CHALMERS | GÖTEBORGS UNIVERSITET

PREPRINT

On Linear Transformations Preserving the Pólya Frequency Property

PETTER BRÄNDÉN

Department of Mathematics
CHALMERS UNIVERSITY OF TECHNOLOGY
GÖTEBORG UNIVERSITY
Göteborg Sweden 2004

Preprint 2004:25

On Linear Transformation Preserving the Pólya Frequency Property

Petter Brändén

CHALMERS | GÖTEBORGS UNIVERSITET

Mathematics
Department of Mathematics
Chalmers University of Technology and Göteborg University
SE-412 96 Göteborg, Sweden
Göteborg, April 2004

NO 2004:25
ISSN 0347-2809

Matematiska Vetenskaper
Göteborg 2004

ON LINEAR TRANSFORMATIONS PRESERVING THE PÓLYA FREQUENCY PROPERTY

Petter Brändén[1]
Matematik, Chalmers tekniska högskola och Göteborgs universitet
S-412 96 Göteborg, Sweden
branden@math.chalmers.se

Abstract. We prove that certain linear operators preserve the Pólya frequency property and real-rootedness, and apply our results to settle some conjectures and open problems in combinatorics proposed by Bóna, Brenti and Reiner-Welker.

1. Introduction

Many sequences encountered in various areas of mathematics, statistics and computer science are known or conjectured to be unimodal or log-concave, see [8, 32, 34]. A sufficient condition for a sequence to enjoy these properties is that it is a Pólya frequency (PF for short) sequence, or equivalently for finite sequences, that its generating function has only real and non-positive zeros. It is often the case that the generating function of a finite PF-sequence has more transparent properties when expanded in a basis other than the standard basis $\{x^i\}_{i\geq 0}$ of $\mathbb{R}[x]$. Therefore it is natural to investigate how PF-sequences translate when expressed in various basis. This amounts to studying properties of the linear operator that maps one basis to another. A systematic study of this was first pursued by Brenti in [7]. This is also the theme of this paper.

In Section 3 we will study linear operators of the type

$$\phi_F = \sum_{k=0}^{n} Q_k(x)\frac{d^k}{dx^k},$$

where $F(x,z) = \sum_{k=0}^n Q_k(x)z^k \in \mathbb{R}[x,z]$. Here we will give sufficient conditions on F for ϕ_F to preserve the PF-property. The results attained generalizes and unifies theorems of Hermite, Poulain, Pólya and Schur. We will also in this section give a sufficient condition for a family of natural $\mathbb{R}$-bilinear forms to preserve the PF-property in both arguments. This generalizes results of Wagner [11, 37, 38].

1991 *Mathematics Subject Classification.* Primary: 05A15, 26C10 ; Secondary: 05A19, 05A05, 20F55.

[1]Research financed by the EC's IHRP Programme, within the Research Training Network "Algebraic Combinatorics in Europe", grant HPRN-CT-2001-00272, while the author was at Universitá di Roma "Tor Vergata", Rome, Italy.

An important linear operator in combinatorics is the operator defined by $\mathcal{E}(\binom{x}{i}) = x^i$, for all $i \in \mathbb{N}$. In Section 4 we will prove that whenever a polynomial f of degree d has nonnegative coefficients when expanded in the basis $\{x^i(x+1)^{d-i}\}_{i=0}^d$ the polynomial $\mathcal{E}(f)$ will have only real, non-positive and simple zeros.

In the remainder of the paper we use the theory developed to settle some conjectures and open problems raised in combinatorics. Highlights are; we prove that the q-Eulerian polynomials, $A_n(x;q)$, defined by Foata and Schützenberger [17] and further studied by Brenti in [10] have only real zeros for all integers q. This settles a conjecture raised by Brenti. We will also continue the study of the W-Eulerian polynomials, defined for any finite Coxeter group W and the q-analog $B_n(x;q)$, initiated by Brenti in [9].

In Section 7 we prove that the h-vectors of a family simplicial complexes associated to finite Weyl groups defined by Fomin and Zelevinski [18] are PF, thus settling an open problem raised by Reiner and Welker [30]. In Section 5 we prove that the numbers $\{W_t(n,k)\}_{k=0}^{n-1}$ of t-stack sortable permutations in $\mathcal{S}_n$ with k descents form PF-sequences when $t = 2, n-2$, and thereby settling two new cases of an open problem proposed by Bóna [2, 3].

2. Notation and preliminaries

In this section we collect definitions, notation and results that will be used frequently in the rest of the paper. Let $\{a_i\}_{i=0}^\infty$ be a sequence of real numbers. It is *unimodal* if there is a number p such that $a_0 \leq a_1 \leq \cdots \leq a_p \geq a_{p+1} \geq \cdots$, and *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for all $i > 0$.

An infinite matrix $A = (a_{ij})_{i,j\geq 0}$ of real numbers is *totally positive*, TP, if all minors of A are nonnegative. An infinite sequence $\{a_i\}_{i=0}^\infty$ of real numbers is a *Pólya frequency sequence*, PF-sequence, if the matrix $(a_{i-j})_{i,j\geq 0}$ is TP. Thus a PF-sequence is by definition log-concave and therefore also unimodal. A finite sequence $a_0, a_1, a_2, \ldots, a_n$ is said to be PF if the infinite sequence $a_0, a_1, a_2, \ldots, a_n, 0, 0, \ldots$ is PF. A sequence $\{a_i\}_{i=0}^\infty$ is said to be PF_r if all minors of size r of $(a_{i-j})_{i,j\geq 0}$ are nonnegative. If the polynomials $\{b_i(x)\}_{i=0}^d$ are linearly independent over $\mathbb{R}$ and $r \in \mathbb{N}$ we define the set $PF_r[\{b_i(x)\}_{i=0}^d]$ to be

$$PF_r[\{b_i(x)\}_{i=1}^d] = \{\sum_{i=0}^d \lambda_i b_i(x) : \{\lambda_i\}_{i=0}^\infty \text{ is } PF_r\},$$

and $PF[\{b_i(x)\}_{i=1}^d] = \bigcap_{r=0}^\infty PF_r[\{b_i(x)\}_{i=1}^d]$.

The following theorem characterizes PF-sequences. It was conjectured by Schoenberg and proved by Edrei [16], see also [24].

Theorem 2.1. *Let $\{a_i\}_{i=0}^\infty$ be a sequence of real numbers with $a_0 = 1$. Then it is a PF-sequence if and only if the generating function can be* *expanded, in a neighborhood of the origin, as*

$$\sum_{i\geq 0} a_i z^i = e^{\gamma z}\frac{\prod_{i\geq 0}(1+\alpha_i z)}{\prod_{i\geq 0}(1-\beta_i z)},$$

where $\gamma \geq 0$, $\alpha_i, \beta_i > 0$ *and* $\sum_{i\geq 0}(\alpha_i+\beta_i) < \infty$.

A consequence of this theorem is that a finite sequence is PF if and only if its generating function is a polynomial with only real non positive zeros.

Let $f, g \in \mathbb{R}[x]$ be real-rooted with zeros: $\alpha_1 \leq \cdots \leq \alpha_i$ and $\beta_1 \leq \cdots \leq \beta_j$, respectively. We say that f *interlaces* g, denoted $f \preceq g$, if $j = i+1$ and

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \cdots \leq \beta_{j-1} \leq \alpha_{j-1} \leq \beta_j.$$

We say that f *alternates left of* g, denoted $f \ll g$, if $i = j$ and

$$\alpha_1 \leq \beta_1 \leq \cdots \leq \beta_{i-1} \leq \alpha_i \leq \beta_i.$$

If in addition f and g have no common zero then we say that f *strictly interlaces* g and f *strictly alternates left of* g, respectively. We also say that two polynomials f and g *alternate* if one of the polynomials alternates left of or interlaces the other. We will need two simple lemmata concerning these concepts. A polynomial is said to be *standard* if its leading coefficient is positive.

Lemma 2.2. *Let g and $\{f_i\}_{i=1}^n$ be real-rooted standard polynomials.*

(i) *If for each $1 \leq i \leq n$ we have either $g \ll f_i$ or $g \preceq f_i$. Then the sum $F = f_1 + f_2 + \cdots + f_n$ is real-rooted with $g \preceq F$ or $g \ll F$, depending on the degree of F.*

(ii) *If for each $1 \leq i \leq n$ we have either $f_i \ll g$ or $f_i \preceq g$. Then the sum $F = f_1 + f_2 + \cdots + f_n$ is real-rooted with $F \preceq g$ or $F \ll g$, depending on the degree of F.*

Proof. The lemma follows easily by counting the sign-changes of F at the zeros of g, see e.g., [39, Prop. 3.5]. □

The next lemma is obvious:

Lemma 2.3. *If $f_0, f_1, \ldots, f_n$ are real-rooted polynomials with $f_0 \ll f_n$ and $f_{i-1} \ll f_i$ for all $1 \leq i \leq n$, then $f_i \ll f_j$ for all $0 \leq i \leq j \leq n$.*

The following theorem is a characterization of alternating polynomials due to Obreschkoff [26] and Dedieu [14]:

Theorem 2.4. *Let $f, g \in \mathbb{R}[x]$. Then f and g alternate (strictly alternate) if and only if all polynomials in the space*

$$\{\alpha f + \beta g : \alpha, \beta \in \mathbb{R}\},$$

have only real (real and simple) zeros.

An immediate but non-trivial consequence of this theorem is:

Corollary 2.5. *Let $\phi : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator. Then ϕ preserves the real-rootedness property (real- and simple-rootedness property) if and only if ϕ preserves the alternating property (strictly alternating property).*

We denote by $\mathbb{N}$ the set of natural numbers $\{0, 1, 2, \ldots\}$. The symmetric group of bijections $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is denoted by $\mathcal{S}_n$. A *descent* in a permutation $\pi \in \mathcal{S}_n$ is an index $1 \leq i \leq n$, such that $\pi(i) > \pi(i+1)$. Let $\operatorname{des}(\pi)$ denote the number of descents in π. The *Eulerian polynomials*, $A_n(x)$, are defined by $A_n(x) = \sum_{\pi\in\mathcal{S}_n} x^{\operatorname{des}(\pi)+1}$ and satisfies, see e.g. [12]

$$\sum_{k\geq 0} k^n x^n = \frac{A_n(x)}{(1-x)^{n+1}}.$$

The binomial polynomials are defined by $\binom{x}{0} = 1$ and $\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}$ for $k \geq 1$.

In several proofs we will implicitly use the fact that the zeros of a polynomial are continuous functions of the coefficients of the polynomial. In particular the limit of a sequence of real-rooted polynomials is again real-rooted. For a treatment of these matters we refer the reader to [25].

3. A class of linear operators preserving the PF-property

For any polynomial $F(x,z) = \sum_{k=0}^n Q_k(x)z^k \in \mathbb{R}[x,z]$ we define a linear operator $\phi_F : \mathbb{R}[x] \to \mathbb{R}[x]$ by,

$$\phi_F(f) := \sum_{k=0}^n Q_k(x)\frac{d^k}{dx^k}f(x).$$

In this section we will investigate for which $F \in \mathbb{R}[x,z]$ the linear operator ϕ_F preserves real-rootedness- and the PF-property .

We will need some terminology and a theorem from [5]. For $\xi \in \mathbb{R}$ let $T_\xi : \mathbb{R}[x] \to \mathbb{R}[x]$ be the translation operator defined by $T_\xi(f(x)) = f(x+\xi)$. For any linear operator $\phi : \mathbb{R}[x] \to \mathbb{R}[x]$ we define a linear transform $\mathcal{L}_\phi : \mathbb{R}[x] \to \mathbb{R}[x,z]$ by

$$\begin{aligned} \mathcal{L}_\phi(f) &:= \phi(T_z(f)) \\ &= \sum_n \phi(f^{(n)})(x)\frac{z^n}{n!} \qquad (3.1)\\ &= \sum_n \frac{\phi(x^n)}{n!}f^{(n)}(z). \end{aligned}$$

Definition 3.1. Let $\phi : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator. Define a function $d_\phi : \mathbb{R}[x] \to \mathbb{N} \cup \{-\infty\}$ by: If $\phi(f^{(n)}) = 0$ for all $n \in \mathbb{N}$, we let

$d_\phi(f) = -\infty$. Otherwise let $d_\phi(f)$ be the smallest integer d such that $\phi(f^{(n)}) = 0$ for all $n > d$. Hence $d_\phi(f) \leq \deg f$ for all $f \in \mathbb{R}[x]$.

The set $\mathscr{A}(\phi)$ is defined as follows: If $d_\phi(f) \in \{-\infty, 0\}$ and $\phi(f)$ is standard real- and simple-rooted, then $f \in \mathscr{A}(\phi)$. Moreover, $f \in \mathscr{A}(\phi)$ if $d = d_\phi(f) \geq 1$ and all of the following conditions are satisfied:

(i) $\phi(f^{(i)})$ all have leading coefficients of the same sign and $\deg(\phi(f^{(i-1)})) = \deg(\phi(f^{(i)})) + 1$ for $1 \leq i \leq d$,
(ii) $\phi(f)$ and $\phi(f')$ have no common real zero,
(iii) $\phi(f^{(d)})$ strictly interlaces $\phi(f^{(d-1)})$,
(iv) for all $\xi \in \mathbb{R}$ the polynomial $\mathcal{L}_\phi(f)(\xi, z)$ is real-rooted.

The following theorem is proved in [5]:

Theorem 3.2. *Let $\phi : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator. If $f \in \mathscr{A}(\phi)$ then $\phi(f)$ is real- and simple-rooted.*

We will also need the following classical theorem of Hermite and Poulain. For a proof see [26].

Theorem 3.3. *Let $f = a_0 + a_1x + \cdots + a_nx^n$ and g be real-rooted polynomials. Then the polynomial*

$$f(\frac{d}{dx})g := a_0g(x) + a_1g'(x) + \cdots + a_ng^{(n)}(x)$$

is real-rooted. Moreover, if $f(\frac{d}{dx})g \neq 0$ then any multiple zero of $f(\frac{d}{dx})g$ is a multiple zero of g.

The following theorem gives a sufficient condition for a polynomial to be mapped onto a real-rooted polynomial.

Theorem 3.4. *Let $F = \sum_{k=0}^n Q_k(x)z^k$ be such that $Q_0 \neq 0$ and*

(I) *For all $\xi \in \mathbb{R}$, $F(\xi, z)$ is real-rooted,*
(II) *Q_0 strictly interlaces or strictly alternates left of Q_1, and $\deg Q_0 = 0$ or Q_0 and Q_1 have leading coefficients of the same sign.*

Suppose that

(III) *f is real- and simple-rooted and that for $0 \leq k \leq \deg f$ the polynomials $\phi_F(f^{(k)})$ have their leading term of the same sign with*
$$\deg \phi_F(f^{(k)}) = \deg Q_0 + \deg f - k.$$

Then $\phi_F(f)$ is real- and simple-rooted.

Proof. We will show that the set of real- and simple-rooted polynomials satisfying (III) is a subset of $\mathscr{A}(\phi_F)$ by verifying conditions (i)-(iv) of Definition 3.1. Condition (i) follows immediately from (III). For condition (iv) note that

$$\mathcal{L}_\phi(f)(\xi, z) = \sum_{k=0}^n Q_k(\xi)f^{(k)}(\xi + z),$$

so by the Theorem 3.3 condition (iv) is satisfied. Suppose that η is a common zero of $\phi_F(f)$ and $\phi_F(f')$. By (3.1) we have that 0 is a multiple zero of $\mathcal{L}_\phi(f)(\eta, z)$. Moreover, since $\mathcal{L}_\phi(f)(\eta, z)$ is not identically equal to zero, by (II), Theorem 3.3 tells us that 0 is a multiple zero of $f(\eta+z)$. This means that η is multiple zero of f contrary to the assumption that f is simple-rooted, and verifies condition (ii).

For condition (iii) we have to show that for all $\alpha \in \mathbb{R}$ such that $x+\alpha$ satisfies (III) the polynomial $\phi_F(1) = Q_0$ strictly interlaces $f(x) := \phi_F(x + \alpha) = (x + \alpha)Q_0 + Q_1$. This follows from (II) when analyzing the sign of $f(x) := \phi_F(x + \alpha)$ at the zeros of Q_0: Let $\alpha_k < \alpha_{k-1} < \cdots < \alpha_1$ be the zeros of Q_0 ordered by size. Suppose that Q_0 and Q_1 are standard and that Q_0 strictly interlaces or strictly alternates left of Q_1. Then $\operatorname{sgn} f(\alpha_i) = \operatorname{sgn} Q_1(\alpha_i) = (-1)^i$ for $1 \leq i \leq k$. By Rolle's theorem we know that f has a zero in each interval (α_i, α_{i+1}). This accounts for $k - 1$ real zeros of f. Since Q_0 has positive sign, so does f by condition (III). Now, because $f(\alpha_1) < 0$ and f is standard, f must have a zero to the right of α_1. We now know that f has k zeros real. The signs at α_i forces the remaining zero to be in the interval $(-\infty, \alpha_k)$. Thus Q_0 strictly interlaces Q_1 as was to be shown.

Now, if $Q_0 = A \in \mathbb{R}$ then $\deg Q_1 \leq 1$. Suppose that $Q_1 = B \in \mathbb{R}$. Then clearly A strictly interlaces $(x + \alpha)A + B$. If $Q_0 = A$ and $Q_1 = Cx + D$ where $A, B, C \in \mathbb{R}$, then $f = (A + C)x + A\alpha + D$, so by (III) we have that Q_0 strictly interlaces f. This concludes the proof. □

In some cases it may be convenient to have sharper hypothesis. Therefore we state the following form of the theorem.

Corollary 3.5. *Let $d \in \mathbb{N}$ be given and let $F = \sum_{k=0}^n Q_k(x)z^k$ be such that $Q_0 \neq 0$ and*

(i) *For all $\xi \in \mathbb{R}$, $F(\xi, z)$ is real-rooted,*
(ii) *Q_0 strictly interlaces or strictly alternates left of Q_1, and $\deg Q_0 = 0$ or Q_0 and Q_1 have leading coefficients of the same sign.*
(iii) *The polynomials $\phi_F(x^k)$, $0 \leq k \leq d$ have the same sign and*
$$\deg \phi_F(x^k) = \deg Q_0 + k.$$

Then $\phi_F(f)$ is real-rooted (real- and simple-rooted) if f is real-rooted (real and simple-rooted) and $\deg(f) \leq d$.

Proof. The case of real- and simple-rooted f follows immediately from Theorem 3.4 since (iii) implies (III). If f is a real-rooted polynomial of degree at most d, then f is the limit of a sequence $\{f_k\}_{k=0}^\infty$ of real- and simple-rooted polynomials of degree at most d. It follows that $\phi_F(f)$ is the limit of $\phi_F(f_k)$, and the thesis follows by continuity. □

In the language of PF-sequences we have:

Theorem 3.6. *Let $d \in \mathbb{N}$ be given and let $F = \sum_{k=0}^n Q_k(x)z^k \in \mathbb{R}[x, z]$ be such that $Q_0 \neq 0$ and*

(i) *For all* $\xi \in \mathbb{R}$, $F(\xi, z)$ *is real-rooted,*
(ii) $\phi_F(1)$ *strictly interlaces* $\phi_F(x)$.
(iii) *For all* $0 \le k \le d$

$$\deg \phi_F(x^k) = \deg Q_0 + k,$$

and $\phi_F(x^k) \in PF_1$.

Then $PF[\{\phi_F(x^i)\}_{i=0}^d] \subseteq PF[x^i]$.

Several old results can be derived from these last few theorems. In [27, p. 163] Pólya gave a theorem which he states probably was the most general theorem on real-rootedness known at the time. "Dieser Satz gehört wohl zu den allgemeinsten bekannten Sätzen über Wurzelrealität.":

Theorem 3.7. *Let* $f(x)$ *be a real-rooted polynomial of degree* n, *and let*

$$b_0 + b_1 x + \cdots + b_{n+m} x^{n+m}, \quad (m \ge 0)$$

be a real-rooted polynomial such that $b_i > 0$ *for* $0 \le i \le n$. *Then the equation*

$$G(x, y) := b_0 f(y) + b_1 x f'(y) + b_2 x^2 f''(y) + \cdots + b_n f^{(n)}(y) = 0,$$

has n *real intersection points, (counted with multiplicity), with the line*

$$sx - ty + u = 0,$$

provided that $s, t \ge 0$, $s + t > 0$ *and* $u \in \mathbb{R}$.

Proof. We may assume that $s, t > 0$ since the other cases follows by continuity when s and/or t tends to zero. Thus we may write the equation as

$$a_0 g(x) + a_1 x g'(x) + a_2 x^2 g''(x) + \cdots + a_n g^{(n)}(x) = 0,$$

where $g(x) = f(st^{-1}x + ut^{-1})$ and $a_i = s^i t^{-i} b_i$. Now, we see that all hypothesis of Corollary 3.5 are satisfied for

$$F(x, z) = a_0 + a_1 xz + a_2 x^2 z^2 + \cdots + a_{n+m} x^{n+m} z^{n+m},$$

when $d = n$. □

We will later need one famous consequence of this theorem, $t = 1, s = u = 0$, due to Schur [31].

Theorem 3.8. *Let* $f = \sum_{k=0}^n a_k x^k$ *and* $g = \sum_{k=0}^m b_k x^k$ *be two real-rooted polynomials such that* g *has all zeros of the same sign. Then the polynomial*

$$(fSg)(x) = \sum_{k \ge 0}^{M} k! a_k b_k x^k,$$

where $M = \min(m, n)$ *has only real zeros.*

3.1. **Multiplier-sequences.** A multiplier-sequence is a sequence $T = \{\gamma_i\}_{i=0}^\infty$ of real numbers such that if a polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ has only real zeros, then the polynomial

$$T[p(x)] := a_0\gamma_0 + a_1\gamma_1 x + \cdots + a_n \gamma_n x^n,$$

also has only real zeros. There is a characterization of multiplier-sequences due to Pólya and Schur [27, p. 100-124]:

Theorem 3.9. *Let* $T = \{\gamma_i\}_{i=0}^\infty$ *be a sequence of real numbers and let* $\phi(x) = T[e^x] = \sum_{k=0}^\infty \gamma_k \frac{x^k}{k!}$ *be its exponential generating function. Then* T *is a multiplier-sequence if and only if* ϕ *is a real entire function which can be written as*

$$\phi(x) = cx^n e^{-\alpha x^2 + \beta x} \prod_{k=1}^\infty (1 + \delta_k x) e^{-\delta_k x},$$

where $c, \beta, \delta_k \in \mathbb{R}$, $c \ne 0$, $\alpha \ge 0$, $n \in \mathbb{N}$ *and* $\sum_{k=1}^\infty \delta_k^2 < \infty$.

The following lemma is well-known but elementary, so we give a proof here.

Lemma 3.10. *A multiplier-sequence is strictly log-concave. In particular, a nonnegative multiplier-sequence has no internal zeros.*

Proof. If $f(x) = a_m x^m + a_{m+1} x^{m+1} + \cdots + a_n x^n$ is real-rooted with $a_m a_n \ne 0$, then the coefficients satisfy (see [21, p. 52]):

$$\frac{a_i^2}{\binom{n}{i}^2} > \frac{a_{i-1}}{\binom{n}{i-1}} \frac{a_{i+1}}{\binom{n}{i+1}} \quad (m < i < n).$$

Now, if $\Gamma = \{\gamma_i\}_{i=0}^\infty$ is a multiplier-sequence, then $\Gamma[(x+1)^n]$ is real-rooted for all $n \in \mathbb{N}$, which implies

$$\gamma_i > \gamma_{i-1}\gamma_{i+1},$$

for all i such that there are integers $m < i < n$ with $\gamma_m \gamma_n \ne 0$. □

Theorem 3.11. *Let* $\{\lambda_k\}_{k=0}^\infty$ *be a non-negative multiplier-sequence, and let* $\alpha < \beta \in \mathbb{R}$ *be given. Define two* $\mathbb{R}$*-bilinear forms* $\mathbb{R}[x] \times \mathbb{R}[x] \to \mathbb{R}[x]$ *by*

$$\begin{aligned} f \cdot g &:= \sum_{k \ge 0} \frac{\lambda_k}{k!} f^{(k)}(x) g^{(k)}(x) (x - \alpha)^k (x - \beta)^k, \\ f \circ g &:= \sum_{k \ge 0} \frac{\lambda_k}{k!} f^{(k)}(x) g^{(k)}(x) (x - \alpha)^k, \end{aligned}$$

If f *is real-rooted and* g *is* $[\alpha, \beta]$*-rooted, then* $f \cdot g$ *is real-rooted. If* f *is real-rooted and* g *is* $[-\infty, \alpha]$*-rooted, then* $f \circ g$ *is real-rooted.*

Proof. We prove the statement for $\cdot$ since the case of $\circ$ is similar. We may assume that $\lambda_0 > 0$. Clearly the theorem is true if $\lambda_i = 0$ for all $i > 0$, so by Lemma 3.10 we may assume that $\lambda_1 > 0$. Let g have all zeros simple and in the interval (α, β), and let ϕ be the linear operator defined by $\phi(f) = f \cdot g$. Then $\phi = \phi_F$, where

$$F(x,z) = \sum_{k \geq 0} \lambda_k \frac{g^{(k)}(x)}{k!}(x-\alpha)^k (x-\beta)^k z^k.$$

Since $\{\lambda_k\}_{k\geq 0}$ is a multiplier sequence $F(\xi, z)$ is real-rooted for all real choices of ξ. Now, $Q_0 = \lambda_0 g(x)$ and $Q_1 = \lambda_1 (x-\alpha)(x-\beta)g'(x)$, so Q_0 strictly interlaces Q_1. Moreover, $\deg \phi(x^k) = \deg Q_0 + k$ for all k, so all the hypothesis of Corollary 3.5 are fulfilled. Since any $[\alpha,\beta]$-rooted polynomial is the limit of polynomial which are (α,β)- and simple-rooted the thesis follows by continuity. $\square$

A sequence of real numbers $\Gamma = \{\gamma_k\}_{k=0}^\infty$ is called an *multiplier n-sequence* if for any real-rooted polynomial $f = a_0 + a_1 x + \cdots + a_n x^n$ of degree at most n the polynomial $\Gamma[f] := a_0\gamma_0 + a_1\gamma_1 x + \cdots + a_n \gamma_n x^n$ is real-rooted. There is a simple algebraic characterization of multiplier n-sequences [13]:

Theorem 3.12. *Let $\Gamma = \{\gamma_k\}_{k=0}^\infty$ be a sequence of real numbers. Then Γ is a multiplier n-sequence if and only if $\Gamma[(x+1)^n]$ is real-rooted with all its zeros of the same sign.*

Recall the definition of the *hypergeometric function* ${}_2F_1$:

$${}_2F_1\,(a,b;c;z) = \sum_{m=0}^\infty \frac{(a)_m (b)_m z^m}{(c)_m m!},$$

where $(\alpha)_0 = 1$ and $(\alpha)_m = \alpha(\alpha+1)\cdots(\alpha+m-1)$ when $m \geq 1$. The Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ can be expressed as follows [28, p. 254]:

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!} {}_2F_1\left(-n, 1+\alpha+\beta+n; 1+\alpha; \frac{1-x}{2}\right), \quad (3.2)$$

We need the following lemma:

Lemma 3.13. *Let n be a positive integer and r a non-negative real number. Then $\Gamma = \{\binom{-n-r}{k}\}_{k=0}^\infty$ is a multiplier n-sequence.*

Proof. Let $r > 0$. Then

$$\begin{aligned} \Gamma[(x+1)^n] &= \sum_{k=0}^n \binom{-n-r}{k}\binom{n}{k} x^k \\ &= {}_2F_1\,(-n, n+r; 1; x) \\ &= P_n^{(0,r-1)}(1-2x), \end{aligned}$$

where the last equality follows from (3.2). Since the Jacobi polynomials are known, see [28], to have all their zeros in $[-1,1]$ when $\alpha, \beta > -1$, we have that $\Gamma[(x+1)^n]$ has all its zeros in $[0,1]$. The case $r = 0$ follows by continuity when we let r tend to zero from above. $\square$

For any real number q let $\Gamma_q := \{q+k\}_{k=0}^\infty$.

Corollary 3.14. *Let $n > 1$ be a positive integer. Then Γ_q is an n-sequence if and only if $q \notin (-n, 0)$.*

Proof. Let $q \in \mathbb{R}$ be given. We have to determine for which $n > 1$ the zeros of $\Gamma_q[(x+1)^n]$ are all real and of the same sign. Now,

$$\Gamma_q[(x+1)^n] = (x+1)^{n-1}\{(n+q)x + q\}.$$

If $q \geq 0$ or $n = -q$ then all zeros are negative so we may assume that $q < 0$ and $n \neq -q$. If $n > -q$ then $q/(n+q)$ is negative so $\Gamma_q[(x+1)^n]$ has zeros of different signs. If on the other hand $n < -q$ then $q/(n+q)$ is positive which gives that all zeros of $\Gamma_q[(x+1)^n]$ are negative, and the lemma follows. $\square$

4. The E-transformation

The *E-transformation* is the invertible linear operator, $\mathcal{E} : \mathbb{R}[x] \to \mathbb{R}[x]$, defined by

$$\mathcal{E}(\binom{x}{i}) = x^i,$$

for all $i \in \mathbb{N}$. The PF-preserving properties of this linear operator was first studied in [7] and later in [38, 39] and [5]. It is important in the theory of (P,ω)-partitions since it maps the order-polynomial of a labeled poset to the E-polynomial of the same labeled poset, see [7, 38]. In, [7] Brenti proved the following theorem. Let $\lambda(f)$ and $\Lambda(f)$ denote the smallest and the largest real zero of the polynomial f, respectively.

Theorem 4.1. *Suppose that $f \in \mathbb{R}[x]$ has only real zeros and that $f(n) = 0$ for all $n \in ([\lambda(f), -1] \cup [0, \Lambda(f)]) \cap \mathbb{Z}$. Then $\mathcal{E}(f)$ has all zeros real and non-positive.*

In this section we will prove the following theorem:

Theorem 4.2. *For all $n \in \mathbb{N}$ we have*

$$PF_1[\{x^i(x+1)^{n-i}\}_{i=0}^n] \subseteq PF[\binom{x}{i}]$$

Moreover if $f \in PF_1[\{x^i(x+1)^{n-i}\}_{i=0}^n]$ then $\mathcal{E}(f)$ has simple zeros and

$$\mathcal{E}((x+1)^d) \ll \mathcal{E}(f) \ll \mathcal{E}(x^d).$$

The *diamond product* of two polynomials in $\mathbb{R}[x]$ is the $\mathbb{R}$-bilinear form defined by

$$(f \diamond g)(x) := \mathcal{E}(\mathcal{E}^{-1}(f)\mathcal{E}^{-1}(g)). \quad (4.1)$$

This product was first studied by Wagner in [38, 39] and further studied in [5]. See also Section 8 of this paper. Using the Vandermonde identity

$$\binom{x}{i}\binom{x}{j} = \sum_{k\geq 0}\binom{k}{k-i,i+j-k,k-j}\binom{x}{k},$$

it follows, see [39], that

$$(f \diamond g)(x) = \sum_{k\geq 0} \frac{f^{(k)}(x)}{k!}\frac{g^{(k)}(x)}{k!}x^k(x+1)^k. \tag{4.2}$$

We will later need a symmetry property of $\mathcal{E}$. Let $\mathcal{R} : \mathbb{R}[x] \to \mathbb{R}[x]$ be the algebra automorphism defined by $\mathcal{R}(x) = -1-x$.

Lemma 4.3.

$$\mathcal{R}\mathcal{E} = \mathcal{E}\mathcal{R}$$

Proof. From (4.2) it follows that

$$\mathcal{R}(f \diamond g) = \mathcal{R}(f) \diamond \mathcal{R}(g) \tag{4.3}$$

Note that $\mathcal{R}\mathcal{E}(f) = \mathcal{E}\mathcal{R}(f)$ whenever f is linear. Now, suppose that f, g are polynomials such that $\mathcal{R}\mathcal{E}(f) = \mathcal{E}\mathcal{R}(f)$ and $\mathcal{R}\mathcal{E}(g) = \mathcal{E}\mathcal{R}(g)$. Then

$$\begin{aligned}
\mathcal{R}\mathcal{E}(fg) &= \mathcal{R}\big(\mathcal{E}(f)\diamond\mathcal{E}(g)\big) \quad \text{by (4.1)}\\
&= \big(\mathcal{R}\mathcal{E}(f)\big)\diamond\big(\mathcal{R}\mathcal{E}(g)\big) \quad \text{by (4.3)}\\
&= \big(\mathcal{E}\mathcal{R}(f)\big)\diamond\big(\mathcal{E}\mathcal{R}(g)\big)\\
&= \mathcal{E}\big(\mathcal{R}(f)\mathcal{R}(g)\big)\\
&= \mathcal{E}\mathcal{R}(fg).
\end{aligned}$$

Since we may view $\mathcal{E}$ and $\mathcal{R}$ as $\mathbb{C}$-linear operators on $\mathbb{C}[x]$, and $\diamond$ as a $\mathbb{C}$-bilinear form on $\mathbb{C}[x]$, the lemma follows from the fundamental theorem of algebra. □

Lemma 4.4. *Let $\alpha \in [-1,0]$ and let f be a polynomial such that $\mathcal{E}(f)$ is $[-1,0]$-rooted. Then $\mathcal{E}((x-\alpha)f)$ is $[-1,0]$-rooted and $\mathcal{E}(f)$ interlaces $\mathcal{E}((x-\alpha)f)$. If $\mathcal{E}(f)$ in addition only has simple zeros, then so does $\mathcal{E}((x-\alpha)f)$.*

Proof. Let $g = \mathcal{E}(f)$ and let $\alpha \in [-1,0]$. By (4.2) we have that

$$\mathcal{E}((x-\alpha)f) = (x-\alpha)g + x(x+1)g'. \tag{4.4}$$

Since g interlaces $(x-\alpha)g$ and $x(x+1)g'$ it also interlaces the sum, by Lemma 2.2. Also, if $x \notin [-1,0]$ then the summands have the same sign so $\mathcal{E}((x-\alpha)f)$ cannot have any zeros outside $[-1,0]$. Suppose that g has only simple zeros. Then by (4.4) the only possible common zeros of g and $\mathcal{E}((x-\alpha)f)$ are 0 and -1. If $\deg(f) \geq 1$ it also follows from (4.4) that the multiplicities of 0 and -1 of $\mathcal{E}((x-\alpha)f)$ are the same as those of g. Hence the (simple) zeros of g separates the zeros of

$\mathcal{E}((x-\alpha)f)$ except possibly at $0, -1$, and we conclude that $\mathcal{E}((x-\alpha)f)$ has only simple zeros. □

Lemma 4.5. *For all integers $n \geq 1$ we have*

$$(x+1)\mathcal{E}(x^n) = x\mathcal{E}((x+1)^n).$$

Proof. We may write

$$x^n = \sum_{k=1}^{n} a_k\binom{x}{k},$$

where $a_k \in \mathbb{R}$. Thus

$$\begin{aligned}
\mathcal{E}((x+1)^n) &= \sum_{k=1}^{n} a_k\mathcal{E}[\binom{x}{k} + \binom{x}{k-1}]\\
&= \sum_{k=1}^{n} a_k(x^k + x^{k-1})\\
&= (x+1)x^{-1}\mathcal{E}(x^n).
\end{aligned}$$

□

For $i \in \mathbb{N}$ and let RR_n denote the set of real-rooted monic polynomials of degree n. We define a partial order $\leq$ on RR_n as follows: If $f, g \in RR_n$ have zeros $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ and $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n$ respectively then $f \leq g$, if $\alpha_i \leq \beta_i$ for $1 \leq i \leq n$.

Theorem 4.6. *Suppose that f and g are $[-1,0]$-rooted with $f \leq g$. Then $\mathcal{E}(f)$ and $\mathcal{E}(g)$ are $[-1,0]$- and simple-rooted, with $\mathcal{E}(f) \ll \mathcal{E}(g)$.*

Proof. By Lemma 4.4 and induction we only have to show that $\mathcal{E}(f) \ll \mathcal{E}(g)$. If f and g have the same zeros except for one, i.e., $f = (x-\alpha)h$ and $g = (x-\beta)h$, where $\alpha < \beta$, then

$$\mathcal{E}(g) = \mathcal{E}(f) - (\beta-\alpha)\mathcal{E}(h),$$

and since $\mathcal{E}(h)$ interlaces $\mathcal{E}(f)$ we have $\mathcal{E}(f) \ll \mathcal{E}(g)$ by Lemma 2.2.

Now, suppose that f and g are $[-1,0]$-rooted polynomials of degree n such that $f \leq g$. Then there are $[-1,0]$-rooted polynomials $\{h_i\}_{i=0}^{M}$ with

$$(x+1)^n = h_0 \leq h_1 \leq \cdots \leq h_M = x^n,$$

such that $f, g \in \{h_i\}_{i=0}^{M}$ and h_{i-1} and h_i only differ in one zero for $1 \leq i \leq n$. We therefore have

$$\mathcal{E}(h_0) \ll \mathcal{E}(h_1) \ll \cdots \ll \mathcal{E}(h_M),$$

and since $\mathcal{E}(h_0) \ll \mathcal{E}(h_M)$, by Lemma 4.5, the theorem follows from Lemma 2.3. □

A consequence of Theorem 4.6 is that if $\{f_i\}_{i=1}^{m}$ is a sequence of standard $[-1,0]$-rooted polynomials of the same degree d, then by Lemma

2.2 and Theorem 4.6, the image under $\mathcal{E}$ of any non-negative sum $F = \sum_{i=1}^m \mu_i f_i$ will be $[-1,0]$-rooted with

$$\mathcal{E}((x+1)^d) \ll \mathcal{E}(F) \ll \mathcal{E}(x^d).$$

It is easy to see that a standard polynomial f of degree d is $[-1,0]$-rooted if and only if f can be written as

$$f(x) = (x+1)^d g(\frac{x}{x+1}),$$

where g is a standard and $(-\infty,0)$-rooted. On the other hand, since $x^i(x+1)^{d-i}$ is $[-1,0]$-rooted we have that F can be written as a non-negative sum of standard $[-1,0]$-rooted polynomial of degree d if and only if

$$F(x) = \sum_{i=0}^d a_i x^i (x+1)^{d-i},$$

where $a_i \geq 0$. This proves Theorem 4.2.

5. t-STACK SORTABLE PERMUTATIONS

For relevant definitions regarding t-stack sortable permutations we refer the reader to [2]. Let $W_t(n,k)$ be the number of t-stack sortable permutations in the symmetric group, $\mathcal{S}_n$, with k descents. Recently, Bóna [1, 3] showed that for fixed n and t the numbers $\{W_t(n,k)\}_{k=0}^{n-1}$ form a unimodal sequence. When $t = n-1$ and $t = 1$ we get the Eulerian and the Narayana numbers (see [36] and [33, Exercise 6.36]), respectively. These are known to be PF-sequences and Bóna [2, 3] has raised the question if this is true for general t. Here we will settle the problem to the affirmative for $t = 2$ and $t = n-2$.

The numbers $W_2(n,k)$ are surprisingly hard to determine despite their compact and simple form. It was recently shown that

$$W_2(n,k) = \frac{(n+k)!(2n-k-1)!}{(k+1)!(n-k)!(2k+1)!(2n-2k-1)!}.$$

See [4, 15, 20, 23] for proofs and more information on 2-stack sortable permutations.

From the case $r = 0$ in Lemma 3.13 and the identity

$$\sum_{k=0}^n \binom{2n-k-1}{n-1}\binom{n}{k} x^k = (-1)^n \sum_{k=0}^n \binom{-n}{k}\binom{n}{k}(-x)^{n-k},$$

it follows that $\binom{2n-k-1}{n-1}$ is an n-sequence.

Theorem 5.1. *For all $n \geq 0$ the sequence $\{W_2(n,k)\}_{k=0}^{n-1}$, which records 2-stack sortable permutations by descents, is PF.*

Proof. We may write $W_2(n,k)$ as

$$W_2(n,k) = \frac{\binom{2n-k-1}{n-1}\binom{n+k}{n-1}\binom{2n}{2k+1}}{n^2\binom{2n}{n}}.$$

An simple consequence of the notion of PF-sequences reads as follows: If $\{a_i\}_{i\geq 0}$ is PF then so is $\{a_{ki}\}_{i\geq 0}$, where k is any positive integer. Applying this to the polynomial $x(1+x)^{2n}$ we see that $\sum_k \binom{2n}{2k+1}x^k$ is real-rooted. Therefore the polynomial,

$$\sum_{k=0}^{n-1}\binom{n+k}{n-1}\binom{2n}{2k+1}x^k = \sum_{k=0}^{n-1}\binom{2n-k-1}{n-1}\binom{2n}{2k+1}x^{n-1-k},$$

is real-rooted. Another application of Lemma 3.13 gives that $W_{n,2}(x)$ is real-rooted. □

It is easy to see that a permutation $\pi \in \mathcal{S}_n$ is $(n-2)$-stack sortable if and only if it is not of the form $\sigma n1$. Thus the generating function satisfies

$$xW_{n,n-2}(x) = A_n(x) - xA_{n-2}(x),$$

where $A_n(x)$ is the nth Eulerian polynomial.

Theorem 5.2. *For all real numbers $t > -2$ and integers $n > 2$, the polynomial*

$$A_n(t,x) = A_n(x) + txA_{n-2}(x),$$

is real- and simple-rooted. Moreover, $A_n(t,x)/x$ strictly interlaces $A_{n+1}(t,x)/x$ for $-2 < t \leq 3$.

Corollary 5.3. *For all $n \geq 2$ we have that $\{W_{n-2}(n,k)\}_{k=0}^{n-1}$ is PF. Moreover, $W_{n,n-2}(x)$ strictly interlaces $W_{n+1,n-1}(x)$.*

Proof of Theorem 5.2. It is well known that $A_{n-1}(x) \ll xA_{n-2}(x)$ and $A_{n-1}(x) \preceq A_n(x)$. So by Lemma 2.2 we have that $A_n(t,x)$ is real- and simple-rooted for $t \geq 0$. However, when $t < 0$ a similar argument does not apply.

Let $E_n(t,x) = A_n(t, \frac{x}{1+x})$. Then

$$E_n(t,x) = E_n(x) + tx(1+x)E_{n-2}(x),$$

where the coefficient to x^k in $E_n(x)$ counts the number of surjections $\sigma : [n] \to [k]$, see [7, 38]. These polynomials satisfy the recursion:

$$E_n(x) = x\frac{d}{dx}((1+x)E_{n-1}(x)),$$

with initial condition $E_1(x) = x$. Thus, if we let $G_n(x) = E_{n+1}(x)/x$ we have the following recursion:

$$G_n(x) = \frac{d}{dx}(x(1+x)G_{n-1}(x)), \tag{5.1}$$

with $G_0(x) = 1$. Obviously $G_n(x)$ is real- and simple-rooted. If we apply (5.1) two times we get the equation:

$$G_n(x) = (1+6x+6x^2)G_{n-2}(x) + 3x(1+2x)(1+x)G'_{n-2}(x) + x^2(1+x)^2G''_{n-2}(x),$$

and

$$G_n(t,x) = (1+(6+t)x+(6+t)x^2)G_{n-2}(x)+3x(1+2x)(1+x)G'_{n-2}(x)+ x^2(1+x)^2G''_{n-2}(x).$$

To apply Theorem 3.4 we need show that for all $\xi \in \mathbb{R}$ and $-2 < t < 0$ the polynomial

$$F(\xi,z) := (1+(6+t)\xi+(6+t)\xi^2)+3\xi(1+2\xi)(1+\xi)z+\xi^2(1+\xi)^2z^2$$

is real-rooted. The discriminant of $F(\xi,z)$,

$$\Delta(F(\xi,z)) = \xi^2(1+\xi)^2(2+t+(3-t)(1+2\xi)^2),$$

is non-negative when $-2 \leq t \leq 3$, so $F(\xi,z)$ real-rooted for these t. Since all the Q_ks are standard it is easy to see that condition (III) in the statement of Theorem 3.4 is satisfied. Moreover, $1+(6+t)x+(6+t)x^2$ strictly interlaces $3x(1+2x)(1+x)$ when $t > -2$ so Theorem 3.4 applies. Since G_n strictly interlaces G_{n+1} we have by Theorem 3.4 and Corollary 2.5 that $\phi_F(G_n)$ strictly interlaces $\phi_F(G_{n+1})$. Thus $A_n(t,x)$ strictly interlaces $A_{n+1}(t,x)$. □

6. q-Eulerian and W-Eulerian polynomials

A q-analog of the Eulerian polynomials was introduced and studied in [17] and further studied in [10]. It is defined by

$$A_n(x;q) := \sum_{\pi\in\mathcal{S}_n} x^{\mathrm{exc}(\pi)}q^{\mathrm{c}(\pi)},$$

where $\mathrm{c}(\pi)$ and $\mathrm{exc}(\pi)$ denotes the number of *cycles* and *excedances* in π respectively. These polynomials satisfy the recursion

$$A_{n+1}(x;q) = (nx+q)A_n(x;q) - x(x-1)\frac{\partial}{\partial x}A_n(x;q),$$

with initial condition $A_0(x;q) := 1$. See [10] for a proof. The following theorem appears in [10].

Theorem 6.1. *Let $q \in \mathbb{R}$, $q > 0$. Then the polynomials $A_n(x,q)$ have only real non-positive simple zeros.*

Brenti also makes the following conjecture:

Conjecture 6.2. *Let $n,m \in \mathbb{N}$. Then $A_n(x;-m)$ has only real zeros.*

In what follows we will prove this conjecture using multiplier n-sequences. For $n \in \mathbb{N}$ define the polynomials $E_n(x;q)$ by:

$$E_n(x;q) := (1+x)^nA_n(\frac{x}{1+x};q).$$

It is clear that $E_n(x;q)$ is real-rooted if and only if $A_n(x;q)$ is real-rooted. These polynomials satisfy a somewhat easier recursion. Namely,

$$E_{n+1}(x;q) = (1+x)\{qE_n(x;q)+x\frac{\partial}{\partial x}E_n(x;q)\}, \tag{6.1}$$

with initial condition $E_0(x;q) = 1$. Now, for $q \in \mathbb{R}$ let $\Gamma_q : \mathbb{R}[x] \to \mathbb{R}[x]$ be the linear operator defined by $\Gamma_q(f(x)) = qf(x) + xf'(x)$. Since $\Gamma_q(x^n) = (q+n)x^n$ we may apply Lemma 3.14.

Theorem 6.3. *Let $q \in \mathbb{R}$ and $n \in \mathbb{N}$. If $q \geq 0$, $n \leq -q$ or $q \in \mathbb{Z}$ then $E_n(x;q)$ has only real zeros.*

Proof. We may write (6.1) as

$$E_{n+1}(x;q) = (x+1)\Gamma_q[E_n(x;q)].$$

The cases $q \geq 0$ and $n \leq -q$ follow from Lemma 3.14 by induction. We may therefore assume that $q = -m$ is a negative integer. We claim that $\deg E_n(x;q) = n$ if $n \leq m$ and $\deg E_n(x;q) = m$ if $n \geq m$. From this the real-rootedness follows by Lemma 3.14 and induction. The case $n \leq m$ is clear since $\Gamma_q[x^{n-1}] = -(m-n+1) < 0$. The case $n > m$ also follows by induction. Suppose that $n \geq m$ and that $\deg E_n(x;q) = m$. Then by the recursion we have that $\deg E_{n+1}(x;q) \leq m+1$. Moreover, since $\Gamma_q[x^m] = 0$ we have that $\deg E_{n+1}(x;q) \leq m$. Let $a \neq 0$ be the coefficient to x^m of $E_n(x;q)$. Then the coefficient to x^m of $E_{n+1}(x;q)$ is $a\Gamma_q[x^{m-1}] = -a$, so $\deg E_{n+1}(x;q) = m$, and the thesis follows. □

The Eulerian polynomial, $P(W,x)$, of a finite Coxeter group W is the polynomial,

$$P(W,x) = \sum_{\sigma\in W} x^{d_W(\sigma)},$$

where $d_W(\sigma)$ is the number of W-descents of σ, see [9]. This polynomial is also the generating function for the h-vector of the Coxeter complex associated to (W,S). For Coxeter groups of type A_n we have that $P(A_n,x) = A_n(x)/x$, the shifted Eulerian polynomial. Also, for Coxeter groups of type B_n it is known, see [9], that $P(B_n,x)$, has only real zeros. It is easy to see that $P(W_1 \times W_2, x) = P(W_1,x)P(W_2,x)$ for finite Coxeter groups W_1 and W_2. Also, the real-rootedness can be checked ad hoc for the exceptional groups. Thus, by the classification of finite irreducible Coxeter groups, to prove that $P(W,x)$ has only real zeros for all finite Coxeter groups it suffices to prove that $P(D_n,x)$ is real-rooted for Coxeter groups of type D_n. The real-rootedness of

$P(D_n, x)$ is conjectured by Brenti in [9]. It is known that the Eulerian polynomials of type A_n, B_n and D_n are related by, see [9, 29, 35]:

$$P(D_n, x) = P(B_n, x) - n2^{n-1}xP(A_{n-1}, x).$$

This relationship was first noticed by Stembridge [35]. One step towards proving the real-rootedness of $P(D_n, x)$ is to learn more about the relationships between the zeros of $P(B_n, x)$ and $P(A_n, x)$.

Brenti [9] introduced a q-analog of $P(B_n, x)$

$$B_n(x; q) = \sum_{\sigma \in B_n} q^{N(\sigma)} x^{d_B(\sigma)}, \tag{6.2}$$

where $d_B(\sigma)$ is the number of B_n-descents of σ and $N(\sigma)$ is the number of negative entries of σ, see [9]. He proved that

$$\sum_{i\geq 0} ((1+q)i+1)^n x^i = \frac{B_n(x;q)}{(1-x)^{n+1}}, \tag{6.3}$$

and that $B_n(x; q)$ is real- and simple-rooted for all $q \geq 0$. Suppose that $f(i)$ is a polynomial in i of degree d, then the polynomial $W(f)$ is defined by

$$\sum_{i\geq 0} f(i)x^i = \frac{W(f)(x)}{(1-x)^{d+1}},$$

One can show, see [7], that $\mathcal{E}(f)$ and $W(f)$ are related by:

$$\mathcal{E}(f)(x) = (1+x)^{\deg(f)} W(f)(\frac{x}{1+x}). \tag{6.4}$$

It follows that $W(f)$ has only real non-positive roots if and only if $\mathcal{E}(f)$ is $[-1, 0]$-rooted. Since $((1 + q)i + 1)^n$ is a $[-1, 0]$-rooted polynomial in i for any $q \geq 0$ it follows from e.g. Theorem 4.2 that $B_n(x; q)$ is real-rooted in x for any fixed $q \geq 0$. It is natural to generalize $B_n(x; q)$ to have $n+1$ parameters as $B_n(x; \mathbf{q}) := W\big(\prod_{i=0}^n ((1+q_i)x+1)\big)$. This polynomial has a nice combinatorial interpretation:

Theorem 6.4. *For all $n \in \mathbb{N}$ we have:*

$$B_n(x, \mathbf{q}) = \sum_{\sigma\in B_n} q_1^{\chi_1(\sigma)} q_2^{\chi_2(\sigma)} \cdots q_n^{\chi_n(\sigma)} t^{d_B(\sigma)},$$

where

$$\chi_i(\sigma) = \begin{cases} 1 & \text{if } \sigma_i < 0, \\ 0 & \text{if } \sigma_i > 0. \end{cases}$$

Proof. The proof is an obvious generalization of the proof of Theorem 3.4 of [9]. □

Note that this theorem gives a semi-combinatorial interpretation of the W-transform of any $[-1, 0)$-rooted polynomial.

Corollary 6.5. *Let $n \in \mathbb{N}$ and let $q_1, q_2, \dots, q_n$ be non-negative real numbers. Then $B_n(x; \mathbf{q})$ has only real and simple zeros.*

We need the following lemma on the degree of $W(f)$.

Lemma 6.6. *Let $f \in \mathbb{R}[x]$. Then*

$$\deg W(f) = \deg f - \mathrm{mult}(-1, \mathcal{E}(f)).$$

Moreover, $\mathrm{mult}(-1, \mathcal{E}(f))$ *is equal to the maximal integer k such that $(x + 1)(x + 2)\cdots(x + k)$ divides f.*

Proof. Since $\deg \mathcal{E}(f) = \deg f$ for all f we have by (6.4) that $\deg W(f) = \deg f - \mathrm{mult}(-1, \mathcal{E}(f))$. If we expand f in the basis $\{\binom{-x-1}{i}\}$ as:

$$\begin{aligned} f(x) &= \sum_{i\geq 0} (-1)^i a_i \binom{-x-1}{i}, \\ &= \sum_{i\geq 0} \frac{a_i}{i!}(x+1)\cdots(x+i), \end{aligned}$$

we have by Lemma 4.3 that

$$\mathcal{E}(f)(x) = \sum_{i\geq 0} a_i (x+1)^i,$$

and the lemma follows. □

We now have more precise knowledge of the location of the zeros of $B_n(x; q)$ for any given $q \geq 0$.

Theorem 6.7. *Let $0 < q < t \in \mathbb{R}$ and $n > 0$ be an integer. Then*

$$B_n(x; 0) \preceq B_n(x; t) \ll B_n(x; q) \ll xB_n(x; 0),$$

where the three first polynomials have no common zeros.

Proof. Let $0 < r < s < 1$. Then by the proof of Lemma 4.4 we have

$$\begin{aligned} &\mathcal{E}(x^n) \ll \mathcal{E}(x(x+r)^{n-1}) \ll_{strict} \mathcal{E}((x+r)^n) \ll_{strict} \mathcal{E}((x+r)^{n-1}(x+s)) \ll \\ &\qquad \mathcal{E}((x+s)^n) \ll_{strict} \mathcal{E}((x+s)^{n-1}(x+1)) \ll \mathcal{E}((x+1)^n), \end{aligned}$$

where $\ll_{strict}$ means strictly alternating left of. Since $(x + 1)\mathcal{E}(x^n) = x\mathcal{E}((x + 1)^n)$ this implies

$$\mathcal{E}(x^n) \ll_{strict} \mathcal{E}((x+r)^n) \ll_{strict} \mathcal{E}((x+s)^n) \ll_{strict} \mathcal{E}((x+1)^n).$$

Now since

$$B_n(x; q) = (q+1)^n W((x+\frac{1}{1+q})^n) = (q+1)^n (1-x)^n \mathcal{E}((x+\frac{1}{1+q})^n)(\frac{x}{1-x}),$$

we see by Lemma 6.6 that $\deg B_n(x; 0) = n - 1$ and $\deg B_n(x; q) = n$ if $q \neq 0$. Moreover, the alternating property is preserved under the operation (6.4) and the theorem follows. □

It follows from (6.2) that $P(B_n, x) = B_n(x; 1)$ and $P(A_n, x) = B_n(x; 0)$.

Corollary 6.8. *For all integers $n \geq 1$ we have that $P(A_n, x)$ strictly interlaces $P(B_n, x)$.*

Since $P(A_n, x) \ll xP(A_{n-1}, x)$ and $P(A_n, x) \preceq P(B_n, x)$, we have by Lemma 2.2 that for all $t \geq 0$ the polynomial $P(B_n, x) + txP(A_{n-1}, x)$ is real-rooted. Unfortunately a similar argument does not apply when $t < 0$.

One can extract more from (6.3). Brenti [9] proved that the polynomial

$$\sum_{\sigma \in B_n, N(\sigma) \in \{k, n-k\}} x^{d_B(\sigma)},$$

is real-rooted for all choices of $0 \leq k \leq n$. Using Theorem 4.6 we can extend this result to:

Corollary 6.9. *Let S be any subset of $[0, n]$. Then the polynomial*

$$P(B_n, S; x) := \sum_{\sigma \in B_n, N(\sigma) \in S} x^{d_B(\sigma)},$$

has only real and simple zeros.

Proof. Comparing the coefficient of q^i in both sides of (6.3) we see that $P(B_n, S; x) = W(f_n(S; x))$ where

$$f_n(S; x) = \sum_{s \in S} \binom{n}{s} x^s (x+1)^{n-s}.$$

So the theorem follows from Theorem 4.2. $\square$

One instance of Theorem 6.9 is particularly interesting. Recall that a Coxeter group of type D_n is isomorphic to the subgroup

$$D_n = \{\sigma \in B_n : \ 2 \mid N(\sigma)\}$$

Hence, we have the following corollary

Corollary 6.10. *For all $n \in \mathbb{N}$ the polynomial*

$$\sum_{\sigma \in D_n} x^{d_B(\sigma)}$$

has only real and simple zeros.

Note that the above polynomial is not $P(D_n, x)$, since B_n-descents and D_n-descents are not the same.

7. The h-vector of a family of simplicial complexes defined by Fomin and Zelevinsky

Fomin and Zelevinsky [18] recently associated to any finite Weyl group W a simplicial complex $\Delta_{FZ}(W)$. For the classical Weyl groups these polynomials are given by

$$\begin{aligned} h(\Delta_{FZ}(A_{n-1}), x) &= \frac{1}{n} \sum_{k=0}^{n-1} \binom{n}{k} \binom{n}{k+1} x^k, \\ h(\Delta_{FZ}(B_n), x) &= \sum_{k=0}^{n} \binom{n}{k} \binom{n}{k} x^k, \\ h(\Delta_{FZ}(D_n), x) &= h(\Delta_{FZ}(B_n), x) - nxh(\Delta_{FZ}(A_{n-2}), x). \end{aligned}$$

It is known that the h-polynomials corresponding to A_n and B_n have only real zeros. We will here show that so has $h(\Delta_{FZ}(D_n), x)$.

Theorem 7.1. *Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha \geq 0, 2\alpha + \beta > 0$ and let $n \geq 2$ be an integer. Then the polynomial*

$$F_n(\alpha, \beta) := \alpha h(\Delta_{FZ}(B_n), x) + \beta nxh(\Delta_{FZ}(A_{n-2}), x),$$

is real- and simple-rooted. Moreover, $h(\Delta_{FZ}(B_{n-1}), x)$ strictly interlaces $F_n(\alpha, \beta)$ if $\alpha > 0$ and strictly alternates left of $F_n(\alpha, \beta)$ if $\alpha = 0$.

Corollary 7.2. *Let W be a finite Weyl group. Then $h(\Delta_{FZ}(W), x)$ has only real and simple zeros. For the classical Weyl groups we have the following relationships:*

$$\begin{aligned} h(\Delta_{FZ}(A_{n-1}), x) &\preceq h(\Delta_{FZ}(A_n), x), \\ h(\Delta_{FZ}(B_{n-1}), x) &\preceq h(\Delta_{FZ}(B_n), x), \\ h(\Delta_{FZ}(B_{n-1}), x) &\preceq h(\Delta_{FZ}(D_n), x), \\ h(\Delta_{FZ}(A_{n-1}), x) &\preceq h(\Delta_{FZ}(B_n), x), \end{aligned}$$

where the interlacing is strict.

Proof. For the exceptional Weyl group one can check the real-rootedness ad hoc, see [30]. That $\{h(\Delta_{FZ}(A_n), x)\}_{n \geq 0}$ form a Sturm sequence is proved in [6]. The other cases follows from Theorem 7.1. $\square$

The *Hadamard product* of two polynomials

$$\begin{aligned} p(x) &= a_0 + a_1 x + \cdots + a_m x^m \\ q(x) &= b_0 + b_1 x + \cdots + b_n x^n \end{aligned}$$

is the polynomial

$$(p \star q)(x) = a_0 b_0 + a_1 b_1 x + \cdots + a_N b_N x^N,$$

where $N = \min(m, n)$. Málo proved that if the zeros of p are real and the zeros of q are real and of the same sign then the zeros of $p \star q$ are real as well. This also follows from Theorem 3.8 since $p \star q = \Gamma[pSq]$ where Γ is the multiplier sequence $\{\frac{1}{k!}\}_{k=0}^{\infty}$. It is known, see e.g. [19], that if f has only real zeros then all zeros of $\Gamma[f]$ are real and simple except for possibly at the origin.

Proof of Theorem 7.1. Let $F_n(\alpha,\beta) = \alpha h(\Delta_{FZ}(B_n),x)+\beta nxh(\Delta_{FZ}(A_{n-2}),x)$. We may write $F_n(\alpha,\beta)$ as

$$F_n(\alpha,\beta) = \alpha(x+1)^n \star (x+1)^n + \beta\big(x(x+1)^{n-1}\big) \star (x+1)^{n-1},$$

where the first summand can be written as

$$(x+1)\big((x+1)^{n-1} \star (x+1)^{n-1}\big) + 2\big(x(x+1)^{n-1}\big) \star (x+1)^{n-1}.$$

Thus $F_n(\alpha,\beta) = \alpha(x+1)f + (2\alpha+\beta)g$ where $f = (x+1)^{n-1}\star(x+1)^{n-1}$ and $g = \big(x(x+1)^{n-1}\big)\star(x+1)^{n-1}$. By the discussion before this proof we have that for all real choices of $\gamma,\delta \in \mathbb{R}$ the polynomial

$$\gamma f + \delta g = \big((\gamma+\delta x)(x+1)^{n-1}\big) \star (x+1)^{n-1},$$

is real- and simple-rooted. By the Obreschkoff theorem we infer that f strictly alternates left of g. Now, since $f \preceq (x+1)f$ and $f \ll g$ we know by Lemma 2.2 that f either interlaces or alternates left of $F_n(\alpha,\beta)$ for all $\alpha,\beta\in\mathbb{R}$ such that $\mathrm{sgn}(\alpha) = \mathrm{sgn}(2\alpha+\beta)$. Moreover, since g and f have no common zeros nor does $F_n(\alpha,\beta)$ and f (provided that $2\alpha+\beta\neq 0$). □

8. Two bilinear forms

There are a few bilinear forms on polynomials that occur frequently in combinatorics. Let $\# : \mathbb{R}[x]\times\mathbb{R}[x]\to\mathbb{R}[x]$ be defined by

$$(f\#g)(x) := \sum_{k\geq 0} f^{(k)}(x)g^{(k)}(x)\frac{x^k}{k!}.$$

This product is important when analyzing how the the zeros of σ-polynomials behave under disjoint union of graphs, see [11].

Theorem 8.1. *Let f be real-rooted and let g have only real zeros of the same sign. Then $f\#g$ is real-rooted.*

Proof. The theorem follows from Theorem 3.11, since $\{1\}_{k=0}^{\infty}$ is trivially a multiplier-sequence. □

This generalizes a result of Wagner, who proved that $f\#g$ is real-rooted whenever f and g have only non-negative zeros, see [11, 37].

Recall the definition, (4.2), of the diamond product. This product is important in the theory of (P,ω)-partitions and the Neggers-Stanley conjecture, see [38]. Applying Theorem 3.11 with the multiplier-sequence $\{\frac{1}{k!}\}_{k\geq 0}$ we get:

Theorem 8.2. *Let f be real-rooted and let g have all zeros in the interval $[-1,0]$. Then $f\diamond g$ is real-rooted.*

This was first proved by Wagner [39] under the additional hypothesis that f has all zeros in $[-1,0]$, and generalized by the present author in [5].

Acknowledgments. The author would like to thank Francesco Brenti for interesting discussions and helpful suggestions on the topic of this paper during the authors stay at Universitá di Roma "Tor Vergata".

References

[1] M. Bóna. Corrigendum: "Symmetry and unimodality in t-stack sortable permutations". *J. Combin. Theory Ser. A*, 99(1):191–194, 2002.

[2] M. Bóna. A survey of stack-sorting disciplines. *Electron. J. Combin.*, 9(2), 2002.

[3] M. Bóna. Symmetry and unimodality in t-stack sortable permutations. *J. Combin. Theory Ser. A*, 98(1):201–209, 2002.

[4] M. Bousquet-Mélou. Multi-statistic enumeration of two-stack sortable permutations. *Electron. J. Combin.*, 5(1):Research Paper 21, 12 pp. (electronic), 1998.

[5] P. Brändén. On operators on polynomials preserving real-rootedness and the Neggers-Stanley conjecture. *J.Algebraic Comb.*, to appear.

[6] P. Brändén. The generating function of two-stack sortable permutations by descents is real-rooted. preprint: arXiv:math.CO/0303149.

[7] F. Brenti. Unimodal, log-concave and Pólya frequency sequences in combinatorics. *Mem. Amer. Math. Soc.*, 81(413):viii+106, 1989.

[8] F. Brenti. Log-concave and unimodal sequences in algebra, combinatorics, and geometry: an update. In *Jerusalem combinatorics '93*, volume 178 of *Contemp. Math.*, pages 71–89. Amer. Math. Soc., Providence, RI, 1994.

[9] F. Brenti. q-Eulerian polynomials arising from Coxeter groups. *European J. Combin.*, 15(5):417–441, 1994.

[10] F. Brenti. A class of q-symmetric functions arising from plethysm. *J. Combin. Theory Ser. A*, 91(1-2):137–170, 2000.

[11] F. Brenti, G. F. Royle, and D. G. Wagner. Location of zeros of chromatic and related polynomials of graphs. *Canad. J. Math.*, 46(1):55–80, 1994.

[12] L. Comtet. *Advanced combinatorics*. D. Reidel Publishing Co., Dordrecht, enlarged edition, 1974. The art of finite and infinite expansions.

[13] T. Craven and G. Csordas. Multiplier sequences for fields. *Illinois J. Math.*, 21(4):801–817, 1977.

[14] J. Dedieu. Obreschkoff's theorem revisited: what convex sets are contained in the set of hyperbolic polynomials? *J. Pure Appl. Algebra*, 81(3):269–278, 1992.

[15] S. Dulucq, S. Gire, and O. Guibert. A combinatorial proof of J. West's conjecture. *Discrete Math.*, 187(1-3):71–96, 1998.

[16] A. Edrei. On the generating functions of totally positive sequences. II. *J. Analyse Math.*, 2:104–109, 1952.

[17] D. Foata and M. Schützenberger. *Théorie géométrique des polynômes eulériens.* Lecture Notes in Mathematics, Vol. 138. Springer-Verlag, Berlin, 1970.

[18] S.V. Fomin and A.V. Zelevinsky. Y-systems and generalized associahedra. *Ann. of Math.*, 158(158):977–1018, 2003.

[19] J. Garloff and D. G. Wagner. Hadamard products of stable polynomials are stable. *J. Math. Anal. Appl.*, 202(3):797–809, 1996.

[20] I. P. Goulden and J. West. Raney paths and a combinatorial relationship between rooted nonseparable planar maps and two-stack-sortable permutations. *J. Combin. Theory Ser. A*, 75(2):220–242, 1996.

[21] G. H. Hardy, J. E. Littlewood, and G. Pólya. *Inequalities.* Cambridge, at the University Press, 1952. 2d ed.

[22] O. J. Heilmann and E. H. Lieb. Theory of monomer-dimer systems. *Comm. Math. Phys.*, 25:190–232, 1972.
[23] B. Jacquard and G. Schaeffer. A bijective census of nonseparable planar maps. *J. Combin. Theory Ser. A*, 83(1):1–20, 1998.
[24] S. Karlin. *Total positivity. Vol. I.* Stanford University Press, Stanford, Calif, 1968.
[25] M. Marden. *Geometry of polynomials.* Second edition. Mathematical Surveys, No. 3. American Mathematical Society, Providence, R.I., 1966.
[26] N. Obreschkoff. *Verteilung und Berechnung der Nullstellen reeller Polynome.* VEB Deutscher Verlag der Wissenschaften, Berlin, 1963.
[27] G. Pólya. *Collected papers.* The MIT Press, Cambridge, Mass.-London, 1974. Vol. II: Location of zeros, Edited by R. P. Boas, Mathematicians of Our Time, Vol. 8.
[28] E. D. Rainville. *Special functions.* The Macmillan Co., New York, 1960.
[29] V. Reiner. Descents and one-dimensional characters for classical Weyl groups. *Discrete Math.*, 140(1-3):129–140, 1995.
[30] V. Reiner and V. Welker. On the Charney-Davis and the Neggers-Stanley conjectures. *http://www.math.umn.edu/~reiner/Papers/papers.html*, 2002.
[31] J. Schur. Zwei sätze über algebraische gleichungen mit lauter reellen wurzeln. *J. Reine Angew. Math.*, 144(2):75–88, 1914.
[32] R. P. Stanley. Log-concave and unimodal sequences in algebra, combinatorics, and geometry. In *Graph theory and its applications: East and West (Jinan, 1986)*, volume 576 of *Ann. New York Acad. Sci.*, pages 500–535. New York Acad. Sci., New York, 1989.
[33] R. P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1999.
[34] R. P. Stanley. Positivity problems and conjectures in algebraic combinatorics. In *Mathematics: frontiers and perspectives*, pages 295–319. Amer. Math. Soc., Providence, RI, 2000.
[35] J. R. Stembridge. Some permutation representations of Weyl groups associated with the cohomology of toric varieties. *Adv. Math.*, 106(2):244–301, 1994.
[36] R. A. Sulanke. The Narayana distribution. *J. Statist. Plann. Inference*, 101(1-2):311–326, 2002. Special issue on lattice path combinatorics and applications (Vienna, 1998).
[37] D. G. Wagner. The partition polynomial of a finite set system. *J. Combin. Theory Ser. A*, 56(1):138–159, 1991.
[38] D. G. Wagner. Enumeration of functions from posets to chains. *European J. Combin.*, 13(4):313–324, 1992.
[39] D. G. Wagner. Total positivity of Hadamard products. *J. Math. Anal. Appl.*, 163(2):459–483, 1992.

Matematik, Chalmers tekniska högskola och Göteborgs universitet, S-412 96 Göteborg, Sweden
E-mail address: branden@math.chalmers.se