THESIS FOR THE DEGREE OF LICENTIATE OF ENGINEERING

Products of Residue Currents of Cauchy-Fantappiè-Leray Type

Elizabeth Wulcan

CHALMERS | GÖTEBORG UNIVERSITY

Department of Mathematics
Chalmers University of Technology and Göteborg University
Göteborg, Sweden 2004

Products of Residue Currents of Cauchy-Fantappiè-Leray Type
ELIZABETH WULCAN

NO 2004:26
ISSN 0347-2809

Department of Mathematics
Chalmers University of Technology and Göteborg University
SE-412 96 Göteborg
Sweden
Telephone +46 (0)31 772 1000

Printed in Göteborg, Sweden 2004

PRODUCTS OF RESIDUE CURRENTS OF CAUCHY-FANTAPPIE-LERAY TYPE

ELIZABETH WULCAN

ABSTRACT

To a given holomorphic section of a Hermitian vector bundle, one can associate a residue current by means of Cauchy-Fantappiè-Leray type formulas. In this thesis we define products of such residue currents. We prove that, in the case of a complete intersection, the product of the residue currents of a tuple of sections coincides with the residue current of the direct sum of the sections.

Keywords: Residue current, Coleff-Herrera current, Bochner-Martinelli formula, Hermitian vector bundle, Cauchy-Fantappiè-Leray formula, complete intersection.

AMS 2000 Subject Classification: 32A26, 32A27, 32C30

ACKNOWLEDGEMENTS

First of all I would like to thank my advisor Mats Andersson for sharing his knowledge with me and guiding me through my first years as a PhD student. I want to thank Elin Götmark and Håkan Samuelsson for valuable comments on the topic of this thesis. They have also been a great support. Also, I want to thank my friends and colleagues at the Department of Mathematics in Göteborg.

Thank You!

1. Introduction and background

Let f be a holomorphic function at 0 in $\mathbb{C}$. One can prove that there exists a distribution U^f such that $fU^f = 1$, as was first done by Schwartz [13]. One way of defining such a U^f, sometimes denoted by $[1/f]$, is as the *principal value distribution* of $1/f$, given by

$$\varphi \mapsto \lim_{\varepsilon \to 0} \int_{|f|>\varepsilon} \frac{\varphi}{f}.$$

Applying the $\bar\partial$ operator to U^f, we obtain a $(0,1)$-current, which we call the *residue current* of f and which we denote by $\bar\partial[1/f]$ or R^f. Clearly, R^f has support on $Y = f^{-1}(0)$ and it is easy to see that a function φ that is holomorphic in a neighborhood of Y belongs to the ideal generated by f precisely when it annihilates R^f, that is when $\varphi R^f = 0$. Observe that these last properties do not depend on the exact definition of U^f, but only on the fact that $fU^f = 1$. By Stokes' theorem, the action of R^f is given by

$$\bar\partial[1/f](\varphi) = \lim_{\varepsilon \to 0} \int_{|f|>\varepsilon} \frac{1}{f}\, \bar\partial\, \varphi = \lim_{\varepsilon \to 0} \int_{|f|>\varepsilon} d\, (\frac{\varphi}{f}) = \lim_{\varepsilon \to 0} \int_{|f|=\varepsilon} \frac{\varphi}{f}, \tag{1.1}$$

where the limit is taken over the regular values of $|f|$.

Given a tuple f of holomorphic functions $f_1, \ldots, f_m$ defined in some domain Ω in $\mathbb{C}^n$, it is natural to ask whether there are analogues to the currents U^f and R^f, that can be used to characterize the ideal (f). In general, definitions of such residue currents lead to integration over singular varieties, and thus the theory of multidimensional residue currents relies heavily on Hironaka's theorem of resolution of singularities from 1964, see [2].

In 1978, Coleff and Herrera [6] used the desingularization to construct a residue current of the tuple f as a product of residue currents of the functions f_i. For a test form $\phi \in \mathcal{D}_{n,n-m}(\Omega)$, consider the *residue integral*

$$I_f^\phi(\varepsilon) = \frac{1}{(2\pi i)^m} \int_{T_\epsilon^m} \frac{\phi}{f_1 \cdots f_m},$$

where $T_\epsilon^m = \{|f_1| = \epsilon_1, \ldots, |f_m| = \epsilon_m\}$ is oriented as the distinguished boundary of the corresponding polyhedron. To see how $\lim_{\varepsilon\to 0} I_f^\phi(\varepsilon)$, if it exists, can be interpreted as the action of a current $\bar\partial[1/f_1]\wedge \ldots \wedge \bar\partial[1/f_m]$, let us consider the case of two functions f_1 and f_2. Assuming f to be regular, that is $df_1 \wedge df_2 \neq 0$, on $Y = f^{-1}(0)$, we can locally choose f_1 and f_2 as coordinates, and for $\phi \in \mathcal{D}_{(n,n-2)}(\Omega)$ we get by Stokes' theorem (compare to (1.1)) that

$$\int_{|f_1|=\varepsilon_1,|f_2|=\varepsilon_2}\frac{\phi}{f_1f_2}=\int_{|f_1|=\varepsilon_1,|f_2|>\varepsilon_2}d_{f_2}(\frac{\phi}{f_1f_2})=\int_{|f_1|=\varepsilon_1,|f_2|>\varepsilon_2}\frac{1}{f_1}\bar{\partial}_{f_2}(\frac{\phi}{f_2})$$
$$=\int_{|f_1|>\varepsilon_1,|f_2|>\varepsilon_2}d_{f_1}(\frac{1}{f_1}\bar{\partial}_{f_2}\,(\frac{\phi}{f_2}))=\int_{|f_1|>\varepsilon_1,|f_2|>\varepsilon_2}\bar{\partial}_{f_1}(\frac{1}{f_1}\bar{\partial}_{f_2}\,(\frac{\phi}{f_2})).$$

In general $I_f^\phi(\varepsilon)$ does not have a limit as ε tends to 0 and it is easy to find examples that illustrate this, see for instance Example 7.1.3 in [8]. However, Coleff and Herrera showed that $\lim_{\varepsilon\to 0} I_f^\phi(\varepsilon)$ does exist if one lets ε tend to zero along a certain path, a so called *admissible trajectory*, for which each ε_j tends to zero faster than any power of ε_{j+1}. In the case of a complete intersection, that is, when the variety Y has codimension m, the limit is independent of the ordering of the functions and therefore, in this case, it is reasonable to expect the limit of $I_f^\phi(\varepsilon)$ to exist unconditionally. There are however counterexamples by Passare and Tsikh [11] and Björk [5] that show that this is not the case.

One way to circumvent the problem of such "bad" trajectories is to consider the residue current as certain averages of $I_f^\phi(\varepsilon)$, as first done by Passare in [9]. In order to avoid integration over the possibly singular varieties $\{|f|=\varepsilon\}$ he modifies the definition of residue currents of Coleff and Herrera [6] somewhat. Regarding the currents as limits of smooth forms he defines $[1/f]$ and $\bar{\partial}[1/f]$ as

$$\lim_{\varepsilon\to 0}\frac{1}{f}\chi(|f|/\varepsilon) \text{ and } \lim_{\varepsilon\to 0}\frac{1}{f}\bar{\partial}\chi(|f|/\varepsilon),$$

respectively, where $\chi(x)$ is a smooth nondecreasing function, equal to 0 in a neighborhood of the origin and equal to 1 for $x>c$ for some $c>0$. One can show that the currents are independent of the particular choice of χ. Further, products of principal value and residue currents are constructed, based on those definitions. Let

$$R^{m'}P^{(m-m')}\Big[\frac{1}{f}\Big](s)=\lim_{\varepsilon\to 0}\frac{1}{f_1}\cdots\frac{1}{f_m}\;\bar{\partial}\chi_1\wedge\ldots\wedge\bar{\partial}\chi_{m'}\chi_{m'+1}\cdots\chi_m,$$

where $\chi_j=\chi(|f_j|/\varepsilon^{s_j})$. Here, $R^{m'}P^{(m-m')}$ should be interpreted as the product of the residue currents of $f_1,\ldots,f_m'$ and the principal value currents of $f_{m'+1},\ldots,f_m$. In particular, $R^m[1/f](s)$ corresponds to $I_f^\phi(\varepsilon^{s_1},\ldots,\varepsilon^{s_m})$. Finally, the product $R^{m'}P^{(m-m')}[\frac{1}{f}]$ is defined as the mean value of $R^{m'}P^{(m-m')}[1/f](s)$ over the simplex

$$\Sigma_m=\{s\in\mathbb{R}^m, s_j>0\sum s_j=1\},$$

that is,

$$R^{m'}P^{(m-m')}[1/f] = \int_{\Sigma_m} R^{m'}P^{(m-m')}[1/f](s)dS_m(s),$$

where S_m is the Lebesgue measure, normalized with respect to Σ_m. In case f defines a complete intersection, the current $R^m[1/f](s)$ is shown to agree with the Coleff-Herrera residue current, as expected since $\lim_{\varepsilon\to 0} I(\varepsilon)$ is then independent of the path along which ε tends to 0 as long as it is an admissible trajectory.

Another way of averaging $I_f^{\phi}(\varepsilon)$, is to take the Mellin transform of $I_f^{\phi}(\varepsilon)$,

$$\Gamma_f^{\phi}(\lambda) = \int_{\mathbb{R}_+^p} I_f^{\phi}(\varepsilon)\varepsilon^{\lambda-1}d\varepsilon,$$

where $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{C}^m$ and $\varepsilon^{\lambda-1}d\varepsilon = \varepsilon_1^{\lambda_1-1}\cdots\varepsilon_m^{\lambda_m-1}d\varepsilon_1\wedge\ldots\wedge d\varepsilon_m$. A computation shows that

$$\Gamma_f^{\phi}(\lambda) = \frac{1}{\lambda_1\cdots\lambda_m}\int \frac{\bar\partial|f_1|^{2\lambda_1}\wedge\ldots\wedge\bar\partial|f_m|^{2\lambda_m}}{f_1\cdots f_m}\wedge\phi.$$

Now, $\Gamma_f^{\phi}(\lambda)$ is meromorphic in $\mathbb{C}^m$, and the polar structure at the origin is related to the limit behaviour of the residue integral, see [10]. In fact, if f is a complete intersection, it holds that

$$\lambda^m\Gamma_f^{\phi}(\lambda,\dots,\lambda)|_{\lambda=0} = \lim_{\varepsilon\to 0} I_f^{\phi}(\varepsilon),$$

where the limit is taken along an admissible trajectory. Thus, this yields an alternative definition of the Coleff-Herrera current. In the particular case when f consists of only two functions, the mapping $\lambda\mapsto\lambda_1\lambda_2\Gamma_f^{\phi}(\lambda)$ is holomorphic at the origin as first shown by Berenstein and Yger, see the proof of Theorem 3.18 in [3]. Although not yet satisfactorily proven, this result is believed to extend to any finite number of functions.

The residue currents we have studied so far are based on the Cauchy kernel $K = 1/(2\pi i)\ d\zeta/\zeta$. From the definition of the one-dimensional residue current, one sees that R^{ζ} times $d\zeta$ can be expressed as $\bar\partial K$, and so $R^f\wedge df = \bar\partial f^*K$. It follows that the Coleff-Herrera current can be regarded as the pull-back under f of the multiple Cauchy kernel

$$1/(2\pi i)^m d\zeta_1/\zeta_1\wedge\ldots\wedge d\zeta_m/\zeta_m,$$

and by that as the formal product of one-dimensional residue currents. Another approach to the multidimensional residue current would be to use other generalizations of the one-dimensional Cauchy kernel. In [12], Passare, Tsikh and Yger define a residue current based on the Bochner-Martinelli kernel. Let

$$b(\zeta) = \frac{\sum \bar{\zeta}_j d\zeta_j}{2\pi i|\zeta|^2}.$$

The Bochner-Martinelli kernel in $\mathbb{C}^m \setminus \{0\}$ is given by

$$B(\zeta) = b(\zeta) \wedge (\bar{\partial} b(\zeta))^{m-1} = \frac{\sum (-1)^{j-1} \bar{\zeta}_j \widehat{d\bar{\zeta}_j}}{(2\pi i)^m |\zeta|^{2m}},$$

where $\widehat{d\bar{\zeta}_j}$ denotes the form $d\bar{\zeta}_1 \wedge \ldots \wedge d\bar{\zeta}_{j-1} \wedge d\bar{\zeta}_{j+1} \wedge \ldots \wedge d\bar{\zeta}_m$. Clearly, $B(\zeta)$ is in L^1_{loc} and it is easy to show that

$$\bar{\partial} B(\zeta) = [0] = \frac{1}{(2\pi i)^m} \bar{\partial} \frac{1}{\zeta_m} \wedge \ldots \wedge \bar{\partial} \frac{1}{\zeta_1} \wedge d\zeta_1 \wedge \ldots \wedge d\zeta_m, \tag{1.2}$$

where the right hand expression should be recognized as the Coleff-Herrera current times $d\zeta_1 \wedge \ldots \wedge d\zeta_m$. If we let $B(f) = f^* B$ then $B(f)$ is clearly smooth outside Y. If f is regular on Y, it is easy to see that $B(f)$ is in L^1_{loc} and that $\bar{\partial} B(f)$ coincides with the Coleff-Herrera current R^f_{CH}. Indeed, after a coordinate change we are back to the case (1.2). Thus, it is natural to expect the current $\bar{\partial} B(f)$ to have meaning also for a more general f. Let

$$R^f_{BM}(\varepsilon) = \int_{|f|^2 = \varepsilon} B(f) \wedge \phi,$$

where $|f|^2 = \sum |f_j|^2$. In [12] it is shown that the limit of $R^f_{BM}(\varepsilon)$ as ε tends to zero, always exists. This is done by proving the existence of the meromorphic continuation of the corresponding Mellin transform. We denote the limit by R^f_{BM}. Moreover we have

Theorem 1.1 (Passare, Tsikh, Yger [12])**.** *Assume that f is a complete intersection. Then*

$$R^f_{BM} = R^f_{CH}.$$

The Bochner-Martinelli residue current has been used for investigations in the noncomplete intersection case. For example, in [4], Berenstein and Yger use the Bochner-Martinelli current to construct Green currents.

In [1], Andersson introduces an alternative approach to R^f_{BM}, based on the Koszul complex. Considering sections of holomorphic vector bundles rather than tuples of functions yields globally defined currents. For further reference, we give a presentation of the construction. As in the one-dimensional case we start out by looking for holomorphic solutions to $fU = 1$, now read as $\sum_{j=1}^m f_j U_j = 1$. We adopt an invariant point of view and assume that f is a holomorphic section of the dual bundle E^* of a holomorphic m-bundle $E \to X$ over a complex manifold X. If $e_1, \ldots, e_m$ is a local holomorphic frame for E and $e_1^*, \dots, e_m^*$ is the dual frame, we can write $f = \sum f_j e_j^*$. On the exterior algebra over E we have mappings

$$\delta_f : \Lambda^{\ell+1} E \to \Lambda^{\ell} E,$$

where δ_f is the interior multiplication by f. In particular, δ_f acts on a section $\psi = \sum \psi_j e_j$ of $E = \Lambda^1 E$ as $\delta_f \psi = \sum f_j \psi_j$, and thus we can formulate the original problem as finding holomorphic solutions to

$$\delta_f \psi = 1. \tag{1.3}$$

This kind of division problem can be solved by means of the Koszul complex. We start by looking for smooth solutions to (1.3). Outside $Y = f^{-1}(0)$ we can easily find one; $u_1 = \sum \bar{f}_j / |f|^2$. However, in general u_1 is not holomorphic, so we need to compensate for that. Now, let us introduce the spaces $\mathcal{E}_{0,k}(X, \Lambda^{\ell} E)$ and $\mathcal{D}'_{0,k}(X, \Lambda^{\ell} E)$ of $(0,k)$ forms and currents, respectively, that take values in $\Lambda^{\ell} E$. Note that δ_f and $\bar\partial$ extend to the exterior algebra over $T^*_{0,1} \oplus E$, where they anticommute. Thus, $\delta_f \bar\partial u_1 = -\bar\partial \delta_f u_1 = 0$, and since the Koszul complex of f is exact outside Y we can find a $u_2 \in \mathcal{E}_{0,1}(X, \Lambda^2 E)$ such that $\delta_f u_2 = \bar\partial u_1$. We proceed by successively solving

$$\delta_f u_k = \bar\partial u_{k-1}, \tag{1.4}$$

where $u_k \in \mathcal{E}_{0,k-1}(X, \Lambda^k E)$. This procedure will terminate after a finite number of steps. In fact, $\bar\partial u_m = 0$, and so, if X is Stein, by successively solving equations

$$\bar\partial v_{k-1} = u_{k-1} + \delta_f v_k \tag{1.5}$$

for $k \le m$, we finally arrive at the desired holomorphic solution

$$\psi = u_1 + \delta_f v_2$$

to (1.3). Now, letting

$$\mathcal{L}^r(X, E) = \bigoplus_{k+\ell=r} \mathcal{D}'_{0,k}(X, \Lambda^{-\ell} E),$$

we introduce

$$\nabla_f = \delta_f - \bar\partial : \mathcal{L}^r \to \mathcal{L}^{r+1}.$$

Note that ∇_f is an antiderivation on the exterior algebra over $T^*_{0,1} \oplus E$. With this notation, the system of equations (1.3) and (1.4) can now be written as

$$\nabla_f u = 1, \tag{1.6}$$

where $u \in \mathcal{L}^{-1}(X, E)$. To find a solution to (1.6) in $X \setminus Y$, let us assume that E is equipped with some Hermitian metric and let s be the section of E

with pointwise minimal norm such that $\delta_f s = |f|^2$. Outside $Y = f^{-1}(0)$, the Cauchy-Fantappiè-Leray form

$$u = \frac{s}{\nabla_f s} = \frac{s}{\delta_f s - \bar\partial s} = \sum_\ell \frac{s \wedge (\bar\partial s)^{\ell-1}}{(\delta_f s)^\ell} = \sum_\ell \frac{s \wedge (\bar\partial s)^{\ell-1}}{|f|^{2\ell}}$$

is well-defined (observe that $\bar\partial s$ is of even degree), and $\nabla u = 1$. In the trivial metric the term of top degree corresponds to the Bochner-Martinelli form. Indeed, $s = \sum_j \bar f_j e_j$, so that

$$u = \sum_\ell \frac{\sum \bar f_j e_j \wedge (\sum \bar\partial \bar f_j e_j)^{\ell-1}}{|f|^{2\ell}}.$$

We wish to extend u to the entire X. If we could find an extension such that (1.6) still holds, by solving equations (1.5), this would imply that 1 belongs to the ideal generated by f, which is clearly not possible in general. Thus, there will appear residues. In [1], the existence of an analytic continuation of $\lambda \mapsto |f|^{2\lambda} u$ to $\operatorname{Re}\lambda > -\epsilon$ is proven. The value at $\lambda = 0$, denoted by U, yields the desired extension of u. (By analogy with the one-dimensional case, we will sometimes refer to U as the principal value current.) Moreover, $\nabla U = 1 - R^f$, where $R^f = \bar\partial |f|^{2\lambda} \wedge u|_{\lambda=0}$ now defines the residue current of f. If f is a complete intersection, then $R^f = R^f_{BM} \wedge e_1 \wedge \ldots \wedge e_m$, where R^f_{BM} is the Bochner-Martinelli current. Furthermore, from the construction it is easily verified that a holomorphic function that annihilates R^f belongs to the ideal generated by f locally.

To sum up, given a tuple f of functions $f_1, \ldots, f_r$, we have discussed essentially two different ways of associating a residue current to f. Either we can consider the product of the residue currents corresponding to each function f_i of the tuple, that is the Coleff-Herrera current, or we can define R^f by means of the Cauchy-Fantappiè-Leray (or Bochner-Martinelli) form of f. Considering this, we are led to ask whether it is possible to define products not only of one function currents but also of the multifunction currents. That is, given two tuples of functions f and g, is there a way to give meaning to the expression $R^f \wedge R^g$? Let us recall that, by the Mellin transform, the Coleff-Herrera current of f is equal to the value at $\lambda = 0$ of

$$\bar\partial |f_1|^{2\lambda} \frac{1}{f_1} \wedge \ldots \wedge \bar\partial |f_r|^{2\lambda} \frac{1}{f_r}.$$

Now, if we assume that each f_i is a section of a line bundle L_i^* with frame e_i^*, the Cauchy-Fantappiè-Leray form is just $u^{f_i} = e_i / f_i$, so in fact this product times the frame elements $e_1 \wedge \ldots \wedge e_r$ can be expressed as

$$\bar\partial |f_1|^{2\lambda} \wedge u^{f_1} \wedge \ldots \wedge \bar\partial |f_r|^{2\lambda} \wedge u^{f_r}. \tag{1.7}$$

In light of this, it is most tempting to extend this definition to include not only sections of line bundles but sections f_i of bundles of arbitrary rank. To be more accurate, we assume that f_i is a section of the dual bundle of a holomorphic m_i-bundle $E_i \to X$. Further, assuming that each E_i is equipped with a Hermitian metric, let s_i be the section of E_i of minimal norm such that $\delta_{f_i} s_i = |f_i|^2$, and let u^{f_i} be the Cauchy-Fantappiè-Leray form defined as above. Since a section f of a bundle E has a natural interpretation as a section of a bundle $E \oplus E'$, that is as $f \oplus 0$, we have given meaning to the expression (1.7) as a form taking values in the exterior algebra over $E = E_1 \oplus \cdots \oplus E_r$. Thus, in accordance with the line bundle case, we can take the value at $\lambda = 0$ of (1.7) as a definition of $R^{f_1} \wedge \ldots \wedge R^{f_r}$, provided that the analytic continuation exists. However, this is assured by Proposition 1.2, where products are defined also of principal value currents.

Proposition 1.2. *Let f_i be a holomorphic section of the Hermitian m_i-bundle $E_i^* \to X$. Let u^{f_i} be the corresponding Cauchy-Fantappiè-Leray form and let $Y_i = f_i^{-1}(0)$. Then*

$$\lambda \mapsto |f_r|^{2\lambda} u^{f_r} \wedge \ldots \wedge |f_{s+1}|^{2\lambda} u^{f_{s+1}} \wedge \bar{\partial}|f_s|^{2\lambda} \wedge u^{f_s} \wedge \ldots \wedge \bar{\partial}|f_1|^{2\lambda} \wedge u^{f_1} \tag{1.8}$$

has an analytic continuation as a current to $\lambda > -\epsilon$.

We define $T = U^{f_r} \wedge \ldots \wedge U^{f_{s+1}} \wedge R^{f_s} \wedge \ldots \wedge R^{f_1}$ as the value at $\lambda = 0$. Then T has support on $\bigcap_{i=1}^{s} Y_i$ and it is alternating with respect to the principal value factors U and commutative with respect to the residue factors R.

Of course there is nothing special about the ordering that we have chosen, that is first the principal value factors and then the residue factors. We can just as well mix U's and R's.

Since the term of top degree of u^{f_i} corresponds to the classical Bochner-Martinelli kernel, the term of top degree of $R^{f_r} \wedge \ldots \wedge R^{f_1}$ can be interpreted as a product of Bochner-Martinelli currents in the sense of [12]. In general, however, there will also occur terms of lower degree. More precisely we have

Proposition 1.3. *Let*

$$T = U^{f_r} \wedge \ldots \wedge U^{f_{s+1}} \wedge R^{f_s} \wedge \ldots \wedge R^{f_1}$$

be defined as above. Let $m = m_1 + \ldots + m_r$. Then $T = T_p + \ldots + T_q$, where $T_\ell \in \mathcal{D}'_{0,\ell}(\Lambda^\bullet E)$, $p = \operatorname{codim} Y_1 \cap \ldots \cap Y_s$ and $q = \min(m, n)$. In particular, if f is a complete intersection, then $R^{f_r} \wedge \ldots \wedge R^{f_1}$ consists of only one term of top degree m.

Notice, that in the particular case when the bundles E_i are all line bundles and thus the sections f_i correspond to single functions, the current

$$R^{f_1} \wedge \ldots \wedge R^{f_r}$$

is just the Coleff-Herrera current of f times $e_1 \wedge \ldots \wedge e_r$. Hence, we can formulate the equality in Theorem 1.1 as

$$R^{f_1 \oplus \ldots \oplus f_r} = R^{f_1} \wedge \ldots \wedge R^{f_r}. \tag{1.9}$$

Now, the obvious question is, does this equality extend to hold for sections of vector bundles of arbitrary rank. Our main result states that this is indeed the case.

Theorem 1.4. *Let f_i be a holomorphic section of the Hermitian m_i-bundle E_i^* and let f denote the section $f_1 \oplus \cdots \oplus f_r$ of $E^* = E_1^* \oplus \cdots \oplus E_r^*$. If f is a complete intersection, that is* $\operatorname{codim} f^{-1}(0) = m_1 + \ldots + m_r$, *then*

$$R^f = R^{f_1} \wedge \ldots \wedge R^{f_r}.$$

That is, in a local perspective, given a tuple of functions split into subtuples, the product of the Bochner-Martinelli currents of each subtuple is equal to the Bochner-Martinelli current of the whole tuple of functions. We give an explicit proof of Theorem 1.4, involving finding a potential to the current $1 - R^{f_1} \wedge \ldots \wedge R^{f_r}$. Indeed we prove

Theorem 1.5. *Let $f = f_1 \oplus \cdots \oplus f_r$ be a section of $E^* = E_1^* \oplus \cdots \oplus E_r^*$. Assume that f is a complete intersection. Then there exists a current V such that*

$$\nabla_f V = 1 - R^{f_1} \wedge \ldots \wedge R^{f_r}, \tag{1.10}$$

and furthermore a current $U^f \wedge V$ such that

$$\nabla_f (U^f \wedge V) = V - U^f. \tag{1.11}$$

At first it might seem a bit peculiar to denote the second potential by $U^f \wedge V$. However, notice that on a formal level, if we were allowed to multiply currents so that ∇_f acted as an antiderivation on the products, then

$$\nabla_f (U^f \wedge V) = (1 - R^f) \wedge V - U^f (1 - R^{f_1} \wedge \ldots \wedge R^{f_r}),$$

since U^f is of odd degree. From Proposition 1.3 we know that R^f and $R^{f_1} \wedge \ldots \wedge R^{f_r}$ are of top degree, m, in $d\bar{z}_j$, since f is a complete intersection, and thus it is reasonable to expect also the products $R^f \wedge V$ and $U^f \wedge R^{f_1} \wedge \ldots \wedge R^{f_r}$ to be of degree m in $d\bar{z}_j$. However, since U^f and V are both in $\mathcal{L}^{-1}(X, E)$, this is not possible unless the products vanish. Thus we are left with $V - U^f$, and the notation is motivated.

Proof of Theorem 1.4. Recall that $\nabla_f U^f = 1 - R^f$. Hence, applying ∇_f twice to $U^f \wedge V$ yields

$$0 = \nabla_f^2 (U^f \wedge V) = \nabla_f (U^f - V) = R^{f_1} \wedge \ldots \wedge R^{f_r} - R^f,$$

and thus we are done. $\square$

The disposition of this paper is as follows. In Section 2 we give proofs of Proposition 1.2 and Proposition 1.3. In Section 3 we prove Theorem 1.5. Finally, in Section 4 we give some examples of products of Cauchy-Fantappiè-Leray currents.

2. Products of residue currents of Cauchy-Fantappiè-Leray type

We start with the proof of Proposition 1.2. For further use a slightly more general formulation is appropriate. Namely, the proof of Theorem 1.5 requires a broader definition of products of currents. We need to allow also products of currents of sections of the bundle E, that are not necessarily orthogonal, at least in certain cases. Thus we give a new, somewhat unwieldy, version of Proposition 1.2 that however covers all the currents that we will be concerned with.

Proposition 2.1. *Let $f = f_1 \oplus \ldots \oplus f_r$ be a holomorphic section of the bundle $E^* = E_1^* \oplus \ldots \oplus E_r^*$, where E_i^* is a Hermitian m_i-bundle. For a subset $I = \{I_1, \ldots, I_p\}$ of $\{1, \ldots, r\}$, let f_I denote the section $f_{I_1} \oplus \ldots \oplus f_{I_p}$ of $E_I^* = E_{I_1}^* \oplus \ldots \oplus E_{I_p}^*$, let u^{f_I} be the corresponding Cauchy-Fantappiè-Leray form, let $Y_I = f_I^{-1}(0)$, and let $m_I = m_{I_1} + \ldots + m_{I_p}$. If $I^1, \ldots, I^t$ are subsets of $\{1, \ldots, r\}$, then*

(2.1)
$$\lambda \mapsto |f_{I^t}|^{2\lambda} u^{f_{I^t}} \wedge \ldots \wedge |f_{I^{s+1}}|^{2\lambda} u^{f_{I^{s+1}}} \wedge \bar\partial |f_{I^s}|^{2\lambda} \wedge u^{f_{I^s}} \wedge \ldots \wedge \bar\partial |f_{I^1}|^{2\lambda} \wedge u^{f_{I^1}}$$

has an analytic continuation to $\lambda > -\epsilon$.

We define $T = U^{f_{I^t}} \wedge \ldots \wedge U^{f_{I^{s+1}}} \wedge R^{f_{I^s}} \wedge \ldots \wedge R^{f_{I^1}}$ as the value at $\lambda = 0$. Then T has support on $\bigcap_{i=1}^s Y_{I^k}$ and it is alternating with respect to the principal value factors U and commutative with respect to the residue factors R.

Note that Proposition 1.2 corresponds to the particular case when each I^j is just a singleton. The proof of Proposition 2.1 is very much inspired by the proof of Lemma 2.2 in [12] and Theorem 1.1 in [1]. It is based on the possibility to resolve singularities by Hironaka's theorem and the following lemma, which is proven essentially by integration by parts.

Lemma 2.2. *Let v be a strictly positive smooth function in $\mathbb{C}$, φ a test function in $\mathbb{C}$, and p a positive integer. Then*

$$\lambda \mapsto \int v^\lambda |s|^{2\lambda} \varphi(s) \frac{ds \wedge d\bar s}{s^p}$$

and

$$\lambda \mapsto \int \bar\partial (v^\lambda |s|^{2\lambda}) \wedge \varphi(s) \frac{ds}{s^p}$$

both have meromorphic continuations to the entire plane with poles at rational points on the negative real axis. At $\lambda = 0$ they are both independent of v, and the second one is a distribution of φ supported at the origin and they only depend on powers of $\partial/\partial s$ of the test function φ. Moreover, if $\varphi(s) = \bar{s}\psi(s)$ or $\varphi = d\bar{s} \wedge \psi$, then the value of the second integral at $\lambda = 0$ is zero.

Proof of Proposition 2.1. We may assume that the bundle $E = E_1 \oplus \cdots \oplus E_r$ is trivial since the statement is clearly local. Note that $f_i = \sum f_{i,j} e^*_{i,j}$, where $e^*_{i,j}$ is the trivial frame. The proof is based on the possibility to resolve singularities locally using Hironaka's theorem. Given a small enough neighborhood $\mathcal{U}$ of a given point in X there exist a n-dimensional manifold $\widetilde{\mathcal{U}}$ and a proper analytic map $\Pi_h : \widetilde{\mathcal{U}} \to \mathcal{U}$ such that if $Z = \{\prod_{i,j} f_{i,j} = 0\}$ and $\widetilde{Z} = \Pi_h^{-1}(Z)$, then $\Pi : \widetilde{\mathcal{U}} \setminus \widetilde{Z} \to \mathcal{U} \setminus Z$ is biholomorphic and such that moreover $\widetilde{Z}$ has normal crossings in $\widetilde{\mathcal{U}}$. This implies that locally in $\widetilde{\mathcal{U}}$ we have that $\Pi_h^* f_{i,j} = a_{i,j}\mu_{i,j}$, where $a_{i,j}$ are non-vanishing and $\mu_{i,j}$ are monomials in some local coordinates τ_k. Further, given a finite number of monomials $\mu_1 \ldots, \mu_m$ in some coordinates τ_k defined in an n-dimensional manifold $\mathcal{U}_t$, there exists a toric manifold $\widetilde{\mathcal{U}_t}$ and a proper analytic map $\Pi_t : \widetilde{\mathcal{U}_t} \to \mathcal{U}_t$ such that Π_t is biholomorphic outside the coordinate axes and moreover, locally it holds that, for some i, $\Pi_t^*\mu_i$ divides all $\Pi_t^*\mu_j$, see [3] and [7]. Clearly, if μ_i divides μ_j in $\mathcal{U}_t$ then $\Pi_t^*\mu_i$ divides $\Pi_t^*\mu_j$ in $\widetilde{\mathcal{U}_t}$. Thus after a number, say q, of such toric resolutions Π_{t_i} we can locally consider each section f_{I^i} as a monomial times a non-vanishing section. More precisely we have that $\Pi^* f_{I^i} = \mu_i f'_{I^i}$, where $\Pi = \Pi_{t_q} \circ \cdots \circ \Pi_{t_1} \circ \Pi_h$, μ_i is a monomial and f'_{I^i} is a non-vanishing section of $E^*_{I^i}$.

Let ϕ be a test form with compact support. After a partition of unity we may assume that it has support in a neighborhood $\mathcal{U}$ as above. Then, since Π_h is proper, the support of $\Pi_h^*\phi$ can be covered by a finite number of neighborhoods in which it holds that $\Pi_h^*\phi = a_{i,j}\mu_{i,j}$. If ψ is a test form with support in such a neighborhood, then the support of $\Pi_{t_1}^*\psi$ can be covered by finitely many neighborhoods in which we have the desired property that the pull-back of one monomial divides some of the other ones, and so on. Thus, for $\operatorname{Re}\lambda > 2\max_i m_{I^i}$, (2.1) is in L^1_{loc}, and since Π is biholomorphic outside a zero-set we have that

$$\int |f_{I^t}|^{2\lambda} u^{f_{I^t}} \wedge \ldots \wedge |f_{I^{s+1}}|^{2\lambda} u^{f_{I^{s+1}}} \wedge \bar{\partial}|f_{I^s}|^{2\lambda} \wedge u^{f_{I^s}} \wedge \ldots \wedge \bar{\partial}|f_{I^1}|^{2\lambda} \wedge u^{f_{I^1}} \wedge \phi$$

is equal to a finite number of integrals of the form

(2.2)
$$\int \Pi^*(|f_{I^t}|^{2\lambda} u^{f_{I^t}} \wedge \ldots \wedge |f_{I^{s+1}}|^{2\lambda} u^{f_{I^{s+1}}} \wedge \bar{\partial}|f_{I^s}|^{2\lambda} \wedge u^{f_{I^s}} \wedge \ldots \wedge \bar{\partial}|f_{I^1}|^{2\lambda} \wedge u^{f_{I^1}}) \wedge \tilde{\phi}.$$

Here

$$\tilde{\phi} = \rho_{t_q} \Pi^*_{t_q}(\dots \rho_{t_1} \Pi^*_{t_1}(\rho_h \Pi^*_h(\phi))),$$

where the $\rho_\bullet$'s are functions from some partitions of unity, so that the test form $\tilde{\phi}$ has support in a neighborhood where it holds that $\Pi^* f_{I^i} = \mu_i f'_{I^i}$. In such a coordinate neighborhood the pullback of s_{I^i} is $\bar{\mu}_i$ times a smooth form, so that $\Pi^*(s_{I^i} \wedge (\bar{\partial} s_{I^i})^{\ell-1})$ is $\bar{\mu}_i^\ell$ times a smooth form. Moreover $\Pi^*|f_{I^i}|^2 = |\mu_i|^2 a_i$, where a_i is a strictly positive smooth function. Thus

$$\Pi^* u^{f_{I^i}} = \sum_\ell \frac{\bar{\mu}_i^\ell \alpha_{i,\ell}}{|\mu_i|^{2\ell}} = \sum_\ell \frac{\alpha_{i,\ell}}{\mu_i^\ell},$$

where $\alpha_{i,\ell}$ are smooth forms (taking values in $\Lambda^\ell E$) and so (2.2) is equal to a finite sum of integrals

$$(2.3) \quad \int |\mu_t|^{2\lambda} a_t^\lambda \frac{\alpha_{t,\ell_t}}{\mu_t^{\ell_t}} \wedge \dots \wedge |\mu_{s+1}|^{2\lambda} a_{s+1}^\lambda \frac{\alpha_{s+1,\ell_{s+1}}}{\mu_{s+1}^{\ell_{s+1}}} \wedge \\ \bar{\partial}(|\mu_s|^{2\lambda} a_s^\lambda) \wedge \frac{\alpha_{s,\ell_s}}{\mu_s^{\ell_s}} \wedge \dots \wedge \bar{\partial}(|\mu_1|^{2\lambda} a_1^\lambda) \wedge \frac{\alpha_{1,\ell_1}}{\mu_1^{\ell_1}} \wedge \tilde{\phi}.$$

Expanding each factor $\bar{\partial}(|\mu_j|^{2\lambda} a_j^\lambda)$ by Leibniz' rule results in a finite sum of terms. Letting $\bar{\partial}$ fall only on the monomials μ_i yields integrals of the form

$$(2.4) \qquad \int a^\lambda |\mu'|^{2\lambda} \frac{\alpha_L}{\mu_L} \wedge \bar{\partial} |\sigma_s^{q_s}|^{2\lambda} \wedge \dots \wedge \bar{\partial} |\sigma_1^{q_1}|^{2\lambda} \wedge \tilde{\phi},$$

where σ_i is one of the coordinate functions τ_j that divide μ_i, $a = a_t \cdots a_1$ is a strictly positive smooth function, μ' and μ'_L are monomials in τ_j not divisible by any σ_i and $\alpha_L = C\alpha_{t,\ell_t} \wedge \dots \wedge \alpha_{1,\ell_1}$ is a smooth form, where C is just a constant that depends on the relation between q_i and the number of σ_i's in μ_i. The remaining integrals, that arise when $\bar{\partial}$ falls on any of the a_i, vanish in accordance with Lemma 2.2. Indeed, consider one of the integrals obtained when $\bar{\partial}$ falls on a_1,

$$\lambda \int a^\lambda |\mu'|^{2\lambda} \frac{\alpha_L}{\mu_L} \wedge \bar{\partial} |\sigma_s^{q_s}|^{2\lambda} \wedge \dots \wedge \bar{\partial} |\sigma_2^{q_2}|^{2\lambda} \wedge \bar{\partial} a_1 \wedge \tilde{\phi}.$$

This is just λ times an integral of the form (2.4), so provided that we can prove the existence of an analytic continuation of (2.4), it must clearly vanish at $\lambda = 0$.

Now an application of Lemma 2.2 for each τ_k that divides any of the μ_j's gives the desired analytic continuation of (2.4) to $\lambda > -\epsilon$. Note that for $\sigma_1, \dots, \sigma_s$ we get integrals of the second type, for the remaining τ_i integrals of the first type, so that the value at $\lambda = 0$ is a current with support on $\{\sigma_s = 0\} \cap \dots \cap \{\sigma_1 = 0\}$. Thus the value of (2.3) at $\lambda = 0$ has support on

$$\{\mu_s = 0\} \cap \dots \cap \{\mu_1 = 0\} = \widetilde{Y}_{I^s} \cap \dots \cap \widetilde{Y}_{I^1},$$

where $\widetilde{Y}_\bullet = \Pi^{-1}Y_\bullet$, and accordingly $U^{f_{I^t}} \wedge \ldots \wedge U^{f_{I^{s+1}}} \wedge R^{f_{I^s}} \wedge \ldots \wedge R^{f_{I^1}}$ is a current with support on $Y_{I^s} \cap \ldots \cap Y_{I^1}$.

Since the form (2.1) is alternating with respect to the factors $|f_{I^i}|^{2\lambda} u^{f_{I^i}}$ and commutative with respect to the factors $\bar\partial |f_{I^i}|^{2\lambda} \wedge u^{f_{I^i}}$, it follows that $U^{f_{I^t}} \wedge \ldots \wedge U^{f_{I^{s+1}}} \wedge R^{f_{I^s}} \wedge \ldots \wedge R^{f_{I^1}}$ is alternating with respect to the principal value factors and commutative with respect to the residue factors. $\square$

We continue with the proof of Proposition 1.3.

Proof of Proposition 1.3. Notice that T_ℓ is the analytic continuation to $\lambda = 0$ of the terms

$$|f_r|^{2\lambda} u^{f_r}_{\ell_r} \wedge \ldots \wedge |f_{s+1}|^{2\lambda} u^{f_{s+1}}_{\ell_{s+1}} \wedge \bar\partial |f_s|^{2\lambda} \wedge u^{f_s}_{\ell_s} \wedge \ldots \wedge \bar\partial |f_1|^{2\lambda} \wedge u^{f_1}_{\ell_1}, \tag{2.5}$$

where

$$u^{f_i}_{\ell_i} = \frac{s_i \wedge (\bar\partial s_i)^{\ell_i - 1}}{|f_i|^{2\ell_i}}$$

and the total degree in $d\bar z_j$ (that is $\ell_1 + \ldots + \ell_r - r + s$) is ℓ. By the notion that a form (or current) is of degree k in $d\bar z_j$, we just mean that it is a $(\bullet, k)$-form. In the same manner, we will say that a form is of degree ℓ in e_j when it takes values in $\Lambda^\ell E$.

Following the proof of Proposition 2.1, a term of the form (2.5), integrated against a test form ϕ, is equal to a sum of terms like

$$\int |\mu_r|^{2\lambda} a_r^\lambda \frac{\alpha_{r,\ell_r}}{\mu_r^{\ell_r}} \wedge \ldots \wedge |\mu_{s+1}|^{2\lambda} a_{s+1}^\lambda \frac{\alpha_{s+1,\ell_{s+1}}}{\mu_{s+1}^{\ell_{s+1}}} \wedge \\ \bar\partial(|\mu_s|^{2\lambda} a_s^\lambda) \wedge \frac{\alpha_{s,\ell_s}}{\mu_s^{\ell_s}} \wedge \ldots \wedge \bar\partial(|\mu_1|^{2\lambda} a_1^\lambda) \wedge \frac{\alpha_{1,\ell_1}}{\mu_1^{\ell_1}} \wedge \tilde\phi, \tag{2.6}$$

where the α_{i,ℓ_i}'s are smooth forms of degree ℓ_i in e_j, the a_i's are non-vanishing functions, the μ_i's are monomials in some local coordinates τ_j and $\tilde\phi$ is as in the previous proof. We can find a toric resolution such that locally one of $\mu_1, \ldots, \mu_s$ divides the other ones, so without loss of generality we may assume that μ_1 divides $\mu_2, \ldots, \mu_s$.

We expand $\bar\partial(|\mu_1|^{2\lambda} a_1^\lambda)$ by Leibniz' rule. Observe that when $\bar\partial$ falls on a_1^λ the integral vanishes as in the proof of Proposition 2.1, and thus it suffices to consider the case when $\bar\partial$ falls on one of the τ_j that divide μ_1, say on $|\sigma|^{2\lambda}$. If $\ell < p$, we claim that this part of (2.6) vanishes when integrating with respect to σ. In fact, we may assume that $\phi = \phi_I \wedge d\bar z_I$, where ϕ_I is an $(n, 0)$-form and $d\bar z_I = d\bar z_{I_1} \wedge \ldots \wedge d\bar z_{I_{n-\ell}}$. Now $d\bar z_I$ vanishes on the variety $Y_1 \cap \ldots \cap Y_s$ of codimension p for degree reasons. Consequently $\Pi^*(d\bar z_I)$ vanishes on $\widetilde{Y}_1 \cap \ldots \cap \widetilde{Y}_s$, and in particular on $\{\sigma = 0\}$. However, this is a form in $d\bar\tau_k$ with antiholomorphic coefficients since Π is holomorphic, and therefore each of its terms contains a factor $d\bar{\sigma}$ or a factor $\bar{\sigma}$. Indeed, if $\Psi(\tau)$ is a form in $d\bar{\tau}_k$ with antiholomorphic coefficients we can write

$$\Psi(\tau) = \Psi'(\tau) \wedge d\bar{\sigma} + \Psi''(\tau),$$

where $\Psi''(\tau)$ does not contain $d\bar{\sigma}$. The first term clearly vanishes on $\{\sigma = 0\}$ since $d\bar{\sigma}$ does. If $\Psi(\tau)$ vanishes on $\{\sigma = 0\}$, then $\Psi''(\tau)$ does, and hence it contains a factor $\bar{\sigma}$ due to antiholomorphicity. In both cases the σ-integral, and thereby (2.6), vanishes according to Lemma 2.2. $\square$

3. The complete intersection case

Recall from Section 1 that if $f = f_1, f_2$ defines a complete intersection, then

$$\lambda_1\lambda_2\Gamma(\lambda_1,\lambda_2) = \int \bar{\partial}|f_1|^{2\lambda_1}\frac{1}{f_1} \wedge \bar{\partial}|f_2|^{2\lambda_2}\frac{1}{f_2} \wedge \phi$$

is holomorphic in λ, and that the result is believed to extend to any finite number of functions f_i.

Thus, taking a closer look at the definition of the currents in Proposition 1.2 (and Proposition 2.1) a natural question is whether

$$t(\lambda) := \bar{\partial}|f_r|^{2\lambda_r} \wedge u^{f_r} \wedge \ldots \wedge \bar{\partial}|f_1|^{2\lambda_1} \wedge u^{f_1} \tag{3.1}$$

is holomorphic in λ.

In general, letting λ tend to 0 along different paths yields different currents as shown by the following example. Let $f_1 = z_1$ and $f_2 = z_1 z_2$ in $\mathbb{C}^2$ and consider $t(\lambda)$ acting on a test form $\phi = \varphi(z_1, z_2)\; dz_1 \wedge dz_2$

$$\int \frac{\bar{\partial}|f_1|^{2\lambda_1} \wedge \bar{\partial}|f_2|^{2\lambda_2}}{f_1 f_2} \wedge \phi = \frac{\bar{\partial}|z_1|^{2\lambda_1} \wedge \bar{\partial}|z_1 z_2|^{2\lambda_2}}{z_1^2 z_2} \wedge \phi.$$

As λ tends to zero the integral approaches the value $\lambda_1/(\lambda_1 + \lambda_2)\varphi_{z_1}(0,0)$. Thus the value at 0 depends on the ratio between λ_1 and λ_2.

In case we have two functions defining a complete intersection though, this phenomenon does not occur. By an integration by parts, we can write $t(\lambda)$ integrated against a test form ϕ as

$$\int |f_2|^{2\lambda_2} u^{f_2} \wedge \bar{\partial}|f_1|^{2\lambda_1} \wedge u^{f_1} \wedge \bar{\partial}\phi.$$

After a resolution of singularities this is equal to a sum of integrals of the form

$$\int |\mu_2|^{2\lambda_2} a_2^{\lambda_2} \frac{\alpha_{2,\ell_2}}{\mu_2^{\ell_2}} \wedge \bar{\partial}(|\mu_1|^{2\lambda_1} a_1^{\lambda_1}) \wedge \frac{\alpha_{1,\ell_1}}{\mu_1^{\ell_1}} \wedge \bar{\partial}\, \tilde{\phi},$$

(see the proof of Proposition 2.1). Let σ be one of the coordinate functions that divides μ_1. If μ_2 does not contain σ, then the σ-integral is clearly independent of λ. If μ_2 does contain σ, at first we seem to end up in a situation similar to the one above, where the result at $\lambda = 0$ depends on the relation between λ_1 and λ_2. However, the σ-integral vanishes for the same reason as T_ℓ vanishes when $\ell < p$ in the proof of Proposition 1.3. Thus, in this case, the definition of the current $T = t(\lambda)|_{\lambda=0}$ is robust in the sense that it does not depend on the particular path along which λ tends to zero.

Provided that the Mellin transform of the residue integral is holomorphic in λ in a neighborhood of $0 \in \mathbb{C}^r$, it is reasonable to believe that also $t(\lambda)$ is. Presuming this to be true, we can give a soft proof of Theorem 1.4 based on Theorem 1.1. Indeed, if $f = \sum_1^m f_j e_j^*$ is a section of a bundle E^* we let

$$t^f_{CFL}(\lambda) = \bar{\partial}|f|^{2\lambda} \wedge u^f,$$

and

$$t^f_{CH}(\lambda) = \bar{\partial}|f_1|^{2\lambda}\frac{1}{f_1} \wedge \ldots \wedge \bar{\partial}|f_m|^{2\lambda}\frac{1}{f_m},$$

where CFL and CH of course stand for Cauchy-Fantappiè-Leray and Coleff-Herrera, respectively. With this notation the equality in Theorem 1.1 can be expressed as

$$t^f_{CFL}(\lambda)|_{\lambda=0} = t^f_{CH}(\lambda)|_{\lambda=0}. \tag{3.2}$$

Now let f and g be sections of the bundles E_1^* and E_2^*, respectively, and assume that $f \oplus g$ is a complete intersection. By definition,

$$R^f \wedge R^g = t^f_{CFL}(\lambda) \wedge t^g_{CFL}(\lambda)|_{\lambda=0},$$

and

$$R^{f\oplus g} = t^{f\oplus g}_{CFL}(\lambda)|_{\lambda=0},$$

so we want to prove that

$$t^f_{CFL}(\lambda) \wedge t^g_{CFL}(\lambda)|_{\lambda=0} = t^{f\oplus g}_{CFL}(\lambda)|_{\lambda=0}.$$

If $\operatorname{Re}\lambda_2$ is large enough, $t^g_{CFL}(\lambda_2)$ is in $\mathcal{L}^1_{\mathrm{loc}}$, and so by (3.2)

$$t^f_{CFL}(\lambda_1) \wedge t^g_{CFL}(\lambda_2)|_{\lambda_1=0} = t^f_{CH}(\lambda_1) \wedge t^g_{CFL}(\lambda_2)|_{\lambda_1=0},$$

and analogously, if $\operatorname{Re}\lambda_1$ is large enough

$$t^f_{CH}(\lambda_1) \wedge t^g_{CFL}(\lambda_2)|_{\lambda_2=0} = t^f_{CH}(\lambda_1) \wedge t^g_{CH}(\lambda_2)|_{\lambda_2=0}.$$

Now, by assumption

$$(\lambda_1, \lambda_2) \mapsto t^f_\bullet(\lambda_1) \wedge t^g_\bullet(\lambda_2),$$

where $\bullet$ stands for either CFL or CH, is holomorphic at the origin, and thus it follows that

$$t^f_{CFL}(\lambda) \wedge t^g_{CFL}(\lambda)|_{\lambda=0} = t^f_{CH}(\lambda) \wedge t^g_{CH}(\lambda)|_{\lambda=0},$$

but the right hand side is, by (3.2), equal to $t^{f\oplus g}_{CFL}(\lambda)|_{\lambda=0}$, and so we obtain Theorem 1.4 for $r = 2$. However, the argument easily extends to arbitrary r.

The way we actually prove Theorem 1.4, that is, as already announced, by Theorem 1.5, is more direct and relies neither on the holomorphicity of the Mellin transform nor on Theorem 1.1. The proof is inspired by Proposition 4.2 in [1], in which potentials were used to prove Theorem 1.1. Our hope is that this construction of potentials will be of use for further investigations in the case of a noncomplete intersection.

Proof of Theorem 1.5. We let

$$V = U^{f_1} + U^{f_2} \wedge R^{f_1} + U^{f_3} \wedge R^{f_2} \wedge R^{f_1} + \ldots + U^{f_r} \wedge R^{f_{r-1}} \wedge \ldots \wedge R^{f_1}.$$

To motivate this choice of V, note that on a formal level

$$\nabla(U^{f_i} \wedge R^{f_{i-1}} \wedge \ldots \wedge R^{f_1}) = R^{f_{i-1}} \wedge \ldots \wedge R^{f_1} - R^{f_i} \wedge \ldots \wedge R^{f_1}, \tag{3.3}$$

(observe that $\nabla = \nabla_f$ acts on U^{f_i} just as ∇_{f_i}, so that $\nabla U^{f_i} = 1 - R^{f_i}$) so that

$$\nabla V = 1 - R^{f_r} \wedge \ldots \wedge R^{f_1}.$$

Thus, to prove the first claim of the theorem we have to make this computation legitimate.

First, notice that if a form $A(\lambda)$, depending on a parameter λ, has an analytic continuation as a current to $\lambda = 0$, then clearly $\nabla A(\lambda)$ has one. The action on a test form ϕ is given by

$$\pm \int A(\lambda) \wedge \nabla\phi.$$

However, by integration by parts with respect to ∇ and due to the uniqueness of analytic continuations, this is equal to

$$\int \nabla A(\lambda) \wedge \phi.$$

To be able to perform the integration by parts in a stringent way we have to regard the currents $T \in \mathcal{D}'_{0,k}(\Lambda^\ell E)$ as functionals on $\mathcal{D}_{n,n-k}(\Lambda^{n-\ell} E \wedge \Lambda^n E^*)$. So far we have been a little sloppy about this.

Thus, to compute ∇V we consider the form

$$v^\lambda = |f_1|^{2\lambda} u^{f_1} + |f_2|^{2\lambda} u^{f_2} \wedge \bar\partial |f_1|^{2\lambda} \wedge u^{f_1} + \ldots \\ \ldots + |f_r|^{2\lambda} u^{f_r} \wedge \bar\partial |f_{r-1}|^{2\lambda} \wedge u^{f_{r-1}} \wedge \ldots \wedge \bar\partial |f_1|^{2\lambda} \wedge u^{f_1},$$

since, by definition, $v^\lambda|_{\lambda=0} = V$, and accordingly $\nabla V = (\nabla v^\lambda)|_{\lambda=0}$. More precisely, to verify (3.3), let us consider (recall that $\nabla u^{f_i} = 1$)

$$\nabla(|f_i|^{2\lambda} u^{f_i} \wedge \bar\partial |f_{i-1}|^{2\lambda} \wedge u^{f_{i-1}} \wedge \ldots \wedge \bar\partial |f_1|^{2\lambda} \wedge u^{f_1}) = \\ - \bar\partial |f_i|^{2\lambda} \wedge u^{f_i} \wedge \bar\partial |f_{i-1}|^{2\lambda} \wedge u^{f_{i-1}} \wedge \ldots \wedge \bar\partial |f_1|^{2\lambda} \wedge u^{f_1} + \\ |f_i|^{2\lambda} \bar\partial |f_{i-1}|^{2\lambda} \wedge u^{f_{i-1}} \wedge \ldots \wedge \bar\partial |f_1|^{2\lambda} \wedge u^{f_1} + \mathcal{R},$$

where $\mathcal{R}$ is a sum of terms of the form

$$|f_i|^{2\lambda}u^{f_i}\wedge\bar\partial|f_{i-1}|^{2\lambda}\wedge u^{f_{i-1}}\wedge\ldots\bar\partial|f_j|^{2\lambda}\wedge\bar\partial|f_{j-1}|^{2\lambda}\wedge\ldots\wedge\bar\partial|f_1|^{2\lambda}\wedge u^{f_1}$$

that arise when ∇ falls on any u^{f_j}, $j<i$. The value at $\lambda=0$ of the first term is just $-R^{f_i}\wedge R^{f_{i-1}}\wedge\ldots\wedge R^{f_1}$, and it follows from Lemma 3.1 that the second term has an analytic continuation to $\lambda=0$ equal to $R^{f_{i-1}}\wedge\ldots\wedge R^{f_1}$. The remaining terms, $\mathcal{R}$, vanish according to Lemma 3.2. Thus (1.10) is proved, and thereby the first part of the theorem.

Furthermore, let

$$\begin{aligned}U^f\wedge V=U^f\wedge U^{f_1}+U^f\wedge U^{f_2}\wedge R^{f_1}+&\\ U^f\wedge U^{f_3}\wedge R^{f_2}\wedge R^{f_1}+\ldots+U^f\wedge U^{f_r}\wedge R^{f_{r-1}}\wedge\ldots\wedge R^{f_1}&.\end{aligned}$$

We compute ∇ of each term. To do this we use a form as above whose analytic continuation to $\lambda=0$ is equal to this particular current. Now, we actually need the extended version of Proposition 1.2, that is Proposition 2.1. Indeed, consider

$$\begin{aligned}&\nabla(|f|^{2\lambda}u^f\wedge|f_i|^{2\lambda}u^{f_i}\wedge\bar\partial|f_{i-1}|^{2\lambda}\wedge u^{f_{i-1}}\wedge\ldots\wedge\bar\partial|f_1|^{2\lambda}\wedge u^{f_1})=\\ &\quad-\bar\partial|f|^{2\lambda}\wedge u^f\wedge|f_i|^{2\lambda}u^{f_i}\wedge\bar\partial|f_{i-1}|^{2\lambda}\wedge u^{f_{i-1}}\wedge\ldots\wedge\bar\partial|f_1|^{2\lambda}\wedge u^{f_1}+\\ &\quad|f|^{2\lambda}|f_i|^{2\lambda}u^{f_i}\wedge\bar\partial|f_{i-1}|^{2\lambda}\wedge u^{f_{i-1}}\wedge\ldots\wedge\bar\partial|f_1|^{2\lambda}\wedge u^{f_1}+\\ &\quad|f|^{2\lambda}u^f\wedge\bar\partial|f_i|^{2\lambda}\wedge u^{f_i}\wedge\bar\partial|f_{i-1}|^{2\lambda}\wedge u^{f_{i-1}}\wedge\ldots\wedge\bar\partial|f_1|^{2\lambda}\wedge u^{f_1}\\ &\quad-|f|^{2\lambda}u^f\wedge|f_i|^{2\lambda}\wedge\bar\partial|f_{i-1}|^{2\lambda}\wedge u^{f_{i-1}}\wedge\ldots\wedge\bar\partial|f_1|^{2\lambda}\wedge u^{f_1}+\\ &\quad|f|^{2\lambda}u^f\wedge\mathcal{R}.\end{aligned}$$

The first term corresponds to $-R^f\wedge U^{f_i}\wedge R^{f_{i-1}}\wedge\ldots\wedge R^{f_1}$. Since f is a complete intersection and R^f therefore is of top degree in $d\bar z_j$ according to Proposition 1.3, it is most reasonable to expect also this product to be of top degree in $d\bar z_j$, but because of the factor $U^{f_i}\in\mathcal{L}^{-1}(E_i)$ that is apparently not possible unless the product vanishes. This is indeed the case, as follows from Lemma 3.3. The second, third and fourth terms have analytic continuations as $U^{f_i}\wedge R^{f_{i-1}}\wedge\ldots\wedge R^{f_1}$, $U^f\wedge R^{f_i}\wedge\ldots\wedge R^{f_1}$ and $-U^f\wedge R^{f_{i-1}}\wedge\ldots\wedge R^{f_1}$, respectively, by Lemma 3.1. The remaining terms vanish according to Lemma 3.2. Hence

$$\begin{aligned}\nabla_f(U\wedge V)=\sum U^{f_i}\wedge R^{f_{i-1}}\wedge\ldots\wedge R^{f_1}&\\ -\sum(U^f\wedge R^{f_{i-1}}\wedge\ldots\wedge R^{f_1}-U^f\wedge R^{f_i}\wedge\ldots\wedge R^{f_1})=&\\ V-U^f+U^f\wedge R^{f_r}\wedge\ldots\wedge R^{f_1}&.\end{aligned}$$

Finally, the term $U^f \wedge R^{f_r} \wedge \ldots \wedge R^{f_1}$ vanishes by Lemma 3.4, and thus taking the lemmas 3.1 to 3.4 for granted, the theorem is proved. $\square$

What remains is the technical part; to prove the lemmas. We have tried to put them as simply as possible. Still the formulations may seem a bit strained though. Hopefully, the remarks will shed some light on what matters.

Lemma 3.1. *Let $f = f_1 \oplus \cdots \oplus f_r$ be a section of $E^* = E_1^* \oplus \cdots \oplus E_r^*$. Assume that f is a complete intersection, let $s < r$ and $r' \leq r$. If $h = f$, or if $h = f_i$ for some $i > s$, then*

$$(3.4)\quad |h|^{2\lambda}|f_{r'}|^{2\lambda}u^{f_{r'}} \wedge \ldots \wedge |f_{s+1}|^{2\lambda}u^{f_{s+1}} \wedge \bar{\partial}|f_s|^{2\lambda} \wedge u^{f_s} \wedge \ldots \wedge \bar{\partial}|f_1|^{2\lambda} \wedge u^{f_1}$$

has an analytic continuation to $\operatorname{Re}\lambda > -\epsilon$, which for $\lambda = 0$ is equal to the current $U^{f_{r'}} \wedge \ldots \wedge U^{f_{s+1}} \wedge R^{f_s} \wedge \ldots \wedge R^{f_1}$.

Moreover,

(3.5)

$$|h|^{2\lambda}|f|^{2\lambda}u^f \wedge |f_{r'}|^{2\lambda}u^{f_{r'}} \wedge \ldots \wedge |f_{s+1}|^{2\lambda}u^{f_{s+1}} \wedge \bar{\partial}|f_s|^{2\lambda} \wedge u^{f_s} \wedge \ldots \wedge \bar{\partial}|f_1|^{2\lambda} \wedge u^{f_1}$$

has an analytic continuation to $\operatorname{Re}\lambda > -\epsilon$, which for $\lambda = 0$ is equal to the current $U^f \wedge U^{f_{r'}} \wedge \ldots \wedge U^{f_{s+1}} \wedge R^{f_s} \wedge \ldots \wedge R^{f_1}$.

Remark 1. The crucial point is that inserting a factor $|h|^{2\lambda}$ has no effect on the value at $\lambda = 0$, as long as

$$\operatorname{codim}\{h = 0\} \cap Y_s \cap \ldots \cap Y_1 > \operatorname{codim} Y_s \cap \ldots \cap Y_1,$$

since then all possibly "dangerous" contributions to the current will vanish for degree reasons as in the proof of Proposition 1.3. There might be a more general formulation of the lemma that catches this behaviour better. Nevertheless, for the proof of Theorem 1.5 it suffices with the cases above. $\square$

Proof. We give a proof of the first claim of the lemma. The second one, concerning (3.5), can be proved along the same lines.

For a compactly supported test form ϕ, we consider

$$\int |h|^{2\lambda}|f_{r'}|^{2\lambda}u^{f_{r'}} \wedge \ldots \wedge |f_{s+1}|^{2\lambda}u^{f_{s+1}} \wedge \bar{\partial}|f_s|^{2\lambda} \wedge u^{f_s} \wedge \ldots \wedge \bar{\partial}|f_1|^{2\lambda} \wedge u^{f_1} \wedge \phi.$$

After a resolution of singularities as described in the proof of Proposition 2.1, for $\operatorname{Re}\lambda$ large enough, this integral is equal to a sum of

$$(3.6)\quad \int |\mu_h|^{2\lambda}|\mu_{r'}|^{2\lambda}a_{r'}^{\lambda}\frac{\alpha_{r',\ell_{r'}}}{\mu_{r'}^{\ell_{r'}}} \wedge \ldots \wedge |\mu_{s+1}|^{2\lambda}a_{s+1}^{\lambda}\frac{\alpha_{s+1,\ell_{s+1}}}{\mu_{s+1}^{\ell_{s+1}}} \wedge \\ \bar{\partial}(|\mu_s|^{2\lambda}a_s^{\lambda}) \wedge \frac{\alpha_{s,\ell_s}}{\mu_s^{\ell_s}} \wedge \ldots \wedge \bar{\partial}(|\mu_1|^{2\lambda}a_1^{\lambda}) \wedge \frac{\alpha_{1,\ell_1}}{\mu_1^{\ell_1}} \wedge \tilde{\phi},$$

where the a_j's are strictly positive functions, the μ_j's are polynomials in some local coordinates τ_j, the α_{j,ℓ_j}'s are smooth forms and $\tilde{\phi}$ is as in the proof of Proposition 2.1. The existence of the analytic continuation to $\operatorname{Re}\lambda > -\epsilon$ follows from Lemma 2.2 as before.

Our aim is to prove that the factor $|h|^{2\lambda}$ does not affect the value at $\lambda = 0$. Let σ be one of the coordinate functions τ_k that divides μ_h. When expanding each factor $\bar{\partial}(|\mu_j|^{2\lambda} a_j^{\lambda})$ by Leibniz' rule we get two different types of terms, integrals with one occurrence of a factor $\bar{\partial}|\sigma^{\alpha}|^{2\lambda}$ for some α, and integrals with no such factors. In the second case the extra factor $|\sigma|^{2\lambda}$ does no harm, since, in fact, the value at $\lambda = 0$ is independent of the number of $|\sigma|^{2\lambda}$'s in the numerator as long as there is no $\bar{\sigma}$ in the denominator. Furthermore, we claim that each integral of the first kind actually vanishes at $\lambda = 0$. The argument is analogous to the one in the proof of Proposition 1.3. Let us first consider the case when $h = f$. Observe that the terms in (3.4) are of degree at most $m_1 + \ldots + m_{r'} - r' + s \leq m - 1$ in $d\bar{z}_j$, where $m = m_1 + \ldots + m_r$. The crucial term -1 appears because of the (at least for the proof) necessary condition that $r < s$, that is that we have at least one factor U. Thus, it is enough to consider test forms of codegree in $d\bar{z}$ at most $m - 1$; by codegree we mean the difference between the dimension n of X and the degree. We assume that $\phi = \phi_I \wedge d\bar{z}_I$, where ϕ_I is a smooth $(n, 0)$-form and $d\bar{z}_I = d\bar{z}_{I_1} \wedge \ldots \wedge d\bar{z}_{I_p}$ where $p \geq n - (m_1 + \ldots + m_r) + 1$. Now, $d\bar{z}_I$ vanishes on the variety $Y = f^{-1}(0)$, since it has codimension m, and accordingly $\Pi^*(d\bar{z}_I)$ vanishes on $\widetilde{Y} = \Pi^{-1}Y$, and in particular on $\{\sigma = 0\}$. Since it is a form in $d\bar{\tau}_j$ with antiholomorphic coefficients, each of its terms contains a factor $\bar{\sigma}$ or $d\bar{\sigma}$ (see the proof of Proposition 1.3), and so in both cases the σ-integrals vanish according to Lemma 2.2.

In the second case, when $h = f_i$, the proof becomes slightly more complicated. We want to prove that the σ-integral vanishes due to the occurrence of a factor $\bar{\sigma}$ or $d\bar{\sigma}$ as above, but now the desired factors $\bar{\sigma}$ and $d\bar{\sigma}$ do not necessarily divide the test form $\tilde{\phi}$. We need to look at a "larger" form than ϕ, in fact at the "largest" possible "σ-free" form. Without loss of generality we may assume that, for some numbers s' and r', $1 \leq s' \leq s \leq r' \leq r$, σ divides $\mu_{s'+1}, \ldots, \mu_s$ and $\mu_{r'+1}, \ldots, \mu_r$ but neither $\mu_1, \ldots, \mu_s'$ nor $\mu_{s+1}, \ldots, \mu_r'$. Recall that $u^{f_i} = \sum_\ell v_{\ell_i}^{f_i}/|f_i|^{2\ell}$, where $v_{\ell_i}^{f_i} = s_i \wedge (\bar{\partial} s_i)^{\ell_i - 1}$. Let the smooth form

$$v_{\ell_r}^{f_r} \wedge \ldots \wedge v_{\ell_{s+1}}^{f_{s+1}} \wedge \bar{\partial}|f_{s'}|^2 \wedge v_{\ell_s'}^{f_s'} \wedge \ldots \wedge \bar{\partial}|f_1|^2 \wedge v_{\ell_1}^{f_1}$$

be denoted by F_ℓ, and let

$$Y' = \{f_{s'+1} = \ldots = f_s = f_{r'+1} + \ldots f_r = h = 0\}.$$

As above we may assume that ϕ consists of only one term $\phi_I \wedge d\bar{z}_I$. Then, by inspection, the form $F_\ell \wedge d\bar{z}_I$ is of codegree at most

$$m_{s'+1} + \ldots + m_s + m_{r'+1} + \ldots + m_r - r + r'$$

in $d\bar{z}_j$, which is strictly less than

$$\operatorname{codim} Y' = m_{s'+1} + \ldots + m_s + m_{r'} + \ldots + m_r + m_h,$$

because of the assumptions of complete intersection. Consequently $F_\ell \wedge d\bar{z}_I$ vanishes on Y', and thus $\Pi^*(F_\ell \wedge d\bar{z}_I)$ vanishes on $\Pi^{-1}Y'$, and in particular on $\{\sigma = 0\}$. Since it is a form in $d\bar{\tau}_j$ with antiholomorphic coefficients, each of its terms contains a factor $\bar{\sigma}$ or a factor $d\bar{\sigma}$. Using that $\bar{\partial}|f|^{2\lambda} = \lambda|f|^{2(\lambda-1)}\bar{\partial}|f|^2$, we can write (3.6) as

$$\pm \int |\mu_h|^{2\lambda}|\mu_r|^{2\lambda}a_r^\lambda \frac{\alpha_{r,\ell_r}}{\mu_r^{\ell_r}} \wedge \ldots \wedge |\mu_{r'+1}|^{2\lambda}a_{r'+1}^\lambda \frac{\alpha_{r'+1,\ell_{r'+1}}}{\mu_{r'+1}^{\ell_{r'+1}}} \wedge$$
$$\bar{\partial}(|\mu_s|^{2\lambda}a_s^\lambda) \wedge \frac{\alpha_{s,\ell_s}}{\mu_s^{\ell_s}} \wedge \ldots \wedge \bar{\partial}(|\mu_{s'+1}|^{2\lambda}a_1^\lambda) \wedge \frac{\alpha_{s'+1,\ell_{s'+1}}}{\mu_{s'+1}^{\ell_{s'+1}}} \wedge$$
$$\frac{|\mu_{r'}|^{2\lambda}a_{r'}^\lambda \cdots |\mu_1|^{2\lambda}a_1^\lambda}{|\mu_{r'}|^{2\ell_{r'}} \cdots |\mu_{s+1}|^{2\ell_{s+1}}} \lambda^{s'} \frac{|\mu_{s'}|^{2\lambda}a_{s'}^{(\lambda-1)} \cdots |\mu_1|^{2\lambda}a_1^{(\lambda-1)}}{|\mu_{s'}|^{2(\ell_{s'}+1)} \cdots |\mu_1|^{2(\ell_1+1)}} \Pi^*(F_\ell) \wedge \tilde{\phi},$$

where the sign depends on the relation between r, r', s and s'. Now the only way a factor $\bar{\sigma}$ in the numerator (more precisely in $\Pi^*(F_\ell) \wedge \tilde{\phi}$) could be cancelled out when λ is small, is by the occurrence of a factor $\bar{\sigma}$ in one of $\mu_1, \ldots, \mu_{s'}$, but that would obviously contradict the assumption made above. Hence each term in the integral must contain a factor $\bar{\sigma}$ or $d\bar{\sigma}$ independently of the value of λ and thus the σ-integral vanishes according to Lemma 2.2. □

Lemma 3.2. *Let $f = f_1 \oplus \cdots \oplus f_r$ be a section of $E^* = E_1^* \oplus \cdots \oplus E_r^*$. Assume that f is a complete intersection. Then*

(3.7)
$$|f_r|^{2\lambda}u^{f_r} \wedge \ldots \wedge |f_{s+1}|^{2\lambda}u^{f_{s+1}} \wedge \bar{\partial}|f_s|^{2\lambda} \wedge u^{f_s} \wedge \ldots \wedge \bar{\partial}|f_t|^{2\lambda} \wedge \ldots \wedge \bar{\partial}|f_1|^{2\lambda} \wedge u^{f_1}$$

has an analytic continuation to $\operatorname{Re}\lambda > -\epsilon$ *that vanishes at* $\lambda = 0$.

Remark 2. Morally, what this lemma says is that when applying Leibniz' rule to ∇ acting on a product of principal value and residue currents, there will be no contributions from ∇ falling on a residue factor. Of course this is expected, since the residue currents are ∇-closed. □

Proof. The result follows from Lemma 3.1 after an integration by parts with respect to ∇. (Recall that ϕ is a form taking values in $\Lambda^{n-\ell}E \wedge \Lambda^n E^*$.) Note that $\bar\partial|f_t|^{2\lambda} = -\nabla|f_t|^{2\lambda}$. By Stokes' theorem,

$$\int |f_r|^{2\lambda}u^{f_r}\wedge\ldots\wedge|f_{s+1}|^{2\lambda}u^{f_{s+1}}\wedge\bar\partial|f_s|^{2\lambda}\wedge u^{f_s}\wedge\ldots$$
$$\ldots\wedge\nabla|f_t|^{2\lambda}\wedge\ldots\wedge\bar\partial|f_1|^{2\lambda}\wedge u^{f_1}\wedge\phi =$$
$$\pm\int(|f_t|^{2\lambda}-1)\nabla(|f_r|^{2\lambda}u^{f_r}\wedge\ldots\wedge|f_{s+1}|^{2\lambda}u^{f_{s+1}}\wedge$$
$$\bar\partial|f_s|^{2\lambda}\wedge u^{f_s}\wedge\ldots\wedge\bar\partial|f_1|^{2\lambda}\wedge u^{f_1}\wedge\phi),$$

so it is enough to prove that this expression vanishes at $\lambda = 0$. However, the form

$$\nabla(|f_r|^{2\lambda}u^{f_r}\wedge\ldots\wedge|f_{s+1}|^{2\lambda}u^{f_{s+1}}\wedge\bar\partial|f_s|^{2\lambda}\wedge u^{f_s}\wedge\ldots\wedge\bar\partial|f_1|^{2\lambda}\wedge u^{f_1})$$

is precisely as in the hypothesis of Lemma 3.1 and f_t is an h of the second kind, so according to the lemma the factor $|f_t|^{2\lambda}$ does not have any effect on the value at $\lambda = 0$. Thus we are done. □

Lemma 3.3. *Let $f = f_1\oplus\cdots\oplus f_r$ be a section of $E^* = E_1^*\oplus\cdots\oplus E_r^*$ of rank m and let $h = f\oplus f'$, where f' is a section of the dual bundle of a holomorphic m'-bundle E'. Assume that h is a complete intersection. If $r > s$, then*

(3.8)
$$\bar\partial|h|^{2\lambda}\wedge u^h\wedge|f_r|^{2\lambda}u^{f_r}\wedge\ldots\wedge|f_{s+1}|^{2\lambda}u^{f_{s+1}}\wedge\bar\partial|f_s|^{2\lambda}\wedge u^{f_s}\wedge\ldots\wedge\bar\partial|f_1|^{2\lambda}\wedge u^{f_1}$$

has an analytic continuation to $\operatorname{Re}\lambda > -\epsilon$ *that vanishes at* $\lambda = 0$.

Remark 3. Notice that the value at $\lambda = 0$ corresponds to the current $R^h\wedge U^{f_r}\wedge\ldots\wedge U^{f_{s+1}}\wedge R^{f_s}\wedge\ldots\wedge R^{f_1}$. Since h is a complete intersection, R^h is of degree $m+m'$ in $d\bar z_j$ according to Proposition 1.3, and therefore it is reasonable to expect also the product to be of degree $m+m'$ in $d\bar z_j$. However, since the product contains at least one principal value factor, the degree in e_j must be strictly larger than the degree in $d\bar z_j$, and so, the product must vanish. We will see that the assumption that $r > s$ is crucial also for the proof. □

Proof. After a resolution of singularities as described in the proof of Proposition 2.1, we can write (3.8) integrated against a test form ϕ as a sum of terms of the type

$$(3.9)\qquad \int\bar\partial(|\mu_h|^{2\lambda}a_h^\lambda)\frac{\alpha_{h,\ell_h}}{\mu_h^{\ell_h}}\wedge|\mu_r|^{2\lambda}a_r^\lambda\frac{\alpha_{r,\ell_r}}{\mu_r^{\ell_r}}\wedge\ldots\wedge|\mu_{s+1}|^{2\lambda}a_{s+1}^\lambda\frac{\alpha_{s+1,\ell_{s+1}}}{\mu_{s+1}^{\ell_{s+1}}}\wedge$$
$$\bar\partial(|\mu_s|^{2\lambda}a_s^\lambda)\wedge\frac{\alpha_{s,\ell_s}}{\mu_s^{\ell_s}}\wedge\ldots\wedge\bar\partial(|\mu_1|^{2\lambda}a_1^\lambda)\wedge\frac{\alpha_{1,\ell_1}}{\mu_1^{\ell_1}}\wedge\tilde\phi,$$

where the α_{i,ℓ_i}'s are smooth forms of degree ℓ_i in e_j, the a_i's are non-vanishing functions and the μ_i's are monomials in some local coordinates τ_k and $\tilde{\phi}$ is as in the previous proofs.

We expand the factor $\bar\partial(|\mu_h|^{2\lambda}a_h^\lambda)$ by Leibniz' rule and consider the term obtained when $\bar\partial$ falls on $|\sigma|^{2\lambda}$, where σ is one of the τ_k's that divide μ_h. We prove that this term vanishes when integrating with respect to σ. The term that arises when $\bar\partial$ falls on a_h^λ clearly vanishes as before (see the proof of Proposition 2.1). Since the rank of $E \oplus E'$ is $m+m'$, the terms in (3.8) are of degree at most $m+m'-1$ in $d\bar z$, since we have at least one U-factor. Thus it is enough to consider test forms of codegree in $d\bar z$ at most $m+m'-1$. As in the previous proofs we may assume that $\phi = \phi_I \wedge d\bar z_I$. It follows that $d\bar z_I$ vanishes on $Y = h^{-1}(0)$ for degree reasons, and thus $\Pi^*(d\bar z_I)$ vanishes on $\Pi^{-1}Y$. Since this is a form in $d\bar\tau_j$ with antiholomorphic coefficients, each of its terms contains a factor $\bar\sigma$ or $d\bar\sigma$ and consequently the σ-integral vanishes according to Lemma 2.2. $\square$

Lemma 3.4. *Let $f = f_1 \oplus \cdots \oplus f_r$ be a section of $E^* = E_1^* \oplus \cdots \oplus E_r^*$. Assume that f is a complete intersection. Let $h = f_{I_1} \oplus \cdots \oplus f_{I_p}$, where $I = \{I_1, \ldots, I_p\} \subset \{1, \ldots, r\}$. Then*

$$|h|^{2\lambda} u^h \wedge \bar\partial |f_r|^{2\lambda} \wedge u^{f_r} \wedge \ldots \wedge \bar\partial |f_1|^{2\lambda} \wedge u^{f_1} \tag{3.10}$$

has an analytic continuation to $\operatorname{Re}\lambda > -\epsilon$ that vanishes at $\lambda = 0$.

Remark 4. The value at $\lambda = 0$ corresponds to the current $U^h \wedge R^{f_1} \wedge \ldots \wedge R^{f_r}$. Since the R-part is of top degree according to Proposition 1.3 this product should formally vanish by arguments similar to those in Remark 3. $\square$

Proof. As in the proofs of the previous lemmas we start by a resolution of singularities. Thus, the form (3.10) integrated against a test form ϕ is equal to a sum of terms of the type

$$\int |\mu_h|^{2\lambda} a_h^\lambda \frac{\alpha_{h,\ell_h}}{\mu_h^{\ell_h}} \wedge \bar\partial(|\mu_r|^{2\lambda} a_r^\lambda) \wedge \frac{\alpha_{r,\ell_r}}{\mu_r^{\ell_r}} \wedge \ldots \wedge \bar\partial(|\mu_1|^{2\lambda} a_1^\lambda) \wedge \frac{\alpha_{1,\ell_1}}{\mu_1^{\ell_1}} \wedge \tilde\phi, \tag{3.11}$$

where α_{i,ℓ_i}, a_i, μ_i and $\tilde\phi$ are as above. Further, we can find a resolution to a certain toric variety so that locally one of the monomials $\mu_1, \ldots, \mu_r$ divides the other ones. Without loss of generality we may assume that μ_1 divides all μ_j's. We expand $\bar\partial(|\mu_1|^{2\lambda} a_1^\lambda)$ by Leibniz' rule. The term obtained when $\bar\partial$ falls on a_1^λ vanishes as in the proof of Proposition 2.1, so it is enough to consider the terms that arise when $\bar\partial$ falls on $|\sigma|^{2\lambda}$, where σ is one of the coordinates in μ_1.

We claim that the σ-integral vanishes at $\lambda = 0$. As usual, we observe that the terms of (3.10) are of degree at most $m-1$ in $d\bar z_j$, where the -1 in this case is due to the factor U^h, so it suffices to consider test forms of codegree at most $m-1$. We assume that $\phi = \phi_I \wedge d\bar{z}_I$, where ϕ_I is an $(n,0)$-form and $d\bar{z}_I = d\bar{z}_{I_1} \wedge \ldots \wedge d\bar{z}_{I_p}$, where $p \leq n-m+1$. Then $d\bar{z}_I$ vanishes on the variety $Y = f^{-1}(0)$ for degree reasons, and accordingly $\Pi^*(d\bar{z})$ vanishes on $\Pi^{-1}Y$, and in particular on $\{\sigma_1 = 0\}$. By arguments as in the proof of Proposition 1.3 it follows that $\Pi^*(d\bar{z})$ must contain a factor $\bar{\sigma}$ or $d\bar{\sigma}$ since it is a form in $d\bar{\tau}_k$ with antiholomorphic coefficients, and hence the σ-integral vanishes as before. □

4. Some examples

We conclude this thesis by giving some examples of our residue currents. Although the sections we have chosen to work with are relatively simple, in fact we have just considered monomials, the computations become quite intricate, including repeated blow-ups. Yet we give a fairly detailed account of the calculations, since they make a nice complement to the proofs above. We are shown the toric resolutions in action and provided an illustration of the rather nontrivial argument about vanishing for degree reasons due to a factor $\bar{\sigma}$ or $d\bar{\sigma}$ (for some σ), which was first used in the proof of Proposition 1.3. Also, we see how combinatorics comes into play in a not always obvious way.

Example 1. We start by illustrating Theorem 1.4 in a very simple case of complete intersection. Let $f_1 = z_1^p$, $f_2 = z_2^p$ and $f_3 = z_3^p$, where p is some integer (> 1 to make the example somewhat interesting), in a neighborhood $\mathcal{U}$ of the origin in $\mathbb{C}^3$. Then $Y = f^{-1}(0)$ is simply the origin so f is a complete intersection.

We want to compute and compare the currents $R^{f_1 \oplus f_2} \wedge R^{f_3}$ and $R^{f_1 \oplus f_2 \oplus f_3}$. However, to warm up and to make the investigation complete we start by computing $R^{f_1} \wedge R^{f_2} \wedge R^{f_3}$. Since each f_i depends only on z_i, the f_i's are separated and thus the current can be regarded as the tensor product $R^{f_1} \otimes R^{f_2} \otimes R^{f_3}$, where

$$R^{f_i} = \bar{\partial}\Big[\frac{1}{f_i}\Big] \wedge e_i = \bar{\partial}\Big[\frac{1}{z_i^p}\Big] \wedge e_i.$$

Here $\bar{\partial}[1/f]$ is merely a short-hand notation for the value at $\lambda = 0$ of $\bar{\partial}|f|^{2\lambda}/f$, which will be used in the sequel. By $[1/f]$ we will mean $|f|^{2\lambda}|_{\lambda=0}$, that is just the principal value of $1/f$. Let $\phi = \varphi(z)dz$. By iterated integration by parts we have that

$$\int_z \bar{\partial}\Big[\frac{1}{z^p}\Big] \wedge \phi = \frac{2\pi i}{(p-1)!}\frac{\partial^{p-1}}{\partial z^{p-1}}\varphi(0). \tag{4.1}$$

Thus, if $\phi = \varphi(z_1, z_2, z_3)\, dz_1 \wedge dz_2 \wedge dz_3$, it follows that

$$R^{f_1} \otimes R^{f_2} \otimes R^{f_3}.\phi = \Big(\frac{-2\pi i}{(p-1)!}\Big)^3 \frac{\partial^{p-1}}{\partial z_1^{p-1}} \frac{\partial^{p-1}}{\partial z_2^{p-1}} \frac{\partial^{p-1}}{\partial z_3^{p-1}}\, \varphi(0,0,0) \wedge e_1 \wedge e_2 \wedge e_3.$$

We continue with the current $R = R^{f_1 \oplus f_2} \wedge R^{f_3}$, that can be considered as $R = R^{f_1 \oplus f_2} \otimes R^{f_3}$ as above. Thus, we need to compute $R^{f_1 \oplus f_2}$. In the trivial metric

$$s = \bar{f}_1 e_1 + \bar{f}_2 e_2 = \bar{z}_1^p e_1 + \bar{z}_2^p e_2,$$

so that

$$u = \frac{s}{|f|^2} + \frac{s \wedge \bar\partial s}{|f|^4} = \frac{\bar{z}_1^p e_1 + \bar{z}_2^p e_2}{|z|^{2p}} + \frac{\bar{z}_1^p \bar\partial(\bar{z}_2^p) - \bar\partial(\bar{z}_1^p)\bar{z}_2^p}{|z|^{4p}} \wedge e_1 \wedge e_2,$$

where $|z|^2 = |z_1|^2 + |z_2|^2$. To find the extension of u_2 across Y we consider the proper mapping $\Pi\colon \widetilde{\mathcal{U}} \to \mathcal{U}$, where $\widetilde{\mathcal{U}}$ is the blow-up at the origin of $\mathcal{U}$. We can cover $\widetilde{\mathcal{U}}$ by two coordinate neighborhoods

$$\Omega_1 = \{\tau;\ (\tau_1, \tau_1\tau_2) = z \in \mathcal{U}\} \text{ and } \Omega_2 = \{\sigma;\ (\sigma_1\sigma_2, \sigma_1) = z \in \mathcal{U}\}.$$

Let us start by computing $\tilde{u} = \Pi^* u$ in Ω_1. Notice that

$$\tilde{s} = \Pi^* s = \bar\tau_1^p e_1 + \bar\tau_1^p \bar\tau_2^p e_2 = \bar\tau_1^p (e_1 + \bar\tau_2^p e_2).$$

Recall that due to homogeneity, if $s = \mu s'$, where μ is a smooth function, then

$$s \wedge (\bar\partial s)^{\ell-1} = \mu^\ell s'(\bar\partial s')^{\ell-1},$$

so it follows that

$$\tilde{u} = \frac{e_1 + \tau_2^p e_2}{\tau_1^p(1 + |\tau_2|^{2p})} + \frac{p\bar\tau_2^{p-1} e_1 \wedge d\bar\tau_2 \wedge e_2}{\tau_1^{2p}(1 + |\tau_2|^{2p})^2}.$$

The extension of $\tilde{u}$ across $\widetilde{Y}$ is simply

$$\Big[\frac{1}{\tau_1^p}\Big] \frac{e_1 + \tau_2^p e_2}{1 + |\tau_2|^{2p}} + \Big[\frac{1}{\tau_1^{2p}}\Big] \frac{p\bar\tau_2^{p-1} e_1 \wedge d\bar\tau_2 \wedge e_2}{(1 + |\tau_2|^{2p})^2},$$

and therefore

$$R^{\Pi^* f_1 \oplus f_2} = \bar\partial\Big[\frac{1}{\tau_1^p}\Big] \wedge \frac{e_1 + \tau_2^p e_2}{1 + |\tau_2|^{2p}} + \bar\partial\Big[\frac{1}{\tau_1^{2p}}\Big] \wedge \frac{p\bar\tau_2^{p-1} e_1 \wedge d\bar\tau_2 \wedge e_2}{(1 + |\tau_2|^{2p})^2}$$

in Ω_1. To compute the action of $R^{f_1 \oplus f_2}$ let ϕ be a test form in $\mathcal{U}$. Then

$$\int_{\mathcal{U}} R^{f_1 \oplus f_2} \wedge \phi = \int_{\widetilde{\mathcal{U}}} R^{\Pi^*(f_1 \oplus f_2)} \wedge \Pi^*\phi. \tag{4.2}$$

Recall that, since f is a complete intersection, the part of degree 1 should vanish according to Proposition 1.3. To see how this happens, assume that $\phi = \phi_I\, dz_1 \wedge dz_2$, where $\phi_I = \varphi^1(z_1, z_2)\, d\bar{z}_1 + \varphi^2(z_1, z_2)\, d\bar{z}_2$. Now, $\Pi^*\phi_I =$

$\varphi^1(\tau_1,\tau_1\tau_2)\,d\bar\tau_1+\varphi^2(\tau_1,\tau_1\tau_2)\,d(\bar\tau_1\bar\tau_2)$, so each term of $\Pi^*\phi$ contains a factor $\bar\tau_1$ or $d\bar\tau_1$ as predicted in the proof of Proposition 1.3, and consequently the $R_1^{\Pi^*(f_1\oplus f_2)}$-part of the integral vanishes.

Further, for the action of $R_2^{\Pi^* f_1\oplus f_2}$, it is enough to consider a $(2,0)$-form $\phi=\varphi(z_1,z_2)dz_1\wedge dz_2$. Then

$$\Pi^*\phi=\varphi(\tau_1,\tau_1\tau_2)\;d\tau_1\wedge d(\tau_1\tau_2)=\tau_1\varphi(\tau_1,\tau_1\tau_2)\;d\tau_1\wedge d\tau_2$$

in Ω_1, so the right hand side integral of (4.2) is equal to

$$\int_\tau \bar\partial\Big[\frac{1}{\tau_1^{2p-1}}\Big]\wedge\frac{p\bar\tau_2^{p-1}e_1\wedge d\bar\tau_2\wedge e_2}{(1+|\tau_2|^{2p})^2}\wedge\varphi(\tau_1,\tau_1\tau_2)\;d\tau_1\wedge d\tau_2.$$

From (4.1) we know that this is equal to

$$-\frac{2\pi i}{(2p-2)!}\int_{\tau_2}\frac{p\bar\tau_2^{p-1}e_1\wedge d\bar\tau_2\wedge e_2}{(1+|\tau_2|^{2p})^2}\wedge\frac{\partial^{2p-2}}{\partial\tau_1^{2p-2}}\varphi(\tau_1,\tau_1\tau_2)|_{\tau_1=0}\;d\tau_2.$$

Observe that

$$\frac{\partial^r}{\partial\tau_1^r}\varphi(\tau_1,\tau_1\tau_2)=\sum_k\binom{r}{k}\tau_2^k\varphi_{1^{r-k}2^k}(\tau_1,\tau_1\tau_2),$$

where $\varphi_{1^j2^k}$ means j derivatives with respect to the first variable and k derivatives with respect to the second one. Now, only the term with a factor τ_2^{p-1} will contribute. The other terms vanish due to antisymmetry, as is easily verified. Thus we get

$$-\frac{2\pi i}{(2p-2)!}\binom{(2p-2)}{(p-1)}\varphi_{1^{p-1}2^{p-1}}(0,0)\int_{\tau_2}\frac{p|\tau_2|^{2(p-1)}e_1\wedge d\bar\tau_2\wedge e_2}{(1+|\tau_2|^{2p})^2}\wedge d\tau_2=$$
$$-\frac{2\pi i}{((p-1)!)^2}\;\varphi_{1^{p-1}2^{p-1}}(0,0)(-2\pi i)\;e_1\wedge e_2=$$
$$\Big(\frac{-2\pi i}{(p-1)!}\Big)^2\frac{\partial^{p-1}}{\partial z_1^{p-1}}\frac{\partial^{p-1}}{\partial z_2^{p-1}}\;\varphi(0,0,0)\;e_1\wedge e_2.$$

Hence

$$R^{f_1\oplus f_2}\otimes R^{f_3}.\phi=\Big(\frac{-2\pi i}{(p-1)!}\Big)^3\frac{\partial^{p-1}}{\partial z_1^{p-1}}\frac{\partial^{p-1}}{\partial z_2^{p-1}}\frac{\partial^{p-1}}{\partial z_3^{p-1}}\;\varphi(0,0,0)\wedge e_1\wedge e_2\wedge e_3$$

in Ω_1. Since the current $R^{\Pi^* f_1\oplus f_2}$ has support only when $\tau_1=0$ we can compute the integral over the entire $\widetilde{\mathcal{U}}$ by extending the integration to $\tau_2\in\mathbb{P}^1$, so this is indeed the total action of the current.

It remains to calculate the current $R^{f_1\oplus f_2\oplus f_3}$. In the trivial metric we have

$$s=\bar f_1e_1+\bar f_2e_2+\bar f_3e_3=\bar z_1^pe_1+\bar z_2^pe_2+\bar z_3^pe_3.$$

To find the extension across Y we need a resolution of the singularity at the origin. Consider the proper mapping $\Pi\colon \widetilde{\mathcal{U}} \to \mathcal{U}$, where $\widetilde{\mathcal{U}}$ is the blow-up of the origin. We can cover $\widetilde{\mathcal{U}}$ by three coordinate neighborhoods

$$\Omega_1 = \{\tau;\ (\tau_1, \tau_1\tau_2, \tau_1\tau_3) = z \in \mathcal{U}\}\ ,\ \Omega_2 = \{\sigma;\ (\sigma_1\sigma_2, \sigma_1, \sigma_1\sigma_3) = z \in \mathcal{U}\}$$
$$\text{and } \Omega_3 = \{\rho;\ (\rho_1\rho_2, \rho_1\rho_3, \rho_1) = z \in \mathcal{U}\}.$$

Let us compute $\tilde{u}_3 = \Pi^* u_3$, that is the part of u of top degree, in one of the neighborhoods; Ω_1. We leave it to the reader to verify that the terms of lower degree vanish. First, notice that

$$\tilde{s} = \Pi^* s = \bar{\tau}_1^p (e_1 + \bar{\tau}_2^p e_2 + \bar{\tau}_3^p e_3),$$

so that

$$\tilde{u}_3 = \frac{p^2 \bar{\tau}_2^{p-1} \bar{\tau}_3^{p-1} e_1 \wedge d\bar{\tau}_2 \wedge e_2 \wedge d\bar{\tau}_3 \wedge e_3}{\tau_1^{3p}(1 + |\tau_2|^{2p} + |\tau_3|^{2p})^3},$$

and thus

$$R_3^{\Pi^*(f_1 \oplus f_2 \oplus f_3)} = \bar{\partial}\Big[\frac{1}{\tau_1^{3p}}\Big] \wedge \frac{p^2 \bar{\tau}_2^{p-1} \bar{\tau}_3^{p-1} e_1 \wedge d\bar{\tau}_2 \wedge e_2 \wedge d\bar{\tau}_3 \wedge e_3}{(1 + |\tau_2|^{2p} + |\tau_3|^{2p})^3}$$

in Ω_1. Now, if $\phi = \varphi(z_1, z_2, z_3) dz_1 \wedge dz_2 \wedge z_3$ is a test form in $\mathcal{U}$ we get

$$\int_{\mathcal{U}} R_3^{f_1 \oplus f_2 \oplus f_3} \wedge \phi = \int_{\widetilde{\mathcal{U}}} R_3^{\Pi^*(f_1 \oplus f_2 \oplus f_3)} \wedge \Pi^* \phi, \tag{4.3}$$

and since

$$\Pi^* \phi = \varphi(\tau_1, \tau_1\tau_2)\, d\tau_1 \wedge d(\tau_1\tau_2) \wedge d(\tau_1\tau_3) = \tau_1^2 \varphi(\tau_1, \tau_1\tau_2)\, d\tau_1 \wedge d\tau_2 \wedge d\tau_3$$

in Ω_1, the right hand side integral of (4.3) is equal to

$$\begin{aligned}(4.4)\quad \int_\tau \bar{\partial}\Big[\frac{1}{\tau_1^{3p-2}}\Big] \wedge \frac{p^2 \bar{\tau}_2^{p-1} \bar{\tau}_3^{p-1} e_1 \wedge d\bar{\tau}_2 \wedge e_2 \wedge d\bar{\tau}_3 \wedge e_3}{(1 + |\tau_2|^{2p} + |\tau_3|^{2p})^3} \wedge \\ \varphi(\tau_1, \tau_1\tau_2, \tau_1\tau_3)\ d\tau_1 \wedge d\tau_2 \wedge d\tau_3 = \\ -\frac{2\pi i}{(3p-3)!} \int_{\tau_2, \tau_3} \frac{p^2 \bar{\tau}_2^{p-1} \bar{\tau}_3^{p-1} e_1 \wedge d\bar{\tau}_2 \wedge e_2 \wedge d\bar{\tau}_3 \wedge e_3}{(1 + |\tau_2|^{2p} + |\tau_3|^{2p})^3} \wedge \\ \frac{\partial^{3p-3}}{\partial \tau_1^{3p-3}} \varphi(\tau_1, \tau_1\tau_2, \tau_1\tau_3)|_{\tau_1 = 0}\ d\tau_2 \wedge d\tau_3.\end{aligned}$$

Moreover, we have that

$$\frac{\partial^r}{\partial \tau_1^r} \varphi(\tau_1, \tau_1\tau_2, \tau_1\tau_3) = \sum_{k,j} \binom{r}{k} \binom{r-k}{j} \tau_2^k \tau_3^j \varphi_{1^{r-k-j} 2^k 3^j}(\tau_1, \tau_1\tau_2, \tau_1\tau_3).$$

The only term of $\frac{\partial^{3p-3}}{\partial \tau_1^{3p-3}}\varphi(\tau_1, \tau_1\tau_2, \tau_1\tau_3)|_{\tau_1=0}$ in (4.4) that does not vanish due to antisymmetry is the one with the factor $\tau_2^{p-1}\tau_3^{p-1}$ so the last integral is equal to

$$-\frac{2\pi i}{(3p-3)!}\binom{(3p-3)}{(p-1)}\binom{(2p-2)}{(p-1)}\varphi_{1^{p-1}2^{p-1}3^{p-1}}(0,0,0)$$
$$\int_{\tau_2\tau_3}\frac{p^2|\tau_2|^{2(p-1)}|\tau_3|^{2(p-1)}e_1\wedge d\bar{\tau}_2\wedge e_2\wedge d\bar{\tau}_3\wedge e_3}{(1+|\tau_2|^{2p}+|\tau_3|^2)^3}\wedge d\tau_2\wedge d\tau_3 =$$
$$-\frac{2\pi i}{((p-1)!)^3}\varphi_{1^{p-1}2^{p-1}3^{p-1}}(0,0,0)(-2\pi i)^2 e_1\wedge e_2\wedge e_3 =$$
$$\left(\frac{-2\pi i}{(p-1)!}\right)^3\frac{\partial^{p-1}}{\partial z_1^{p-1}}\frac{\partial^{p-1}}{\partial z_2^{p-1}}\frac{\partial^{p-1}}{\partial z_3^{p-1}}\varphi(0,0,0)\ e_1\wedge e_2\wedge e_3.$$

Since the current $R^{f_1\oplus f_2\oplus f_3}$ has support at the origin, and thus $R^{\Pi^*(f_1\oplus f_2\oplus f_3)}$ has support where $\tau_1 = 0$, we can compute the integral over the entire $\widetilde{\mathcal{U}}$ simply by extending the integration to $\tau_2, \tau_3 \in \mathbb{P}^2$, so we have actually computed the total action of the current.

Hence we have shown that the three currents $R^{f_1}\wedge R^{f_2}\wedge R^{f_3}$, $R^{f_1\oplus f_2}\wedge R^{f_3}$ and $R^{f_1\oplus f_2\oplus f_3}$, in this case, are the same in accordance with Theorem 1.4. $\square$

When generalizing Theorem 1.1, or rather its line bundle formulation (1.9), to sections of bundles of arbitrary rank, it is not obvious how one should interpret the assumption of f being a complete intersection. In the formulation of Theorem 1.4 we require the codimension of $f^{-1}(0)$ to be equal to the rank of the bundle E. A less strong hypothesis would be to just demand the f_i's to intersect properly, that is that $\operatorname{codim} f^{-1}(0) = p_1 + \ldots + p_r$ if $p_i = \operatorname{codim} f_i$. However, the following example shows that Theorem 1.4 does not extend to this case.

Example 2. Let $f_1 = z_1^2$, $f_2 = z_1z_2$ and $g = z_2z_3$ be defined in a neighborhood of the origin in $\mathbb{C}^3$. Then

$$Y_f = \{f_1 = f_2 = 0\} = \{z_1 = 0\},$$
$$Y_g = \{g = 0\} = \{z_2 = 0\} \cup \{z_3 = 0\},$$

and

$$Y = Y_f \cap Y_g = \{z_1 = z_2 = 0\} \cup \{z_1 = z_3 = 0\},$$

that is the union of the z_3-axis and the z_2-axis. Evidently, Y_f and Y_g have codimension 1, and Y has codimension 2, so f and g intersect properly, although they do not define a complete intersection.

We want to compute and compare the currents $R^f \wedge R^g$ and $R^{f\oplus g}$. However, we confine ourselves to computing the parts of the currents of degree 2, since we find them the most interesting. The top degree parts do not differ very much from the ones in Example 1. Therefore it suffices to consider test forms of bidegree $(3,1)$. Throughout this example we let $\phi = \phi_I \wedge d\bar{z}$, where $d\bar{z} = d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3$ and $\phi_I = \varphi^1(z)d\bar{z}_1 + \varphi^2(z)d\bar{z}_2 + \varphi^3(z)d\bar{z}_3$.

We start with the current $R^f \wedge R^g$. Adopting the trivial metric we get

$$s^f = \bar{f}_1 e_1 + \bar{f}_2 e_2 = \bar{z}_1(\bar{z}_1 e_1 + \bar{z}_2 e_2)$$

and

$$|f|^2 = |z_1|^2(|z_1|^2 + |z_2|^2),$$

so that

$$u_1^f = \frac{\bar{z}_1 e_1 + \bar{z}_2 e_2}{z_1(|z_1|^2 + |z_2|^2)}$$

(for the part of $R^f \wedge R^g$ of degree 2 we only need to care about u_1). If we restrict ourselves to test forms with support outside $\{z_2 = 0\}$, where $|z_1|^2 + |z_2|^2$ is non-vanishing, the action on ϕ is given by

$$\int \bar\partial(|z_1|^{2\lambda}(|z_1|^2+|z_2|^2)^\lambda)\frac{\bar{z}_1 e_1 + \bar{z}_2 e_2}{z_1(|z_1|^2 + |z_2|^2)} \wedge \bar\partial|z_2 z_3|^{2\lambda}\frac{e_3}{z_2 z_3} \wedge \phi|_{\lambda=0} =$$
$$-\int \bar\partial\Big[\frac{1}{z_1}\Big]\Big[\frac{1}{z_2}\Big] \wedge \bar\partial\Big[\frac{1}{z_3}\Big] \wedge e_2 \wedge e_3 \wedge \phi.$$

In particular, this is a current with support on the z_2-axis, which is to be expected, since Y is the union of the z_2- and z_3-axes and ϕ now has support outside the z_3-axis. More explicitly we can write this expression as

$$\int_{z_2} \frac{1}{z_2}\varphi^2(0,z_2,0)\, e_2 \wedge e_3 \wedge d\bar{z}_2 \wedge dz_2.$$

To include also test forms with support over $\{z_2 = 0\}$ we need to resolve the singularity of f at the z_3-axis. By the blow-up of the origin in the $z_1 z_2$-plane, using the notation of Example 1, we get

$$\tilde{s}^f = \Pi^* s^f = \bar{\tau}_1^2(e_1 + \bar{\tau}_2 e_2)$$

in Ω_1, so that

$$\tilde{u}_1^f = \frac{e_1 + \bar{\tau}_2 e_2}{\tau_1^2(1+|\tau_2|^2)}.$$

The action of $R^{\Pi^* f} \wedge R^{\Pi^* g}$ on $\tilde\phi$ is given by

$$\int \bar\partial|\tau_1|^{4\lambda} \wedge \frac{e_1 + \bar{\tau}_2 e_2}{\tau_1^2(1+|\tau_2|^2)} \wedge\ \bar\partial|\tau_1\tau_2 z_3|^{2\lambda} \wedge \frac{e_3}{\tau_1\tau_2 z_3} \wedge \tilde\phi|_{\lambda=0} =$$
$$\frac{2}{3}\int \bar\partial|\tau_1|^{6\lambda} \wedge \frac{e_1 + \bar{\tau}_2 e_2}{\tau_1^3(1+|\tau_2|^2)} \wedge\ \bar\partial|\tau_2 z_3|^{2\lambda} \wedge \frac{e_3}{\tau_2 z_3} \wedge \tilde\phi|_{\lambda=0}. \tag{4.5}$$

Note that we have omitted the non-vanishing factor $(1+|\tau_2|^2)^\lambda$ from the numerator. This is allowed since it has no effect on the value at $\lambda=0$ according to Lemma 2.2. Before proceeding, let us take a closer look at $\phi=\Pi^*\phi_I\wedge\Pi^*dz$. Observing that

$$\Pi^*dz = d\tau_1\wedge d(\tau_1\tau_2)\wedge dz_3=\tau_1 d\tau_1\wedge d\tau_2\wedge dz_3,$$

and that

$$\Pi^*\phi_I=\varphi^1(\tau_1,\tau_1\tau_2,z_3)\ d\bar\tau_1+\varphi^2(\tau_1,\tau_1\tau_2,z_3)\ d(\bar\tau_1\bar\tau_2)+\varphi^3(\tau_1,\tau_1\tau_2,z_3)\ d\bar z_3,$$

we notice that if the second $\bar\partial$ in the right hand side integral of (4.5) falls on $|z_3|^{2\lambda}$, the only part of ϕ that survives is the φ^2-part. However, this contains a factor $d(\bar\tau_1\bar\tau_2)$, which implies that the τ_1 integral vanishes according to Lemma 2.2. Thus, this $\bar\partial$ must fall on $|\tau_2|^{2\lambda}$ and therefore the integral is equal to

$$\begin{aligned}&\frac{2}{3}\int\bar\partial\Big[\frac{1}{\tau_1^3}\Big]\wedge e_1\wedge\bar\partial\Big[\frac{1}{\tau_2}\Big]\Big[\frac{1}{z_3}\Big]\wedge e_3\wedge\tilde\phi=\\ &\frac{2}{3}\int\bar\partial\Big[\frac{1}{\tau_1^2}\Big]\wedge e_1\wedge\bar\partial\Big[\frac{1}{\tau_2}\Big]\Big[\frac{1}{z_3}\Big]\wedge e_3\wedge\varphi^3(\tau_1,\tau_1\tau_2,z_3)d\bar z_3\wedge d\tau_1\wedge d\tau_2\wedge dz_3=\\ &\qquad(2\pi i)^2\frac{2}{3}\int\varphi_1^3(0,0,z_3)\frac{1}{z_3}e_1\wedge e_3\wedge d\bar z_3\wedge dz_3=\\ &\qquad\qquad-\frac{2}{3}\int\bar\partial\Big[\frac{1}{z_1^2}\Big]\wedge\bar\partial\Big[\frac{1}{z_2}\Big]\Big[\frac{1}{z_3}\Big]\wedge e_1\wedge e_3\wedge\tilde\phi.\end{aligned}$$

For the last two equalities we have used (4.1) from Example 1.

Next, we compute the action of $R^{\Pi^*f}\wedge R^{\Pi^*g}$ in Ω_2. Now,

$$\tilde s^f=\bar\sigma_1^2\bar\sigma_2(\sigma_2 e_1+e_2),$$

and so

$$\tilde u_1^f=\frac{\bar\sigma e_1+e_2}{\sigma_1^2\sigma_2(1+|\sigma_2|^2)}.$$

Thus, the action on $\tilde\phi$ is given by

$$\int\bar\partial|\sigma_1^2\sigma_2|^{2\lambda}\wedge\frac{\bar\sigma_2 e_1+e_2}{\sigma_1^2\sigma_2(1+|\sigma_2|^2)}\wedge\ \bar\partial|\sigma_1 z_3|^{2\lambda}\wedge\frac{e_3}{\sigma_1 z_3}\wedge\tilde\phi|_{\lambda=0}.$$

If the first $\bar\partial$ falls on $|\sigma_1|^{2\lambda}$, the second $\bar\partial$ must fall on $|z_3|^{2\lambda}$, and then the only part of $\tilde\phi_I$ that comes into play is $\bar\sigma_1\varphi^2 d\bar\sigma_2$, so the σ_1-integral will vanish due to the factor $\bar\sigma_1$. Thus the first $\bar\partial$ must fall on $|\sigma_2|^{2\lambda}$. The term that appears when the second $\bar\partial$ falls on $|z_3|^{2\lambda}$ is exactly the current we obtained when we assumed that $z_2\neq 0$, as can be easily checked. Therefore, what remains is

$$\frac{1}{3}\int \bar\partial|\sigma_2|^{2\lambda}\wedge\frac{\bar\sigma_2 e_1+e_2}{\sigma_1^2\sigma_2(1+|\sigma_2|^2)}\wedge\ \bar\partial|\sigma_1|^{6\lambda}\wedge\frac{e_3}{\sigma_1 z_3}\wedge\tilde\phi|_{\lambda=0}=$$

$$\frac{1}{3}\int\bar\partial\Big[\frac{1}{\sigma_2}\Big]\wedge e_2\wedge\bar\partial\Big[\frac{1}{\sigma_1^3}\Big]\Big[\frac{1}{z_3}\Big]\wedge e_3\wedge\tilde\phi=$$

$$(2\pi i)^2\frac{1}{3}\int\varphi_2^3(0,0,z_3)\frac{1}{z_3}\wedge e_2\wedge e_3\wedge d\bar z_3\wedge dz_3=$$

$$\frac{1}{3}\int\bar\partial\Big[\frac{1}{z_1}\Big]\wedge\bar\partial\Big[\frac{1}{z_2^2}\Big]\Big[\frac{1}{z_3}\Big]\wedge e_2\wedge e_3\wedge\tilde\phi.$$

To sum up so far, the action of $R_1^f\wedge R^g$ consists of two parts, one of which is an integration of the test form along the z_2-axis, and another of which is an integration along the z_3-axis of derivatives of the test form.

We continue with $R^{f\oplus g}$. Now

$$s=s^{f\oplus g}=\bar z_1^2e_1+\bar z_1\bar z_2e_2+\bar z_2\bar z_3e_3.$$

To be able to compute the current we need a resolution of singularities. Recall the blow-up of the origin in $\mathbb{C}^3$ from Example 1. In the coordinate neighborhood Ω_1 we get

$$\tilde s=\Pi^*s=\bar\tau_1^2s',\text{ where } s'=e_1+\bar\tau_2e_2+\bar\tau_2\bar\tau_3e_3,$$

and thus the action of $R_2^{\Pi^*(f\oplus g)}$ on $\tilde\phi$ there is given by

$$\int\bar\partial|\tau_1|^{4\lambda}\frac{s'\wedge\bar\partial s'}{\tau_1^4(1+|\tau_2|^2+|\tau_2\tau_3|^2)^2}\wedge\tilde\phi|_{\lambda=0}. \tag{4.6}$$

However, all terms of

$$\Pi^*\phi=\phi^1d\bar\tau_1+\phi^2d(\bar\tau_1\bar\tau_2)+\phi^3d(\bar\tau_1\bar\tau_3)$$

contain a factor $\bar\tau_1$ or $d\bar\tau_1$ and thus (4.6) vanishes at $\lambda=0$ according to Lemma 2.2.

Morally, this can be explained by the fact that the image of Ω_1 under Π is $\mathcal{U}$ minus the plane $\{z_1=0\}$ (plus the origin). However we know that $R^f\wedge R^g$ should have support in this plane (more precisely on the axes), and thus it is not very surprising that we get no contribution from Ω_1.

We go on to the second coordinate neighborhood Ω_2. According to the intuitive argument above, since the image of Ω_2 is $\mathcal{U}$ minus the plane $\{z_2=0\}$, we expect to obtain something with support on the z_2-axis. First, note that

$$\tilde s=\bar\sigma_1^2(\bar\sigma_2^2e_1+\bar\sigma_2e_2+\bar\sigma_3e_3).$$

To calculate the current close to the z_2-axis, that is, where σ_2 and σ_3 are both zero, we need to refine the resolution by a blow-up along the axis. For a neighborhood $\mathcal{U}_\sigma$ of the origin in the $\sigma_2\sigma_3$-plane we consider the proper mapping $\Pi_\sigma\colon \widetilde{\mathcal{U}}_\sigma \to \mathcal{U}_\sigma$, where $\widetilde{\mathcal{U}}_\sigma$ is the blow-up of the origin in the $\sigma_2\sigma_3$-plane. We cover $\widetilde{\mathcal{U}}_\sigma$ by two coordinate neighborhoods

$$\{x;\ (x_1, x_1x_2) = (\sigma_2, \sigma_3) \in \mathcal{U}_\sigma\} \ \text{ and } \{y;\ (y_1y_2, y_1) = (\sigma_2, \sigma_3) \in \mathcal{U}_\sigma\}.$$

In the second one we have

$$\tilde{s} = \bar{\sigma}_1^2 \bar{y}_1 s', \text{ where } s' = \bar{y}_1\bar{y}_2^2 e_1 + \bar{y}_2 e_2 + e_3,$$

so that the action on $\tilde{\phi}$ here is given by

$$\int \bar{\partial}|\sigma_1^2 y_1|^{2\lambda} \wedge \frac{s' \wedge \bar{\partial} s'}{\sigma_1^4 y_1^2 (1 + |y_2^2 y_1|^2 + |y_2|^2)^2} \wedge \tilde{\phi}|_{\lambda=0}.$$

Now $\bar{\partial}$ must fall on the factor y_1, since otherwise the integral will vanish by the usual arguments, so we get

$$\int \bar{\partial}\Big[\frac{1}{y_1^2}\Big]\Big[\frac{1}{\sigma_1^4}\Big] \wedge \frac{e_3 \wedge d\bar{y}_2 \wedge e_2}{\sigma_1^4 y_1^2 (1 + |y_2^2 y_1|^2 + |y_2|^2)^2} \wedge \tilde{\phi}.$$

Now let us take a closer look at $\tilde{\phi}$. The pull-back of dz is equal to $\sigma_1^2 y_1\, dy_2 \wedge d\sigma_1 \wedge y_1$ and the only part of ϕ_I that matters is $\varphi^2(\sigma_1 y_1 y_2, \sigma_1, \sigma_1 y_1)\, d\bar{\sigma}_1$. By an integration in y_2 we get

$$-(2\pi i)^2 \int \frac{1}{\sigma_1^2} \varphi^2(0, \sigma_1, 0)\ e_2 \wedge e_3 \wedge d\bar{\sigma}_1 \wedge d\sigma_1 =$$
$$- \int \bar{\partial}\Big[\frac{1}{z_1}\Big]\Big[\frac{1}{z_2^2}\Big] \wedge \bar{\partial}\Big[\frac{1}{z_3}\Big] \wedge \phi \wedge e_2 \wedge e_3.$$

Some remarks are in order. First note that the integration in y_2 might be extended to $\mathbb{P}^1$ so that we have actually calculated the action in $\widetilde{\mathcal{U}}_2$. Also, let us just notice that this part of $R^{f\oplus g}$ coincides with the corresponding part of $R^f \wedge R^g$.

Finally we compute the contribution from Ω_3. One realizes that a further resolution along the z_1-axis is necessary. For a neighborhood $\mathcal{U}_3$ of the origin in the $\rho_2\rho_3$-plane, we consider $\Pi_\sigma\colon \widetilde{\mathcal{U}}_\rho \to \mathcal{U}_\rho$, where $\widetilde{\mathcal{U}}_\rho$ is the blow-up of the origin. We cover this by two coordinate neighborhoods

$$\{x;\ (x_1, x_1x_2) = (\rho_2, \rho_3) \in \mathcal{U}_\rho\} \ \text{ and } \{y;\ (y_1y_2, y_1) = (\rho_2, \rho_3) \in \mathcal{U}_\rho\}.$$

It is easily verified that there will be no contribution from the second neighborhood. As usual, the pull-back of the test form will contain factors that kill the current. So what remains is the first neighborhood. It turns out that we actually need a third resolution, now of the origin in the x_1x_2-plane. The blow-up $\widetilde{\mathcal{U}}_x$ of a neighborhood $\mathcal{U}_x$ of this point is covered by two coordinate neighborhoods

$$\{\xi;\ (\xi_1, \xi_1\xi_2) = (x_1, x_2) \in \mathcal{U}_x\} \ \text{ and } \{\eta;\ (\eta_1\eta_2, \eta_1) = (x_1, x_2) \in \mathcal{U}_x\}.$$

In the first one,

$$s = \bar{\rho}_1^2 \bar{\xi}_1^2 s', \text{ where } s' = e_1 + \bar{\xi}_1 \bar{\xi}_2 e_2 + \bar{\xi}_2 e_3,$$

so that the action on $\widetilde{\phi}$ is given by

$$\int \bar{\partial} |\rho_1^2 \xi_1^2|^{2\lambda} \wedge \frac{s' \wedge \bar{\partial} s'}{\rho_1^4 \xi_1^4 (1 + |\xi_1 \xi_2|^2 + |\xi_2|^2)^2} \wedge \tilde{\phi}|_{\lambda=0}.$$

By the usual degree arguments, the $\bar{\partial}$ must fall on $|\xi_1|^{2\lambda}$, and since $e_1 \wedge d\bar{\xi}_2 \wedge e_3$ is the only part of $s' \wedge \bar{\partial} s'$ that does not contain $\bar{\xi}_1$ or $d\bar{\xi}_1$, the integral is equal to

$$\int \bar{\partial} \Big[\frac{1}{\xi_1^4}\Big] \Big[\frac{1}{\rho_1^4}\Big] \wedge \frac{e_1 \wedge d\bar{\xi}_2 \wedge e_3}{(1 + |\xi_1 \xi_2|^2 + |\xi_2|^2)^2} \wedge \tilde{\phi}.$$

The pull-back of dz is equal to $\rho_1^2 \xi_1^2 \, d\xi_1 \wedge d\xi_2 \wedge d\rho_1$, and the only surviving part of ϕ_I is $\varphi^3(\rho_1 \xi_1, \rho_1 \xi_1 \xi_2, \rho_1) \; d\bar{\rho}_1$, so by an integration in ξ_2, we get

$$(2\pi i)^2 \int \frac{1}{\rho_1^2} \; \varphi_1^3(0, 0, \rho_1) \; e_1 \wedge e_3 \wedge d\bar{\rho}_1 \wedge d\rho_1 = \\ \int \bar{\partial} \Big[\frac{1}{z_1^2}\Big] \wedge \bar{\partial} \Big[\frac{1}{z_2^2}\Big] \Big[\frac{1}{z_3}\Big] \wedge e_1 \wedge e_3 \wedge \phi.$$

Here we have used that the ξ_1-derivative must fall on the first variable, since otherwise we get something that vanishes due to antisymmetry. By extending the integration to $\xi_2 \in \mathbb{P}^1$, we can compute the integral over the entire $\widetilde{\mathcal{U}}_x$, and therefore we have actually computed the action in Ω_3.

Knowing the result of these computations, we could have skipped the first blow-up of the origin in $\mathbb{C}^3$. It would have been sufficient to resolve the singularities along the axes. However to be sure about what happens at the origin, the resolutions we made were necessary.

To conclude, the parts of the currents with support on the z_2-axis coincide, while the parts with support on the z_3-axis differ in the way they differentiate the test form. Hence Theorem 1.4 does not generalize to the case of proper intersections. □

References

[1] M. Andersson: *Residue currents and ideals of holomorphic functions*, Bull. Sci. Math. (to appear).

[2] M. F. Atiyah: *Resolution of singularities and division of distributions*, Com. Pure. Appl. Math. **23** (1970), 145–150.

[3] C. A. Berenstein & R. Gay & A. Vidras & A. Yger: *Residue currents and Bezout identities*, Progress in Mathematics **114** Birkhäuser Yerlag (1993).

[4] C. A. Berenstein & A. Yger: *Analytic residue theory in the non-complete intersection case*, J. Reine Angew. Math. **527** (2000), 203–235.

[5] J-E. Björk: *Residue currents and $\mathcal{D}$-modules on complex manifolds*, Preprint Stockholm (1996).

[6] N. Coleff & M. Herrera: *Les Courants Résiduels Associés à une Forme Mèromorphe*, Lecture Notes in Mathematics **633** Springer Verlag, Berlin, 1978.

[7] G. Kempf & F. Knudsen & D. Mumford & B. Saint-Donat: *Toroidal Embeddings I*, Lecture Notes in Mathematics **339** Springer Verlag, New York, 1973.

[8] M. Passare: *Residues, currents, and their relation to ideals of holomorphic functions*, Math. Scand. **62** (1988), no. 1, 75–152.

[9] M. Passare: *A calculus for meromorphic currents*, J. Reine Angew. Math. **392** (1988), 37–56.

[10] M. Passare & A. Tsikh: *Residue integrals and their Mellin transforms*, Canadian J. of Math. **47** (1995), 1037–50.

[11] M. Passare & A. Tsikh: *Defining the residue of a complete intersection. Complex Analysis, Harmonic Analysis and Applications*, Pitman Res. Notes Math. Ser. **347**, Longman, Harlow, 1996, 250–267.

[12] M. Passare & A. Tsikh & A. Yger: *Residue currents of the Bochner-Martinelli type*, Publ. Mat. **44** (2000), 85–117.

[13] L. Schwartz: *Division par une fonction holomorphe sur une variété analytique complexe*, Summa Brasil. Math **3** (1955), 181–209.

Department of Mathematics, Chalmers University of Technology and the University of Göteborg, S-412 96 GÖTEBORG, SWEDEN

E-mail address: `wulcan@math.chalmers.se`