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On Linear Transformations Preserving the Pólya Frequency Property

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ABSTRACT. We prove that certain linear operators preserve the
Pólya frequency property and real-stability, and apply our re-
sults to settle open conjectures and open problems in combinatorics
proposed by Bóna, Brenti and Reiner/Welker.

1. INTRODUCTION

Many sequences encountered in various areas of mathematics, sta-
tistics and computer science are known or conjectured to be unimodal
or log-concave, see [8, 32, 34]. A sufficient condition for a sequence to
enjoy these properties is that it is a Pólya frequency (PF) sequence, or
equivalently for finite sequences, that its generating function has only
real and non-positive zeros. It is often the case that the generating
function of a finite PF-sequence has more transparent properties
when expanded in a basis of her than the standard basis \( \{x^k\}_{k=0}^{\infty} \) of \( \mathbb{R}[x] \).
Therefore it is natural to investigate how PF sequences translate when
expressed in various bases. This amounts to studying properties of
the linear operator that maps one basis to another. A systematic study of
this was first pursued by Brenti in [7]. This is also the theme of this paper.

In Section 3 we will study linear operators of the type
\[
\phi_F = \sum_{k=0}^n Q_k(x) x^k,
\]
where \( F(x) = \sum_{k=0}^n Q_k(x) x^k \in \mathbb{R}[x] \). Here we will give sufficient
conditions on \( F \) for \( \phi_F \) to preserve the PF-property. The results
attained generalize and unify theorems of Hermite, Poulain, Pólya and
Schur. We will also in this section give a sufficient condition for a fam-
ily of natural bi-linear forms to preserve the PF-property in both argu-
ments. This generalizes results of Wagner [11, 37, 38].

2. NOTATION AND PRELIMINARIES

In this section we collect definitions, notation and results that will
be used frequently in the rest of the paper. Let \( \{a_i\}_{i=0}^{\infty} \) be a sequence of
real numbers. It is **unimodal** if there is a number \( p \) such that \( a_0 \leq a_1 \leq \cdots \leq a_p \geq a_{p+1} \geq \cdots \), and **log-concave** if \( a_i^2 \geq a_{i-1}a_{i+1} \) for all
\( i > 0 \).

An infinite matrix \( A = (a_{ij})_{j=0}^{\infty} \) of real numbers is **totally positive**, \( TP \), if all minors of \( A \) are nonnegative. An infinite sequence \( \{a_i\}_{i=0}^{\infty} \) of
real numbers is a **Pólya frequency sequence**, \( PF \)-sequence, if the matrix
\( (a_{ij})_{j=0}^{\infty} \) is \( TP \). Thus a \( PF \)-sequence is by definition log-concave and therefore also unimodal. A finite sequence \( a_0, a_1, a_2, \ldots, a_n \) is said to be \( PF \) if the infinite sequence \( a_0, a_1, a_2, \ldots, a_0, 0, 0, \ldots \) is \( PF \).
A sequence \( \{a_i\}_{i=0}^{\infty} \) is said to be \( PF \) if all minors of size \( r \) of \( (a_{ij})_{j=0}^{\infty} \) are nonnegative. If the polynomials \( h_i(x) \) are linearly independent over \( \mathbb{R} \) then \( r \in \mathbb{N} \) we define the set \( PF(F) = \{h_i(x)\}_{i=0}^{\infty} \) to be
\[
PF(F) = \{h_i(x)\}_{i=0}^{\infty} = \sum_{i=0}^{\infty} \lambda_i h_i(x) : (\lambda_i)_{i=0}^{\infty} \text{ is } PF, \}
\]
and \( PF(F) = \{h_i(x)\}_{i=0}^{\infty} = \bigcap_{r=0}^{\infty} PF(F) \).

The following theorem characterizes \( PF \)-sequences. It was con-
jectured by Schornagel and proved by Edrei [16], see also [24].

**Theorem 2.1.** Let \( \{a_i\}_{i=0}^{\infty} \) be a sequence of real numbers with \( a_0 = 1 \). Then it is a \( PF \)-sequence if and only if the generating function can be
expanded, in a neighborhood of the origin, as
\[ \sum_{i \geq 0} a_i z^i = e^\gamma \prod_{i \geq 0} (1 + a_i z) \prod_{i \geq 0} (1 - \beta_i z), \]
where \( \gamma \geq 0 \), \( a_i, \beta_i > 0 \) and \( \sum_{i \geq 0} (a_i + \beta_i) < \infty \).

A consequence of this theorem is that a finite sequence is PF if and only if its generating function is a polynomial with only real non-positive zeros.

Let \( f, g \in \mathbb{R}[x] \) be real rooted with zeros: \( a_1 \leq \cdots \leq a_i \) and \( \beta_1 \leq \cdots \leq \beta_j \), respectively. We say that \( f \) interlaces \( g \), denoted \( f \leq g \), if \( j = i + 1 \) and
\[ \beta_1 \leq a_1 \leq \beta_2 \leq \cdots \leq \beta_{j-1} \leq a_j \leq \beta_j, \]
We say that \( f \) alternates left of \( g \), denoted \( f \ll g \), if \( i = j \) and
\[ a_1 \leq \beta_1 \leq \cdots \leq \beta_{j-1} \leq a_j \leq \beta_j. \]

If in addition \( f \) and \( g \) have no common zero then we say that \( f \) strictly interlaces \( g \) and \( f \) strictly alternates left of \( g \), respectively. We also say that two polynomials \( f \) and \( g \) alternate if one of the polynomials alternates left of or interlaces the other. We will need two simple lemmas concerning these concepts. A polynomial is said to be standard if its leading coefficient is positive.

**Lemma 2.2.** Let \( f \) and \( \{ f_i \}_{i=1}^\infty \) be real rooted standard polynomials,

(i) If for each \( 1 \leq i \leq n \) we have either \( g \ll f_i \) or \( g \leq f_i \) then the sum \( F = f_1 + f_2 + \cdots + f_n \) is real rooted with \( g \leq F \) or \( g \ll F \), depending on the degree of \( F \).

(ii) If for each \( 1 \leq i \leq n \) we have either \( f_i \ll g \) or \( f_i \leq g \), then the sum \( F = f_1 + f_2 + \cdots + f_n \) is real rooted with \( F \leq g \) or \( F \ll g \), depending on the degree of \( F \).

**Proof.** The lemma follows easily by counting the sign changes of \( F \) at the zeros of \( g \), see eg., [39, Prop. 3.5].

The next lemma is obvious:

**Lemma 2.3.** If \( f_0, f_1, \ldots, f_n \) are real rooted polynomials with \( f_0 \ll f_n \) and \( f_{i-1} \ll f_i \) for all \( 1 \leq i \leq n \), then \( f_i \ll f_j \) for all \( 0 \leq i < j \leq n \).

The following theorem is a characterization of alternating polynomials due to Oblomkov [38] and Denef [14]:

**Theorem 2.4.** Let \( f, g \in \mathbb{R}[x] \). Then \( f \) and \( g \) alternate (strictly alternate) if and only if all polynomials in the space
\[ \{ \alpha f + \beta g : \alpha, \beta \in \mathbb{R} \}, \]
have only real (not and simple) zeros.

An immediate but non-trivial consequence of this theorem is:

**Corollary 2.5.** Let \( \phi : \mathbb{R}[x] \to \mathbb{R}[x] \) be a linear operator. Then \( \phi \) preserves the real rootedness property (real and simple rootedness property) if and only if \( \phi \) preserves the alternating property (strictly alternating property).

We denote by \( \mathbb{N} \) the set of natural numbers \( \{0, 1, 2, \ldots \} \). The symmetric group of bijections \( \pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\} \) is denoted by \( S_n \). A descent in a permutation \( \pi \in S_n \) is an index \( 1 \leq i \leq n \) such that \( \pi(i) > \pi(i+1) \). Let \( \text{des}(\pi) \) denote the number of descents in \( \pi \). The Eulerian polynomials, \( A_n(x) \), are defined by \( A_n(x) = \sum_{\pi \in S_n} x^{\text{des}(\pi) + \tau(\pi)} \) and satisfy, see eg., [14]
\[ \sum_{k \geq 0} k^nx^n = \frac{A_n(x)}{(1-x)^{n+1}}. \]
The binomial polynomials are defined by \( A_n(1) = 1 \) and \( A_n(x) = \sum_{k \geq 0} \binom{n}{k} \frac{x^{n-k}}{k!(n-k)^{k+1}} \) for \( k \geq 1 \).

In several proofs we will implicitly use the fact that the zeros of a polynomial are continuous functions of the coefficients of the polynomial. In particular the limit of a sequence of real-rooted polynomials is again real-rooted. For a treatment of these matters we refer the reader to [25].

3. A CLASS OF LINEAR OPERATORS PRESERVING THE PF-PROPERTY

For any polynomial \( F(x, z) = \sum_{n=0}^\infty Q_n(x)z^n \in \mathbb{R}[x, z] \) we define a linear operator \( \phi_{F} : \mathbb{R}[x] \to \mathbb{R}[x] \) by
\[ \phi_{F}(f) := \sum_{k=0}^n Q_k(x) \frac{d^k}{dx^k} f(x). \]
In this section we will investigate for which \( F \in \mathbb{R}[x, z] \) the linear operator \( \phi_F \) preserves real-rootedness and the PF-property.

We will need some terminology and a theorem from [3]. For \( \xi \in \mathbb{R} \) let \( T_z : \mathbb{R}[x] \to \mathbb{R}[x] \) be the translation operator defined by \( T_z(f)(x) = f(x + \xi) \). For any linear operator \( \phi : \mathbb{R}[x] \to \mathbb{R}[x] \) we define a linear transform \( L_{\phi} : \mathbb{R}[x] \to \mathbb{R}[x] \) by
\[ L_{\phi}(f) := \phi(T_z(f)) \]
\[ = \sum_{n} \phi(f^{(n)}(x)) \frac{x^n}{n!} \quad (3.1) \]
\[ = \sum_{n} \phi(f^{(n)}(x)) \frac{x^n}{n!} \]

**Definition 3.1.** Let \( \phi : \mathbb{R}[x] \to \mathbb{R}[x] \) be a linear operator. Define a function \( d_{\phi} : \mathbb{R}[x] \to \mathbb{N} \cup \{-\infty\} \) by: If \( \phi(f^{(n)}) = 0 \) for all \( n \in \mathbb{N} \) we let

Let \( d_0(f) = -\infty \). Otherwise let \( d_0(f) \) be the smallest integer \( d \) such that 
\[
\phi(f(x)) = 0 \quad \text{for all } n > d, \quad \text{Hence } d_0(f) \leq \deg f \quad \text{for all } f \in \mathbb{R}[x],
\]

The set \( \mathcal{A}(\phi) \) is defined as follows: If \( d_0(f) = -\infty \) or \( \phi(f) \) is standard real and simple-rooted, then \( f \in \mathcal{A}(\phi) \). Moreover, \( f \in \mathcal{A}(\phi) \) if \( d = d_0(f) \geq 1 \) and all of the following conditions are satisfied:

(i) \( \phi(f^{(i)}) \) all have leading coefficients of the same sign and \( \deg(\phi(f^{(i)})) = \deg(\phi(f^{(i-1)})) + 1 \) for \( 1 \leq i \leq d \),

(ii) \( \phi(f) \) and \( \phi(f') \) have no common real zero,

(iii) \( \phi(f^{(d)}) \) strictly interchanges \( \phi(f^{(d-1)}) \),

(iv) for all \( \xi \in \mathbb{R} \) the polynomial \( L_0(f)(\xi, z) \) is real-rooted.

The following theorem is proved in [5]:

**Theorem 3.2.** Let \( \phi : \mathbb{R}[x] \to \mathbb{R}[x] \) be a linear operator. If \( f \in \mathcal{A}(\phi) \) then \( \phi(f) \) is real and simple-rooted.

We will also need the following classical theorem of Hermite and Poulain. For a proof see [26].

**Theorem 3.3.** Let \( f = a_0 + a_1 x + \cdots + a_n x^n \) and \( g \) be real-rooted polynomials. Then the polynomial

\[
f(x) = \frac{d}{dx} g(x) = a_0 g(x) + a_1 g'(x) + \cdots + a_n g^n(x)
\]

is real-rooted. Moreover, if \( f(\frac{d}{dx}) g \neq 0 \) then any multiple zero of \( f(\frac{d}{dx}) \) is a multiple zero of \( g \).

The following theorem gives a sufficient condition for a polynomial to be mapped onto a real-rooted polynomial.

**Theorem 3.4.** Let \( F = \sum_{k=0}^{n} Q_k(x) z^k \) be such that \( Q_0 \neq 0 \)

(i) For all \( \xi \in \mathbb{R} \), \( F(\xi, z) \) is real-rooted,

(ii) \( Q_0 \) strictly interchanges or strictly alternates left of \( Q_1 \), and \( \deg Q_0 = 0 \) or \( Q_0 \) and \( Q_1 \) have leading coefficients of the same sign.

Suppose that

(iii) \( f \) is real and simple-rooted and that for \( 0 \leq k \leq \deg f \) the polynomials \( \phi(f)(x) \) have their leading term of the same sign with

\[
\deg \phi(f)(x) = \deg Q_0 + \deg f - k.
\]

Then \( \phi(f) \) is real and simple-rooted.

**Proof.** We will show that the set of real and simple-rooted polynomials satisfying (III) is a subset of \( \mathcal{A}(\phi) \) by verifying conditions (i) (iv) of Definition 3.1. Condition (i) follows immediately from (III). For condition (iv) note that

\[
L_0(f)(\xi, z) = \sum_{k=0}^{n} Q_k(\xi) f^{(k)}(\xi + z).
\]

so by the Theorem 3.3 condition (iv) is satisfied, Suppose that \( \eta \) is a common zero of \( \phi(\psi(f)) \) and \( \phi(f') \). By (3.1) we have that \( 0 \) is a multiple zero of \( L_0(f)(\eta, z) \). Moreover, since \( L_0(f)(\eta, z) \) is not identically equal to zero, by (II), Theorem 3.3 tells us that \( 0 \) is a multiple zero of \( f(\eta, z) \). This means that \( \eta \) is multiple zero of \( f \) contrary to the assumption that \( f \) is simple-rooted, and verifies condition (ii).

For condition (iii) we have to show that for all \( \alpha \in \mathbb{R} \) such that \( x + \alpha \) satisfies (II) the polynomial \( \phi(1) = Q_0 \) strictly interchanges \( f(x) = \phi(x + \alpha) = (x + \alpha) Q_0 + Q_1 \). This follows from (II) when analyzing the sign of \( f(x) = \phi(x + \alpha) \) at the zeros of \( Q_0 \). Let \( a_0 < a_0 < \cdots < a_i \) be the zeros of \( Q_0 \) ordered by size. Suppose that \( Q_0 \) and \( Q_1 \) are standard and that \( Q_0 \) strictly interchanges or strictly alternates left of \( Q_1 \). Then \( \phi(a_i) = \phi_Q(a_i) = (x + \alpha) \) for \( 1 \leq i \leq k \). By Rolle’s theorem we know that \( f \) has a zero in each interval \( (a_i, a_{i+1}) \).

This accounts for \( k-1 \) real zeros of \( f \). Since \( Q_0 \) has positive sign, so does \( d \) by condition (III). Now, because \( f(a_1) < 0 \) and \( f \) must have a zero to the right of \( a_1 \), We now know that \( f \) has \( k \) zeros real. The signs at \( a_i \) forces the remaining zeros to be in the interval \((\infty, a_1)\), Thus \( Q_0 \) strictly interchanges \( Q_1 \) as was to be shown.

Now, if \( Q_0 = A \in \mathbb{R} \) then \( \deg Q_1 \leq 1 \). Suppose that \( Q_1 = B \in \mathbb{R} \), Then clearly \( A \) strictly interchanges \( x + \alpha) A + B \). If \( Q_0 = A \) and \( Q_1 = Cx + D \) where \( A, B, C \in \mathbb{R} \) then \( \phi(f(x) + A + D) \) by (III) we have that \( Q_0 \) strictly interchanges \( f \). This concludes the proof.

In some cases it may be convenient to have sharper hypotheses. Therefore we state the following facts of the theorem.

**Corollary 3.5.** Let \( d \in \mathbb{N} \) be given and let \( F = \sum_{k=0}^{n} Q_k(x) z^k \) be such that \( Q_0 \neq 0 \)

(i) For all \( \xi \in \mathbb{R} \), \( F(\xi, z) \) is real-rooted,

(ii) \( Q_0 \) strictly interchanges or strictly alternates left of \( Q_1 \), and \( \deg Q_0 = 0 \) or \( Q_0 \) and \( Q_1 \) have leading coefficients of the same sign,

(iii) The polynomials \( \phi(f)(x), 0 \leq k \leq d \) have the same sign and

\[
\deg \phi(f)(x) = \deg Q_0 + k.
\]

Then \( \phi(f) \) is real rooted (real and simple rooted) if \( f \) is real rooted (real and simple rooted) and \( \deg(f) \leq d \).

**Proof.** The case of real and simple rooted \( f \) follows immediately from Theorem 3.4 since (iii) implies (II). If \( f \) is a real polynomial of degree \( m \), then \( f \) is the limit of a sequence \( \{ f_k \}_{k=0}^{\infty} \) of real polynomials of degree at most \( m \). It follows that \( \phi(f) \) is the limit of \( \phi(f_k) \) and the thesis follows by continuity.

In the language of PF-sequences we have

**Theorem 3.6.** Let \( d \in \mathbb{N} \) be given and let \( F = \sum_{k=0}^{n} Q_k(x) z^k \in \mathbb{R}[x, z] \) be such that \( Q_0 \neq 0 \) and
(i) For all $\xi \in \mathbb{R}$, $F(\xi, z) = z$ is real rooted, (ii) $\phi_F(1)$ strictly intersects $\phi_F(x)$, (iii) For all $0 \leq k \leq d$
\[
\deg \phi_F(x^k) = \deg Q_0 + k,
\]
and $\phi_F(x^k) \in PF$, Then $PF[\phi_F(x^k)] \subseteq PF[x]$.

Several old results can be derived from these last few theorems. In
[27, p. 163] Pólya gave a theorem which he states probably was the
most general theorem on real rootedness known at the time, "Dieser
Satz gehört wohl zu den allgemeinsten bekannten Sätzen über Warzel
realität;"

**Theorem 3.7.** Let $f(x)$ be a real-rooted polynomial of degree $n$, and let
\[
b_0 + b_1 x + \cdots + b_{n+m} x^{n+m}, \quad (m \geq 0)
\]
be a real-rooted polynomial such that $b_i > 0$ for $0 \leq i < n$. Then the
equation
\[
G(x, y) := b_0 f(y) + b_1 x f'(y) + b_2 x^2 f''(y) + \cdots + b_n f^{(n)}(y) = 0,
\]
has $n$ real intersection points, (counted with multiplicity), with the line
\[sx - ty + u = 0,
\]
provided that $s, t > 0, s + t > 0$ and $u \in \mathbb{R}$.

**Proof.** We may assume that $s, t > 0$ since the other cases follow by
continuity when $s$ and/or $t$ tends to zero. Thus we may write the equation
\[
a_0 g(x) + a_1 x g'(x) + a_2 x^2 g''(x) + \cdots + a_n g^{(n)}(x) = 0,
\]
where $g(x) = f(st^{-1}x + ut^{-1})$ and $a_i = s^{i-1}b_i$. Now, we see that all
hypothesis of Corollary 3.5 are satisfied for
\[
F(x, z) = a_0 + a_1 x z + a_2 x^2 z^2 + \cdots + a_{n+m} x^{n+m} z^{n+m},
\]
when $d = n$.

We will later need one famous consequence of this theorem, $t =
1, s = u = 0$ due to Schur[31].

**Theorem 3.8.** Let $f = \sum_{k=0}^{n} a_k x^k$ and $g = \sum_{k=0}^{m} b_k x^k$ be two real
rooted polynomials such that $g$ has all zeros of the same sign. Then the
polynomial
\[
(fSg)(x) = \sum_{k=0}^{M} c_k a_k b_k x^k,
\]
where $M = \min(m, n)$ has only real zeros.
Proof. We prove the statement for $\cdot$ since the case of $\circ$ is similar. We may assume that $\lambda_0 > 0$. Clearly the theorem is true if $\lambda_i = 0$ for all $i > 0$, so by Lemma 3.10 we may assume that $\lambda_1 > 0$. Let $g$ have all zeros simple and in the interval $(\alpha, \beta)$, and let $\phi$ be the linear operator defined by $\phi(f) = f \cdot g$. Then $\phi = \phi_f$, where

$$F(x, z) = \sum_{k=0}^{\infty} \lambda_k \frac{g^{(k)}(x)}{k!} (x - \alpha)^k (x - \beta)^{k-1}.$$ 

Since $\{\lambda_k\}_{k=0}^\infty$ is a multiplicity $n$ sequence for any real rooted polynomial $f = a_0 + a_1 x + \cdots + a_n x^n$ of degree at most $n$ the polynomial $G[f] = a_0 \lambda_0 + a_1 \lambda_1 x + \cdots + a_n \lambda_n x^n$ is real rooted. There is a simple algebraic characterization of multiplicity $n$ sequences [13]:

**Theorem 3.12.** Let $\Gamma = \{\gamma_k\}_{k=0}^\infty$ be a sequence of real numbers. Then $\Gamma$ is a multiplicity $n$ sequence if and only if $\Gamma[(x + 1)^n]$ is real-rooted with all its zeros of the same sign.

Recall the definition of the hypergeometric function $2F_1$:

$$2F_1(a, b; c; z) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m z^m}{(c)_m m!},$$

where $(a)_0 = 1$ and $(a)_m = a(a+1) \cdots (a+m-1)$ when $m \geq 1$. The Jacobi polynomials $P^{[\alpha, \beta]}_n(x)$ can be expressed as follows [28, p. 254]:

$$P^{[\alpha, \beta]}_n(x) = \frac{(1+\alpha)_n}{n!} 2F_1 \left( -n, 1+\alpha + \beta + n; 1+\alpha; \frac{1-x}{2} \right).$$

We need the following lemma:

**Lemma 3.13.** Let $n$ be a positive integer and $r$ a non-negative real number. Then $\Gamma = \{(\cdot)^r\}_{k=0}^\infty$ is a multiplicity $n$ sequence.

**Proof.** Let $r > 0$, then

$$\Gamma[(x + 1)^n] = \sum_{k=0}^{n} \binom{-n-r}{k} \frac{(n-r)_k}{k!} x^k = 2F_1 \left( -n, n+r; 1; x \right) = P^{[0,0]}_n(1-2x),$$

where the last equality follows from (3.2). Since the Jacobi polynomials are known, see [28], to have all their zeros in $[-1, 1]$ when $\alpha, \beta > -1$.

4. The $E$-transformation

The $E$-transformation is the invertible linear operator, $E : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$, defined by

$$E \left( \frac{x}{1-x} \right) = x^r,$$

for all $i \in \mathbb{N}$. The $PF$-preserving properties of this linear operator was first studied in [7] and later in [38, 39] and [5]. It is important in the theory of $(P, \omega)$-partitions since it maps the order-polynomial of a labeled poset to the $E$-polynomial of the same labeled poset, see [7, 38, 39]. In [7] Bimbi proved the following theorem, Let $\lambda(f)$ and $\Lambda(f)$ denote the smallest and the largest real zero of the polynomial $f$, respectively.

**Theorem 4.1.** Suppose that $f \in \mathbb{R}[x]$ has only real zeros and that $f(n) = 0$ for all $n \in (\lambda(f), -1) \cup [0, \Lambda(f)) \cap \mathbb{Z}$. Then $E(f)$ has all zeros real and non-positive.

In this section we will prove the following theorem:

**Theorem 4.2.** For all $n \in \mathbb{N}$ we have

$$PF_{\mathbb{R}}[(x^r(1-x)^{-1})] \subseteq PF_{\mathbb{R}} \left( \frac{x}{1-x} \right)$$

Moreover if $f \in PF_{\mathbb{R}}[(x^r(1-x)^{-1})]$ then $E(f)$ has simple zeros and $E(f(x+1)) \approx E(f(x^r))$.

The diamond product of two polynomials in $\mathbb{R}[x]$ is the $\mathbb{R}$-linear form defined by

$$(f \diamond g)(x) := E^{-1}(E(f)E^{-1}(g)).$$

(4.1)
This product was first studied by Wagner in [38, 39] and further studied
in [5]. See also Section 8 of this paper. Using the Vandermonde
identity
\[
\binom{x}{j} = \sum_{k=0}^{j} \binom{k}{k-j} x^j
\]

it follows, see [39], that
\[
(f \circ g)(x) = \sum_{k=0}^{n} \frac{n!}{k!} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x) x^k.
\]  

We will later need a symmetry property of \( \mathcal{E} \). Let \( \mathcal{R} : \mathbb{R}[x] \to \mathbb{R}[x] \) be
the algebra automorphism defined by \( \mathcal{R}(x) = -1 - x \).

**Lemma 4.3.**
\[
\mathcal{R} \mathcal{E} = \mathcal{E} \mathcal{R}
\]

**Proof.** From (4.2) it follows that
\[
\mathcal{R}(f) = \mathcal{R}(f) \circ \mathcal{R}(g)
\]  

Note that \( \mathcal{R} \mathcal{E}(f) = \mathcal{E} \mathcal{R}(f) \) whenever \( f \) is linear. Now, suppose that
\( f, g \) are polynomials such that \( \mathcal{R} \mathcal{E}(f) = \mathcal{E} \mathcal{R}(f) \) and \( \mathcal{R} \mathcal{E}(g) = \mathcal{E} \mathcal{R}(g) \).

Then
\[
\mathcal{R} \mathcal{E}(f) = \mathcal{R}(f) \circ \mathcal{E}(g) = \mathcal{E} \mathcal{R}(f) \circ \mathcal{R}(g)
\]

Since we may view \( \mathcal{E} \) and \( \mathcal{R} \) as \( \mathbb{C} \)-linear operators on \( \mathbb{C}[x] \), the lemma follows from the fundamental
theorem of algebra. \( \square \)

**Lemma 4.4.** Let \( \alpha \in [-1,0] \) and let \( f \) be a polynomial such that \( \mathcal{E}(f) \)
is \([-1,0]\)-rooted. Then \( \mathcal{E}(\mathcal{E}(x-\alpha) f) \) is \([-1,0]\)-rooted and \( \mathcal{E}(f) \)
interlaces \( \mathcal{E}(\mathcal{E}(x-\alpha) f) \). If \( \mathcal{E}(f) \) in addition only has simple zeros, then so does
\( \mathcal{E}(\mathcal{E}(x-\alpha) f) \).

**Proof.** Let \( y = \mathcal{E}(f) \) and \( \alpha \in [-1,0] \). By (4.2) we have that
\[
\mathcal{E}(\mathcal{E}(x-\alpha) f) = (x-\alpha) g + x(x+1) y' \quad \text{and} \quad \mathcal{E}(f) = \mathcal{E}(x-\alpha) f.
\]

Since \( g \) interlaces \( (x-\alpha) g \) and \( x(x+1) y' \) it also interlaces the sum, by Lemma 2.2. Also, if \( x \notin [-1,0] \) then the summands have the same
sign so \( \mathcal{E}(\mathcal{E}(x-\alpha) f) \) cannot have any zeros outside \([-1,0]\). Suppose
that \( g \) has only simple zeros, then by (4.4) the only possible common
zeros of \( g \) and \( \mathcal{E}(\mathcal{E}(x-\alpha) f) \) are \( 0 \) and \(-1\). If \( \deg(y) \geq 1 \) it also follows
from (4.4) that the multiplicities of \( 0 \) and \(-1\) of \( \mathcal{E}(\mathcal{E}(x-\alpha) f) \) are
the same as those of \( g \). Hence the (simple) zeros of \( g \) separates the zeros of
\( \mathcal{E}(\mathcal{E}(x-\alpha) f) \).

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\[ \mathcal{E}(x-\alpha) f \quad \text{except possibly at } 0, -1, \text{ and we conclude that } \mathcal{E}(x-\alpha) f \text{ has only simple zeros.} \]

**Lemma 4.5.** For all integers \( n \geq 1 \) we have
\[
(x+1)^n \mathcal{E}(x^n) = x \mathcal{E}(x+1)^n.
\]

**Proof.** We may write
\[
x^n = \sum_{k=1}^{n} a_k \binom{x}{k},
\]

where \( a_k \in \mathbb{R} \). Thus
\[
\mathcal{E}((x+1)^n) = \sum_{k=1}^{n} a_k \mathcal{E} \binom{x}{k} + \left( \frac{x}{k-1} \right)
\]

\[
= \sum_{k=1}^{n} a_k (x^k + x^{k-1}) = (x+1)^n \mathcal{E}(x^n).
\]

For \( i \in \mathbb{N} \) and let \( RR_i \) denote the set of real-rooted monic polynomials of degree \( n \). We define a partial order \( \leq \) on \( RR_i \) as follows: If \( f, g \in RR_i \), have zeros \( \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \) and \( \beta_1 \leq \beta_2 \leq \cdots \leq \beta_n \) respectively then \( f \leq g \), if \( \alpha_i \leq \beta_i \) for \( 1 \leq i \leq n \).

**Theorem 4.6.** Suppose that \( f \) and \( g \) are \([-1,0]\)-rooted with \( f \leq g \). Then \( \mathcal{E}(f) \) and \( \mathcal{E}(g) \) are \([-1,0]\)-rooted with \( \mathcal{E}(f) \leq \mathcal{E}(g) \).

**Proof.** By Lemma 4.4 and induction we only have to show that \( \mathcal{E}(f) \leq \mathcal{E}(g) \). If \( f \) and \( g \) have the same zeros except for one, i.e., \( f = (x-\alpha) h \) and \( g = (x-\beta) h \) where \( \alpha < \beta \), then
\[
\mathcal{E}(g) = \mathcal{E}(f) - (\beta - \alpha) \mathcal{E}(h),
\]

and since \( \mathcal{E}(h) \) interlaces \( \mathcal{E}(f) \) we have \( \mathcal{E}(f) \leq \mathcal{E}(g) \) by Lemma 2.2.

Now, suppose that \( f \) and \( g \) are \([-1,0]\)-rooted polynomials of degree \( n \) \( f \leq g \). Then there are \([-1,0]\)-rooted polynomials \( \{h_i\}_{i=0}^{n-1} \) with
\[
(x+1)^n = h_0 \leq h_1 \leq \cdots \leq h_{n-1} = x^n,
\]

such that \( f, g \in \{h_i\}_{i=0}^{n-1} \) and \( h_{i-1} \) and \( h_i \) only differ in one zero for \( 1 \leq i \leq n \). We therefore have
\[
\mathcal{E}(h_0) \leq \mathcal{E}(h_1) \leq \cdots \leq \mathcal{E}(h_{n-1}),
\]

and since \( \mathcal{E}(h_0) \leq \mathcal{E}(h_{n-1}) \), by Lemma 4.5, the theorem follows from Lemma 2.3. \( \square \)

A consequence of Theorem 4.6 is that if \( \{f_i\}_{i=0}^{n} \) is a sequence of standard \([-1,0]\)-rooted polynomials of the same degree \( d \), then by Lemma
2, and Theorem 4.6, the image under $E$ of any nonnegative sum $F = \sum_{i=1}^m a_i f_i$ will be $[-1, 0]$-rooted with $E((x+1)^d) \preceq E(F) \preceq E(x^d)$.

It is easy to see that a standard polynomial $f$ of degree $d$ is $[-1, 0]$-rooted if and only if $f$ can be written as

$$f(x) = (x+1)^{d} g(x),$$

where $g$ is a standard and $(-\infty, 0]$-rooted. On the other hand, since $x^d(x+1)^{d-i}$ is $[-1, 0]$-rooted we have that $F$ can be written as a nonnegative sum of standard $[-1, 0]$-rooted polynomial of degree $d$ if and only if

$$F(x) = \sum_{i=0}^d a_i x^{d-i},$$

where $a_i \geq 0$. This proves Theorem 4.2.

5. 1-STACK SORTABLE PERMUTATIONS

For relevant definitions regarding $t$ stack sortable permutations we refer the reader to [2]. Let $W_t(n, k)$ be the number of $t$ stack sortable permutations in the symmetric group, $S_n$, with $k$ descents. Recently, Bona [1, 3] showed that for fixed $n$ and $t$ the numbers $\{W_t(n, k)\}_{k=0}^{n-1}$ form a unimodal sequence. When $t = n-1$ and $t = 1$ we get the Eulerian and the Narayana numbers [see [30] and [33, Exercise 6.36]], respectively. These are known to be $PF$ sequences and Bona [2, 3] has raised the question of whether $W_t(n, k)$ is also $PF$. Here we will settle the problem to the affirmative for $t=2$ and $t=n-2$.

The numbers $W_2(n, k)$ are surprisingly hard to determine despite their compact and simple form. It was recently shown that

$$W_2(n, k) = \frac{(n+k)!(2n-k-1)!}{(k+1)!(n-k)!(2k+1)!(2n-2k-1)!}.$$

See [4, 15, 20, 23] for proofs and more information on 2 stack sortable permutations.

From the case $r = 0$ in Lemma 3.13 and the identity

$$\sum_{k=0}^n \binom{2n-k-1}{n-k} n^k = (-1)^n \sum_{k=0}^n \binom{n}{k} (-n)^{n-k},$$

it follows that $\{W_2(n, k)\}_{k=0}^{n-1}$ is a $PF$ sequence.

**Theorem 5.1.** For all $n \geq 0$ the sequence $\{W_2(n, k)\}_{k=0}^{n-1}$, which counts 2-stack sortable permutations by descents, is $PF$.

**Proof.** We may write $W_2(n, k)$ as

$$W_2(n, k) = \frac{(2n-k-1)!(n-k)!(2n-k+1)!}{n^2!}.$$

An simple consequence of the notion of $PF$-sequences reads as follows: If $\{a_k\}_{k=0}^\infty$ is $PF$ then so is $\{a_k\}_{k=0}^\infty$, where $k$ is any positive integer.

Applying this to the polynomial $x(1+x)^m$ we see that $\sum_{k=0}^m (2n-k) x^k$ is real-rooted. Therefore the polynomial

$$\sum_{k=0}^m (n+k)(2n-k-1) x^k = \sum_{k=0}^m (2n-k-1)(2n-k+1)x^{n-k},$$

is real-rooted. Another application of Lemma 3.13 gives that $W_{n-2}(x)$ is real-rooted.

It is easy to see that a permutation $\pi \in S_n$ is $(n-2)$-stack sortable if and only if it is not of the form $\sigma 1\sigma$, where $\sigma$ is the $(n-1)$th Eulerian polynomial.

**Theorem 5.2.** For all real numbers $t > -2$ and integers $n \geq 2$, the polynomial

$$A_n(t, x) = A_n(x) + tx A_{n-2}(x),$$

is real and simple rooted. Moreover, $A_n(t, x)/x$ strictly interlaces $A_{n+1}(t, x)/x$ for $-2 < t \leq 3$.

**Corollary 5.3.** For all $n \geq 2$ we have that $\{W_{n-2}(n, k)\}_{k=0}^{n-1}$ is $PF$. Moreover, $W_{n-2}(x)$ strictly interlaces $W_{n+1,n-1}(x)$.

Proof of Theorem 5.2. It is well known that $A_{n+1}(x) \preceq x A_{n-2}(x)$ and $A_{n-1}(x) \preceq A_{n-3}(x)$. So by Lemma 3.2 we have that $A_n(x)$ is real and simple rooted for $t \geq 0$. However, when $t < 0$ a similar argument does not apply.

Let $E_n(x) = A_n(x) + \frac{x}{1+x}$. Then

$$E_n(x) = E_0(x) + tx (1+x) E_{n-1}(x),$$

where the coefficient to $x^k$ in $E_n(x)$ counts the number of surjections $\sigma : [n] \rightarrow [k]$, see [7, 38]. These polynomials satisfy the recursion:

$$E_n(x) = x \frac{d}{dx} (1+x) E_{n-1}(x),$$

with initial condition $E_0(x) = x$. Thus, if we let $G_n(x) = E_{n+1}(x)/x$ we have the following recursion:

$$G_n(x) = \frac{d}{dx} (x(1+x) G_{n-1}(x)), \quad (5.1)$$
with \( G_0(x) = 1 \). Obviously \( G_n(x) \) is real and single-rooted. If we apply (5.1) two times we get the equation:
\[
G_n(x) = (1 + 6x + 6x^2)G_{n-2}(x) + 3x(1 + 2x)(1 + x)G_{n-2}(x) + x^3(1 + x)^2G_{n-4}(x),
\]
and
\[
G_n(t,x) = (1 + (6t + x)(6t + x^2))G_{n-2}(x) + 3x(1 + 2x)(1 + x)G_{n-2}(x) + x^3(1 + x)^2G_{n-4}(x).
\]
To apply Theorem 3.4 we need show that for all \( \xi \in \mathbb{R} \) and \(-2 < t < 0\) the polynomial
\[
F(t, z) := (1 + (6t + x)(6t + x^2)) + 3x(1 + 2x)(1 + x) + x^3(1 + x)^2z^2
\]
is real-rooted, that is, it is real for all \( t \) in the interval \((-2, 0)\). Since all the coefficients of \( F(t, z) \) are real and non-negative, then \( F(t, z) \) is real-rooted for all \( t \) in the interval \((-2, 0)\).

**Theorem 3.4.** Let \( q \in \mathbb{R} \) and \( n \in \mathbb{N} \). If \( q \geq 0 \), then \( A_n(x, q) \) has only real zeros.

**Proof.** We may write (6.1) as
\[
E_{n+1}(x; q) = (x + 1)E_n(x; q).
\]
The cases \( q \geq 0 \) and \( q < 0 \) follow from Lemma 3.1 by induction. We may therefore assume that \( q = -m \). We claim that \( \deg E_n(x; q) = n \) if \( n \leq m \) and \( \deg E_n(x; q) = m \) if \( n > m \). From this the result follows by Lemma 3.1 and induction. The case \( n < m \) is clear since \( \Gamma_n(x^{m-n}) = (n-m+1)^{-1} \times 1 \). The case \( n > m \) also follows by induction. Suppose that \( n \geq m \) and that \( \deg E_n(x; q) = m \). Then by the recurrence we have that \( \deg E_{n+1}(x; q) = m \) as well. Moreover, since \( \Gamma_n(x^{m-n}) = 0 \) we have that \( \deg E_{n+1}(x; q) = m \). Let \( a \neq 0 \) be the coefficient of \( x^m \) of \( E_n(x; q) \). Then the coefficient of \( x^{m-1} \) of \( E_{n+1}(x; q) \) is \( a \Gamma_{n-1}(x^{m-1}) = -a \), so \( \deg E_{n+1}(x; q) = m \), and the thesis follows.

The Eulerian polynomial,
\[
P(W, x) \quad \text{of a finite Coxeter group } W \text{ is the polynomial},
\]
\[
P(W, x) = \sum_{\sigma \in W} a_{\sigma} x^{|\sigma|},
\]
where \( a_{\sigma} \) is the number of \( W \)-descents of \( \sigma \), see [6]. This polynomial is also the generating function for the \( h \)-vector of the Coxeter complex associated to \( (W, S) \). For Coxeter groups of type \( A_n \), we have that \( P(A_n, x) = A_n(x) \), the shifted Eulerian polynomial. Also, for Coxeter groups of type \( B_n \), it is known, see [8], that \( P(B_n, x) \), has only real zeros. It is easy to see that \( P(W_1 \times W_2, x) = P(W_1, x)P(W_2, x) \) for finite Coxeter groups \( W_1 \) and \( W_2 \). Also, the real-rootedness can be checked ad hoc for the exceptional groups. Thus, by the classification of finite irreducible Coxeter groups, to prove that \( P(W, x) \) has only real zeros for all finite Coxeter groups \( W \) it suffices to prove that \( P(D_n, x) \) is real-rooted for Coxeter groups of type \( D_n \). The real-rootedness of
Let $P(D_n, x)$ be conjectured by Brenti in [9]. It is known that the Eulerian polynomials of type $A_n$, $B_n$, and $D_n$ are related by, see [9, 29, 35]:

$$P(D_n, x) = P(B_n, x) - n2^{n-1}xP(A_n-1, x).$$

This relationship was first noticed by Stembridge [35]. One step towards proving the real rootedness of $P(D_n, x)$ is to learn more about the relationships between the zeros of $P(B_n, x)$ and $P(A_n, x)$.

Brenti [9] introduced a $q$ analogue of $P(B_n, x)$

$$B_n(x; q) = \sum_{\sigma \in B_n} q^{N(\sigma)} x^{d_2(\sigma)}, \quad (6.2)$$

where $d_2(\sigma)$ is the number of $B_n$-descents of $\sigma$ and $N(\sigma)$ is the number of negative entries of $\sigma$, see [9]. He proved that

$$\sum_{\sigma \in B_n} ((1 + q)i + 1)^{x'} = B_n(x; q) \frac{1}{(1 - x)^{n+1}}, \quad (6.3)$$

and that $B_n(x; q)$ is real and simple rooted for all $q \geq 0$. Suppose that $f(i)$ is a polynomial in $i$ of degree $\ell$, then the polynomial $W(f)$ is defined by

$$\sum_{i \geq 0} f(i)x' = \frac{W(f)(x)}{(1 - x)^{\ell+1}}.$$  

One can show, see [7], that $E(f)$ and $W(f)$ are related by:

$$E(f)(x) = (1 + x)^{\deg(f)}W(f)(x)\frac{x}{1 + x}.$$  

It follows that $W(f)$ has only real non-positive roots if and only if $E(f)$ is $[-1, 0]$-rooted. Since $((1 + q)i + 1)^{x'}$ is a $[-1, 0]$-rooted polynomial in $i$ for any $q \geq 0$ it follows from eq. 6.2 that $B_n(x; q)$ is real-rooted in $x$ for any fixed $q \geq 0$. It is natural to generalize $B_n(x; q)$ to have $n+1$ parameters as $B_n(x; q) := W(\prod_{i=0}^{n} ((1 + q)i + 1))$. This polynomial has a nice combinatorial interpretation:

Theorem 6.4. For all $n \in \mathbb{N}$ we have:

$$B_n(x; q) = \sum_{\sigma \in B_n} \chi_i(\sigma) q_{x_1(\sigma)}^1 \cdots q_{x_n(\sigma)}^n x^{d_2(\sigma)},$$

where

$$\chi_i(\sigma) = \begin{cases} 1 & \text{if } \sigma_i < 0, \\ 0 & \text{if } \sigma_i > 0. \end{cases}$$

Proof. The proof is an obvious generalization of the proof of Theorem 3.4 of [9]. \hfill \Box

Note that this theorem gives a semi-combinatorial interpretation of the $W$-transform of any $[-1, 0]$-rooted polynomial.

Corollary 6.5. Let $n \in \mathbb{N}$ and let $q_1, q_2, \ldots, q_n$ be non-negative real numbers. Then $B_n(x; q)$ has only real and simple zeros.

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We need the following lemma on the degree of $W(f)$.

Lemma 6.6. Let $f \in \mathbb{R}[x]$. Then

$$\deg W(f) = \deg f - \mu(-1, E(f)).$$

Moreover, $\mu(-1, E(f))$ is equal to the maximal integer $k$ such that $(x + 1)(x + 2) \cdots (x + k)$ divides $f$.

Proof. Since $\deg E(f) = \deg f$ for all $f$ we have by [6, 4] that $\deg W(f) = \deg f - \mu(-1, E(f))$. If we expand $f$ in the basis $\left\{ \left( \frac{-1}{i} \right)^{\ell} \right\}$ as

$$f(x) = \sum_{i \geq 0} (-1)^i a_i \left( \frac{x - 1}{i} \right),$$

we have by Lemma 4.3 that

$$E(f)(x) = \sum_{i \geq 0} a_i(x + 1)^i,$$

and the lemma follows. \hfill \Box

We now have more precise knowledge of the location of the zeros of $B_n(x; q)$ for any given $q \geq 0$.

Theorem 6.7. Let $0 < q < t \in \mathbb{R}$ and $n > 0$ be an integer. Then

$$B_n(x; 0) \leq B_n(x; t) \leq B_n(x; q) \leq xB_n(x; 0),$$

where the three first polynomials have no common zeros.

Proof. Let $0 < r < s < 1$, then by the proof of Lemma 4.4 we have

$$E(x^n) \ll E((x + r)^{n-1}) \ll E((x + r)^n) \ll E((x + s)^n) \ll E((x + s)^{n+1}) \ll E((x + 1)^n),$$

where $\ll$ means strictly alternating left of, since $(x + 1)E(x^n) = xE((x + 1)^n)$ this implies

$$E(x^n) \ll E((x + r)^n) \ll E((x + s)^n) \ll E((x + 1)^n).$$

Now since

$$B_n(x; q) = (q + 1)^nW((x + 1 + q)^n) = (q + 1)^n(1 - x)^n E((x + 1 + q)^n),$$

we see by Lemma 6.6 that $\deg B_n(x; 0) = n - 1$ and $\deg B_n(x; q) = n$ if $q \neq 0$. Moreover, the alternating property is preserved under the operation $(6.4)$ and the theorem follows. \hfill \Box

It follows from $(6.2)$ that $P(B_n, x) = B_n(x; 1)$ and $P(A_n, x) = B_n(x; 0)$.

Corollary 6.8. For all integers $n \geq 1$ we have that $P(A_n, x)$ strictly interlaces $P(B_n, x)$.
Since $P(A_n, x) \leq xP(A_{n-1}, x)$ and $P(A_n, x) \preceq P(B_n, x)$, we have by Lemma 2.2 that for all $t \geq 0$ the polynomial $P(B_n, x) + txP(A_{n-1}, x)$ is real-rooted. Unfortunately a similar argument does not apply when $t < 0$.

One can extend more from (6.3), Bessenyei [8] proved that the polynomial

$$\sum_{\sigma \in B_n, N(\sigma) \leq (n-1)} x^{d_\sigma(\sigma)},$$

is real-rooted for all choices of $0 \leq k \leq n$, Using Theorem 4.6 we can extend this result to

**Corollary 6.9.** Let $S$ be any subset of $[0, n]$. Then the polynomial

$$P(B_n, S, x) := \sum_{\sigma \in B_n, N(\sigma) \leq S} x^{d_\sigma(\sigma)},$$

has only real and simple zeros.

**Proof.** Comparing the coefficient of $q^i$ in both sides of (6.3) we see that $P(B_n, S, x) = W(f_n(S, x))$ where

$$f_n(S, x) = \sum_{\sigma \in S} \binom{n}{k} x^k(x + 1)^{n-k}.$$

So the theorem follows from Theorem 4.2. \qed

One instance of Theorem 6.9 is particularly interesting. Recall that a Coxeter group of type $D_n$ is isomorphic to the subgroup

$$D_n = \{ \sigma \in B_n : 2 \mid N(\sigma) \}.$$

Hence, we have the following corollary

**Corollary 6.10.** For all $n \in \mathbb{N}$ the polynomial

$$\sum_{\sigma \in D_n} x^{d_\sigma(\sigma)}$$

has only real and simple zeros.

Note that the above polynomial is not $P(D_n, x)$, since $B_n$ descents and $D_n$ descents are not the same.

7. The $h$-vector of a family of simplicial complexes defined by Fomin and Zelevinsky

Fomin and Zelevinsky [18] recently associated to any finite Weyl group $W$ a simplicial complex $\Delta_{FZ}(W)$. For the classical Weyl groups these polynomials are given by

$$h(\Delta_{FZ}(A_{n-1}), x) = \prod_{k=0}^{n-1} \binom{n}{k} \left( x^{k+1} \right),$$

$$h(\Delta_{FZ}(B_n), x) = \prod_{k=0}^{n} \binom{n}{k} x^{k},$$

$$h(\Delta_{FZ}(D_n), x) = h(\Delta_{FZ}(B_n), x) - nh(\Delta_{FZ}(A_{n-2}), x).$$

It is known that the $h$-polynomials corresponding to $A_n$ and $B_n$ have only real zeros, We will here show that so has $h(\Delta_{FZ}(D_n), x)$.

**Theorem 7.1.** Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha \geq 0, 2\alpha + \beta > 0$ and let $n \geq 2$ be an integer. Then the polynomial

$$F_n(\alpha, \beta) := ah(\Delta_{FZ}(B_n), x) + \beta nh(\Delta_{FZ}(A_{n-2}), x),$$

is real and simple-rooted. Moreover, $h(\Delta_{FZ}(B_n), x)$ strictly interlaces $F_n(\alpha, \beta)$ if $\alpha > 0$ and strictly alternates left of $F_n(\alpha, \beta)$ if $\alpha = 0$.

**Corollary 7.2.** Let $W$ be a finite Weyl group. Then $h(\Delta_{FZ}(W), x)$ has only real and simple zeros. For the classical Weyl groups we have the following relationships:

$$h(\Delta_{FZ}(A_{n-1}), x) \preceq h(\Delta_{FZ}(A_n), x),$$

$$h(\Delta_{FZ}(B_{n-1}), x) \preceq h(\Delta_{FZ}(B_n), x),$$

$$h(\Delta_{FZ}(D_{n-1}), x) \preceq h(\Delta_{FZ}(D_n), x),$$

$$h(\Delta_{FZ}(A_{n-1}), x) \preceq h(\Delta_{FZ}(B_n), x),$$

where the interlacing is strict.

**Proof.** For the exceptional Weyl group one can check the real-rootedness as before, see [36], That $\{h(\Delta_{FZ}(A_{n-1}), x)\}_{n \geq 0}$ form a Sturm sequence is proved in [6], The other cases follows from Theorem 7.1. \qed

The Hadamard product of two polynomials

$$p(x) = a_0 + a_1x + \cdots + a_mx^m,$$

$$q(x) = b_0 + b_1x + \cdots + b_nx^n,$$

is the polynomial

$$(p \ast q)(x) = a_0b_0 + a_1b_1x + \cdots + a_nb_nx^{N},$$

where $N = \min(m, n)$. Malo proved that if the zeros of $p$ are real and the zeros of $q$ are real and of the same sign then the zeros of $p \ast q$ are real as well, This also follows from Theorem 3.8 since $p \ast q = \Gamma[pq]$ where $\Gamma$ is the multiplier sequence $\{\frac{1}{t}\}_{t=1}^{\infty}$. It is known, see e.g. [19], that if $f$ has only real zeros then all zeros of $\Gamma[f]$ are real and simple except for possibly at the origin,
Proof of Theorem 7.1. Let $F_0(\alpha, \beta) = \alpha h(\Delta x(B_0), x) + \beta \lambda h(\Delta x(A_0, x)).$
We may write $F_0(\alpha, \beta)$ as
$$F_0(\alpha, \beta) = \alpha (x + 1)\* (x + 1)^{n-1} \* (x + 1)^{n-1},$$
where the first summand can be written as
$$(x + 1)^{n-1} \* (x + 1)^{n-1} + 2(x + 1)^{n-1} \* (x + 1)^{n-1}.$$  
Thus $F_0(\alpha, \beta) = \alpha (x + 1)^{n} + (2\alpha + \beta)g$ where $g = (x + 1)^{n-1} \* (x + 1)^{n-1}$ and $g = (x + 1)^{n-1} \* (x + 1)^{n-1}$. By the discussion before this proof we have that for all real choices of $\gamma, \delta \in \mathbb{R}$ the polynomial
$$\gamma f + \delta g = ((\gamma + \delta)x + 1)^{n-1}$$
is real and simple-rooted. By the Oberschlepp theorem we infer that $f$ strictly alternates left of $g$. Now, since $f \preceq (x + 1)^{n}$ and $f \preceq g$ we know by Lemma 2.2 that $f$ either alternates or alternates left of $F_0(\alpha, \beta)$ for all $\alpha, \beta \in \mathbb{R}$ such that $\text{sgn}(\alpha) = \text{sgn}(2\alpha + \beta)$. Moreover, since $g$ and $f$ have no common zeros nor does $F_0(\alpha, \beta)$ and $f$ (provided that $2\alpha + \beta \neq 0$).

8. TWO-BILINEAR FORMS

There are few bilinear forms on polynomials that occur frequently in combinatorics. Let $\mathbb{R}[x] \times \mathbb{R}[x] \to \mathbb{R}$ be defined by\n
$$(f \# g)(x) := \sum_{k \geq 0} f^{(k)}(x)g^{(k)}(x)\frac{x^k}{k!}.$$

This product is important when analyzing how the the zeros of $\sigma$-polynomials behave under disjoint union of graphs, see [11].

Theorem 8.1. Let $f$ be real rooted and let $g$ have only real zeros of the same sign. Then $f \# g$ is real rooted.

Proof. The theorem follows from Theorem 3.11, since $(1)_{\mathbb{R}_{\geq 0}}$ is trivially a multipliers-sequence.

This generalizes a result of Wagner, who proved that $f \# g$ is real-rooted whenever $f$ and $g$ have only non-negative zeros, see [11, 37].

Recall the definition, (4.2), of the diamond product. This product is important in the theory of $(P, \omega)$-partitions and the Neggers-Stanley conjecture, see [38]. Applying Theorem 3.11 with the multipliers-sequence $(\frac{1}{x})_{\mathbb{R}_{\geq 0}}$ we get:

Theorem 8.2. Let $f$ be real rooted and let $g$ have all zeros in the interval $[-1, 0]$. Then $f \# g$ is real rooted.

This was first proved by Wagner [39] under the additional hypothesis that $f$ has all zeros in $[-1, 0]$, and generalized by the present author in [5].

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