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Abstract

We classify the $*$-representation types for the radical $*$-doubles of finite-dimensional associative algebras over the field of complex numbers.

1 Introduction

Let $\mathbb{C}$ denote the field of complex numbers and $\bar{\cdot} : \mathbb{C} \to \mathbb{C}$ denote the complex conjugation. All algebras we consider in the present paper are over $\mathbb{C}$ and are assumed to have a unit element. All tensor products and dimensions are taken over $\mathbb{C}$.

Recall that for two complex associative algebras A and B the map $\varphi : A \to B$ is called an *anti-homomorphism* provided that $\varphi(\lambda a + \mu b) = \bar{\lambda}\varphi(a) + \bar{\mu}\varphi(b)$ and $\varphi(ab) = \varphi(b)\varphi(a)$ for all $\lambda, \mu \in \mathbb{C}$ and $a, b \in A$.

Let A be a finite-dimensional algebra with n generators. Consider two free associative $\mathbb{C}$-algebras $\mathcal{A}_n^{(\varepsilon)}$, $\varepsilon = 1, 2$, with respective generators $x_1^{(\varepsilon)}, \dots, x_n^{(\varepsilon)}$. Denote by $\sigma : \mathcal{A}_n^{(1)} \to \mathcal{A}_n^{(2)}$ the unique anti-homomorphism satisfying $\sigma(x_j^{(1)}) = x_j^{(2)}$ for all $j = 1, \dots, n$. Let I be an ideal of $\mathcal{A}_n^{(1)}$, such that $A \simeq \mathcal{A}_n^{(1)}/I$. Then the set $\sigma(I)$ is an ideal in $\mathcal{A}_n^{(2)}$ and we can consider the algebra $A^* = \mathcal{A}_n^{(2)}/\sigma(I)$. It is easy to see that A^* does not depend on the presentation of A up to an isomorphism.

Construct now a new algebra, $A(*)$, which is the quotient of the free product $\mathcal{A}_n^{(1,2)}$ of $\mathcal{A}_n^{(1)}$ and $\mathcal{A}_n^{(2)}$ (i.e., the free algebra with $2n$ generators $x_1^{(1)}, \dots, x_n^{(1)}$, $x_1^{(2)}, \dots, x_n^{(2)}$), modulo the ideal J, which is generated by I and $\sigma(I)$. The algebra $A(*)$ is identified with the free product (over $\mathbb{C}$) of A and A^* in a natural way. The algebra $\mathcal{A}_n^{(1,2)}$ possesses a natural $*$-structure, defined by $(x_j^{(1)})^* = x_j^{(2)}$, $j = 1, \dots, n$, and one sees that J is a $*$-ideal with respect to this structure. Hence, $A(*)$ inherits a $*$-structure and the corresponding $*$-algebra is called the $*$-*double of* A, see [MT]. It is easy to see that, up to a $*$-isomorphism, the algebra $A(*)$ does not depend on the presentation of A. The $*$-representation types of $*$-doubles of finite-dimensional algebras were classified in [MT]. It was shown that $A(*)$ is $*$-finite if and only if $A \cong \mathbb{C}$, $A(*)$ is of type $\mathbf{I}$ if and only if $\dim(A) \leq 2$, and $A(*)$ is $*$-wild (in the sense of [OS2]) in all other cases.

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In the present paper we study the $*$-representation types of a more subtle construction, which we call the radical $*$-doubling. The difference is that for the usual $*$-doubling we add independent $*$-adjoints to all elements of the original algebra, whereas for the radical $*$-doubling we add independent $*$-adjoints only to the elements from the Jacobson radical of the algebra, preserving the natural $*$-structure on a maximal semi-simple subalgebra. We classify the $*$-representation type for the radical $*$-doubles of all finite-dimensional algebras, and the answer we obtain is much more interesting than that of [MT]. The principal advantage of the new construction is that the $*$-representation type of the radical $*$-doubles happens to be a Morita invariant of the original algebra. The list of those $\mathcal{A}$, whose radical $*$-doubles are of type $\mathbf{I}$, is also much more interesting and contains all semi-simple algebras and all finite-dimensional algebras, the length of indecomposable modules over which is bounded by 2. As a consequence we also obtain a tame-wild dichotomy for our problem (which is not automatic in the $*$-case in contrast with the usual finite-dimensional associative algebras, for which it was proved by Drozd, [Dr], in a very general setup). Some analogous problems were earlier considered in [Be1, Be2, Se], see also [OS2] and the references therin.

The paper is organized as follows: in the next section we present a rigorous definition of the radical $*$-double of a finite-dimensional algebra, in Section 3 we recall basic facts about the $*$-representation types, in Section 4 we formulate our main result, which classifies the $*$-representation types of the radical $*$-doubles of finite-dimensional algebras. The rest of the paper is devoted to the proof of the main result, which is spread over three sections. In Section 5 we collected several auxiliary lemmas classifying the $*$-representation types of the radical $*$-doubles of certain finite-dimensional algebras. In Section 6 we establish the Morita invariance of the $*$-representation types of the radical $*$-doubles. The latter study has led us to a very interesting question, which seems to be both quite natural and rather nontrivial: describe, up to unitary equivalence, all projections in the full matrix algebra $M_n(C^*(\mathcal{F}_2))$, where $\mathcal{F}_2$ is a free group with 2 generators. The answer to this question would substantially clarify the notion of $*$-wildness in the sense of [OS2], see Remark 4. Finally, the proof of our main result is completed in Section 7.

2 Radical $*$-doubling

In this section we give a thorough definition of the intuitive construction of the radical $*$-doubling, described in Section 1.

Let A be a finite-dimensional associative complex algebra, S be a maximal semi-simple subalgebra of A and Rad (A) be the Jacobson radical of A. Then A decomposes, as a complex vector space, into a direct sum $A = S \oplus \text{Rad }(A)$. Note that A is not isomorphic to the direct sum $S \oplus \text{Rad }(A)$ of associative algebras.

Being semi-simple, the algebra S admits a decomposition into a direct sum of full matrix algebras $M_{n_i}(\mathbb{C})$, $n_i \in \mathbb{N}$, by the Wedderburn-Artin Theorem. Every $M_{n_i}(\mathbb{C})$ has a natural $*$-structure associated with the transposition of a matrix. In every $M_{n_i}(\mathbb{C})$ we can choose a standard basis, consisting of matrix units. Let $\{b_1, \dots, b_s\}$ be a list of all diagonal matrix units in all $M_{n_i}(\mathbb{C})$ (they are self-dual with respect to $*$, i.e. $b_i^* = b_i$, $i = 1, \dots, s$), and $\{c_1, \dots, c_t\}$ be a list of all upper triangular matrix units in all $M_{n_i}(\mathbb{C})$. Then $\{c_1^*, \dots, c_t^*\}$

will be a list of all lower triangular matrix units in all $M_{n_i}(\mathbb{C})$. Note that the basis, constructed above, is closed with respect to $*$.

Fix some basis, $\{a_1, \ldots, a_k\}$, in Rad (A), and let $\mathbb{B}$ denote the basis of A, formed as the union of the bases for S and Rad (A), which we have just fixed. For $x, y, z \in \mathbb{B}$ let $\alpha^z_{x,y} \in \mathbb{C}$ be the corresponding structural constant, i.e. for $x, y \in \mathbb{B}$ we have

$$xy = \sum_{z\in\mathbb{B}} \alpha^z_{x,y} z.$$

Denote by $A(Rad - *)$ the associative algebra, generated over $\mathbb{C}$ by the elements from $\mathbb{B} \cup \{a_1^*, \ldots, a_k^*\}$, subject to the following relations:

$$xy = \sum_{z\in\mathbb{B}} \alpha^z_{x,y} z, \qquad x, y \in \mathbb{B};$$

$$xy = \sum_{z\in\mathbb{B}} \overline{\alpha^z_{y^*,x^*}} z^*, \qquad x \in \mathbb{B} \setminus \{a_1, \ldots, a_k\}, y \in \{a_1^*, \ldots, a_k^*\};$$

$$xy = \sum_{z\in\mathbb{B}} \overline{\alpha^z_{y^*,x^*}} z^*, \qquad x \in \{a_1^*, \ldots, a_k^*\}, y \in \mathbb{B} \setminus \{a_1, \ldots, a_k\}.$$

We will call the algebra $A(Rad - *)$ the *radical $*$-double* of the algebra A. It is straightforward that $A(Rad - *)$ inherits a natural $*$-structure from that on $\mathbb{B} \cup \{a_1^*, \ldots, a_k^*\}$. It is an easy (but quite lengthy) exercise to show that, up to a $*$-isomorphism, $A(Rad - *)$ does not depend neither on the presentation of A, nor on the choice of S, nor on the choice of $\{a_1, \ldots, a_k\}$.

Both A and A^* are subalgebra of $A(Rad - *)$ in a natural way. However, in contrast with $A(*)$, $A(Rad - *)$ is no longer a free product of A and A^* over $\mathbb{C}$, but rather a free product of A and A^* over the "common subalgebra" S. Remark that S can be arbitrary semi-simple finite-dimensional algebra. In particular, S can be non-commutative. Neither is S central in A in general. However $A(Rad - *) \cong A(*)$ (as $*$-algebras) in the case when A is local and basic.

3 Basic definitions and facts about the $*$-representation types

In this section we list some notation and definitions related to $*$-wild and $*$-tame algebras. In this exposition we follow [KS2, OS2]. All $*$-algebras considered here are unital with the unit 1 and representations of $*$-algebras are unital $*$-homomorphisms into $B(H)$, the $*$-algebra of all linear bounded operators on a separable Hilbert space H. For a $*$-algebra, $\mathcal{A}$, we denote by Rep $(\mathcal{A})$ the category of all $*$-representation of $\mathcal{A}$. Given a $*$-algebra, $\mathcal{A}$, of operators on H, denote by $\mathcal{A}'$ its commutant, i.e. $\mathcal{A}' = \{C \in B(H) \mid [C, A] = 0 \text{ for every } A \in \mathcal{A}\}$.

Definition 1. Let $\mathcal{A}$ be a $*$-algebra. A pair, $(\tilde{\mathcal{A}}; \varphi\colon \mathcal{A} \to \tilde{\mathcal{A}})$, where $\tilde{\mathcal{A}}$ is a $*$-algebra and φ is a unital $*$-homomorphism, is called an *enveloping $*$-algebra* of the algebra $\mathcal{A}$ if for any

$*$-representation $\pi\colon \mathcal{A} \to B(H)$ of $\mathcal{A}$ there exists a unique $*$-representation $\tilde{\pi}\colon \tilde{\mathcal{A}} \to B(H)$ such that the diagram

$$\begin{array}{ccc} \tilde{\mathcal{A}} & & \\ {\scriptstyle\varphi}\uparrow & \overset{\tilde{\pi}}{\searrow} & \\ \mathcal{A} & \xrightarrow{\pi} & B(H) \end{array}$$

is commutative, and any operator $X : H_1 \to H_2$ which intertwines representations $\pi_1 : \mathcal{A} \to B(H_1)$ and $\pi_2 : \mathcal{A} \to B(H_2)$ of $\mathcal{A}$ is also an intertwining operator for the representations $\tilde{\pi}_1$ and $\tilde{\pi}_2$ of the algebra $\tilde{\mathcal{A}}$.

It is easy to see that $(\mathcal{A}; \mathrm{Id}\colon \mathcal{A} \to \mathcal{A})$ is an enveloping $*$-algebra of $\mathcal{A}$.

Let $M_n(\mathcal{A})$ $(= M_n(\mathbb{C}) \otimes \mathcal{A})$ be the full matrix algebra over $\mathcal{A}$ with the natural $*$-structure. If $\mathcal{A}$ is a C^*-algebra then $M_n(\mathcal{A})$ carries also the structure of a C^*-algebra. Any representation $\pi : \mathcal{A} \to B(H)$ of $\mathcal{A}$ induces the representation $\pi_n\colon M_n(\mathcal{A}) \to B(H \oplus \ldots \oplus H)$ of the algebra $M_n(\mathcal{A})$. The representation π_n determines the representation $\tilde{\pi}_n$ of an enveloping algebra, $(\tilde{M}_n(\mathcal{A}), \varphi)$, of $M_n(\mathcal{A})$ on the same Hilbert space. If ψ is a unital $*$-homomorphism of a $*$-algebra $\mathcal{B}$ to the algebra $\tilde{M}_n(\mathcal{A})$ then $\tilde{\pi}_n \circ \psi$ defines a representation of $\mathcal{B}$. So we can define a functor, $F_\psi\colon \mathrm{Rep}\ (\mathcal{A}) \to \mathrm{Rep}\ (\mathcal{B})$, in the following natural way:

- $F_\psi(\pi) = \tilde{\pi}_n \circ \psi$, for every $\pi \in \mathrm{Rep}\ (\mathcal{A})$,
- $F_\psi(c) = diag(c, \ldots, c)$ for a morphism, $c : \pi_1 \to \pi_2$, of representations π_1, π_2 of $\mathcal{A}$.

Definition 2. We say that a $*$-algebra, $\mathcal{B}$, *majorizes* a $*$-algebra, $\mathcal{A}$, denoted by $\mathcal{B} \succ \mathcal{A}$, if there exist $n \in \mathbb{N}$, an enveloping algebra, $\tilde{M}_n(\mathcal{A})$, of the algebra $M_n(\mathcal{A})$, and a $*$-homomorphism, $\psi\colon \mathcal{B} \to \tilde{M}_n(\mathcal{A})$, such that the functor $F_\psi\colon \mathrm{Rep}\ (\mathcal{A}) \to \mathrm{Rep}\ (\mathcal{B})$ is full.

We say that $\mathcal{B}$ *strongly majorizes* $\mathcal{A}$ ($\mathcal{B} \succ^s \mathcal{A}$) if there exist $n \in \mathbb{N}$ and a $*$-homomorphism, $\psi\colon \mathcal{B} \to M_n(\mathcal{A})$, such that the functor $F_\psi\colon \mathrm{Rep}\ (\mathcal{A}) \to \mathrm{Rep}\ (\mathcal{B})$ is full.

Note that to define strong majorization we consider $M_n(\mathcal{A})$ as an enveloping $*$-algebra of $M_n(\mathcal{A})$.

Clearly, $\mathcal{B} \succ^s \mathcal{A} \Rightarrow \mathcal{B} \succ \mathcal{A}$. Note that both the majorization and the strong majorization are quasi-order relations: $\mathcal{C} \succ \mathcal{B}$ and $\mathcal{B} \succ \mathcal{A}$ imply $\mathcal{C} \succ \mathcal{A}$, and $\mathcal{C} \succ^s \mathcal{B}$ and $\mathcal{B} \succ^s \mathcal{A}$ imply $\mathcal{C} \succ^s \mathcal{A}$.

It follows easily from the definition that if $\mathcal{B} \succ \mathcal{A}$ then two representations π_1, π_2 of $\mathcal{A}$ are unitarily equivalent if and only if the representations $F_\psi(\pi_1)$, $F_\psi(\pi_2)$ of $\mathcal{B}$ are unitarily equivalent, a representation π of $\mathcal{A}$ is irreducible if and only if the representation $F_\psi(\pi)$ is irreducible. Thus the problem of unitary classification of the representations of the $*$-algebra $\mathcal{B}$ contains, as a subproblem, the problem of unitary classification of the representations of the $*$-algebra $\mathcal{A}$.

Practically, in order to verify that the functor F_ψ is full, it is sufficient to show that for each representation $\pi \in \mathrm{Rep}\ \mathcal{A}$ on H and $C \in B(H)$ the inclusion $C \in F_\psi(\pi)(\mathcal{B})'$ implies $C = diag(c, \ldots, c)$, where $c \in \pi(\mathcal{A})'$.

Let $\mathbb{C}[\mathcal{F}_2]$ denote the group $*$-algebra of the free group $\mathcal{F}_2$ with two generators, u, v, and involution defined on the generators in the usual way: $u^* = u^{-1}$, $v^* = v^{-1}$. Let $C^*(\mathcal{F}_2)$ be the full C^*-algebra of $\mathcal{F}_2$, i.e. the completion of $\mathbb{C}[\mathcal{F}_2]$ with respect to the norm

$$||a|| = \sup\{\pi(a) : \pi \in \text{Rep }(\mathbb{C}[\mathcal{F}_2])\}.$$

Definition 3. A $*$-algebra, $\mathcal{A}$, is called *$*$-wild* if $\mathcal{A} \succ C^*(\mathcal{F}_2)$. We say that $\mathcal{A}$ is *strongly $*$-wild* if $\mathcal{A}$ strongly majorizes the group $*$-algebra $\mathbb{C}[\mathcal{F}_2]$.

Clearly, any strongly $*$-wild algebra is $*$-wild. A motivation for such definition of $*$-wildness was a result proved in [KS1, KS2] saying that $C^*(\mathcal{F}_2)$ majorizes any finitely-generated $*$-algebra.

Since the majorization is a quasi-order, to prove that a $*$-algebra, $\mathcal{A}$, is $*$-wild it is enough to find some $*$-wild algebra which majorizes the algebra $\mathcal{A}$. One very important $*$-wild algebra, which we will frequently use in the paper, is the following. Let $\mathfrak{S}_2 = \mathbb{C}\langle a_1, a_2 \mid a_1 = a_1^*, a_2 = a_2^*\rangle$. Consider, for some fixed $0 < m < n$, the semi-norm $||a|| = ||a||_{m,n} = \sup \pi(a)$ on the $*$-algebra $\mathfrak{S}_2$, where the supremum is taken over all representation π of $\mathfrak{S}_2$ such that $mI \leq \pi(a_i) \leq nI$, $i = 1, 2$, I being the identity operator. Denote by

$$\mathfrak{C} = \mathfrak{C}_{m,n} = C^*(a_1, a_2 : m \leq a_i = a_i^* \leq n, i = 1, 2)$$

the C^*-algebra which is obtained by the completion of $\mathfrak{S}_2/(a : ||a|| = 0)$ with respect to $||\cdot||$. Clearly, the elements a_1 and a_2 become invertible in $\mathfrak{C}$ and positive in every bounded representation. The following statement was proved in [MT, Lemma 4], but the formulation there contained only the first part of the statement below.

Lemma 1. *The C^*-algebra $\mathfrak{C}$ is $*$-wild. Moreover, there exists a homomorphism $\psi : \mathfrak{C} \to M_4(C^*(\mathcal{F}_2))$ such that $\psi(a_i) \in M_4(\mathbb{C}[\mathcal{F}_2])$ and the corresponding functor F_ψ is full.*

Remark 1. From Lemma 1 it follows that a finitely generated $*$-algebra $\mathcal{A}$ is strongly $*$-wild if $\mathcal{A}$ majorizes $\mathfrak{C}$ and the corresponding homomorphism ψ is such that the image $\psi(\mathcal{A})$ is contained in $M_n(\langle a_1, a_2\rangle)$, where $\langle a_1, a_2\rangle$ is the (not completed) $*$-subalgebra of $\mathfrak{C}$.

Definition 4. A $*$-algebra is called *$*$-finite* if it has only finitely many irreducible representations up to unitary equivalence, and *$*$-tame* if it is of type **I** (see [Di, Chapter 9]) and not $*$-finite.

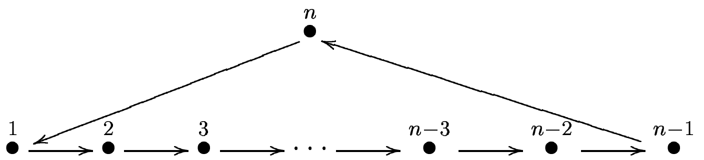
Remark 2. A finitely generated $*$-algebra, $\mathcal{A}$, is of type **I** if and only if for any irreducible representation π of the algebra $\mathcal{A}$ on a Hilbert space, $\mathcal{H}_\pi$, the operator closure $\overline{\pi(\mathcal{A})}$ contains a compact operator, and therefore contains all compact operators on $\mathcal{H}_\pi$ ([Di, Theorem 9.1,Corollary 4.1.10]). Clearly, if a $*$-algebra has only finite-dimensional irreducible representations, it is of type **I**.

4 Main Result

To formulate the main theorem we have to introduce the following notation: for every positive integer n we denote by A_n and $\tilde{A}_n$ respectively the path algebras of the quivers

$$A_n : \quad \overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \overset{3}{\bullet} \longrightarrow \cdots \longrightarrow \overset{n-2}{\bullet} \longrightarrow \overset{n-1}{\bullet} \longrightarrow \overset{n}{\bullet} \qquad \text{Rad}^{\,2}(A_n) = 0,$$

$\tilde{A}_n$:



Rad $^2(A_n) = 0$

modulo the relation that the radical square of the algebra is zero. In particular, the algebra A_1 is isomorphic to $\mathbb{C}$ and the algebra $\tilde{A}_1$ is isomorphic to $\mathbb{C}[x]/(x^2)$.

Theorem 1. *(I) Let A be a finite-dimensional indecomposable associative complex algebra and $A(Rad - *)$ be its radical $*$-double.*

1 $A(Rad - *)$ is $*$-finite if and only if A is simple if and only if $A \simeq M_n(\mathbb{C})$ for some n if and only if A is Morita equivalent to $\mathbb{C}$.

2 $A(Rad - *)$ is $*$-tame if and only if A is Morita equivalent to either A_n for some $n > 1$ or to $\tilde{A}_n$ for some n.

3 $A(Rad - *)$ is $*$-wild if and only if A is not Morita equivalent to any of A_n or $\tilde{A}_n$ for all n.

(II) Let A and B be two finite dimensional algebras.

1 $(A \oplus B)(Rad - *)$ is $*$-finite if and only if both $A(Rad - *)$ and $B(Rad - *)$ are $*$-finite.

2 $(A \oplus B)(Rad - *)$ is $*$-tame if and only if it is not $*$-finite and both $A(Rad - *)$ and $B(Rad - *)$ are either $*$-finite or $*$-tame.

3 $(A \oplus B)(Rad - *)$ is $*$-wild if and only if at least one of $A(Rad - *)$ and $B(Rad - *)$ is $*$-wild.

Remark 3. We remark that the algebras A_n and $\tilde{A}_n$ are precisely those finite-dimensional indecomposable algebras, which do not have indecomposable representations of length (= the number of simple subquotients) greater than 2. This is very well-known and can be proved for example using the following argument. Let A be a basic algebra, which does not have indecomposable representations of dimension greater than 2. Consider the quiver of A. First one shows that any vertex x of the quiver is a starting point of at most one arrow and is an ending point of at most one arrow, since otherwise the idempotent, representing x, and elements of A representing two different arrows starting from (ending at) x define a 3-dimensional indecomposable representation of A. This implies that the quiver of A is a disjoint union of quivers of A_n and $\tilde{A}_n$. If Rad $^2(A) \neq 0$ we get that there are two arrows, in the quiver, whose product is non-zero. With the idempotent, representing the common vertex, we again get a 3-dimensional indecomposable representation of A. This implies that A is a direct sum of algebras of type A_n and $\tilde{A}_n$.

Hence Theorem 1 can be reformulated as follows.

Corollary 1. *The radical $*$-double of an indecomposable associative complex finite-dimensional algebra A is $*$-finite or $*$-tame (that is of type* **I***) if and only if A does not have indecomposable representations of length greater than* 2.

5 Preparatory lemmas: tame and wild collections

Lemma 2. *The radical $*$-double of the algebra A_n and $\tilde{A}_n$ is $*$-tame.*

Proof. Let $e_1, \ldots, e_n$ be the orthogonal primitive idempotents of A_n (or $\tilde{A}_n$) corresponding to vertexes of the quiver A_n ($\tilde{A}_n$ respectively) and let $x_{i,i+1}$ be the element of A_n ($\tilde{A}_n$ respectively) which corresponds to the arrow $\overset{i}{\bullet} \longrightarrow \overset{i+1}{\bullet}$.

Let π be a non-zero irreducible representation of $A_n(Rad - *)$. Denoting $p_i = \pi(e_i)$, $X_i = \pi(x_{i,i+1})$, we have that $p_i \neq 0$ for some i, $X_i : p_iH \to p_{i+1}H$, $i = 1, \ldots, n-1$, and $X_i p_j H = 0$ if $j \neq i$. Choose the smallest i such that $p_i \neq 0$. If $i \neq 1$ we have $X_j = 0$ for any $j = 1, \ldots, i-1$. If $X_i = 0$, then any subspace $U \subset p_iH$ is invariant with respect to π and therefore π is one-dimensional. If $X_i \neq 0$, one can easily show that for any subspace $U \subset p_iH$ which is invariant with respect to $X_i^*X_i$, the direct sum $U \oplus X_iU$ is invariant with respect to π. Then using the fact that π is irreducible we get that U is necessarily one-dimensional and generated by an eigenvector of $X_i^*X_i$ and the representation π is two-dimensional. This shows that $A_n(Rad - *)$ is $*$-tame.

Since radical $*$-double of $\tilde{A}_1$ coincides with its $*$-double and $\tilde{A}_1 \simeq \mathbb{C}[x]/(x^2)$ we have, by [MT], that $\tilde{A}_1(Rad - *)$ is $*$-tame. Consider now $\tilde{A}_n(Rad - *)$, $n > 1$. Let π be its non-zero irreducible representation. Keeping the above notation we have that $X_{i-1}X_{i-1}^*X_i^*X_i = 0 = X_i^*X_iX_{i-1}X_{i-1}^*$ and therefore either $\ker X_{i-1}X_{i-1}^* = \ker X_{i-1}^*$ or $\ker X_i^*X_i = \ker X_i$ is non-zero for some i. If $\ker X_{i-1}^* \neq \{0\}$ (resp. $\ker X_i \neq \{0\}$) then this kernel is invariant with respect to $X_i^*X_i$ (resp. $X_{i-1}X_{i-1}^*$) and for any subspace $U \subset \ker X_{i-1}^*$ (resp. $U \subset \ker X_i$), which is invariant with respect to $X_i^*X_i$ (resp. $X_{i-1}X_{i-1}^*$), we have that $U \oplus X_iU$ (resp. $U \oplus X_{i-1}^*U$) is invariant with respect to π. Since π is irreducible, using the same arguments as above we conclude that U is one-dimensional and the representation π is one or two-dimensional. Therefore we have that $\tilde{A}_n(Rad - *)$ is $*$-tame for any n. $\square$

Lemma 3. *The radical $*$-double $\mathcal{A}$ of the quiver algebra $\bullet \xleftarrow{x} \bullet \circlearrowleft y$, with the relations $y^2 = xy = 0$ is strongly $*$-wild*

Proof. We let f to be the primitive idempotent, corresponding to the right point. The homomorphism $\psi : \mathcal{A} \to M_3(\mathfrak{C})$, defined by

$$\psi(f) = \begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \psi(y) = \begin{pmatrix} 0 & a_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \psi(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_2 & 0 \end{pmatrix}.$$

generates a full functor $F_\psi : \mathrm{Rep}(\mathfrak{C}) \to \mathrm{Rep}(\mathcal{A})$. In fact, let π be a representation of $\mathfrak{C}$. To prove that F_ψ is full it is enough to show that any operator $C = C^* = [c_{ij}]_{i,j=1}^3$, which intertwines the representation $\pi_3 \circ \psi$ of $\mathcal{A}$, is $diag(c, c, c)$, where c intertwines the representation π of $\mathfrak{C}$. If $[C, \pi_3(\psi(f))] = 0$ then $C = \begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ 0 & 0 & c_{33} \end{pmatrix}$. Taking into account that $\pi(a_i)$, $i = 1, 2$, are invertible one gets from $[C, \pi_3(\psi(y))] = 0$ that $c_{12} = c_{21} = 0$ and $c_{11}\pi(a_1) = \pi(a_1)c_{22}$, $c_{22}\pi(a_1) = \pi(a_1)c_{11}$. Since c_{11} and c_{22} are necessarily self-adjoint, we obtain from this that $c_{11}\pi(a_1)^2 = \pi(a_1)^2c_{11}$ and therefore, by the positivity of the operator $\pi(a_1)$, $c_{11}\pi(a_1) = \pi(a_1)c_{11}$. Thus we have $\pi(a_1)c_{22} = c_{11}\pi(a_1) = \pi(a_1)c_{11}$ and, using invertibility of $\pi(a_1)$, it yields $c_{11} = c_{22}$. Similarly, from $[C, \pi_3(\psi(x))] = 0$ we have $c_{22} = c_{33}$, giving $\mathcal{A} \succ \mathfrak{C}$. Since $\psi(\mathcal{A}) \subset \langle a_1, a_2 \rangle \subset \mathfrak{C}$, the $*$-algebra $\mathcal{A}$ is strongly $*$-wild by Remark 1. $\square$

Lemma 4. *The radical $*$-doubles of the following quiver algebras are strongly $*$-wild:*

(a) $\bullet \xrightarrow{x} \bullet \xrightarrow{y} \bullet$,

(b) $\bullet \xleftarrow{x} \bullet \xrightarrow{y} \bullet$,

(c) $\bullet \xrightarrow{x} \bullet \xleftarrow{y} \bullet$,

(d) $\bullet \underset{y}{\overset{x}{\rightleftarrows}} \bullet$, *with the relation* $xy = 0$,

(e) $\bullet \underset{y}{\overset{x}{\rightrightarrows}} \bullet$,

(f) $\bullet \xleftarrow{x} \bullet \circlearrowleft y$, *with the relation* $y^2 = 0$,

(g) $\bullet \xrightarrow{x} \bullet \circlearrowleft y$, *with the relation* $y^2 = 0$.

Proof. We shall only give homomorphisms ψ from the corresponding $*$-algebras, $\mathcal{A}$, to $M_n(\langle a_1, a_2 \rangle) \subset M_n(\mathfrak{C})$ which generate full functors $F_\psi : \mathrm{Rep}(\mathfrak{C}) \to \mathrm{Rep}(\mathcal{A})$. We denote by f_1, f_2 and f_3 the primitive idempotents for the quiver algebras, which correspond to the points, counted from the left.

(a)

$$\psi(f_1) = \begin{pmatrix} e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \psi(f_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\psi(x) = \begin{pmatrix} 0 & 0 & 0 \\ a_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \psi(y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_2 & 0 \end{pmatrix}.$$

Similar for the case (b) and (c).

(d)

$$\psi(f_1) = \begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \psi(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 & 0 & 0 \end{pmatrix}, \qquad \psi(y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Similar for the case (e).

(g) The strongly $*$-wild algebra from Lemma 3 is a factor-algebra of $\mathcal{A}$. Therefore $\mathcal{A}$ is strongly $*$-wild.

(f) is similar to Lemma 3 and (g). $\square$

Lemma 5. *1 A direct sum of $*$-algebras is $*$-finite if and only if all summands are $*$-finite.*

2 A direct sum of $*$-algebras is of type **I** *if and only if all summands are of type* **I**.

3 A direct sum of $*$-algebras is $*$-wild if and only if some of the summands is $*$-wild.

Proof. The first statement of the lemma is obvious. To prove the rest, it is certainly enough to consider the case $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$, where $\mathcal{A}_1$, $\mathcal{A}_2$ are $*$-algebras.

Let π be a representation of $\mathcal{A}$ and let $1_{\mathcal{A}_i}$ denote the unit in $\mathcal{A}_i$. Then $p = \pi(1_{\mathcal{A}_1} \oplus 0)$ is a self-adjoint projection commuting with any element of $\mathcal{A}$. Therefore $\pi = \pi_1 \oplus \pi_2$, where $\pi_1(a) = \pi(a)p$, $\pi_2(a) = \pi(a)(1-p)$, $a \in \mathcal{A}$. By [Di, 5.4.3], π is of type **I** if and only if both π_1 and π_2 are of type **I**. We have $\pi_1(a_1 \oplus a_2) = \pi(a_1)$ and $\pi_2(a_1 \oplus a_2) = \pi(a_2)$, $a_i \in \mathcal{A}_i$. If $\mathcal{A}_i$, $i = 1, 2$, are both of type **I**, the restrictions of π to each $\mathcal{A}_i$ are representations of type **I** and therefore the representation π itself also is of type **I**.

Let $\mathcal{A}$ be a type **I** algebra. Assuming that, say $\mathcal{A}_1$ is not of type **I**, we have that there exists a representation π_1 of $\mathcal{A}_1$ such that the von-Neumann algebra generated by $\pi_1(\mathcal{A}_1)$ is not of type **I**. Now, setting $\pi(a_1 \oplus a_2) = \pi_1(a_1)$, $a_i \in \mathcal{A}_i$, we get a non-type **I** representations of $\mathcal{A}$ giving a contradiction.

Assume that $\mathcal{A}$ is a $*$-wild. Let $\varphi : \mathcal{A} \to M_n(C^*(\mathcal{F}_2))$ be a $*$-homomorphism generating the full functor F_φ : Rep $(C^*(\mathcal{F}_2)) \to$ Rep $(\mathcal{A})$. Then $\pi(\varphi(A))' = \pi(M_n(C^*(\mathcal{F}_2))'$ for any representation $\pi \in$ Rep $(M_n(C^*(\mathcal{F}_2)))$ (see, for example, the proof of [OS2, Theorem 50]). By [OS2, Lemma 14], $\overline{\varphi(A)} = M_n(C^*(\mathcal{F}_2))$, where bar indicates the closure in the C^*-algebra $M_n(C^*(\mathcal{F}_2))$. It is well-known that $M_n(C^*(\mathcal{F}_2))$ is an irreducible algebra. In fact, it is well-known that $C^*(\mathcal{F}_2)$ has a faithful irreducible representation, π, (see, for example cite[Theorem VII.6.5]davidson), and hence so does $M_n(C^*(\mathcal{F}_2))$: $id \otimes \pi$. On the other hand, $\varphi(\mathcal{A}) = \varphi(\mathcal{A}_1 \oplus 0) \oplus \varphi(0 \oplus \mathcal{A}_2)$. Therefore, either $\varphi(\mathcal{A}_1 \oplus 0)$ or $\varphi(0 \oplus \mathcal{A}_2)$ is zero implying that either $\mathcal{A}_1$ or $\mathcal{A}_2$ is $*$-wild with the corresponding $*$-homomorphisms $\varphi_i : \mathcal{A}_i \to M_n(C^*(\mathcal{F}_2))$ defined via $\varphi_1(a_1) = \varphi(a_1 \oplus 0)$, $a_1 \in \mathcal{A}_1$, and $\varphi_2(a_2) = \varphi(0 \oplus a_2)$, $a_2 \in \mathcal{A}_2$, respectively. The converse statement is trivial. □

6 Preparatory lemmas: Morita equivalence

Let $\mathcal{A}$ be a $*$-algebra and let $1 = e_1 + e_2 + \ldots + e_n$ be a decomposition of the identity of the algebra $\mathcal{A}$ into a sum of pairwise orthogonal projections. Set $\mathcal{A}_{ij} = e_i \mathcal{A} e_j$ and consider a vector space $\mathcal{B} = \oplus_{i,j=1}^n \mathcal{B}_{ij}$ where $\mathcal{B}_{ij} = \mathbb{C}^{m_i} \otimes \mathcal{A}_{ij} \otimes \mathbb{C}^{m_j}$, $\{m_i\}$ are positive integers. We write elements of $\mathcal{B}$ as matrices (b_{ij}), $b_{ij} \in \mathcal{B}_{ij}$. $\mathcal{B}$ possesses an algebra structure: if $b_{ij} = f_i \otimes a_{ij} \otimes g_j$ and $c_{ij} = u_i \otimes d_{ij} \otimes w_j$ with $f_i, u_i \in \mathbb{C}^{m_i}$, $g_j, w_j \in \mathbb{C}^{m_j}$, $a_{ij}, d_{ij} \in \mathcal{A}_{ij}$, we define a product of (b_{ij}) and (c_{ij}) by

$$(b_{ij}) \cdot (c_{ij}) = (s_{ij}),$$
$$s_{ij} = \sum_k b_{ik} \cdot c_{kj}, \qquad b_{ik} \cdot c_{kj} = (g_k, \bar{u}_k) f_i \otimes a_{ik} v_{kj} \otimes w_j,$$

where bar indicates the complex conjugation and $(g_k, \bar{u}_k)$ is the scalar product in $\mathbb{C}^{m_k}$ of g_k, $\bar{u}_k$.

$\mathcal{B}$ is a $*$-algebra with involution defined as follows:

$$(b_{ij})^* = (b^*_{ji}), \quad b^*_{ji} = \bar{g}_i \otimes a^*_{ji} \otimes \bar{f}_j.$$

Taking a trivial decomposition of the identity ($n = 1$) we get $\mathcal{B} \simeq M_{m_1}(\mathbb{C}) \otimes \mathcal{A}$ with an isomorphism φ given by $\varphi(f_i \otimes a \otimes f_j) = e_{ij} \otimes a$, where $\{f_i\}$ is the standard basis in $\mathbb{C}^{m_1}$ and e_{ij} are the matrix units in $M_{m_1}(\mathbb{C})$. In general, considering in $M_{m_1+m_2+\ldots+m_n}(\mathcal{A})$ the projection $p = diag(e_1 \otimes I_{m_1}, e_2 \otimes I_{m_2}, \ldots, e_n \otimes I_{m_n})$, where I_{m_i} is the identity matrix in $M_{m_i}(\mathbb{C})$, we have $\mathcal{B} \simeq pM_N(\mathcal{A})p$, where $N = m_1 + m_2 + \ldots + m_n$.

Let ρ be a $*$-representation of $\mathcal{A}$ on H and set $q_i = \rho(e_i)$. ρ generates a $*$-representation $\Pi(\rho)$ of $\mathcal{B}$ on $\mathcal{H} = \oplus_{i=1}^n \mathcal{H}_i$, where $\mathcal{H}_i = \mathbb{C}^{m_i} \otimes q_i H$: if $h_i = v_i \otimes w_i$, with $v_i \in \mathbb{C}^{m_i}$ and $w_i \in q_i H$ and $b_{ij} = f_i \otimes a_{ij} \otimes g_j$,

$$\Pi(\rho)((b_{ij}))(h_1, h_2, \ldots, h_n) = (u_1, u_2, \ldots, u_n), \quad u_i = \sum_{j=1}^n (g_j, \bar{v}_j) f_i \otimes \rho(a_{ij}) w_j.$$

The representation $\Pi(\rho)$ comes from the following representation $\tilde{\Pi}(\rho)$ of the $*$-algebra $pM_N(\mathcal{A})p$: The representation ρ naturally induces the representation ρ_N of $M_N(\mathcal{A})$. Let $\tilde{\mathcal{H}} = \oplus_{i=1}^n \mathbb{C}^{m_i} \otimes H$. Then $\mathcal{H} = \rho_N(p)\tilde{\mathcal{H}}$. For $a \in M_N(\mathcal{A})$, we set $\tilde{\Pi}(\rho)(pap) = \rho_N(pa)|_{\mathcal{H}}$ as a representation on the Hilbert space $\mathcal{H}$.

Lemma 6. *Any $*$-representation π of $\mathcal{B}$ is unitarily equivalent to $\Pi(\rho)$ for some $*$-representation ρ of $\mathcal{A}$.*

Proof. Let π be a $*$-representation of $\mathcal{B}$ on $\mathcal{H}$ and let p_i be the projection onto the subspace $\pi(\mathcal{B}_{ii})\mathcal{H}$. Since $b_{ii} \cdot b_{jj} = 0$ for $b_{ii} \in \mathcal{B}_{ii}$, $b_{jj} \in \mathcal{B}_{jj}$ and $i \neq j$, we have $p_i p_j = 0$ if $i \neq j$ and $\sum_{i=1}^n p_i = 1$ so that $\mathcal{H} = \oplus_{i=1}^n \mathcal{H}_i$, where $\mathcal{H}_i = p_i \mathcal{H}$. We also have that $\pi(\mathcal{B}_{ij})\mathcal{H}_k = 0$ if $k \neq j$ and $\pi(\mathcal{B}_{ij})\mathcal{H}_j \subset \mathcal{H}_i$. Since each $\mathcal{B}_{ii}$ is an algebra isomorphic to $M_{m_i}(\mathbb{C}) \otimes \mathcal{A}_{ii}$ with the unit $I_{m_i} \otimes e_i$ and any representation of the later is unitarily equivalent to $id \otimes \rho_i$, where ρ_i is a representation of $\mathcal{A}_{ii}$ on a Hilbert space H_i, we have that there exists a unitary operator $V : \oplus_{i=1}^n \mathcal{H}_i \to H = \oplus_{i=1}^n \mathbb{C}^{m_i} \otimes H_i$ such that for the representation $\pi' = V \pi V^{-1}$,

$$\pi'(b_{ii})|_{\mathbb{C}^{m_i} \otimes H_i} = (id \otimes \rho_i)(\varphi_i(b_{ii})) = \Pi(\rho_i)(b_{ii}),$$

where $b_{ii} \in \mathcal{B}_{ii}$ and $\varphi_i : \mathcal{B}_{ii} \to M_{m_i}(\mathbb{C}) \otimes \mathcal{A}_{ii}$ is the isomorphism of the corresponding algebras defined above.

Let $\{f_i^k\}$ be the standard basis in $\mathbb{C}^{m_i}$. Then Ran $\pi'(f_i^k \otimes p_i \otimes f_i^k) \subset f_i^k \otimes H_i$, and we have

$$\pi'(f_i^k \otimes a_{ij} \otimes f_j^l) f_j^s \otimes w_j = \pi'(f_i^k \otimes p_i \otimes f_i^k) \pi'(f_i^k \otimes a_{ij} \otimes f_j^l) \pi'(f_j^l \otimes p_j \otimes f_j^l) f_j^s \otimes w_j =$$
$$\pi'(f_i^k \otimes p_i \otimes f_i^k) \pi'(f_i^k \otimes a_{ij} \otimes f_j^l)(f_j^l, \bar{f}_j^s) f_j^l \otimes w_j = (f_j^l, \bar{f}_j^s) f_i^k \otimes \tilde{w}_i.$$

for some $\tilde{w}_i \in H_i$. Since $f_i^r \otimes a_{ij} \otimes f_j^t = (f_i^r \otimes p_i \otimes f_i^k) \cdot (f_i^k \otimes a_{ij} \otimes f_j^l) \cdot (f_j^l \otimes p_j \otimes f_j^t)$ one easily checks that $\pi'(f_i^r \otimes a_{ij} \otimes f_j^t) f_j^s \otimes w_j = (f_j^t, \bar{f}_j^s) f_i^r \otimes \tilde{w}_i$ so that $\tilde{w}_i$ depends only on w_j and a_{ij}.

Let $X_{a_{ij}}$ be the mapping from H_j to H_i which sends w_j to $\tilde{w}_i$. It is easy to check that it is a linear bounded operator and that

$$\pi'(f_i \otimes a_{ij} \otimes g_j) u_j \otimes w_j = (g_j, \bar{u}_j) f_i \otimes X_{a_{ij}} w_j$$

for arbitrary $f_i \in \mathbb{C}^{m_i}$, $g_j, u_j \in \mathbb{C}^{m_j}$, and $w_j \in H_j$. We extend $X_{a_{ij}}$ to the whole space H in the trivial way and denote the resulting mapping by the same letter. What is left to prove is that $\rho(a) := \sum_{i,j=1}^{n} X_{p_i a p_j}$, $a \in \mathcal{A}$, is a $*$-representation of $\mathcal{A}$ on $\oplus_{i=1}^{n} H_i$. Direct verification shows that $X_{a_{ij}+b_{ij}} = X_{a_{ij}} + X_{b_{ij}}$ and $X_{\lambda a_{ij}} = \lambda X_{a_{ij}}$ implying $\rho(a+b) = \rho(a) + \rho(b)$ and $\rho(\lambda a) = \lambda \rho(a)$.

$$\rho(a)\rho(b) = \left(\sum_{i,j=1}^{n} X_{p_i a p_j}\right)\left(\sum_{i,j=1}^{n} X_{p_i b p_j}\right) = \sum_{i,j=1}^{n}\left(\sum_{k=1}^{n} X_{p_i a p_k} X_{p_k b p_j}\right)$$

and, if $h_k \in \mathbb{C}^{m_k}$ such that $(h_k, \bar{h}_k) = 1$,

$$\pi'(f_i \otimes p_i a b p_j \otimes g_j)(u_j \otimes w_j) = \pi'\left(f_i \otimes \sum_{k=1}^{n}(p_i a p_k \cdot p_k b p_j) \otimes g_j\right)(u_j \otimes w_j) =$$

$$= \sum_{k=1}^{n} \pi'(f_i \otimes p_i a p_k \otimes h_k)\pi'(h_k \otimes p_k b p_j \otimes g_j)(u_j \otimes w_j) =$$

$$= \sum_{k=1}^{n} \pi'(f_i \otimes p_i a p_k \otimes h_k)((g_j, \bar{u}_j) h_k \otimes X_{p_k b p_j} w_j) =$$

$$= (g_j, \bar{u}_j) f_i \otimes \left(\sum_{k=1}^{n} X_{p_i a p_k} X_{p_k b p_j} w_j\right).$$

On the other hand, $\pi'(f_i \otimes p_i a b p_j \otimes g_j)(u_j \otimes w_j) = (g_j, \bar{u}_j) f_i \otimes X_{p_i a b p_j} w_j$, giving us

$$X_{p_i a b p_j} = \sum_{k=1}^{n} X_{p_i a p_k} X_{p_k b p_j}$$

and

$$\rho(ab) = \sum_{i,j=1}^{n} X_{p_i a b p_j} = \sum_{i,j=1}^{n}\sum_{k=1}^{n} X_{p_i a p_k} X_{p_k b p_j} = \rho(a)\rho(b).$$

Since $\rho(a^*) = \sum_{i,j=1}^{n} X_{p_i a^* p_j}$ and $\rho(a)^* = \sum_{i,j=1}^{n} X^*_{p_i a p_j}$, to show that ρ is a $*$-representation we have to prove that $X_{p_i a^* p_j} = X^*_{p_j a p_i}$. The verification of this is an easy task and we left it to the reader. □

Lemma 7. *Any idempotent in the algebra $M_n(\mathbb{C}[\mathcal{F}_2])$ is equivalent to an idempotent of the the form $q \otimes e$, where q is an idempotent in $M_n(\mathbb{C})$ and e is the unit in $\mathbb{C}[\mathcal{F}_2]$.*

Proof. By [Co, Corollary 3], the algebra $\mathbb{C}[\mathcal{F}_2]$ is a free ideal ring. Hence the statement of the lemma follows from [Co, Lemma 2.5]. □

Lemma 8. *1 $\mathcal{A}$ is $*$-finite if and only if $\mathcal{B}$ is $*$-finite.*

2 $\mathcal{A}$ is of type **I** *if and only if $\mathcal{B}$ is of type* **I**.

3 If $\mathcal{A}$ is strongly $*$-wild then $\mathcal{B}$ is $*$-wild.

Proof. The first statement follows from Lemma 6.

Assume $\mathcal{A}$ is of type **I**. Then so is the algebra $M_N(\mathcal{A})$. In fact, any $*$-representation of $M_N(\mathbb{C}) \otimes \mathcal{A}$ is unitarily equivalent to $\rho = id \otimes \pi$, where π is a $*$-representation of $\mathcal{A}$. Therefore the von-Neumann algebra generated by $\rho(M_N(\mathbb{C}) \otimes \mathcal{A})$ is $M_N(\mathbb{C})\bar{\otimes}\mathcal{N}$, where $\mathcal{N}$ is the von-Neumann algebra generated by $\pi(\mathcal{A})$. Since $\mathcal{N}$ and $M_N(\mathbb{C})$ are of type **I** so is $M_N(\mathbb{C})\bar{\otimes}\mathcal{N}$ ([Ta, Theorem 2.30]). Like $\mathcal{A}$, $M_N(\mathcal{A})$ is a finitely generated algebra. Therefore, by Remark 2, if ρ is an irreducible representation of $M_N(\mathcal{A})$ on $\tilde{\mathcal{H}}$, the closure $\overline{\rho(M_N(\mathcal{A}))}$ contains a compact operator K. Let p be the projection given the isomorphism $pM_N(\mathcal{A})p \simeq \mathcal{B}$. Clearly, for $P = \rho(p)$, the operator PKP is compact as an operator from $B(\mathcal{H})$, where $\mathcal{H} = P\tilde{\mathcal{H}}$. Thus for the representation $\tilde{\Pi}(\rho)$ of the algebra $pM_N(\mathcal{A})p$, the operator closure of the image $\tilde{\Pi}(\rho)(pM_N(\mathcal{A})p)$ contains a compact operator. Since, by Lemma 6, any representation of $pM_N(\mathcal{A})p$ is unitarily equivalent to $\tilde{\Pi}(\rho)$ for some $\rho \in \operatorname{Rep} \mathcal{A}$, this implies that $pM_N(\mathcal{A})p$ and therefore $\mathcal{B}$ is of type **I**. We leave the converse statement (which we do not need) to the reader.

Assume now that $\mathcal{A}$ is a strongly $*$-wild algebra. To prove $*$-wildness of $\mathcal{B} \simeq pM_N(\mathcal{A})p$ it is enough to show that there exists a $*$-homomorphism $\psi : pM_N(\mathcal{A})p \to M_n(C^*(\mathcal{F}_2))$ such that $\psi(pM_N(\mathcal{A})p)$ is dense in $M_n(C^*(\mathcal{F}_2))$. In fact, since $(\pi_n(M_n(C^*(\mathcal{F}_2)))' = M_n(\mathbb{C})' \otimes (\pi(C^*(\mathcal{F}_2)))' = \mathbb{C}I_n \otimes (\pi(C^*(\mathcal{F}_2)))'$ for any $\pi \in \operatorname{Rep}(C^*(\mathcal{F}_2))$, we would have in this case that $C \in F_\psi(\pi)(pM_N(\mathcal{A})p)' = (\pi_n(M_n(C^*(\mathcal{F}_2)))' = \mathbb{C}I_n \otimes (\pi(C^*(\mathcal{F}_2)))'$, giving the statement. We know the existence of a $*$-homomorphism $\varphi : \mathcal{A} \to M_K(C^*(\mathcal{F}_2))$ satisfying this density condition (see the corresponding arguments in the proof of Lemma 5) and, moreover, the condition $\varphi(\mathcal{A}) \subset M_K(\mathbb{C}[\mathcal{F}_2])$. Let φ_N be the $*$-homomorphism $M_N(\mathcal{A}) \to M_{NK}(\mathbb{C}[\mathcal{F}_2])$ induced by φ. Then

$$\varphi_N(pM_N(\mathcal{A})p) = \varphi_N(p)\varphi_N(M_N(\mathcal{A}))\varphi_N(p)$$

is dense in $\varphi_N(p)M_{NK}(C^*(\mathcal{F}_2))\varphi_N(p)$. $\varphi_N(p)$ is a projection in $M_{NK}(\mathbb{C}([\mathcal{F}_2]))$ and therefore, by Lemma 7, is equivalent to a projection of type $q \otimes e$, where q is a projection (say, of rank n) in $M_{NK}(\mathbb{C})$. Let $T \in M_{NK}(\mathbb{C}([\mathcal{F}_2]))$ be an invertible element giving the equivalence. Then

$$\begin{aligned}\varphi_N(p)M_{NK}(C^*(\mathcal{F}_2))\varphi_N(p) &= T^{-1}(q \otimes e)TM_{NK}(C^*(\mathcal{F}_2))T^{-1}(q \otimes e)T = \\ &= T^{-1}(q \otimes e)M_{NK}(C^*(\mathcal{F}_2))(q \otimes e)T.\end{aligned}$$

Since $(q \otimes e)M_{NK}(C^*(\mathcal{F}_2))(q \otimes e) \simeq M_n(C^*(\mathcal{F}_2))$, we have $\varphi_N(p)M_{NK}(C^*(\mathcal{F}_2))\varphi_N(p) \simeq M_n(C^*(\mathcal{F}_2))$. If δ is the corresponding isomorphism, $\delta \circ \varphi_N : pM_N(\mathcal{A})p \to M_n(C^*(\mathcal{F}_2))$ is the required $*$-homomorphism $\psi : pM_N(\mathcal{A})p \to M_n(C^*(\mathcal{F}_2))$. $\square$

Remark 4. Using similar arguments one can prove that if $\mathcal{B}$ is strongly $*$-wild, then $\mathcal{A}$ is $*$-wild. We do not know if strong $*$-wildness can be replaced by $*$-wildness. This would be true if we could prove that any projection in $M_n(C^*(\mathcal{F}_2))$ is unitarily equivalent to an elementary one, that is $q \otimes e$, where q is a projection in $M_n(\mathbb{C})$. However, at the moment we do not know how to prove the last statement (and we do not know if it is correct).

7 Proof of the main result

The second statement of the theorem follows from Lemma 5. For the first statement we can assume that the algebra A is indecomposable and basic using Lemma 8.

If A is basic and is isomorphic to A_1, then it is obvious that $A(Rad - *) \simeq A$ is $*$-finite. In fact it has (up to a $*$-isomorphism) only one irreducible $*$-representation, which is the trivial one.

By Lemma 2, the radical $*$-double of both A_n and $\tilde{A}_n$ is $*$-tame. In Remark 3 we have seen that the algebras A_n and $\tilde{A}_n$ are characterized as those finite-dimensional basic indecomposable algebras whose dimensions of indecomposable representations do not exceed 2. To complete the proof it is now left to show that the radical $*$-double of a basic indecomposable algebra A, admitting a 3-dimensional indecomposable representation, π say, is strongly $*$-wild. Let us denote by B the quotient of A modulo the annihilator of π. We note that B is basic and indecomposable and that π induces a 3-dimensional indecomposable representation of B. It is of course enough to show that the radical $*$-double of B is strongly $*$-wild. We will now show that this essentially reduces to Lemma 4.

We have the following three possibilities for π:

(I) π has exactly one 1-dimensional subrepresentation which we denote by π_1, and exactly one 1-dimensional quotient representation which we denote by π_2.

(II) π has exactly one 1-dimensional subrepresentation which we denote by π_1, but more than one 1-dimensional quotient representations.

(III) π has exactly one 1-dimensional quotient representation which we denote by π_2, but more than one 1-dimensional subrepresentations.

In what follows we are going to study all possibilities for B case by case.

Assume first that 1 is a primitive idempotent. Then the radical $*$-double of B and the usual $*$-double of B in the sense of [MT] coincide and the statement follows from [MT, Corollary 1] (remark that all $*$-wild $*$-doubles in [MT] are in fact strongly $*$-wild by the constructions used in [MT] and Remark 1).

Let us now assume that B has two non-equivalent orthogonal primitive idempotents f and $1-f$. Since both these elements do not annihilate π, we can assume that the image of f is 2-dimensional and the image of $1-f$ is thus 1-dimensional. In the case (I) we have three possibilities:

1. $1-f$ is not annihilated by π_1. In this case the algebra B is (via π) the algebra of the following matrices:

$$B = \left\{ \begin{pmatrix} b & c & d \\ 0 & a & l \\ 0 & 0 & a \end{pmatrix} : a, b, c, d, l \in \mathbb{C} \right\},$$

   and one easily constructs an isomorphism to the algebra of Lemma 4(f). Using the latter lemma we conclude that $B(Rad - *)$ is strongly $*$-wild.

2. $1-f$ is not annihilated by π_2. In the same way as above, it is easy to see that in this case B is isomorphic to the algebra of Lemma 4(g) and hence $B(Rad - *)$ is strongly $*$-wild.

3. $1 - f$ is annihilated by both π_1 and π_2. It is easy to see that in this case B is isomorphic to the algebra of Lemma 4(d) and hence $B(Rad - *)$ is strongly $*$-wild.

In the case (II) we have two possibilities:

1. $1 - f$ is not annihilated by π_1. It is easy to see that in this case B is isomorphic to the algebra of Lemma 4(e) and hence $B(Rad - *)$ is strongly $*$-wild.
2. $1 - f$ is annihilated by π_1. It is easy to see that in this case B is isomorphic to the algebra of Lemma 4(g) and hence $B(Rad - *)$ is strongly $*$-wild.

In the case (III) we have two possibilities:

1. $1 - f$ is not annihilated by π_2. It is easy to see that in this case B is isomorphic to the algebra of Lemma 4(e) and hence $B(Rad - *)$ is strongly $*$-wild.
2. $1 - f$ is annihilated by π_2. It is easy to see that in this case B is isomorphic to the algebra of Lemma 4(f) and hence $B(Rad - *)$ is strongly $*$-wild.

Finally, let us assume that B has three non-equivalent pairwise orthogonal primitive idempotents e, f and $1 - f - e$. Then the rank of each of them under π is 1-dimensional. Hence in the case (I) we get that B is isomorphic to the algebra of Lemma 4(a) and hence $B(Rad - *)$ is strongly $*$-wild. In the case (II) we get that B is isomorphic to the algebra of Lemma 4(c) and hence $B(Rad - *)$ is strongly $*$-wild. In the case (III) we get that B is isomorphic to the algebra of Lemma 4(b) and hence $B(Rad - *)$ is strongly $*$-wild. This completes the proof.

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