, x\rangle$, $x\in H_1\otimes H_2$. We may assume that

$$x=\sum_j v_j\otimes w_j$$

for some $v_1,\dots,v_n\in H_1$ and $w_1,\dots,w_n\in H_2$ since those elements are dense in $H_1\otimes H_2$. We get that the inner product is

$$\begin{aligned}
&\Big\langle (A\otimes B)\Big(\sum_j v_j\otimes w_j\Big), \Big(\sum_j v_j\otimes w_j\Big)\Big\rangle\\
=\ &\sum_{i,j}\langle Av_j, v_i\rangle_1\langle Bw_j, w_i\rangle_2\\
=\ &\sum_{i,j}\langle A^{1/2}v_j, A^{1/2}v_i\rangle_1\langle B^{1/2}w_j, B^{1/2}w_i\rangle_2\\
=\ &\Big\langle \sum_j (A^{1/2}\otimes B^{1/2})(v_j\otimes w_j), \sum_j (A^{1/2}\otimes B^{1/2})(v_j\otimes w_j)\Big\rangle \geq 0
\end{aligned}$$

This proves the lemma. □

Remark 1. Since $B^t(z,z)$ is positive on V' we have now $\otimes^s B^t(z,z)$ is positive for $s=0,1,2,\cdots$.

2.3 The norm of z^α in the Bergman space $L_a^2(d\iota_\nu)$

Let $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_d)$ denote ordered d-tuples of non-negative integers α_i and denote $|\alpha|=\alpha_1+\cdots+\alpha_d$. Then the polynomials $\{z^\alpha\}$ forms an orthogonal basis for $L_a^2(d\iota_\nu)$ and

$$||z^\alpha||_\nu^2=\int_B |z_1^{\alpha_1}\cdot z_2^{\alpha_2}\cdots z_d^{\alpha_d}|^2\, d\iota_\nu(z)=\frac{\alpha_1!\alpha_2!\cdots\alpha_d!}{(\nu)_{|\alpha|}} \tag{11}$$

where $(\nu)_{|\alpha|}=\nu(\nu+1)\cdots(\nu+|\alpha|-1)=\Gamma(\nu+|\alpha|)/\Gamma(\nu)$, $(\nu)_0=1$, is the Pochammer symbol.

2.4 Some remarks on boundedness, compactness and $\mathcal{S}_2$

Consider the bilinear Hankel form H_F^s with symbol F. First observe that the operator norm of the corresponding operator $\overline{A}_F^s$ equals

$$||H_F^s||=\sup_{||f||_\nu=||g||_\nu=1}|H_F^s(f,g)|\,.$$

If $\overline{A}_F^s$ is compact and $\{g_n\}_{n=1}^{\infty} \subset L_a^2(d\iota_\nu)$, with $||g_n||_\nu = 1$, $g_n \to 0$ weakly as $n \to \infty$, then there is a sequence $\{c_n\}_{n=1}^{\infty}$ of positive numbers such that

$$|H_F^s(f, g_n)| \leq c_n ||f||_\nu$$

for all n. Also $c_n \to 0$ as $n \to \infty$. On the other hand, if $\{A_n\}_{n=1}^{\infty}$ is a sequence of compact bilinear forms on $L_a^2(d\iota_\nu) \otimes L_a^2(d\iota_\nu)$ such that $A_n \to H_F^s$ in operator norm, then H_F^s is compact. Also H_F^s is of Hilbert-Schmidt class $\mathcal{S}_2$ if and only if

$$||H_F^s||_{\mathcal{S}_2}^2 = \sum_{|\alpha|=0}^{\infty} \sum_{|\beta|=0}^{\infty} |H_F^s(e_\alpha, e_\beta)|^2 < \infty$$

where $e_\alpha = z^\alpha / ||z^\alpha||_\nu$. In addition, if A is a bilinear form on $L_a^2(d\iota_\nu) \otimes L_a^2(d\iota_\nu)$ of Hilbert-Schmidt class, then A is compact.

3 The Banach space $\mathcal{H}_{\nu,s}^p$

Denote by $\mathcal{H}_{\nu,s}^p = \mathcal{H}_{\nu,s}^p(B, \odot^s V')$, $2 \leq p < \infty$, the Banach space of all holomorphic functions $S : B \to \odot^s V'$ such that

$$||S||_{\nu,s,p} = \left(\int_B \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) S(z), S(z) \right\rangle^{p/2} d\iota(z) \right)^{1/p} < \infty .$$

3.1 Transformation properties of H_F

Define an action π_ν of G on $L_a^2(d\iota_\nu)$ by

$$\pi_\nu : \ g \in G \ , \ f(w) \to f(g^{-1}w) \left(J_{g^{-1}}(w) \right)^{\nu/(d+1)} . \tag{12}$$

Then $\pi_\nu : g \to \pi_\nu(g)$ is a projective unitary representation on $L_a^2(d\iota_\nu)$, that is $||\pi_\nu(g) f||_\nu = ||f||_\nu$ and $\pi_\nu(g_1 g_2) = C(g_1, g_2) \pi_\nu(g_1) \pi_\nu(g_2)$ for some constant $C(g_1, g_2)$. This yields the following equality of two operator norms

$$||H_F^s \left(\pi_\nu(g))(\cdot), \pi_\nu(g)(\cdot) \right) || = ||H_F^s||. \tag{13}$$

Define an action $\pi_{\nu,s}$ on $\mathcal{H}_{\nu,s}^2$ by

$$\pi_{\nu,s} : \ g \in G \ , \ S(z) \quad \to \quad \left(\otimes^s \left(dg^{-1}(z) \right)^t \right) S(g^{-1}z) \left(J_{g^{-1}}(z) \right)^{2\nu/(d+1)} . \tag{14}$$

Then

$$H_F^s(\pi_\nu(g) f_1, \pi_\nu(g) f_2) = H_S^s(f_1, f_2) \tag{15}$$

where $S(z) = \pi_{\nu,s}(g^{-1}) F(z)$. Equation (15) is a consequence of Lemma 3 below. Define an action $\pi_\nu(\cdot) \otimes \pi_\nu(\cdot)$ on $L_a^2(d\iota_\nu) \otimes L_a^2(d\iota_\nu)$ by

$$\pi_\nu \otimes \pi_\nu : \ g \in G \ , \ (f_1(w_1), f_2(w_2)) \\ \to f_1(g^{-1}w_1) f_2(g^{-1}w_2) \left(J_{g^{-1}}(w_1) \right)^{d+1} \left(J_{g^{-1}}(w_2) \right)^{d+1} . \tag{16}$$

The following invariance property of the Transvectant is proved in [P1], see also [PZ].

Lemma 3. *Let $\pi_{\nu,s}$ and $\pi_\nu(\cdot)\otimes\pi_\nu(\cdot)$ be the representations given by (14) and (16) respectively. Let $g \in G$. Then*

$$\mathcal{T}_s\left(\pi_\nu(g)\otimes\pi_\nu(g)\right)(f_1,f_2) = \pi_{\nu,s}(g)\mathcal{T}_s(f_1,f_2)\,.$$

Remark 2. It follows from Theorem 4 that $\mathcal{T}_s$ takes values in $\mathcal{H}^2_{\nu,s}$. In fact, Theorem 4 shows that $\mathcal{T}_s : L^2_a(d\iota_\nu)\otimes L^2_a(d\iota_\nu) \to \mathcal{H}^2_{\nu,s}$ is a bounded bilinear form.

Remark 3. As a consequence of Lemma 3 we have (15), namely

$$\begin{aligned} & H^s_F\Big(\left(\pi_\nu(g)\otimes\pi_\nu(g)\right)(f_1,f_2)\Big) \\ = \;& \langle \mathcal{T}_s\left(\pi_\nu(g)\otimes\pi_\nu(g)\right)(f_1,f_2), F\rangle_{\nu,s,2} \\ = \;& \langle \pi_{\nu,s}(g)\mathcal{T}_s(f_1,f_2), F\rangle_{\nu,s,2} \\ = \;& \left\langle \mathcal{T}_s(f_1,f_2), \pi_{\nu,s}(g^{-1})F\right\rangle_{\nu,s,2} \end{aligned}$$

which gives the result if we observe that $S = \pi_{\nu,s}(g^{-1})F$.

3.2 Reproducing kernel of the space $\mathcal{H}^2_{\nu,s}$

Lemma 4. *The reproducing kernel of $\mathcal{H}^2_{\nu,s}$ is, up to a nonzero constant,*

$$K_{\nu,s}(z,w) = (1-\langle z,w\rangle)^{-2\nu}\otimes^s\left(B^t(z,w)\right)^{-1}\,.$$

Namely, for any $f\in\mathcal{H}^2_{\nu,s}$ and any $v\in\odot^s V'$ it holds that

$$\begin{aligned}\langle f(z),v\rangle &= c\,\langle f(\cdot),K_{\nu,s}(\cdot,z)v\rangle_{\nu,s,2} \\ &= c\int_B \left\langle (1-|w|^2)^{2\nu}\otimes^s B^t(w,w)f(w), K_{\nu,s}(w,z)v\right\rangle\, d\iota(w).\end{aligned}$$

Proof. For any $v\in\odot^s V'$ we prove that $f\to\langle f(z),v\rangle$ is a continuous functional on $\mathcal{H}^2_{\nu,s}$. It follows then by Riesz lemma that there exists a function $R(z,w)$: $\odot^s V'\to\odot^s V'$ such that $\langle f(z),v\rangle = \langle f,R(\cdot,z)v\rangle_{\nu,s,2}$. Let $\{f_n\}_{n=1}^\infty\subset\mathcal{H}^2_{\nu,s}$ with $f_n\to f\in\mathcal{H}^2_{\nu,s}$ and let $z\in B$. It is enough to show that $\{f_n(z)\}$ is Cauchy in $\odot^s V'$. Since $z\to\|f_n(z)-f_m(z)\|$ is subharmonic then

$$\|f_n(z)-f_m(z)\|\le C_{d,r,\nu}\int_{z+rB}\|f_n(w)-f_m(w)\|\,d\iota_{2\nu}(w)$$

so by Jensen's inequality

$$\|f_n(z)-f_m(z)\|^2\le C'_{d,r,\nu}\int_{z+rB}\|f_n(w)-f_m(w)\|^2\,d\iota_{2\nu}(w)$$

if $\overline{z+rB} \subset B$. On the other hand, there is a constant $d_r > 0$ such that $d_r I \leq \otimes^s B^t(w,w)$ for all $w \in \overline{z+rB}$. Hence

$$\|f_n(z) - f_m(z)\|^2 \leq$$
$$D_{d,r,\nu} \int_{z+rB} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(w,w)\,(f_n(w) - f_m(w))\,, f_n(w) - f_m(w) \right\rangle\, d\iota(w)$$

so that $\{f_n(z)\}$ is Cauchy in $\odot^s V'$. Then the reproducing property at $z=0$ reads as

$$\langle f(0), v\rangle = \langle f(\cdot), R(\cdot,0)v\rangle_{\nu,s,2}\,.$$

On the other hand, the space of $\odot^s V'$-valued polynomials is dense in $\mathcal{H}^2_{\nu,s}$ and $\langle p(\cdot), v\rangle_{\nu,s,2} = 0$ for all homogeneous polynomials of degree ≥ 1. Thus if

$$f(z) = \sum_{m=0}^{\infty} f_m(z)$$

where f_m are homogeneous polynomials of degree m, then

$$\langle f(\cdot), v\rangle_{\nu,s,2} = \langle f_0(\cdot), v\rangle_{\nu,s,2} = \langle f(0), v\rangle_{\nu,s,2} = c'\langle f(0), v\rangle\,.$$

Therefore

$$\langle f(\cdot), R(\cdot,0)v\rangle_{\nu,s,2} = \langle f(0), v\rangle = \frac{1}{c'}\,\langle f(\cdot), v\rangle_{\nu,s,2}$$

so that $R(\cdot,0) = cI$ with $c \neq 0$. Next we prove that $R(z,w)$ transforms under G as follows

$$R(gz,gw) =$$
$$\left(\otimes^s dg(z)^t\right)^{-1} R(z,w) \left(\otimes^s \left(dg(w)^t\right)^*\right)^{-1} (J_g(z))^{-2\nu/(d+1)} \left(\overline{J_g(w)}\right)^{-2\nu/(d+1)} \quad (17)$$

where $g \in G$. Indeed, for all $F \in \mathcal{H}^2_{\nu,s}$

$$\langle F(z), v\rangle = \int_B \left\langle (1-|w|^2)^{2\nu} \otimes^s B^t(w,w)F(w), R(w,z)v\right\rangle\, d\iota(w)$$

from which it follows that for all $f \in L^2_a(d\iota_\nu)$

$$\left\langle J_g(z)^{2\nu/(d+1)} \otimes^s dg(z)^t f(gz), v\right\rangle$$
$$= \int_B \left\langle (1-|w|^2)^{2\nu} \otimes^s B^t(w,w) J_g(w)^{2\nu/(d+1)} \otimes^s dg(w)^t f(gw), R(w,z)v\right\rangle\, d\iota(w)\,. \quad (18)$$

On the other hand, it follows from (9)

$$
\begin{aligned}
&\left\langle f(gz), \left(\overline{J_g(z)}\right)^{2\nu/(d+1)} \otimes^s \left(dg(z)^t\right)^* v\right\rangle \\
= \quad & \int_B \left\langle \otimes^s B^t(w,w) f(w), R(w,gz) \left(\overline{J_g(z)}\right)^{2\nu/(d+1)} \otimes^s \left(dg(z)^t\right)^* v\right\rangle \frac{d\iota_{2\nu}(w)}{c_{2\nu}} \\
= \quad & \int_B \left\langle \otimes^s B^t(gw,gw) f(gw), R(gw,gz) \left(\overline{J_g(z)}\right)^{2\nu/(d+1)} \otimes^s \left(dg(z)^t\right)^* v\right\rangle \\
& \qquad \cdot \left|\left(J_g(w)\right)^{2\nu/(d+1)}\right|^2 \frac{d\iota_{2\nu}(w)}{c_{2\nu}} \\
= \quad & \int_B \Big\langle (1-|w|^2)^{2\nu} \otimes^s B^t(w,w) \left(J_g(w)\right)^{2\nu/(d+1)} \otimes^s dg(w)^t f(gw), \\
& \qquad \otimes^s dg(w)^t R(gw,gz) \otimes^s \left(dg(z)^t\right)^* v \Big\rangle \\
& \qquad \cdot \left(J_g(z)\right)^{2\nu/(d+1)} \left(\overline{J_g(w)}\right)^{2\nu/(d+1)} d\iota(w)\,.
\end{aligned}
$$

Comparing this with (18) we get (17). Now both $R(z,w)/c$ and $K_{\nu,s}(z,w)$ satisfy the same transformation rule (17) and are identity operator at $z=0$. Thus they are the same for all $z, w \in B$. This completes the proof of the lemma. □

4 The Besov space $\mathcal{B}_{\nu,s}$

Let $s = 1, 2, 3, \ldots$ and define

$$
\mathcal{B}_{\nu,s} = \left\{ f : B \to \mathbb{C} \text{ holomorphic },\ \int_B \left\langle \otimes^s B^t(z,z) \partial^s f(z), \partial^s f(z)\right\rangle d\iota_\nu(z) < +\infty \right\}.
$$

The space $\mathcal{B}_{\nu,s}$ is called a Besov space. It is a Hilbert space, equipped with the inner product $\langle\cdot,\cdot\rangle_{\nu,s}$ given by

$$
\begin{aligned}
\langle f, g\rangle_{\nu,s} = f(0)\overline{g(0)} \quad &+ \quad \cdots + \left\langle \left(\partial^{(s-1)} f\right)(0), \left(\partial^{(s-1)} g\right)(0)\right\rangle + \\
&+ \quad \int_B \left\langle \otimes^s B^t(z,z) \partial^s f(z), \partial^s g(z)\right\rangle d\iota_\nu(z)\,.
\end{aligned}
$$

Actually $\mathcal{B}_{\nu,s} = L^2_a(d\iota_\nu)$, namely they are equal as sets and their norms are equivalent, as is shown below.

Theorem 4. *There exist constants $C_{\nu,s}, D_{\nu,s} > 0$ such that*

$$
C_{\nu,s} \cdot ||f||_\nu \le ||f||_{\nu,s} \le D_{\nu,s} \cdot ||f||_\nu
$$

for all holomorphic $f : B \to \mathbb{C}$.

We need first some elementary lemmas.

Lemma 5. *Let f_m and f_n be homogeneous holomorphic polynomials of degree m and n respectively, with $m \neq n$. Then $\langle f_m, f_n \rangle_{\nu,s} = 0$.*

Proof. Let $0 < \theta < 2\pi$. Then $e^{i\theta} \neq 1$. Since f_m is a homogeneous polynomial of degree m we have $f_m(e^{i\theta}z) = e^{im\theta} f_m(z)$. Given m and n with $m \neq n$, it is enough to prove that

$$\langle f_m, f_n \rangle_{\nu,s} = e^{i(m-n)\theta} \langle f_m, f_n \rangle_{\nu,s} \tag{19}$$

The case $s = 0$ follows directly from the homogeneity. Now consider the case $s = 1$. It is easy to see that $B^t(z,z) = B^t(e^{-i\theta}z, e^{-i\theta}z)$. By the chain rule and homogeneity it follows that

$$(\partial f_m)(e^{i\theta}w) = e^{-i\theta}\partial(f_m(e^{i\theta}\cdot))(w) = e^{i(m-1)\theta}(\partial f_m)(w)$$

so that the equation (19) holds for $s = 1$. The cases $s = 2, 3, \dots$ now follow in the same way if we first notice that $(\partial^s f_m)(e^{i\theta}w) = e^{i(m-s)\theta}(\partial^s f_m)(w)$. This completes the proof. □

We recall now a result from Rudin (Theorem 12.2.8 in [Ru]). Consider the space $\mathcal{P}_m$ of all homogeneous holomorphic polynomials of degree m on B with the natural group action of the unitary group $\mathcal{U}(d)$:

$$(\pi_g f)(z) = f(g^{-1}z), \quad f \in \mathcal{P}_m, \quad g \in \mathcal{U}(d).$$

Then $(\mathcal{P}_m, \pi_g)$ is a unitary irreducible representation of $\mathcal{U}(d)$. As a consequence of Schur's lemma (Theorem 1.10 in [BD]) we have

Lemma 6. *Let m be a non-negative integer. Then there exists a positive constant $C_{\nu,s,m}$ such that*

$$||f_m||_{\nu,s} = C_{\nu,s,m} \cdot ||f_m||_\nu$$

for all $f_m \in \mathcal{P}_m$. .

Remark 4. Actually, this lemma is a special case of the result in exercise 1.16.7 in [BD].

Now we can prove the norm-equivalence of $\mathcal{B}_{\nu,s}$ and $L^2_a(d\iota_\nu)$.

Proof of Theorem 4. It is enough to prove the theorem for f with $f(0) = \cdots = \partial^{s-1}f(0) = 0$. Write $f = \sum_{m=0}^\infty f_m$ where $f_m \in \mathcal{P}_m$. By Lemma 5 we have that $\{f_m\}_{m=0}^\infty$ is an orthogonal set in both $L^2_a(d\iota_\nu)$ and $\mathcal{B}_{\nu,s}$. Also, by Lemma 6 we have $||f_m||_{\nu,s} = C_{\nu,s,m} \cdot ||f_m||_\nu$ where $C_{\nu,s,m}$ does not depended on f_m of degree m. We compute $C_{\nu,s,m}$ and prove that there exist positive constants $C_{\nu,s}$ and $D_{\nu,s}$ such that

$$C_{\nu,s} \leq C_{\nu,s,m} \leq D_{\nu,s} \tag{20}$$

for all m. We may assume that $m \geq s$. Take $f_m(z) = z_1^m$. We shall calculate

$$||f_m||^2_{\nu,s} = \int_B \left\langle \otimes^s B^t(z,z)\partial^s f_m(z), \partial^s f_m(z)\right\rangle \, d\iota_\nu(z).$$

First observe that

$$\begin{aligned}
& \left\langle \otimes^s B^t(z,z)\partial^s f_m(z), \partial^s f_m(z)\right\rangle \\
= & \left\langle \otimes^s B^t(z,z)\,(\partial_1^s z_1^m) \otimes^s dz_1, (\partial_1^s z_1^m) \otimes^s dz_1\right\rangle \\
= & \left\langle B^t(z,z)(\partial_1^s z_1^m) dz_1, (\partial_1^s z_1^m) dz_1\right\rangle \cdot \left\langle B^t(z,z) dz_1, dz_1\right\rangle^{s-1} \\
= & \frac{\Gamma(m+1)^2}{\Gamma(m-s+1)^2}(1-|z|^2)^s(1-|z_1|^2)^s|z_1|^{2(m-s)}.
\end{aligned}$$

We have

$$C_\nu \int_B |z_1|^{2(m-s)}(1-|z_1|^2)(1-|z|^2)^{\nu+s}\, d\iota(z) =$$

$$\int_{|z_1|<1} |z_1|^{2(m-s)}(1-|z_1|^2)^s \int_{|z'|<\sqrt{1-|z'|^2}} (1-|z_1|^2-|z'|^2)^{\nu+s-d-1} dm(z') dm(z_1)$$

and

$$\int_{|z'|<\sqrt{1-|z'|^2}} (1-|z_1|^2-|z'|^2)^{\nu+s-d-1} dm(z') = C'_\nu \cdot (1-|z_1|^2)^{\nu+s-2}\,.$$

Since

$$\int_{|z_1|<1} |z_1|^{2(m-s)}(1-|z_1|^2)^s(1-|z|^2)^{s+\nu-d-1}\, dm(z_1) = C''_\nu \cdot \frac{\Gamma(m-s+1)\Gamma(\nu+2s-1)}{\Gamma(m+s+\nu)}$$

we get

$$||f_m||^2_{\nu,s} = a_\nu \cdot \frac{\Gamma(m+1)^2\Gamma(\nu+2s-1)}{\Gamma(m-s+1)\Gamma(m+s+\nu)}\,.$$

On the other hand

$$||f_m||^2_\nu = \frac{\Gamma(m+1)\Gamma(\nu)}{\Gamma(m+\nu)}$$

so that

$$C^2_{\nu,s,m} = \frac{||f_m||^2_{\nu,s}}{||f_m||^2_\nu} = a_\nu \cdot \frac{\Gamma(m+1)\Gamma(\nu+2s-1)\Gamma(m+\nu)}{\Gamma(m-s+1)\Gamma(m+s+\nu)\Gamma(\nu)}\,.$$

For $m \geq s$ we have

$$\begin{aligned}
\frac{\Gamma(m+1)\Gamma(m+\nu)}{\Gamma(m-s+1)\Gamma(m+s+\nu)} &= \frac{m(m-1)\cdots(m-s+1)}{(m+s+\nu-1)\cdots(m+\nu)} \\
&= \frac{\left(1-\frac{1}{m}\right)\cdots\left(1-\frac{s-1}{m}\right)}{\left(1+\frac{s+\nu-1}{m}\right)\cdots\left(1+\frac{\nu}{m}\right)}
\end{aligned}$$

so that

$$b_{\nu,s} = a_\nu \cdot \frac{\left(1-\frac{1}{s}\right)\cdots\left(1-\frac{s-1}{s}\right)}{\left(1+\frac{s+\nu-1}{s}\right)\cdots\left(1+\frac{\nu}{s}\right)} \leq a_\nu \cdot \frac{\Gamma(m+1)\Gamma(m+\nu)}{\Gamma(m-s+1)\Gamma(m+s+\nu)} \leq a_\nu\,.$$

So (20) follows by putting

$$C_{\nu,s} = \sqrt{\frac{a_\nu \cdot b_{\nu,s} \cdot \Gamma(\nu+2s-1)}{\Gamma(\nu)}}$$

and

$$D_{\nu,s} = \sqrt{\frac{a_\nu \cdot \Gamma(\nu+2s-1)}{\Gamma(\nu)}}\,.$$

□

5 Boundedness

5.1 The Banach space $\mathcal{H}^\infty_{\nu,s}$

Denote by $L^\infty_{\nu,s}$ the space of functions $F : B \to \odot^s V'$ such that

$$\|F\|_{\nu,s,\infty} = \sup_{z\in B} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) F(z), F(z) \right\rangle^{1/2} < \infty\,.$$

If we write $\|F\|_{\nu,s,\infty} = \sup_{z\in B} \|S(z)\|_{\mathcal{B}}$ where

$$\|S(z)\|_{\mathcal{B}} = \left\| \left((1-|z|^2)^{2\nu} \otimes^s B^t(z,z) \right)^{1/2} F(z) \right\|$$

and $\mathcal{B} = \odot^s V'$, then $L^\infty_{\nu,s}$ is a Banach space since it is easy to see that, if $S_n : B \to \mathcal{B}$ satisfies

$$\sum_{n=1}^{\infty} \sup_{z\in B} \|S_n(z)\|_{\mathcal{B}} < \infty$$

then there is a $S : B \to \mathcal{B}$ with $\sup_{z\in B} \|S(z)\|_{\mathcal{B}} < \infty$ such that

$$\sup_{z\in B} \left\| S(z) - \sum_{n=1}^{N} S_n(z) \right\|_{\mathcal{B}} \to 0 \quad \text{as} \quad N \to \infty\,.$$

Denote by $\mathcal{H}^\infty_{\nu,s}$ the space of all holomorphic functions in $L^\infty_{\nu,s}$. Then $\mathcal{H}^\infty_{\nu,s}$ is a closed subspace of $L^\infty_{\nu,s}$ which yields that $\mathcal{H}^\infty_{\nu,s}$ is a Banach space.

5.2 Proof of Theorem 1 (a)

Proof of sufficiency. The Hankel form in (7) can be written as a sum of certain integrals, we estimate each one, as follows,

$$\left| \int_B \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) \partial^k f(z) \otimes \partial^{s-k} g(z), F(z) \right\rangle \, d\iota(z) \right| \leq$$
$$\|F\|_{\nu,s,\infty} \cdot \int_B \left\langle \otimes^s B^t(z,z) \partial^k f(z) \otimes \partial^{s-k} g(z), \partial^k f(z) \otimes \partial^{s-k} g(z) \right\rangle^{1/2} \frac{d\iota_\nu(z)}{c_\nu}$$

and

$$\left\langle \otimes^s B^t(z,z)\partial^k f(z)\otimes\partial^{s-k}g(z), \partial^k f(z)\otimes\partial^{s-k}g(z)\right\rangle =$$
$$\left\langle \otimes^k B^t(z,z)\partial^k f(z), \partial^k f(z)\right\rangle \cdot \left\langle \otimes^{s-k} B^t(z,z)\partial^{s-k}g(z), \partial^{s-k}g(z)\right\rangle$$

so that

$$\left|\int_B \left\langle (1-|z|^2)^{2\nu}\otimes^s B^t(z,z)\partial^k f(z)\otimes\partial^{s-k}g(z), F(z)\right\rangle\, d\iota(z)\right|$$
$$\leq c'_\nu \cdot ||F||_{\nu,s,\infty}\cdot ||f||_{\nu,k}\cdot ||g||_{\nu,s-k} \leq C_{\nu,s}\cdot ||F||_{s,\infty}\cdot ||f||_\nu \cdot ||g||_\nu\,,$$

where the last inequality follows from Theorem 4. □

For notational convenience we denote

$$\langle u,v\rangle_z = \left\langle \otimes^s B^t(z,z)u, v\right\rangle$$

where $u, v\in \odot^s V'$, and it defines an inner product on $\odot^s V'$.

Proof of necessity. Let $v \in \odot^s V'$. By Lemma 4 we have

$$\langle F(0), v\rangle = c\int_B \left\langle (1-|z|^2)^{2\nu}\otimes^s B^t(w,w)F(w), v\right\rangle\, d\iota(w)\,.$$

We may write

$$v = \sum_{|i|=s} v_i\, e_1^{i_1}\odot\cdots\odot e_d^{i_d}$$

where $i=(i_1,\ldots,i_d)$ and $v_i\in\mathbb{C}$. Take

$$f(w) = \sum_{|i|=s} w_1^{i_1}\cdots w_d^{i_s}\cdot v_i \quad \text{and} \quad g(w)=1\,.$$

Then $f, g\in L^2_a(d\iota_\nu)$. By (6)

$$\mathcal{T}_s(f,g)(w) = \sum_{k=0}^{s}\binom{s}{k}(-1)^k\frac{\partial^k f(w)\odot\partial^{s-k}g(w)}{(\nu)_k(\nu)_{s-k}} = \binom{s}{0}(-1)^s\frac{\partial^s f(w)\odot g(w)}{(\nu)_s(\nu)_0}$$

where

$$\partial^s f(w) \;=\; \sum_{|i|=s}\partial^s\left(w_1^{i_1}\cdots w_d^{i_d}\right)\cdot v_i \;=\; \sum_{|i|=s} s!\,\cdot\, v_i\, e_1^{i_1}\,\odot\,\cdots\,\odot\, e_d^{i_d} \;=\; s!\,v$$

so that

$$\mathcal{T}_s(f,g)(w) = \frac{(-1)^s s!}{(\nu)_s}\,v\,.$$

Hence

$$|\langle F(0),v\rangle|^2 = c^2(\nu)_s^2\cdot\frac{1}{(s!)^2}\,|H_F^s(f,g)|^2 \tag{21}$$

so that

$$|\langle F(0), v\rangle|^2 \leq C_{\nu,s}\|H_F^s\|^2\|f\|_\nu^2\|g\|_\nu^2 \leq C_{\nu,s}\|H_F^s\|^2\|v\|^2 \,. \tag{22}$$

Define

$$S(w) = (\pi_{\nu,s}(\phi_z)F)(w) = \left(\otimes^s \phi_z'(w)^t\right) F(\phi_z(w)) \left(J_{\phi_z}(w)\right)^{2\nu/(d+1)} \,.$$

Then $S : B \to \odot^s V'$ is holomorphic. Also by equations (13) and (15)

$$\|H_S^s\| = \|H_F^s\| < \infty \,,$$

so by (22) with F replaced by S

$$|\langle S(0), v\rangle|^2 \leq C\|H_S^s\|^2\|v\|^2 = C\|H_F^s\|^2\|v\|^2 \,. \tag{23}$$

Now

$$S(0) = \left(\otimes^s \phi_z'(0)^t\right) F(z) \left(J_{\phi_z}(0)\right)^{2\nu/(d+1)} \,.$$

Since $-\phi_z'(0)^t = s_z^2 P_{\bar{z}} + s_z Q_{\bar{z}} \geq 0$ then $\left(-\phi_z'(0)^t\right)^2 = B^t(z,z)$ and by the uniqueness of positive square root $B^t(z,z)^{1/2} = -\phi_z'(0)^t$. Thus

$$\begin{aligned} \left(\otimes^s B^t(z,z)\right)^{1/2} &= \otimes^s B^t(z,z)^{1/2} \\ &= (-1)^s \otimes^s \phi_z'(0)^t \,. \end{aligned}$$

Hence

$$S(0) = (-1)^{d+1+s}(1-|z|^2)^\nu \left(\otimes^s B^t(z,z)\right)^{1/2} F(z)$$

so that (23) becomes

$$\left|\left\langle F(z), \left(\otimes^s B^t(z,z)\right)^{1/2} v\right\rangle\right|^2 \leq C\|H_F^s\|^2 \left\|\left(\otimes^s B^t(z,z)\right)^{-1/2} v\right\|_z^2 (1-|z|^2)^{-2\nu} \,.$$

Observe that

$$\left\langle F(z), \left(\otimes^s B^t(z,z)\right)^{1/2} v\right\rangle = \left\langle F(z), \left(\otimes^s B^t(z,z)\right)^{-1/2} v\right\rangle_z$$

so the result follows from Riesz lemma, for the inner product $\langle\cdot,\cdot\rangle_z$. □

6 Compactness and Hilbert-Schmidt properties

6.1 Compactness

In this subsection we prove Theorem 1 (b).

Remark 5. Let $\{e_1, \dots, e_d\}$ be a basis for V'. Then we can write

$$F(z) = \sum_{i_1+\cdots+i_d=s} F_i(z)\, e_1^{i_1} \odot \cdots \odot e_d^{i_d}$$

where $i = (i_1, \dots, i_d)$ and $F_i : B \to \mathbb{C}$ are holomorphic. Also

$$F_i(z) = \sum_{m=0}^{\infty} p_m^{(i)}(z)$$

where $p_m^{(i)}$ are homogeneous holomorphic polynomials of degree m.

To prove the sufficiency of Theorem 1 (b) we need the following result.

Lemma 7. *Let $F : B \to \odot^s V'$ be holomorphic with the property*

$$\left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z)F(z), F(z) \right\rangle \to 0 \quad \textit{if} \quad |z| \nearrow 1\,.$$

Let $\varepsilon > 0$ be given. Then there exists a number r' with $0 < r' < 1$ and a natural number N such that

$$\|F - P_N\|_{\nu,s,\infty} < \varepsilon$$

where

$$P_N(z) = \sum_{|i|=s} \sum_{m=0}^{N} p_m^{(i)}(r'z)\, e_1^{i_1} \odot \cdots \odot e_d^{i_d}\,.$$

Remark 6. Remember that we have already defined

$$\|F\|_{\nu,s,\infty} = \sup_{z \in B} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z)F(z), F(z) \right\rangle^{1/2}$$

for holomorphic $F : B \to \odot^s V'$.

Remark 7. Let H_1 and H_2 be Hilbert spaces and let $A_1, B_1 : H_1 \to H_1$ and $A_2, B_2 : H_2 \to H_2$ be positive operators. Then

$$(A_1 - B_1) \otimes (A_2 + B_2) + (A_1 + B_1) \otimes (A_2 - B_2) = 2\,(A_1 \otimes A_2 - B_1 \otimes B_2)\ . \tag{24}$$

Thus it follows from (24) that

$$A_1 \geq B_1 \quad , \quad A_2 \geq B_2 \quad \Longrightarrow \quad A_1 \otimes A_2 \geq B_1 \otimes B_2\,. \tag{25}$$

Proof of Lemma 7. Let $\varepsilon > 0$ be given. Then there exists $0 < r_0 < 1$ such that

$$\sup_{r_0 < |z| < 1} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z)F(z), F(z) \right\rangle < \frac{\varepsilon^2}{32}\,.$$

Define $F_r(z) = F(rz)$ where $0 < r < 1$. Since $P_{r\bar{z}} = P_{\bar{z}}$ then

$$B^t(rz, rz) = (1 - r^2|z|^2)(I - r^2|z|^2 P_{r\bar{z}}) \geq B^t(z,z)$$

for all $0 < r < 1$. By (25) it then follows that

$$\otimes^s B^t(rz, rz) \geq \otimes^s B^t(z, z)$$

for all $0 < r < 1$. Hence,

$$\begin{aligned} &\left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) F_r(z), F_r(z) \right\rangle \\ &\qquad \leq \left\langle (1-|rz|^2)^{2\nu} \otimes^s B^t(rz,rz) F(rz), F(rz) \right\rangle . \end{aligned}$$

Then it follows from the inequalities

$$\begin{aligned} &\left\langle \otimes^s B^t(z,z) \left(F(z) - F_r(z)\right), F(z) - F_r(z) \right\rangle \\ &\qquad \leq \left\langle \otimes^s B^t(z,z) F(z), F(z) \right\rangle + \left\langle \otimes^s B^t(z,z) F_r(z), F_r(z) \right\rangle \\ &\qquad\qquad + 2 \left| \left\langle \otimes^s B^t(z,z) F(z), F_r(z) \right\rangle \right| \end{aligned}$$

and

$$\begin{aligned} &\left| \left\langle \otimes^s B^t(z,z) F(z), F_r(z) \right\rangle \right| \\ &\qquad \leq \left\langle \otimes^s B^t(z,z) F(z), F(z) \right\rangle^{1/2} \left\langle \otimes^s B^t(z,z) F_r(z), F_r(z) \right\rangle^{1/2} \end{aligned}$$

that, if $1 > r > r_1 = 2r_0/(1+r_0)$ and $R_0 = (1+r_0)/2$,

$$\sup_{R_0 < |z| < 1} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) \left(F(z) - F_r(z)\right), F(z) - F_r(z) \right\rangle < \frac{\varepsilon^2}{8} ,$$

since, if $r_1 < r < 1$,

$$\begin{aligned} &\sup_{R_0 < |z| < 1} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) F(rz), F(rz) \right\rangle \\ \leq\; &\sup_{R_0 r < |rz| < r} \left\langle (1-|rz|^2)^{2\nu} \otimes^s B^t(rz,rz) F(rz), F(rz) \right\rangle \\ \leq\; &\sup_{r_0 < |rz| < 1} \left\langle (1-|rz|^2)^{2\nu} \otimes^s B^t(rz,rz) F(rz), F(rz) \right\rangle < \frac{\varepsilon^2}{32} . \end{aligned}$$

As $F_r \to F$ uniformly, $r \to 1$, on every compact subset of B, there is a number r_2 such that if $r_2 < r < 1$, then

$$\sup_{|z| \leq R_0} \langle F(z) - F_r(z), F(z) - F_r(z) \rangle < \frac{\varepsilon^2}{8} .$$

Since $B^t(z,z) \leq (1-|z|^2) I \leq I$ then (25) yields $\otimes^s B^t(z,z) \leq \otimes^s I$ so that if $r_2 < r < 1$, then

$$\begin{aligned} &\sup_{|z| \leq R_0} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) \left(F(z) - F_r(z)\right), F(z) - F_r(z) \right\rangle \\ &\qquad \leq \sup_{|z| \leq R_0} \langle F(z) - F_r(z), F(z) - F_r(z) \rangle < \frac{\varepsilon^2}{8} . \end{aligned}$$

Hence for $\max(r_1, r_2) < r < 1$ it holds that

$$\|F - F_r\|^2_{\nu,s,\infty} \leq \sup_{|z|\leq R_0} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) \left(F(z) - F_r(z)\right), F(z) - F_r(z) \right\rangle$$

$$+ \sup_{R_0<|z|<1} \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) \left(F(z) - F_r(z)\right), F(z) - F_r(z) \right\rangle < \frac{\varepsilon^2}{4}\,.$$

Now, take r' such that $\max(r_1, r_2) < r' < 1$. The sum $\sum_{|i|=s} \sum_{m=0}^{\infty} p_m^{(i)}(r'z) e_1^{i_1} \odot \cdots \odot e_d^{i_d}$ converges uniformly to $F_{r'}(z)$ on B. Hence there exists a natural number N such that

$$\|F_{r'} - P_N\|^2_{\nu,s,\infty} \leq \sup_{z\in B} \langle F_{r'}(z) - P_N(z), F_{r'}(z) - P_N(z)\rangle < \frac{\varepsilon^2}{4}$$

where $P_N(z) = \sum_{|i|=s} \sum_{m=0}^{N} p_m^{(i)}(r'z)\, e_1^{i_1} \odot \cdots \odot e_d^{i_d}$. This yields

$$\|F - P_N\|_{\nu,s,\infty} \leq \|F - F_{r'}\|_{\nu,s,\infty} + \|F_{r'} - P_N\|_{\nu,s,\infty} < \varepsilon$$

which completes the proof of the lemma. □

Now we can prove the sufficiency.

Proof of sufficiency of Theorem 1 (b). Let $\varepsilon > 0$ be given. Then it follows from Lemma 7 that there is a P_N such that $\|F - P_N\|_{\nu,s,\infty} < \varepsilon$. Then the bilinear Hankel form $H^s_{F-P_N} = H^s_F - H^s_{P_N}$ with $F - P_N$ is bounded. In fact, the operator norm $\|\cdot\|$ satisfies

$$\|H^s_F - H^s_{P_N}\| \leq C\|F - P_N\|_{\nu,s,\infty} < C\varepsilon\,.$$

If we can prove that $H^s_{P_N}$ is compact then we are done. Actually we shall find that $H^s_{P_N}$ is of Hilbert-Schmidt class $\mathcal{S}_2$ and thus especially compact. By construction (see Lemma 7) P_N is a linear combination of terms $z^{\gamma'} e^{\gamma} = z^{\gamma'} e_1^{\gamma_1} \odot \cdots \odot e_d^{\gamma_d}$ so it is enough to prove that $H^s_{z^{\gamma'} e^{\gamma}} \in \mathcal{S}_2$. Consider

$$H^s_{z^{\gamma'} e^{\gamma}}(z^\alpha, z^\beta) = \int_B \left\langle \otimes^s B^t(w,w) \mathcal{T}_s(z^\alpha, z^\beta)(w), w^{\gamma'} e_1^{\gamma_1} \odot \cdots \odot e_d^{\gamma_d} \right\rangle d\iota_{2\nu}(w)\,.$$

First we observe that

$$\left\langle \otimes^s B^t(w,w) \mathcal{T}_s\left(z^\alpha, z^\beta\right)(w), w^{\gamma'} e_1^{\gamma_1} \odot \cdots \odot e_d^{\gamma_d} \right\rangle$$

is a linear combination of terms

$$\Bigg\langle \otimes^s B^t(w,w) \left(\partial_1^{i_1} \cdots \partial_d^{i_d}\right)(w^\alpha) \left(\partial_1^{j_1} \cdots \partial_d^{j_d}\right)(w^\beta)\; u_1 \otimes u_2 \otimes \cdots \otimes u_s,$$

$$w^{\gamma'} v_1 \otimes v_2 \otimes \cdots \otimes v_s \Bigg\rangle \quad (26)$$

where $u_1 \otimes \cdots \otimes u_s$ and $v_1 \otimes \cdots \otimes v_s$ contains $i_k + j_k$ copies and γ_k copies of e_k respectively. We may assume that $\alpha_k \geq i_k$ and $\beta_k \geq j_k$ for $k = 1, 2, \ldots d$. Denote $i = (i_1, \ldots, i_d)$ and $j = (j_1, \ldots, j_d)$. Then the term (26) equals

$$C_{i,j}(1 - |w|^2)^s w^{(\alpha+\beta)-(i+j)} \bar{w}^{\gamma'} \prod_{m=1}^{s} \left(\langle u_m, v_m \rangle - \langle u_m, \bar{w} \rangle \langle \bar{w}, v_m \rangle \right) .$$

But this term yields a nonzero integral only for those α and β with $|\alpha+\beta| \leq |\gamma'|+s$. Thus

$$\|H^s_{z^{\gamma'} e^{\gamma}}\|^2_{\mathcal{S}_2} = \sum_{\alpha,\beta} \frac{\left| H^s_{z^{\gamma'} e^{\gamma}} \left(z^\alpha, z^\beta \right) \right|^2}{\|z^\alpha\|^2_\nu \|z^\beta\|^2_\nu}$$

with a finite sum. Hence $H^s_{z^{\gamma'} e^{\gamma}} \in \mathcal{S}_2$ so that $H^s_{P_N} \in \mathcal{S}_2$. □

Now we prove the necessity.

Proof of the necessity of Theorem 1 (b). Let F be a symbol such that H^s_F is compact. Since $\odot^s V'$ is a finite dimensional Hilbert space we need only to prove that $\langle u_n, v \rangle \to 0$ as $n \to \infty$ where

$$u_n = \left((1 - |z_n|^2)^{2\nu} \otimes^s B^t(z_n, z_n) \right)^{1/2} F(z_n)$$

and $|z_n| \nearrow 1$ as $n \to \infty$, for any $v \in \odot^s V'$. As in the proof of the necessity of Theorem 1 (a) we write

$$v = \sum_{|i|=s} v_i \, e_1^{i_1} \odot e_2^{i_2} \odot \cdots \odot e_d^{i_d}$$

and let

$$f(w) = \sum_{|i|=s} w_1^{i_1} \cdots w_d^{i_s} \cdot v_i \quad \text{and} \quad g(w) = 1 \, .$$

So for any symbol S we have

$$|\langle S(0), v \rangle| = C_{\nu,s} |H^s_S(f, g)| \, ,$$

by the same arguments as for (21) in the proof of the necessity of Theorem 1 (a). Let

$$S(w) = \pi_{\nu,s} \left((\phi_{z_n}) \, F \right) (w) \otimes^s \phi'_{z_n}(w)^t F(\phi_{z_n}(w)) \left(J_{\phi_{z_n}}(w) \right)^{2\nu/(d+1)}$$

so that

$$S(0) = \otimes^s \phi'_{z_n}(0)^t F(z_n) \left(J_{\phi_{z_n}}(0) \right)^{2\nu/(d+1)} . \tag{27}$$

By Proposition 1,

$$J_{\phi_{z_n}}(0) = (-1)^d (1 - |z_n|^2)^{(d+1)/2} \quad \text{and} \quad B^t(z_n, z_n)^{1/2} = -\phi'_{z_n}(0)^t$$

so that

$$|\langle S(0), v\rangle| = |\langle u_n, v\rangle| \,. \tag{28}$$

On the other hand

$$H_S^s(f,g) = H_F^s\left(f\circ\phi_{z_n}\cdot J_{\phi_{z_n}}^{\nu/(d+1)}, k_{z_n}\right)$$

where

$$k_{z_n}(w) = (g\circ\phi_{z_n})(w)\left(J_{\phi_{z_n}}(w)\right)^{\nu/(d+1)} = (-1)^d\frac{(1-|z_n|^2)^{\nu/2}}{(1-\langle w, z_n\rangle)^\nu}$$

so that $k_{z_n}(w)\to 0$ weakly as $n\to\infty$ and $\|k_{z_n}\|_\nu = 1$. Since H_F^s is compact then there is a sequence $\{c_n\}_{n=0}^\infty$ of positive numbers such that $c_n\to 0$ and

$$|H_F^s(h, k_{z_n})| \le c_n\|h\|_\nu$$

for all $h\in L_a^2(d\iota_\nu)$. Let $h = f\circ\phi_{z_n}\cdot J_{\phi_{z_n}}^{\nu/(d+1)} = \pi_\nu\left(\phi_{z_n}\right)f$ which yields

$$\|h\|_\nu^2 = \|f\|_\nu^2 \,.$$

Thus

$$|\langle u_n, v\rangle| \le C_{\nu,s}c_n\|f\|_\nu \le C'_{\nu,s}c_n\|v\|$$

so that $\langle u_n, v\rangle\to 0$ as $n\to\infty$, which, combined with the equalities (27) and (28), implies that

$$\left\langle (1-|z_n|^2)^{2\nu}\otimes^s B^t(z_n,z_n)F(z_n), F(z_n)\right\rangle \to 0 \quad \text{as} \quad |z_n|\nearrow 1 \,.$$

□

6.2 Hilbert-Schmidt properties

In this subsection we prove Theorem 2. Denote by $\mathcal{H}'_{\nu,s}$ the space of all holomorphic functions $F : B\to \odot^s V'$ such that the corresponding bilinear Hankel form on $L_a^2(d\iota_\nu)\otimes L_a^2(d\iota_\nu)$

$$H_F^s(f,g) = \int_B \left\langle \otimes^s B^t(z,z)\mathcal{T}_s(f,g)(z), F(z)\right\rangle\, d\iota_{2\nu}(z)$$

is of Hilbert-Schmidt class $\mathcal{S}_2$. By Proposition 8, it is a Hilbert space with an inner product $\langle F, S\rangle'_{\nu,s} = \langle H_F^s, H_S^s\rangle_{\mathcal{S}_2}$ where

$$\langle H_F^s, H_S^s\rangle_{\mathcal{S}_2} = \sum_{|\alpha|=0}^\infty\sum_{|\beta|=0}^\infty H_F^s(e_\alpha, e_\beta)\overline{H_S^s(e_\alpha, e_\beta)}$$

and $e_\alpha = z^\alpha/\|z^\alpha\|_\nu$.

Lemma 8. *The space $\mathcal{H}'_{\nu,s}$ is a Hilbert space.*

Proof. Let $\{F_n\}_{n=0}^{\infty}$ be a Cauchy sequence in $\mathcal{H}'_{\nu,s}$. Then $\{H^s_{F_n}\}_{n=0}^{\infty}$ is Cauchy in operator norm so that $\{F_n\}_{n=0}^{\infty}$ is Cauchy in $\|\cdot\|_{\nu,s,\infty}$. Then there is a $F \in \mathcal{H}^{\infty}_{\nu,s}$ such that $F_n \to F$ in $\|\cdot\|_{\nu,s,\infty}$. Thus $H^s_{F_n} \to H^s_F$ in operator norm. On the other hand, the space of all bilinear forms of Hilbert-Schmidt class $\mathcal{S}_2$ is a Hilbert space so that $H^s_{F_n} \to H \in \mathcal{S}_2$ in $\|\cdot\|_{\mathcal{S}_2}$. Then $H^s_{F_n} \to H$ in operator norm so that $H = H^s_F$. Thus $F \in \mathcal{H}'_{\nu,s}$ and $F_n \to F$ in $\|\cdot\|_{\mathcal{S}_2}$. □

We now shall see that $\mathcal{H}'_{\nu,s} = \mathcal{H}^2_{\nu,s}$, namely they are equal as sets and the norms are equivalent, as is shown below. Actually, Theorem 2 is a direct consequence of Theorem 5.

Theorem 5. *There is a constant $C_{\nu,s} > 0$ such that*

$$\|F\|'_{\nu,s} = C_{\nu,s}\|F\|_{\nu,s,2}$$

for all holomorphic $F : B \to \odot^s V'$.

To prove Theorem 5 we need some lemmas.

Lemma 9. *Let $\{e_1, \dots, e_d\}$ be an orthonormal basis for V'. Then the spaces $\mathcal{H}'_{\nu,s}$ and $\mathcal{H}^2_{\nu,s}$ contains the element $e_1^s = e_1 \otimes \cdots \otimes e_1$.*

Proof. Clearly $e_1^s \in \mathcal{H}^2_{\nu,s}$. The fact that $e_1^s \in \mathcal{H}'_{\nu,s}$ follows from (26), letting $\gamma' = 0$ and $\gamma_j = s \cdot \delta_{1j}$ for $j = 1, \dots, d$. □

Lemma 10. *The action $\pi_{\nu,s}$, defined in (14), is unitary on both $\mathcal{H}'_{\nu,s}$ and $\mathcal{H}^2_{\nu,s}$.*

Proof. Clearly, $\pi_{\nu,s}$ is unitary on $\mathcal{H}^2_{\nu,s}$. That $\pi_{\nu,s}$ is also unitary on $\mathcal{H}'_{\nu,s}$ follows from Lemma 3 and the fact that π_ν, defined in (12), is unitary on $L^2_a(d\iota_\nu)$. □

Lemma 11. *The space $\mathcal{H}^2_{\nu,s}$ is irreducible with respect to the action $\pi_{\nu,s}$, defined in (14).*

Proof. Let $\mathcal{H}_0 \subset \mathcal{H}^2_{\nu.s}$ be invariant under the action $\pi_{\nu,s}(g)$, $g \in G$, and assume that $h \in \mathcal{H}_0$ for some $h \neq 0$. We may assume, by replacing h by an action of $\pi_{\nu,s}(g)$ on h if necessary, that $h(0) \neq 0$. We need to prove

$$f \in \mathcal{H}^2_{\nu,s} \quad , \quad f \perp \mathcal{H}_0 \quad \Longrightarrow \quad f = 0\,. \tag{29}$$

Take such an $f \in \mathcal{H}^2_{\nu,s}$. Since $e^{i\theta} : z \to e^{i\theta}z$ is in G and

$$\mathcal{H}_0 \ni \left(\pi_{\nu,s}(e^{i\theta})h\right)(z) = \left(e^{-i\theta d}\right)^{2\nu/(d+1)} \cdot e^{-i\theta s} \cdot h(e^{i\theta}z)$$

then $h(e^{i\theta}z) \in \mathcal{H}_0$. Hence, by the mean value property,

$$h(0) = \int_0^{2\pi} h(e^{i\theta}z)\,d\theta \in \mathcal{H}_0\,.$$

Then we have found a nonzero element in $\odot^s V'$ which is also contained in $\mathcal{H}_0$. Then $v \in \mathcal{H}_0$ for any $v \in \odot^s V'$ (by Theorem 12.2.8 in [Ru]). Then $[\pi_{\nu,s}(\phi_w)v](z) = c \cdot K(z,w)v$ is in $\mathcal{H}_0$, for any $v \in \odot^s V'$, where $K(\cdot,w)$ is the reproducing kernel for $\mathcal{H}^2_{\nu,s}$ and c is a nonzero constant. Hence

$$f \perp K(\cdot\,, w)v$$

so that

$$f(w) = 0 \quad \text{for all} \quad w \in B$$

by the reproducing property. This proves (29). □

Now we can prove Theorem 5.

Proof of Theorem 5. As a consequence of Theorem VI.23 in [RS] we can make the following identification of the space $\mathcal{S}_2(L^2_a(d\iota_\nu), L^2_a(d\iota_\nu))$ of Hilbert-Schmidt bilinear forms on $L^2_a(d\iota_\nu)$ with the tensor product, that is,

$$\mathcal{S}_2\left(L^2_a(d\iota_\nu), L^2_a(d\iota_\nu)\right) = L^2_a(d\iota_\nu) \otimes L^2_a(d\iota_\nu)\,.$$

Moreover $L^2_a(d\iota_\nu) \otimes L^2_a(d\iota_\nu)$ can be decomposed into irreducible subspaces $\tilde{\mathcal{H}}_{\nu,s}$ of Hankel forms of weight s with an intertwining operator $T : \mathcal{H}^2_{\nu,s} \to \tilde{\mathcal{H}}_{\nu,s}$, (see [HLZ]). Also, H_F defined in (7) is a Hankel form of weight s and by Lemma 9 there is a nonzero element in $\mathcal{H}^2_{\nu,s}$ which yields a nonzero element in $\mathcal{H}'_{\nu,s}$. Thus

$$\mathcal{H}'_{\nu,s} = \mathcal{H}^2_{\nu,s}$$

whose norms are the same up to a constant, by Corollary 8.13 in [K]. □

7 Schatten-von Neumann properties

In this section we prove Theorem 3. Let $L^p_{\nu,s}$, for $2 \le p < \infty$, be the space of measurable functions $S : B \to \odot^s V'$ such that

$$\|S\|_{\nu,s,p} = \left(\int_B \langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) S(z), S(z)\rangle^{p/2}\, d\iota(z)\right)^{1/p} < \infty\,.$$

Then $L^p_{\nu,s}$ is a Banach space and $\mathcal{H}^p_{\nu,s}$ is a closed subspace of $L^p_{\nu,s}$. Also, the spaces $\mathcal{A}_1 = L^2_{\nu,s} + L^\infty_{\nu,s}$ and $\mathcal{A}_2 = \mathcal{H}^2_{\nu,s} + \mathcal{H}^\infty_{\nu,s}$ are Banach spaces with the norms

$$\|F\|_{\mathcal{A}_i} = \inf\left\{\|F_2\|_{\nu,s,2} + \|F_\infty\|_{\nu,s,\infty} \,:\, F = F_2 + F_\infty \in \mathcal{A}_i\right\},$$

$i = 1, 2$, respectively, by Lemma 2.3.1 in [BL]. Denote by $\mathcal{F}_i = \mathcal{F}(\mathcal{A}_i)$, $i = 1, 2$, the space of all functions with values in $\mathcal{A}_i$, which are bounded and continuous on the strip

$$S = \{z \in \mathbb{C} : 0 \le \Re z \le 1\}$$

and holomorphic on the open strip

$$S_0 = \{z \in \mathbb{C} : 0 < \Re z < 1\}$$

and moreover, the functions $t \to f(j+it)$ are continuous functions from the real line into $L^2_{\nu,s}$, $L^\infty_{\nu,s}$ and $\mathcal{H}^2_{\nu,s}$, $\mathcal{H}^\infty_{\nu,s}$ respectively, which tends to zero as $|t| \to \infty$. Then $\mathcal{F}_i$, $i = 1, 2$, are Banach spaces with the same norm

$$\|f\|_{\mathcal{F}} = \max\left(\sup \|f(it)\|_{\nu,s,2}\, ,\, \sup \|f(1+it)\|_{\nu,s,\infty}\right),$$

by Lemma 4.1.1 in [BL]. Now let $0 < \theta < 1$ and denote by $\left(L^2_{\nu,s}, L^\infty_{\nu,s}\right)_{[\theta]}$ and $\left(\mathcal{H}^2_{\nu,s}, \mathcal{H}^\infty_{\nu,s}\right)_{[\theta]}$ the space of all $S \in \mathcal{A}_i$ such that

$$\|S\|_{i,[\theta]} = \inf\left\{\|f\|_{\mathcal{F}} \,:\, f(\theta) = S\, ,\, f \in \mathcal{F}_i\right\} < \infty\,,$$

$i = 1, 2$, respectively. As a direct consequence, the space $\left(\mathcal{H}^2_{\nu,s}, \mathcal{H}^\infty_{\nu,s}\right)_{[1/p]}$ consists of holomorphic functions and

$$\left(\mathcal{H}^2_{\nu,s}, \mathcal{H}^\infty_{\nu,s}\right)_{[1/p]} \subset \left(L^2_s, L^\infty_s\right)_{[1/p]}$$

for $2 < p < \infty$.

If we claim that

$$\left(L^2_{\nu,s}, L^\infty_{\nu,s}\right)_{[1/p]} = L^p_{\nu,s} \quad , \quad 2 < p < \infty\,, \tag{30}$$

then we have the following lemma.

Lemma 12. *If* $2 < p < \infty$*, then*

$$\left(\mathcal{H}^2_{\nu,s}, \mathcal{H}^\infty_{\nu,s}\right)_{[1/p]} \subset \mathcal{H}^p_{\nu,s}\,.$$

The identity (30) can be proved by slightly modifying Theorem 5.1.1 in [BL] using

$$\|F\|_{\nu,s,p} = \sup\left\{\left|\int_B \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z)F(z), S(z)\right\rangle d\iota(z)\right| :\right.$$
$$\left. S \text{ bounded with compact support}\ ,\ \|S\|_{\nu,s,q} = 1\right\} \tag{31}$$

where $1/p + 1/q = 1$. Indeed, to prove (31) let $F : B \to \odot^s V'$ be measurable. Then

$$H = \left((1-|\cdot|^2)^{2\nu} \otimes^s B^t(\cdot,\cdot)\right)^{1/2} F : B \to \odot^s V'$$

is measurable and we may write $H = (H_1, \ldots, H_N)$, where $\dim(\odot^s V') = N$. For $1 \le j \le N$ we can find bounded functions b^j_n with compact support in B such that $|b^j_n| \nearrow |H_j|$. Let

$$s^j_n = |b^j_n| \cdot e^{iArgH_j}\,.$$

Then s_n^j are bounded with compact support and

$$H_j \cdot \overline{s_n^j} = |H_j| \cdot |b_n^j| .$$

Let $s_n = (s_1^1, \ldots, s_n^N)$ and put

$$t_n(z) = \left((1-|z|^2)^{2\nu} \otimes^s B^t(z,z)\right)^{-1/2} s_n(z) .$$

Then $t_n : B \to \odot^s V'$ is measurable and

$$\left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) t_n(z), t_n(z)\right\rangle = \sum_{j=1}^N s_n^j(z) \cdot \overline{s_n^j(z)} = \sum_{j=1}^N |b_n^j(z)| \cdot |b_n^j(z)|$$

$$\leq \sum_{j=1}^N |H_j(z)| \cdot |b_n^j(z)| = \sum_{j=1}^N H_j(z) \cdot \overline{s_n^j(z)} = \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) F(z), t_n(z)\right\rangle . \tag{32}$$

Now, let

$$S_n(z) = \frac{\left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) t_n(z), t_n(z)\right\rangle^{(q-2)/2} \cdot t_n(z)}{\|t_n\|_{\nu,s,q}^{q-1}} .$$

Then $S_n : B \to \odot^s V'$ is measurable,

$$\|S_n\|_{\nu,s,q} = 1 \quad \text{and} \quad \int_B \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) S_n(z), t_n(z)\right\rangle d\iota(z) = \|t_n\|_{\nu,s,p}$$

so by (32)

$$\begin{aligned} \|F\|_{\nu,s,p} \leq \underline{\lim}\, \|t_n\|_{\nu,s,p} &= \underline{\lim} \int_B \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) S_n(z), t_n(z)\right\rangle d\iota(z) \\ &\leq \underline{\lim} \int_B \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) S_n(z), F(z)\right\rangle d\iota(z) \leq M_p(F) \end{aligned}$$

where

$$M_p(F) = \sup \left\{ \left| \int_B \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) F(z), S(z)\right\rangle d\iota(z) \right| : \right. \\ \left. S \text{ bounded with compact support} \, , \ \|S\|_{\nu,s,q} = 1 \right\} .$$

On the other hand

$$\left| \int_B \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) F(z), G(z)\right\rangle d\iota(z) \right| \leq \|F\|_{\nu,s,p} \cdot \|S\|_{\nu,s,q}$$

which proves (31). The rest is almost the same as in [BL] loc. cit., only replacing the usual absolute value $|g(z)|$ of scalar functions $g(z)$ by the norm $\|S(z)\|_z =$

$\| \left((1-|z|^2)^{2\nu} \otimes^s B^t(z,z)\right)^{1/2} S(z)\|$ of vector-valued functions $S(z)$, also $E(z) = \langle f(z), g(z)\rangle$ by

$$H(z) = \int_B \left\langle (1-|z|^2)^{2\nu} \otimes^s B^t(z,z) F(z), \overline{S(z)} \right\rangle d\iota(z)\,.$$

Also, by Theorem 1 (a), Theorem 2 and Theorem 2.10 in [S],

$$H_F^s \in \mathcal{S}_p = (\mathcal{S}_2, \mathcal{S}_\infty)_{[1/p]} \quad \Longleftrightarrow \quad F \in \left(\mathcal{H}_{2\nu,s}, L^\infty_{a,s}\right)_{[1/p]} \tag{33}$$

if $2 < p < \infty$. Then Lemma 12 together with (33) yield Theorem 3.

8 Further work

In the previous section we proved a necessary condition for the Hankel forms to be in Schatten-von Neumann class $\mathcal{S}_p$, $2 < p < \infty$. A natural question is to ask whether this condition is sufficient or not. This seems harder to answer at the moment.

Another problem is whether there is an atomic decomposition for the space of symbols or not. If we can find one, then it could give rise to sufficient and necessary conditions for the Hankel forms to be in trace class $\mathcal{S}_1$, see [Pe1] and [R2].

Also, one might be interested in finding a Kronecker theorem for Hankel forms of higher weights, see [R1].

References

[BL] J. Bergh and J. Löfström, *Interpolation spaces. An introduction. Grundlehren der Mathematischen Wissenschaften*, No. 223, Springer-Verlag, Berlin-New York. 1976.

[BD] T. Bröcker and T. Dieck, *Representations on Compact Lie Groups*, Springer-Verlag, 1985.

[FK] J. Faraut and A. Koranyi, *Analysis on Symmetric Cones*, Clarendon Press, 1994.

[FR] S. Fergusson and R. Rochberg, *Higher order Hilbert-Schmidt Hankel forms and tensors of analytic kernels*, preprint, 2003.

[HLZ] S. Hwang, Y. Liu and G. Zhang, *Hilbert spaces of tensor-valued holomorphic functions on the unit ball of* $\mathbb{C}^n$, Pacific J. Math, 2004.

[JP] S. Janson and J. Peetre, *A new generalization of Hankel operators (the case of higher weights)*, Math. Nachr. **132** (1987), 313-328.

[JPR] S. Janson, J. Peetre and R. Rochberg, *Hankel forms and the Fock space*, Revista Mat. Iberoamer. **3** (1986), 61-138.

[K] A. Knapp, *Representation Theory of Semisimple Groups*, Princeton University Press, Princeton, New Jersey (1986).

[L] O. Loos, *Bounded symmetric domains and Jordan pairs*, The University of California at Irvine, 1977.

[P1] J. Peetre, *Hankel kernels of higher weight for the ball*, Nagoya Math. J. **130** (1993), 183-192.

[P2] J. Peetre, *Hankel forms of arbitrary weight over a symmetric domain via the Transvectant*, Rocky Mountain J. Math. **24** (1994), no. 3, 1065-1085.

[Pe1] V. V. Peller, *Hankel operators and their applications*, Springer-Verlag, 2002.

[Pe2] V. V. Peller, *Vectorial Hankel operators, commutators and related operators of the Schatten-von Neumann class* γ_p, Integral Equations Operator Theory **5** (1982), no. 2, 244-272.

[PZ] L. Peng and G. Zhang, *Tensor product of holomorphic representations and bilinear differential operators*, J. Funct. Analysis, to appear.

[RS] M. Reed and B. Simon, *Methods of modern mathematical physics I: Functional analysis*, Academic press, 1980.

[R1] R. Rochberg, *A Kronecker theorem for higher order Hankel forms*, Proc. Amer. Math. Soc. **123** (1995), no. 10, 3113-3118.

[R2] R. Rochberg, *Trace ideal criteria for Hankel operators and commutators*, Indiana Univ. Math. J. **31** (1982), no. 6, 913-925.

[Ro] H. Rosengren, *Multilinear Hankel forms of higher order and orthogonal polynomials*, Math. Scand. **82** (1998), 53-88.

[Ru] W. Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$, Springer-Verlag, 1980.

[S] B. Simon, *Trace ideals and their applications*, Cambridge University Press, Cambridge-London-New York-Melbourne, 1979.

[Zh] K. Zhu, *Operator theory in function spaces*, Marcel Dekker, Inc., New York, 1990.