Thesis for the Degree of Licentiate of Engineering

On the Pricing of Cliquet Options with Global Floor and Cap

Mats Kjaer

CHALMERS | GÖTEBORG UNIVERSITY

Department of Mathematics
Chalmers University of Technology and Göteborg University
SE-412 96 Göteborg, Sweden
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MATS KJAER

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Department of Mathematics
Chalmers University of Technology and Göteborg University
SE-412 96 Göteborg
Sweden
Telephone +46 (0)31-772 1000

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Abstract

In this thesis we present two methods for the pricing and hedging of cliquet options with global floor and/or cap within a Black-Scholes market model with fixed dividends and time dependent volatilities and interest rates.

The first is a *Fourier transform method* giving integral formulas for the price and the greeks. A numerical integration scheme is proposed for the evaluation of these formulas.

Using Ito's Lemma it is proved that the vanilla Black-Scholes PDE is valid. In addition to giving us the gamma for free, it forms the basis for an *explicit finite difference method.*

Both methods outperform Monte Carlo simulation in terms of computational time, with the Fourier method in most cases being the faster one for a given level of accuracy. This tendency is amplified as the number of reset periods increases.

Potential future research includes local volatility models and early exercise features for the finite difference method and Levy-process market models for the Fourier method.

Preface

About this thesis

This licentiate thesis concludes the five semester ECMI postgraduate programme in applied mathematics. ECMI is an acronym for the European Consortium for Mathematics in Industry and is a collaboration between universities in Europe.

The work has partly been performed in cooperation with the financial software company Front Capital Systems AB in Stockholm, Sweden. Some of the results originating from this thesis have been implemented in the C programming language and will be available in the February 2004 release of the Front Arena Trading System.

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Contents

Chapter 1

Introduction

This Millennium started with a recession and rapidly falling stock markets. Investors who had relied on annual returns on investments exceeding 20% suddenly became aware of the risk inherent in owning shares and almost coherently turned their attention to safer investments like bonds and ordinary bank accounts. As an attempt to capitalize on this fear of losses, a variety of equity linked products with capital guarantees were introduced on the market. Among the most successful is the so called *cliquet option with global floor*, which is usually packaged with a bond and sold to retail investors under names *like equity linked bond with capital guarantee* or *equity index bond*.

This thesis introduces and evaluates two methods for pricing and hedging these cliquet options with global floor.

Readers of this thesis should be familiar with basic derivatives pricing theory and stochastic calculus since we will not prove everything from scratch. A good introduction to derivatives pricing can be found in [16], while [18] and [12] cover stochastic calculus.

This thesis is organized as follows. Chapter 2 introduces cliquets with global floor in detail and gives a summary of the mathematical theory of option pricing. This includes a description of the market and dividend models as well as revisions of some fundamental concepts and theorems of option pricing.

In Chapter 3 Fourier integral formulas for the price and greeks are derived. This is followed by a numerical integration scheme for fast and robust evaluation of these integrals. At the end we show how the method can be extended to the case with a global cap.

Chapter 4 deals with the derivation and numerical solution of a partial differential equation for the price. At the end we indicate how this allows us to add early exercise features to the derivative.

In Chapter 5 the Fourier and PDE methods are compared with the classical Monte-Carlo method in numerical tests.

Conclusions are presented in Chapter 6 and suggestions for future research in Chapter 7.

Chapter 2

Background

In this chapter we introduce cliquet options with global floor and describe the market model used. In addition we state some fundamental concepts and theorems of option pricing.

2.1 Floored cliquet options

Let T be a future point in time, and divide the interval $[0, T]$ into N subintervals called *reset periods* of length $\Delta T_n = T_n - T_{n-1}$, where $\{T_n\}_{n=0}^{N}, T_0 = 0, T_N = T$ are called the *reset days*. The return of an asset with price process S_t over a reset period $[T_{n-1}, T_n)$ is then defined as

$$R_n = \frac{S_{T_n}}{S_{T_{n-1}}} - 1.$$

Truncated returns, $\overline{R}_n = \max(\min(R_n, C), F)$ are returns truncated at some floor and cap levels F and C respectively with $F < C$ as illustrated in Figure 2.1 below. Absence of floor and/or cap corresponds to $F = -1$ and $C = +\infty$. A general cliquet option has a payoff Y at time T of

$$Y = B \times \min(\max(\sum_{n=1}^{N} \overline{R}_n, F_g), C_g)$$

where the *global floor* F_g and *global cap* C_g are minimum and maximum returns respectively and B is a notional amount which is set to one for the remainder of this thesis. For F_g and C_g to be of interest, they must satisfy $NF < F_g < C_g < NC$.

This very general form of cliquet option is not very common on the market but removing the global cap C_g gives the popular *cliquet option with global floor*, which pays the holder

$$Y = \max(\sum_{n=1}^{N} \overline{R}_n, F_g)$$

at time T.

Variants of this derivative being offered at the market include the cliquet with global floor and coupon credit K

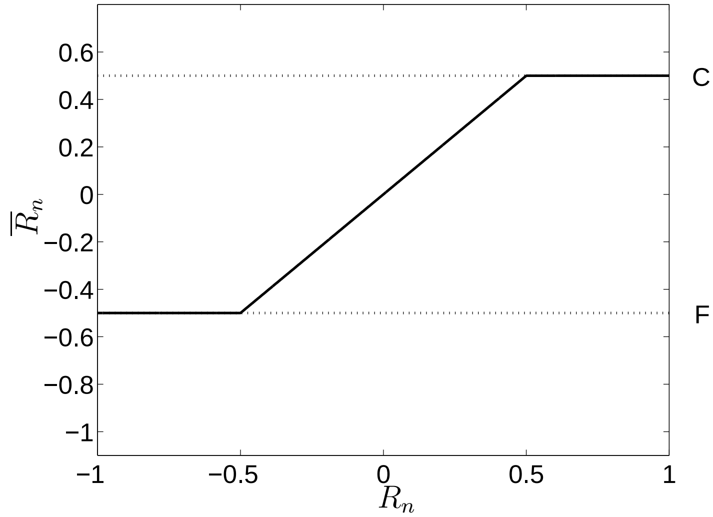

Figure 2.1: A truncated return $\overline{R}_n$ with $F = -0.50$ and $C = 0.50$.

$$Y = \max(\sum_{n=1}^{N} \overline{R}_n - K, F_g)$$

and the reversed cliquet with global floor

$$Y = \max(C_g - \sum_{n=1}^{N} R_n^-, F_g) = \max(C_g + \sum_{n=1}^{N} \overline{R}_n, F_g)$$

with $C = 0, F = -1$, i.e. negative returns are added to an original maximum payoff C_g.

The cliquet with coupon credit can be rewritten as

$$Y = \max(\sum_{n=1}^{N} \overline{R}_n - K, F_g) = \max(\sum_{n=1}^{N} \overline{R}_n, F_g + K) - K$$

and the reversed cliquet as

$$Y = \max(C_g + \sum_{n=1}^{N} \overline{R}_n, F_g) = \max(\sum_{n=1}^{N} \overline{R}_n, F_g - C_g) + C_g$$

proving that once we have a pricing method for the first type of floored cliquet we can price all three derivatives presented above. An interesting article about cliquet options and their pricing can be found in [17].

2.2 Market Models

In order to price a derivative like the cliquet with global floor introduced in the previous section we need a model for the share price S_t. We start by describing the market model used in this thesis and continue with some possible generalizations.

2.2.1 Market model used in this thesis

In this thesis we consider a Black-Scholes market model with one stock and one risk free asset. The interest rate r_t and volatility σ_t are assumed to be positive, deterministic and piecewise constant processes. The volatility and interest rate in period n are denoted by σ_n and r_n respectively and estimated from option market prices and yield curves.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, P)$ be a filtered probability space, where $\mathcal{F}_t$ is the P-completion of the natural filtration $\sigma(W_u, 0 \leq u \leq t)$ of the Wiener process W_t. For simplicity, we choose P to be the equivalent martingale measure, under which the price processes of the stock S_t and bond B_t satisfy the SDEs

$$\begin{cases} dS_t = S_t(r_t dt + \sigma_t dW_t) \\ dB_t = r_t B_t dt, \end{cases}$$

where

$$\int_0^T r_u du < \infty \quad \textit{and} \quad \int_0^T \sigma_u^2 du < \infty.$$

The equations above have the solutions

$$\begin{cases} S_t = S_0 \exp(\int_0^t (r_u - \frac{\sigma_u^2}{2})du + \int_0^t \sigma_u dW_u) \\ B_t = B_0 \exp(\int_0^t r_u du). \end{cases}$$

The main reason for choosing a Black-Scholes model with time-dependent coefficients is that floored cliquets may have time to maturities T of two to five years, during which booth volatilities and interest rates may change significantly.

In order to save space we define the average interest rate $\tilde{r}_t$ over the interval $[t, T]$ as

$$\tilde{r}_t = \frac{1}{T-t}\Big(r_m(T_m - t) + \sum_{n=m+1}^{N} r_n(T_n - T_{n-1})\Big),$$

where we have assumed $T_{m-1} \leq t < T_m$. This enables us to write the discounting factor as

$$e^{-r_m(T_m-t)-\sum_{n=m+1}^{N} r_n(T_n-T_{n-1})} = e^{-\tilde{r}_t(T-t)}.$$

As a solution to an SDE with piecewise constant coefficients, it can be shown that S_t possesses the strong Markov property and independent increments. A consequence of this is that returns R_n over disjoint periods of time are independent random variables with $1 + R_n$ being log-normal.

In the following we write $X \sim Y$ if the random variables X and Y have the same distribution. Assuming $T_{m-1} \le t < T_m$, it is easily seen from the expression for S_t above that for $n > m$, $R_n \sim e^{a_n + b_n X_n} - 1$, where $a_n = (r_n - \frac{\sigma_n^2}{2})(T_n - T_{n-1})$, $b_n = \sigma_n \sqrt{T_n - T_{n-1}}$ and $\{X_n\}_{n=m+1}^N$ are i.i.d. (independent, identically distributed) $N(0,1)$ random variables. In the current reset period m this expression is modified to $R_m \sim \frac{s}{\bar{s}} e^{a_m + b_m X_m} - 1$, where $s = S_t$, $\bar{s} = S_{T_{m-1}}$, $a_m = (r_m - \frac{\sigma_m^2}{2})(T_m - t)$, $b_m = \sigma_n \sqrt{T_m - t}$ and $X_m \sim N(0,1)$ is independent of $\{X_n\}_{n=m+1}^N$. Moreover, $\{X_n\}_{n=m}^N$ and $\mathcal{F}_t$ are independent. Below we write R_m^t instead of R_m to indicate that s is known at time t and define $\overline{R}_m^t = \max(\min(R_m^t, C), F)$ in order to be consistent with previous notation.

2.2.2 More general market models

The market model proposed in the previous section cannot model smile, skew and fat tails. For this reason we will briefly review some more advanced market models in this section. Their usefulness in connection with pricing cliquet options with global floor will be discussed in Chapter 7.

The terms smile and skew arise from the fact that European call options defined in Section 2.4 with the same maturity date T do not have the same implied volatility for all strike prices K. Instead, if plotted against the strike, the implied volatility tends to form a smile shaped curve. If the smile is skewed, the phenomenon is called skew. Probability density functions with more mass in the tails than that of a normal distribution are said to have fat tails.

One way of incorporating smile and skew are the *local volatility models* by Brigo et al or Derman et al presented in [2] and [6]. In these the share price follows the dynamics

$$dS_t/S_t = r_t dt + \sigma(t, S_t) dW_t$$

and they differ only in the way $\sigma(t,s)$ is computed. Given some technical conditions on $\sigma(s,t)$, this model is complete and hence allows perfect hedging. Share prices are still Markov, but returns over disjoint periods of time may not be independent any more. The marginal densities of the share price at all times t are known explicitly in the Brigo model but not in the Derman model.

Another way of achieving smile and skew is to use a two-factor or *stochastic volatility model* like the one by Heston presented in [15], where the share price S_t and squared volatility ν_t satisfy the following system of SDE's.

$$\begin{cases} d\nu_t = \kappa(\theta - \nu_t)dt + \sigma\sqrt{\nu_t} dW_t^{(1)} \\ dS_t/S_t = r_t dt + \sqrt{\nu_t} dW_t^{(2)}. \end{cases}$$

Here $W_t^{(1)}$ and $W_t^{(2)}$ are two Wiener processes with correlation ρ that jointly generate the filtration $(\mathcal{F}_t)_{t \ge 0}$. Returns over disjoint periods of time are not independent and the model does not allow perfect hedging using bonds and shares only.

Levy-process market models is a third option for getting fat tails, smile and skew. In this model the share price is given by

$$dS_t/S_t = rdt + dL_t,$$

where L_t is a Levy-process with zero mean under some equivalent martingale measure P. Due to the independent increments of the Levy-process, returns over disjoint periods of time are independent. The main open questions regarding this model are of a very fundamental nature. Selection of martingale measure P and how to (partially) hedge are still unresolved issues for all derivatives in Levy-market models. An introduction to Levy-processes in finance can be found in [19].

2.3 Dividends

There are several ways of incorporating dividend payments into the market model described in Section 2.2.1, none of which can be said to be the right one for all situations. Recent articles, for example [5] and [9], cover the problem of selecting dividend model and that this may not be trivial is highlighted by the following quote.

> *"Modelling Dividends is almost more difficult than modelling the share price itself"*
>
> Professor Ralf Korn, University of Kaiserslauten
> PhD. dissertation speech at Chalmers University of Technology
> September 26, 2003

We will start by briefly reviewing two main classes of dividend models, the *discrete dividend yield* and *fixed dividend* models. Their feasibility in the context of cliquet options with global floor will also be assessed. Then we will propose a combination of the two models that will be used in this thesis.

In the discrete dividend yield model, a dividend of size $\alpha_n S_{\tau_n^-}$ is paid to the shareholder at a dividend date $\tau_n \in (T_{n-1}, T_n)$ for a fixed reset period n. Here we have assumed that dividends are distributed to the shareholders during the trading day, such that dividend and reset times do not coincide. In order to avoid arbitrage, the share price has to drop $\alpha_n S_{\tau_n^-}$ when the dividend is paid, if taxes and transaction costs are not considered. Thus

$$\tilde{S}_t = \begin{cases} (1-\alpha_n)S_t, & T_{n-1} \le t < \tau_n < T_n \\ S_t, & T_{n-1} < \tau_n \le t < T_n \end{cases}$$

is an exponential Brownian motion with continuous trajectories satisfying

$$\tilde{S}_t = \tilde{S}_{T_{n-1}} \exp\left(\int_{T_{n-1}}^t (r_u - \frac{\sigma_u^2}{2})du + \int_{T_{n-1}}^t \sigma_u dW_u\right), \ \ T_{n-1} \le t < T_n. \tag{2.1}$$

Equation 2.1 implies that the returns R_n are still of the analytically tractable form $R_n = e^{a_n + b_n X} - 1$ with $X \sim N(0,1)$, provided that

$$\begin{cases} a_n &= (r_n - \frac{\sigma_n^2}{2})(T_n - T_{n-1}) + \log(1-\alpha_n) \\ b_n &= \sigma_n \sqrt{T_n - T_{n-1}} \end{cases} \tag{2.2}$$

for $n > m$. This also holds for $n = m$, $t \in [T_{m-1}, T_m)$, if we set

$$\begin{cases} a_m &= (r_m - \frac{\sigma_m^2}{2})(T_m - t) + \log(1-\alpha_m^t) + \log(s/\bar{s}) \\ b_m &= \sigma_m \sqrt{T_m - t}. \end{cases} \tag{2.3}$$

Here the notation α_m^t emphasizes that in the current reset period m, we should only consider future dividends in the interval (t, T_m) as a consequence of Equation 2.1.

The discrete dividend yield approach is particulary good in two situations. Firstly, it is the obvious choice when modelling the impact of stock splits and issuing of new shares. Secondly it can be used for fixed cash dividends occurring half a year ahead or more. This is because the exact dividend amount to be paid to the shareholders is decided only a few months in advance of the payment date. If the dividend occurs later, it has to be estimated by analysts. This task becomes increasingly difficult as the time to the dividend payment increases. However, dividend payments are related to financial performance, which in turn is related to the share price. Thus analysts estimate dividend yields rather than amounts. Also, at the bottom line, the yield is also what matters to investors. For an example of an analyst report about dividends, see [10].

One major weakness of the model presented is that it is not clear whether it provides a good description for fixed dividends occurring in the near future or not. A common way to cover this situation is the fixed dividend model originally proposed by Heath and Jarrow in [11]. To avoid arbitrage, a dividend of D_n EUR (or whatever unit the share price is quoted in) paid at time τ_n must be followed by a drop in the share price of the same amount, again disregarding taxes and transaction costs. If $D_{n,t} = D_n \exp(-\int_t^{\tau_n} r_u du)$ denotes the present value of D_n, it is assumed that

$$\tilde{S}_t = \begin{cases} S_t - D_{n,t}, & T_{n-1} \le t < \tau_n < T_n \\ S_t, & T_{n-1} < \tau_n \le t < T_n. \end{cases}$$

is an exponential Brownian motion called *the volatile part* with trajectories satisfying

$$\tilde{S}_t = \tilde{S}_{T_{n-1}} \exp(\int_{T_{n-1}}^t (r_u - \frac{\sigma_u^2}{2})du + \int_{T_{n-1}}^t \sigma_u dW_u), \;\; T_{n-1} \le t < T_n.$$

In the case of multiple dividends $D_n^{(j)}$ at times $\tau_n^{(j)} \in (T_{n-1}, T_n)$ this generalizes to

$$\begin{aligned} S_t &= \tilde{S}_t + D_{n,t} \\ D_{n,t} &= \sum_{j=1}^{J_n} \chi_{[T_{n-1}, \tau_n^{(j)})}(t) D_n^{(j)} \exp(-\int_t^{\tau_n^{(j)}} r_u du) \end{aligned}$$

where χ_E denotes the indicator function of the set E.

The return R_n of reset period n is of the form

$$R_n = \frac{S_{T_n}}{S_{T_{n-1}}} - 1 = \frac{\tilde{S}_{T_n}}{\tilde{S}_{T_{n-1}} + D_{n,T_{n-1}}} - 1,$$

where only dividends occurring within the reset period are considered which explains why $D_{n,T_n} = 0$. The return R_n above does not have an analytic expression for its distribution function, which leaves computationally heavy methods like Monte Carlo simulation as the only option to evaluate the price of the cliquet option with global floor. For this reason we approximate the returns above with a linear Taylor series expansion.

$$R_n \approx \frac{\tilde{S}_{T_n}}{\tilde{S}_{T_{n-1}}} \left(1 - \frac{D_{n,Tn-1}}{\tilde{S}_{T_{n-1}}} \right) - 1$$

This can be interpreted as the return of a share paying a stochastic discrete dividend yield in reset period n of $\hat{\alpha}_n = \frac{D_{n,Tn-1}}{\tilde{S}_{T_{n-1}}}$. The validity of the expansion comes from the fact that dividends are almost always small compared to the share price. Finally, given $n > m$, we estimate $\tilde{S}_{T_{n-1}}$ by $S_t \exp\left\{ r_m(T_m - t) + \sum_{l=m+1}^{n-1} r_l(T_l - T_{l-1}) \right\}$ and obtain the dividend yield α_n for reset period n as

$$\alpha_n = \frac{D_{n,T_{n-1}}}{S_t} \exp\left\{ -r_m(T_m - t) - \sum_{l=m+1}^{n-1} r_l(T_l - T_{l-1}) \right\}. \tag{2.4}$$

For the current reset period $(n = m)$ there is no need to estimate $S_{T_{m-1}}$, so we have

$$\alpha_m^t = \frac{D_{m,t}}{S_{T_{m-1}}}.$$

Thus we are back at the discrete dividend yield setting described earlier and *this will be the way dividends are handled for the remainder of this thesis.* For this model to be useful, it should give option prices very close to those of the fixed dividend model for dividends occurring in the near future. For dividends further ahead in time it has the advantage of needing yield rather than fixed amount predictions.

A numerical comparison between the two dividend approaches is presented in Section 5.1.

2.4 Some results from general option pricing theory

The results in this Section mainly come from [16] and [13]. We start with a precise mathematical definition of a European contingent claim.

Definition 1. Let $T \geq 0$. A contingent T-claim is an $\mathcal{F}_T$ measurable random variable $Y \geq 0$.

In the case of our cliquet wtih global floor, it is trivial to prove that the payoff is $\mathcal{F}_T$ measurable.

Theorem 1. *Let $T \geq 0$ be fixed and Y a T-contingent claim. Then the arbitrage free price V_t of Y at time $0 \leq t \leq T$ is given by*

$$V_t = e^{-\int_t^T r_u du} E[Y|\mathcal{F}_t]$$

where the expectation is taken with respect to P.

For a proof of Theorem 1, see [13].

For risk management purposes, it is essential to compute partial derivatives of the price V_t with respect to different parameters. They are referred to as the greeks and defined in Definition 2 below.

Definition 2. The greeks Δ, Θ and Γ of a derivative with price $V = V_t$ are defined as

$$\begin{aligned} \Delta &= \frac{\partial V}{\partial s} \\ \Theta &= \frac{\partial V}{\partial t} \\ \Gamma &= \frac{\partial^2 V}{\partial s^2}. \end{aligned}$$

A *European call option* with strike price K and maturity T pays the holder $Y = \max(S_T - K, 0)$ at time T. The price and greeks of this option are given in Lemma 1. Here $\Phi(x)$ denotes the distribution function of a normally distributed random variable with mean zero and unit variance. Moreover, $\phi = \Phi'$.

Lemma 1. *The price $c = c(t, s, K, T, \sigma, r)$ and greeks Δ_c, Θ_c and Γ_c of a European call option with strike K, maturity $T > t$ and no dividends are given by*

$$\begin{aligned} c &= s\Phi(d_K + \sigma\sqrt{T-t}) - Ke^{-r(T-t)}\Phi(d_K) \\ \Delta_c &= \Phi(d_K + \sigma\sqrt{T-t}) \\ \Theta_c &= -\frac{s\sigma\phi(d_K + \sigma\sqrt{T-t})}{2\sqrt{T-t}} - rKe^{-r(T-t)}\Phi(d_K) \\ \Gamma_c &= \frac{\phi(d_K + \sigma\sqrt{T-t})}{s\sigma\sqrt{T-t}} \end{aligned}$$

where $d_K = \Big(\log(s/K) + (r - \sigma^2/2)(T-t)\Big)/\sigma\sqrt{T-t}$.

A proof can be found in [13].

Chapter 3

A Fourier integral method

In this section we derive integral formulas for the price and greeks of a cliquet option with global floor together with a numerical integration scheme for fast and accurate evaluation of these integrals. Finally we show that the more general case with a global cap added can be treated as a simple extension.

3.1 Integral Formulas for the Price and Greeks

Before stating and proving the Fourier integral pricing formula, we fix some notation. The payoff Y at time T is defined in Section 2.1 as

$$Y = \max(\sum_{n=1}^{N} \overline{R}_n, F_g)$$

Assuming that $t \in [T_{m-1}, T_m)$, we define

$$z = \sum_{n=1}^{m-1} \overline{R}_n$$

and

$$A = MC - F_g + z,$$

where $M = N - m + 1$ denotes the number of remaining reset periods including the remainder of the present one.

Finally, the characteristic function of a random variable X is written $\varphi_X(\xi) = E[e^{i\xi X}]$.

The form of the price formula can be divided into the following three cases, two of which have trivial solutions whereas the third one requires a more thorough analysis.

- **Case I:** $A \leq 0$: Performance has been so poor that the payoff will be F_g independent of future share price development.

- **Case II:** $MF + z \geq F_g$: Performance has been so good that the payoff will be higher than F_g independent of future share price development. This results in the analytical formulas for the price and the greeks in Proposition 1.
- **Case III:** $A > 0$. The formula in Proposition 2 is valid. Case II is included in this case but we prefer to treat it separately due to the existence of the analytical formulas for the price and the greeks in Proposition 1.

3.1.1 Case I

Since the payoff at time T will be F_g independent of future stock market development, we have for the price and greeks

$$\begin{aligned} V_t &= e^{-\tilde{r}_t(T-t)} F_g \\ \Delta &= 0 \\ \Theta &= r_m e^{-\tilde{r}_t(T-t)} F_g \\ \Gamma &= 0 \end{aligned}$$

3.1.2 Case II

We start with a Lemma, again assuming that $T_{m-1} \leq t < T_m$ and that $c(t, s, K, T, \sigma, r)$ is the price of a European call given in Lemma 1.

Lemma 2. *We have*

$$\begin{aligned} E[\overline{R}_m | \mathcal{F}_t] &= F + e^{r_m(T_m - t)} \Big\{ c(t, (1 - \alpha_m^t) s/\bar{s}, 1 + F, T_m, \sigma_m, r_m) \\ &\quad - c(t, (1 - \alpha_m^t) s/\bar{s}, 1 + C, T_m, \sigma_m, r_m) \Big\} \\ E[\overline{R}_n] &= F + e^{r_n(T_n - T_{n-1})} \Big\{ c(T_{n-1}, 1 - \alpha_n, 1 + F, T_n, \sigma_n, r_n) \\ &\quad - c(T_{n-1}, 1 - \alpha_n, 1 + C, T_n, \sigma_n, r_n) \Big\}, m < n \leq N \end{aligned}$$

where α_n is the size of a possible dividend yield within reset period n.

PROOF: We can write

$$\overline{R}_n = \max(\min(R_n, C), F) = F + \max(R_n - F, 0) - \max(R_n - C, 0).$$

A derivative with payoff

$$\max(R_n - F, 0) = \max\left(\frac{S_{T_n}}{S_{T_{n-1}}} - (1 + F), 0 \right)$$

is called a *forward performance option* with strike $K = (1 + F)$. This derivative is traded on the market either as a stand alone product or a part of a cliquet option structure without global floor. If $T_{m-1} \leq t < T_m$, with a possible dividend yield of size α_m^t, Lemma 1, Equation 2.3, and Theorem 1 give

$$e^{-r_m(T_m-t)}E[\max(R_m - F, 0)|\mathcal{F}_t] = c(t, (1-\alpha_m^t)s/\bar{s}, 1+F, T_m, \sigma_m, r_m).$$

For $m < n \leq N$ we have that

$$\begin{aligned} c(T_{n-1}, 1-\alpha_n, 1+F, T_n, \sigma_n, r_n) &= e^{-r_n(T_n-T_{n-1})}E[\max(R_n - F, 0)|\mathcal{F}_{T_{n-1}}] \\ &= e^{-r_n(T_n-T_{n-1})}E[\max(R_n - F, 0)] \end{aligned}$$

because $\sigma(R_n)$ and $\mathcal{F}_{T_{n-1}}$ are independent. The lemma now follows by repeating the arguments above with $K = 1 + C$ and the linearity of the conditional expectation operator.

□

Proposition 1. *If $MF + z \geq F_g$, the price V_t and greeks Δ, Θ and Γ of the cliquet option with global floor are given by*

$$\begin{aligned} V_t &= e^{-\tilde{r}_t(T-t)}\Big\{z + MF \\ &\quad + e^{r_m(T_m-t)}\Big[c(t, (1-\alpha_m^t)s/\bar{s}, 1+F, T_m, \sigma_m, r_m) \\ &\quad - c(t, (1-\alpha_m^t)s/\bar{s}, 1+C, T_m, \sigma_m, r_m)\Big] \\ &\quad + \sum_{n=m+1}^{N} e^{r_n(T_n-T_{n-1})}\Big[c(T_{n-1}, 1-\alpha_n, 1+F, T_n, \sigma_n, r_n) \\ &\quad - c(T_{n-1}, 1-\alpha_n, 1+C, T_n, \sigma_n, r_n)\Big]\Big\}, \\ \Delta &= e^{-\sum_{n=m+1}^{N} r_n(T_n-T_{n-1})}\Big[\Delta_c(t, (1-\alpha_m^t)s/\bar{s}, 1+F, T_m, \sigma_m, r_m) \\ &\quad - \Delta_c(t, (1-\alpha_m^t)s/\bar{s}, 1+C, T_1, \sigma_m, r_m)\Big]\frac{(1-\alpha_m^t)}{\bar{s}}, \\ \Theta &= r_m V_t + e^{-\tilde{r}_t(T-t)}e^{r_m(T_m-t)}\Big\{ - r_m[c(t, (1-\alpha_m^t)s/\bar{s}, 1+F, T_m, \sigma_m, r_m) \\ &\quad - c(t, (1-\alpha_m^t)s/\bar{s}, 1+C, T_m, \sigma_m, r_m)] \\ &\quad + [\Theta_c(t, (1-\alpha_m^t)s/\bar{s}, 1+F, T_m, \sigma_m, r_m) - \Theta_c(t, (1-\alpha_m^t)s/\bar{s}, 1+C, T_m, \sigma_m, r_m)]\Big\}, \\ &\quad \text{if } t \text{ not a dividend date, and} \\ \Gamma &= e^{-\sum_{n=m+1}^{N} r_n(T_n-T_{n-1})}\Big[\Gamma_c(t, (1-\alpha_m^t)s/\bar{s}, 1+F, T_m, \sigma_m, r_m) \\ &\quad - \Gamma_c(t, (1-\alpha_m^t)s/\bar{s}, 1+C, T_m, \sigma_m, r_m)\Big]\frac{(1-\alpha_m^t)^2}{\bar{s}^2}. \end{aligned}$$

PROOF: Again we assume that $T_{m-1} \leq t < T_m$. From Theorem 1 we have that

$$V_t = e^{-\tilde{r}_t(T-t)} E[\max(\sum_{n=1}^{N} \overline{R}_n, F_g)|\mathcal{F}_t].$$

The condition $MF + z \geq F_g$ implies that

$$\begin{aligned} V_t &= e^{-\tilde{r}_t(T-t)} E[\sum_{n=1}^{N} \overline{R}_n|\mathcal{F}_t] \\ &= e^{-\tilde{r}_t(T-t)} \left\{ z + E[\overline{R}_m|\mathcal{F}_t] + \sum_{n=m+1}^{N} E[\overline{R}_n] \right\} \end{aligned}$$

due to the independence of $\sigma(R_n)$ and $\mathcal{F}_t$ for $m < n \leq N$. We note that s and t are only present in the formula for $\overline{R}_m$ which together with Lemmas 1 and 2 prove the proposition.

□

3.1.3 Case III

Before stating and proving the formulas for the price and greeks, we recall that

$$\begin{aligned} R_m &\sim \frac{s}{\bar{s}} e^{a_m + b_m X_m} - 1 \\ R_n &\sim e^{a_n + b_n X_n} - 1, \end{aligned}$$

where $\{X\}_{n=m}^{N}$ are i.i.d. $N(0,1)$ random variables independent of $\mathcal{F}_t$ and

$$a_n = \begin{cases} (r_m - \frac{\sigma_m^2}{2})(T_m - t) + \log(1 - \alpha_m^t) + \log(s/\bar{s}), & n = m, \\ (r_n - \frac{\sigma_n^2}{2})(T_n - T_{n-1}) + \log(1 - \alpha_n), & N \geq n > m, \end{cases}$$

$$b_n = \begin{cases} \sigma_m \sqrt{T_m - t}, & n = m, \\ \sigma_n \sqrt{T_n - T_{n-1}}, & N \geq n > m. \end{cases}$$

Again $s = S_t$, $\bar{s} = S_{T_{m-1}}$ and α_n the discrete dividend yield computed from fixed dividends as shown in Equation 2.4 of Section 2.3.

Finally we introduce the random variables $\tilde{R}_n = C - \overline{R}_n$ and $\tilde{R}_m^t = C - \overline{R}_m^t$, which are non-negative, and define

$$\text{sinc}(x) = \begin{cases} 1, & x = 0, \\ \frac{\sin(x)}{x}, & x \neq 0. \end{cases}$$

Proposition 2. *If $A > 0$, the price V_t of a cliquet with global floor is given by*

$$V_t = e^{-\tilde{r}_t(T-t)} \left\{ F_g + A^2 \int_{-\infty}^{\infty} \text{sinc}^2(\frac{\xi A}{2}) \times \varphi_{\tilde{R}_m^t}(\xi) \times \prod_{n=m+1}^{N} \varphi_{\tilde{R}_n}(\xi) \frac{d\xi}{2\pi} \right\}$$

where

$$\varphi_{\tilde{R}^t_m}(\xi) = e^{i\xi(C-F)} - i\xi \int_0^{C-F} \Phi\left(\frac{a_m - \log(1+C-x)}{b_m}\right) e^{i\xi x} dx,$$

and

$$\varphi_{\tilde{R}_n}(\xi) = e^{i\xi(C-F)} - i\xi \int_0^{C-F} \Phi\left(\frac{a_n - \log(1+C-x)}{b_n}\right) e^{i\xi x} dx, N \geq n > m.$$

PROOF: Analogous to the proof of Proposition 1 we have

$$\begin{aligned} V_t &= e^{-\tilde{r}_t(T-t)} E[\max(\sum_{n=1}^{N} \overline{R}_n, F_g)|\mathcal{F}_t] \\ &= e^{-\tilde{r}_t(T-t)} E[F_g + \max(z - F_g + (\overline{R}^t_m + \sum_{n=m+1}^{N} \overline{R}_n), 0)] \end{aligned}$$

since the returns R_n are independent. Using the relations $\overline{R}_n = C - \tilde{R}_n$ and $\overline{R}^t_m = C - \tilde{R}^t_m$ yields

$$\begin{aligned} V_t &= e^{-\tilde{r}_t(T-t)} \left\{ F_g + E[\max(MC - F_g + z - (\tilde{R}^t_m + \sum_{n=m+1}^{N} \tilde{R}_n), 0)] \right\} \\ &= e^{-\tilde{r}_t(T-t)} \left\{ F_g + E[\max(A - (\tilde{R}^t_m + \sum_{n=m+1}^{N} \tilde{R}_n), 0)] \right\}. \end{aligned}$$

By Fourier analysis, see [8] for details, we have

$$\Lambda_A(x) = \max(A - |x|, 0) = A^2 \int_{-\infty}^{\infty} \operatorname{sinc}^2(\frac{\xi A}{2}) e^{i\xi x} \frac{d\xi}{2\pi}. \tag{3.1}$$

Using this result with $x = \tilde{R}^t_m + \sum_{n=m+1}^{N} \tilde{R}_n$, which is non-negative by construction, gives

$$\begin{aligned} V_t &= e^{-\tilde{r}_t(T-t)} \left\{ F_g + E[A^2 \int_{-\infty}^{\infty} \operatorname{sinc}^2(\frac{\xi A}{2}) e^{i\xi(\tilde{R}^t_m + \sum_{n=m+1}^{N} \tilde{R}_n)} \frac{d\xi}{2\pi}] \right\} \\ &= e^{-\tilde{r}_t(T-t)} \left\{ F_g + A^2 \int_{-\infty}^{\infty} \operatorname{sinc}^2(\frac{\xi A}{2}) E[e^{i\xi(\tilde{R}^t_m + \sum_{n=m+1}^{N} \tilde{R}_n)}] \frac{d\xi}{2\pi} \right\} \end{aligned}$$

by the Fubini theorem. Independence of returns implies that $E[e^{i\xi(\tilde{R}^t_m + \sum_{n=m+1}^{N} \tilde{R}_n)}] = E[e^{i\xi \tilde{R}^t_m}] \times \prod_{n=m+1}^{N} E[e^{i\xi \tilde{R}_n}]$ and consequently

$$V_t = e^{-\tilde{r}_t(T-t)} \left\{ F_g + A^2 \int_{-\infty}^{\infty} \operatorname{sinc}^2(\frac{\xi A}{2}) E[e^{i\xi \tilde{R}^t_m}] \prod_{n=m+1}^{N} E[e^{i\xi \tilde{R}_n}] \frac{d\xi}{2\pi} \right\}.$$

To arrive at the formula in Proposition 2 it remains to compute $E[e^{i\xi\tilde{R}^t_m}]$ and $E[e^{i\xi\tilde{R}_n}]$. But

$$E[e^{i\xi\tilde{R}_n}] = e^{i\xi(C-F)} \cdot P(R_n \le F) + \int_0^{C-F} e^{i\xi x} dP(C - R_n \le x) + 1 \cdot P(R_n > C)$$

where

$$\begin{aligned} P(R_n \le F) &= P(e^{a_n + b_n X} - 1 \le F) \\ &= P(a_n + b_n X \le \log(1+F)) \\ &= \Phi\left(\frac{\log(1+F) - a_n}{b_n}\right), \\ P(R_n \ge C) &= \Phi\left(\frac{a_n - \log(1+C)}{b_n}\right), \\ P(C - R_n \le x) &= \Phi\left(\frac{a_n - \log(1+C-x)}{b_n}\right) \end{aligned}$$

and

$$dP(C - R_n \le x) = \frac{1}{b_n(1+C-x)}\phi\left(\frac{a_n - \log(1+C-x)}{b_n}\right) dx.$$

Hence

$$\begin{aligned} E[e^{i\xi\tilde{R}_n}] &= e^{i\xi(C-F)} \cdot \Phi\left(\frac{\log(1+F) - a_n}{b_n}\right) + \Phi\left(\frac{a_n - \log(1+C)}{b_n}\right) \\ &\quad + \int_0^{C-F} \frac{1}{b_n(1+C-x)}\phi\left(\frac{a_n - \log(1+C-x)}{b_n}\right) e^{i\xi x} dx \\ &= e^{i\xi(C-F)} - i\xi \int_0^{C-F} \Phi\left(\frac{a_n - \log(1+C-x)}{b_n}\right) e^{i\xi x} dx \end{aligned}$$

by partial integration. $E[e^{i\xi\tilde{R}^t_m}]$ is computed analogously.

□

The idea to make a Fourier expansion of a put-style payoff function is taken from [3], where it is proposed for a reversed cliquet.

By direct differentiation under the integral of the price formula in Proposition 2 we get the expressions for the greeks stated in Propositions 3 to 5.

Proposition 3. *If $A > 0$ and $S_t = s$, then the delta of a cliquet with global floor is given by*

$$\Delta = e^{-\tilde{r}_t(T-t)} A^2 \int_{-\infty}^{\infty} \operatorname{sinc}^2(\frac{\xi A}{2}) \times \frac{\partial \varphi_{\tilde{R}^t_m}}{\partial s}(\xi) \times \prod_{n=m+1}^{N} \varphi_{\tilde{R}_n}(\xi) \frac{d\xi}{2\pi}$$

where

$$\frac{\partial \varphi_{\tilde{R}^t_m}}{\partial s}(\xi) = -\frac{i\xi}{b_m s} \int_0^{C-F} \phi\left(\frac{a_m - \log(1+C-x)}{b_m}\right) e^{i\xi x} dx.$$

Proposition 4. *If $A > 0$, t is not a dividend date, and V_t denotes the price, then the theta of a cliquet with global floor is given by*

$$\Theta = r_m V_t + e^{-\tilde{r}_t(T-t)} A^2 \int_{-\infty}^{\infty} \operatorname{sinc}^2(\frac{\xi A}{2}) \times \frac{\partial \varphi_{\tilde{R}_m^t}}{\partial t}(\xi) \times \prod_{n=m+1}^{N} \varphi_{\tilde{R}_n}(\xi) \frac{d\xi}{2\pi}$$

where

$$\begin{aligned} \frac{\partial \varphi_{\tilde{R}_m^t}}{\partial t}(\xi) &= i\xi \int_0^{C-F} \phi\left(\frac{a_m - \log(1+C-x)}{b_m}\right) \\ &\quad \times \left\{\frac{1}{b_m}(r_m - \frac{\sigma_m^2}{2}) + \frac{\sigma_m^2}{2b_m^3}[\log(1+C-x) - a_m]\right\} e^{i\xi x} dx. \end{aligned}$$

Proposition 5. *If $A > 0$ and $S_t = s$ denotes the price, then the gamma of a cliquet with global floor is given by*

$$\Gamma = e^{-\tilde{r}_t(T-t)} A^2 \int_{-\infty}^{\infty} \operatorname{sinc}^2(\frac{\xi A}{2}) \times \frac{\partial^2 \varphi_{\tilde{R}_m^t}}{\partial s^2}(\xi) \times \prod_{n=m+1}^{N} \varphi_{\tilde{R}_n}(\xi) \frac{d\xi}{2\pi}$$

where

$$\begin{aligned} \frac{\partial^2 \varphi_{\tilde{R}_m^t}}{\partial s^2}(\xi) &= \frac{i\xi}{b_m s^2} \int_0^{C-F} \phi\left(\frac{a_m - \log(1+C-x)}{b_m}\right) \\ &\quad \times \left\{1 + \frac{1}{b_m^2}[a_m - \log(1+C-x)]\right\} e^{i\xi x} dx. \end{aligned}$$

In Section 4.1 we show that the Black-Scholes equation is valid for a cliquet with global floor, which means that the gamma can be obtained for free provided that the price, delta and theta have been computed. Keen readers may want to verify this claim directly by inserting the formulas from Propositions 2 to 5 into the Black-Scholes equation.

3.2 Properties of the price

In this section we investigate some aspects of the price regarding analytical upper and lower bounds, convergence when the local cap $C \to \infty$, and volatility dependence.

3.2.1 Upper and lower bounds

In order to have a control mechanism in the computation of the price, analytical upper and lower bounds are derived.

Proposition 6. *The price V_t is bounded from below by*

$$\begin{aligned} V_t \geq \; & e^{-\tilde{r}_t(T-t)} \max \Big(z + MF \\ & + e^{r_m(T_m-t)} \big[c(t, (1-\alpha_m^t)s/\bar{s}, 1+F, T_m, \sigma_m, r_m) \\ & - c(t, (1-\alpha_m^t)s/\bar{s}, 1+C, T_m, \sigma_m, r_m) \big] \\ & + \sum_{n=m+1}^{N} e^{r_n(T_n-T_{n-1})} \big[c(T_{n-1}, 1-\alpha_n, 1+F, T_n, \sigma_n, r_n) \\ & - c(T_{n-1}, 1-\alpha_n, 1+C, T_n, \sigma_n, r_n) \big], F_g \Big) \end{aligned}$$

Proof: The Jensen inequality for conditional expectations yields

$$\begin{aligned} V_t &= e^{-\tilde{r}_t(T-t)} E[\max(\sum_{n=1}^{N} \overline{R}_n, F_g)|\mathcal{F}_t] \\ &\geq e^{-\tilde{r}_t(T-t)} \max(E[\sum_{n=1}^{N} \overline{R}_n|\mathcal{F}_t], F_g) \\ &= e^{-\tilde{r}_t(T-t)} \max(z + E[\overline{R}_m|\mathcal{F}_t] + \sum_{n=m+1}^{N} E[\overline{R}_n], F_g) \end{aligned}$$

The result now follows from Lemma 2.

□

Proposition 7. *Let $A > 0$, $z + MF \leq F_g$ and $\tilde{F} = \frac{F_g - z}{M}$. Then V_t is bounded from above by*

$$\begin{aligned} V_t \leq \; & e^{-\tilde{r}_t(T-t)} \Big\{ F_g + e^{r_m(T_m-t)} \big[c(t, (1-\alpha_m^t)s/\bar{s}, 1+\tilde{F}, T_n, \sigma_m, r_m) \\ & - c(t, (1-\alpha_m^t)s/\bar{s}, 1+C, T_m, \sigma_m, r_m) \big] \\ & + \sum_{n=m+1}^{N} e^{r_n(T_n-T_{n-1})} \big[c(T_{n-1}, 1-\alpha_n, 1+\tilde{F}, T_n, \sigma_n, r_n) \\ & - c(T_{n-1}, 1-\alpha_n, 1+C, T_n, \sigma_n, r_n) \big] \Big\} \end{aligned}$$

with equality at least if $z + MF = F_g$.

PROOF: We have by convexity

$$\begin{aligned}
\max(z+\sum_{n=m}^{N}\overline{R}_n, F_g) &= z+\max(\sum_{n=m}^{N}\overline{R}_n, F_g-z)\\
&= z+M\cdot\max(\sum_{n=m}^{N}\frac{1}{M}\overline{R}_n, \frac{F_g-z}{M})\\
&\leq z+M\cdot\frac{1}{M}\sum_{n=m}^{N}\max(\overline{R}_n, \frac{F_g-z}{M})\\
&= z+\sum_{n=m}^{N}\max(\overline{R}_n, \frac{F_g-z}{M})
\end{aligned}$$

Thus

$$\begin{aligned}
V_t &\leq e^{-\tilde{r}_t(T-t)}\Big\{z+E[\max(\overline{R}_m, \frac{F_g-z}{M})|\mathcal{F}_t]\\
&\quad+\sum_{n=m+1}^{N}E[\max(\overline{R}_n, \frac{F_g-z}{M})]\Big\}.
\end{aligned}$$

The assumption $F \leq \frac{F_g-z}{M}$ means that we can replace F by $\tilde{F}=(F_g-z)/M$ and remove the outer maximum. Hence

$$V_t \leq e^{-\tilde{r}_t(T-t)}\left\{z+E[\bar{\bar{R}}_m|\mathcal{F}_t]+\sum_{n=m+1}^{N}E[\bar{\bar{R}}_n]\right\}$$

where $\bar{\bar{R}}_n=\max(\min(R_n, C), \tilde{F})$.
The bound in Proposition 7 then follows from Lemma 2 and the statement about equality by noting that if $F_g=MF+z$, the bound and the analytical formula for the price in Proposition 1 (which is now valid!) coincide. Note that this does not exclude the possibility of equality in other cases.

□

3.2.2 Modifications when $C=\infty$

It is possible that a broker wants to construct a derivative with $C=\infty$, in which case the formulas in Propositions 2 to 5 are not valid. However, by inserting a large virtual local cap, the formulas can still be used to compute arbitrarily good approximations. This section will deal with the issue of automatically setting this cap by proving an analytical formula for the maximum truncation error.

In this section $\hat{R}_n=\max(R_n, F)$ represents a return without local cap. We start with a central lemma.

Lemma 3. *Let V_t^C and V_t^∞ be the prices of floored cliquet options with local caps $C < \infty$ and $C = \infty$ respectively. Then*

$$V_t^\infty - V_t^C \leq 2e^{-\tilde{r}_t(T-t)}\Big\{\big(E[\hat{R}_m^t] - E[\overline{R}_m^t]\big) + \sum_{n=m+1}^{N}\big(E[\hat{R}_n] - E[\overline{R}_n]\big)\Big\}.$$

Proof: Following the first steps of the derivation of Proposition 2, the truncation error can be written as

$$\begin{aligned}(V_t^\infty - V_t^C)/e^{-\tilde{r}_t(T-t)} &= E[\max(\sum_{n=1}^{N}\hat{R}_n, F_g)|\mathcal{F}_t] - E[\max(\sum_{n=1}^{N}\overline{R}_n, F_g)|\mathcal{F}_t] \\ &= E[\max(\hat{R}_m^t + \sum_{n=m+1}^{N}\hat{R}_n, F_g - z) - \max(\overline{R}_m^t + \sum_{n=m+1}^{N}\overline{R}_n, F_g - z)]\end{aligned}$$

If $x \geq y$ we have

$$\max(x,a) - \max(y,a) = \begin{cases} 0, & x \leq a, \\ x-a, & x \geq a \geq y, \\ x-y, & y \geq a. \end{cases}$$

Using this with $X = \hat{R}_m^t + \sum_{n=m+1}^{N}\hat{R}_n$, $Y = \overline{R}_m^t + \sum_{n=m+1}^{n}\overline{R}_n$ and $a = F_g - z$ yields

$$\begin{aligned}(V_t^\infty - V_t^C)/e^{-\tilde{r}_t(T-t)} &= E[X-a; X \geq a \geq Y] + E[X-Y; Y \geq a] \\ &\leq E[X-Y; X \geq a \geq Y] + E[X-Y; Y \geq a],\end{aligned}$$

where the inequality follows from the fact that $a \geq Y$ on $X \geq a \geq Y$. Furthermore, since $X \geq Y$, the integrands are non-negative, enabling the estimates

$$\begin{aligned}(V_t^\infty - V_t^C)/e^{-\tilde{r}_t(T-t)} &\leq E[X-Y] + E[X-Y] \\ &= 2E[(\hat{R}_m^t - \overline{R}_m^t) + \sum_{n=m+1}^{N}\hat{R}_n - \sum_{n=m+1}^{N}\overline{R}_n].\end{aligned}$$

□

Proposition 8. *Let V_t^C and V_t^∞ be the price of floored cliquet options with local caps $C < \infty$ and $C = \infty$ respectively. Then*

$$\begin{aligned}V_t^\infty - V_t^C \leq\; & 2e^{-\tilde{r}_t(T-t)}\Big\{e^{r_m(T_m-t)}c(t,(1-\alpha_m^t)s/\bar{s}, 1+C, T_m, \sigma_m, r_m) \\ &+ \sum_{n=m+1}^{N} e^{r_n(T_n-T_{n-1})}c(T_{n-1},(1-\alpha_n), 1+C, T_n, \sigma_n, r_n)\Big\}.\end{aligned}$$

Proof: This is merely a simple combination of Lemma 3 in this section and Lemma 2 in Section 3.1.2, where $\Phi(-\infty) = 0$ has been inserted.

□

The error bound in Proposition 8 gives a simple rule for inserting a local cap with error control.

3.2.3 Volatility dependence

Proposition 9. *Assume constant volatility, $\sigma_t = \sigma$ and $A > 0$. Then*

$$\lim_{\sigma\to\infty} V_t = e^{-\tilde{r}_t(T-t)}\left\{F_g + \Lambda_A\big(M(C-F)\big)\right\}.$$

Proof: Since

$$\frac{a_n - \log(1 + C - x)}{b_n} \approx \frac{-\sigma^2/2(T_n - T_{n-1})}{\sigma\sqrt{T_n - T_{n-1}}} \to -\infty$$

when $\sigma \to \infty$, the dominated convergence theorem implies that

$$\begin{aligned}
\lim_{\sigma\to\infty} \varphi_{\tilde{R}_n}(\xi) &= e^{i\xi(C-F)} - i\xi \lim_{\sigma\to\infty} \int_0^{C-F} \Phi\left(\frac{a_n - \log(1 + C - x)}{b_n}\right) e^{i\xi x} dx \\
&= e^{i\xi(C-F)} - i\xi \int_0^{C-F} \lim_{\sigma\to\infty} \Phi\left(\frac{a_n - \log(1 + C - x)}{b_n}\right) e^{i\xi x} dx \\
&= e^{i\xi(C-F)},
\end{aligned}$$

and analogously for $\varphi_{\tilde{R}^t_m}$.
By Equation 3.1 we have that

$$A^2 \int_{-\infty}^{\infty} \operatorname{sinc}^2\left(\frac{\xi A}{2}\right) e^{i\xi M(C-F)} \frac{d\xi}{2\pi} = \Lambda_A\big(M(C-F)\big)$$

and Proposition 2 finally implies

$$\lim_{\sigma\to\infty} V_t = e^{-\tilde{r}_t(T-t)}\left\{F_g + \Lambda_A\big(M(C-F)\big)\right\}.$$

□

3.3 A numerical integration scheme

In this section we develop a numerical integration scheme for computation of the formulas in Propositions 2 to 4. Throughout this section , it is assumed that $A > 0$.

3.3.1 Initial reduction of required computations

It is possible to achieve a 75% reduction of the computations needed to compute the price by noting that the real part of the integrand is even, the imaginary part is odd and the domain of integration is symmetric. This follows from the fact that $\text{sinc}^2(A\xi/2)$ is an even function of ξ and $E[e^{i\xi \tilde{R}_m^t}]\prod_{n=m+1}^{M} E[e^{i\xi \tilde{R}_n}]$ is a characteristic function with even real part and odd imaginary part. Thus

$$\begin{aligned} V_t &= e^{-\tilde{r}_t(T-t)}\Big\{F_g + A^2\int_{-\infty}^{\infty}\text{sinc}^2(\frac{\xi A}{2})\times E[e^{i\xi\tilde{R}_m^t}]\times\prod_{n=m+1}^{N}E[e^{i\xi\tilde{R}_n}]\frac{d\xi}{2\pi}\Big\} \\ &= e^{-\tilde{r}_t(T-t)}\Big\{F_g + A^2\int_{0}^{\infty}\text{sinc}^2(\frac{\xi A}{2})\times \text{Re}\,\Big\{E[e^{i\xi\tilde{R}_m^t}]\times\prod_{n=m+1}^{N}E[e^{i\xi\tilde{R}_n}]\Big\}\frac{d\xi}{\pi}\Big\}. \end{aligned}$$

Since differentiating with respect to a parameter and taking real parts commute, this type of reduction extends to the computation of the greeks as well.

3.3.2 Setting the upper limit of integration

In order to compute the price integral numerically, an artificial upper limit of integration ξ_{max} is needed. Characteristic functions have a modulus less or equal to one which together with the fact that $\text{sinc}^2(A\xi/2) \geq 0$ gives the following estimate of the truncation error $E(\xi_{max})$.

$$\begin{aligned} |E(\xi_{max})| &= e^{-\tilde{r}_t(T-t)}A^2\Big|\int_{\xi_{max}}^{\infty}\text{sinc}^2(\frac{\xi A}{2})\times\text{Re}\,\Big\{E[e^{i\xi\tilde{R}_m^t}]\times\prod_{n=m+1}^{N}E[e^{i\xi\tilde{R}_n}]\Big\}\frac{d\xi}{\pi}\Big| \\ &\leq e^{-\tilde{r}_t(T-t)}A^2\int_{\xi_{max}}^{\infty}\text{sinc}^2(\frac{\xi A}{2})\frac{d\xi}{\pi} \\ &= e^{-\tilde{r}_t(T-t)}\frac{2A}{\pi}\int_{\xi_{max}A/2}^{\infty}\text{sinc}^2(x)dx. \end{aligned}$$

The integral in the last line is computed numerically for different values of $A\xi_{max}/2$ and presented in the table below.

$A\xi_{max}/2$	10	20	50	100	200	400
$\int_{\xi_{max}A/2}^{\infty}\text{sinc}^2(x)dx$	0.0521	0.0254	0.0099	0.0040	0.0022	0.0010

3.3.3 Evaluation of the characteristic function

In the previous section we showed that

$$E[e^{i\xi\tilde{R}_n}] = e^{i\xi(C-F)} - i\xi\int_0^{C-F}\Phi\left(\frac{a_n - \log(1+C-x)}{b_n}\right)e^{i\xi x}dx.$$

In the computation of the price formula of Proposition 2, this quantity has to be evaluated M times for each ξ. Due to the rapid oscillation of the integrand above for large ξ, this would be computationally very heavy if done directly by numerical integration.

The monotonicity and high degree of smoothness of $\Phi(\frac{a_n - \log(1+C-x)}{b_n})$ suggests that interpolation with complete cubic splines over the interval $[0, C-F]$ may be a good idea. Initially this interval is divided into N_p equally long subintervals $[x_n, x_{n+1}]$, $n = 0, \ldots, N_p$ and a cubic polynomial $p_3^{(n)}(x) = c_3^{(n)}x^3 + c_2^{(n)}x^2 + c_1^{(n)}x + c_0^{(n)}$ is assigned to each one of them. The coefficients are then chosen such that they interpolate the function at the *spline knots* x_n, $n = 0, \ldots, N_p + 1$ and have continuous first and second derivatives. In addition we require that the derivative of the spline and the function to be interpolated coincide at the endpoints 0 and $C - F$. For more details about complete cubic spline construction, see [4] pp. 53-55.

In the case of C large and/or F close to -1, $\Phi(\frac{a_n - \log(1+C-x)}{b_n})$ is close either to 0 and/or 1 for some x. By splining only on the the smaller interval $[x_{min}, x_{max}] \subseteq [0, C-F]$ and letting the approximation $\hat{\Phi}$ be one or zero outside, the number of spline knots N_p needed for a small sup-norm truncation error can be kept small.

To summarize, the cubic spline approximation $\hat{\Phi}$ of Φ can be written as

$$\hat{\Phi}\left(\frac{a_n - \log(1+C-x)}{b_n}\right) = \begin{cases} 0, & 0 \leq x \leq x_{min}, \\ \sum_{n=0}^{N_p-1} \chi_{[x_n x_{n+1}]}(x) p_3^{(n)}(x), & x_{min} \leq x \leq x_{max}, \\ 1, & x_{max} \leq x \leq C - F. \end{cases}$$

Inserting this approximation into the expression for the characteristic function yields

$$\begin{aligned} &\int_0^{C-F} \hat{\Phi}\left(\frac{a_n - \log(1+C-x)}{b_n}\right) e^{i\xi x} dx \\ =\ & \int_{x_{min}}^{x_{max}} [\sum_{n=0}^{N_p-1} \chi_{[x_n x_{n+1}]}(x) p_3^{(n)}(x)] e^{i\xi x} dx + \int_{x_{max}}^{C-F} 1 \cdot e^{i\xi x} dx \\ =\ & \sum_{n=0}^{N_p-1} [\int_{x_n}^{x_{n+1}} p_3^{(n)}(x) e^{i\xi x} dx] + \int_{x_{max}}^{C-F} 1 \cdot e^{i\xi x} dx \\ =\ & \sum_{n=0}^{N_p-1} [c_3^{(n)} \int_{x_n}^{x_{n+1}} x^3 e^{i\xi x} dx + c_2^{(n)} \int_{x_n}^{x_{n+1}} x^2 e^{i\xi x} dx \\ & + c_1^{(n)} \int_{x_n}^{x_{n+1}} x e^{i\xi x} dx + c_0^{(n)} \int_{x_n}^{x_{n+1}} e^{i\xi x} dx] + \int_{x_{max}}^{C-F} 1 \cdot e^{i\xi x} dx \end{aligned}$$

$$
\begin{aligned}
= & \sum_{n=0}^{N_p-1}\left\{c_3^{(n)}\left[\frac{e^{i\xi x}}{(i\xi)^4}((i\xi x)^3-3(i\xi x)^2+6i\xi x-6)\right]_{x_n}^{x_{n+1}}\right. \\
& +c_2^{(n)}\left[\frac{e^{i\xi x}}{(i\xi)^3}((i\xi x)^2-2i\xi x+2)\right]_{x_n}^{x_{n+1}} \\
& \left.+c_1^{(n)}\left[\frac{e^{i\xi x}}{(i\xi)^2}(i\xi x-1)\right]_{x_n}^{x_{n+1}}+c_0^{(n)}\left[\frac{e^{i\xi x}}{i\xi}\right]_{x_n}^{x_{n+1}}\right\}+\left[\frac{e^{i\xi x}}{i\xi}\right]_{x_{max}}^{C-F}
\end{aligned}
$$

Despite its horrible appearance, the formula is very fast to evaluate on a computer.

To compute the distribution function of a normal random variable at the spline knots, a fractional approximation proposed in [16] is used, which promises five to six correct decimals.

The next proposition gives two upper bounds of the error which is useful in selecting the number of spline knots. The first is good for ξ small, whereas the other for ξ large. For transparency we assume x_{min}=0 and $x_{max} = C - F$ and start by stating a lemma, which proof can be found in [4] on pp. 68-69.

Lemma 4. *If $f(x) \in C^{(4)}$, $h = x_n - x_{n-1}$ and $p_3(x)$ is the cubic spline approximation of f on $[a, b]$, then*

$$|f(x) - p_3(x)| \leq \frac{5h^4}{384} \sup_{x\in[a,b]} \left|\frac{d^4 f}{dx^4}\right|$$

and

$$|f'(x) - p_3'(x)| \leq \frac{h^3}{24} \sup_{x\in[a,b]} \left|\frac{d^4 f}{dx^4}\right|.$$

Proposition 10. *Let $\hat{\varphi}_{\tilde{R}_n}(\xi)$ be the approximation of $\varphi_{\tilde{R}_n}(\xi)$ and N_p the number of spline intervals of length $h = (C - F)/N_p$. Then*

$$|\hat{\varphi}_{\tilde{R}_n}(\xi) - \varphi_{\tilde{R}_n}(\xi)| \leq \xi\frac{5h^4}{384}(C - F) \sup_{x\in[0,C-F]} \left|\frac{d^4}{dx^4}\Phi\left(\frac{a_n - \log(1 + C - x)}{b_n}\right)\right|$$

and

$$|\hat{\varphi}_{\tilde{R}_n}(\xi) - \varphi_{\tilde{R}_n}(\xi)| \leq \frac{h^3}{24}(C - F) \sup_{x\in[0,C-F]} \left|\frac{d^4}{dx^4}\Phi\left(\frac{a_n - \log(1 + C - x)}{b_n}\right)\right|.$$

PROOF: Let $E(x) = \hat{\Phi}\left(\frac{a-\log(1+C-x)}{b}\right) - \Phi\left(\frac{a-\log(1+C-x)}{b}\right)$. Then by Lemma 4

$$
\begin{aligned}
|\hat{\varphi}_{\tilde{R}_n}(\xi) - \varphi_{\tilde{R}_n}(\xi)| &= \left|-i\xi\int_0^{C-F} E(x)e^{i\xi x}dx\right| \\
&\leq \xi(C - F) \sup_{x\in[0,C-F]} |E(x)| \\
&\leq \frac{5h^4\xi}{384}(C - F) \sup_{x\in[0,C-F]} \left|\frac{d^4}{dx^4}\Phi\left(\frac{a_n - \log(1 + C - x)}{b_n}\right)\right|
\end{aligned}
$$

On the other hand,

$$\begin{aligned}
|\hat{\varphi}_{\tilde{R}_n}(\xi) - \varphi_{\tilde{R}_n}(\xi)| &= \left| -i\xi \int_0^{C-F} E(x) e^{i\xi x} dx \right| \\
&= \left| -i\xi \left[E(x) \frac{e^{i\xi x}}{i\xi} \right]_0^{C-F} + i\xi \int_0^{C-F} E'(x) \frac{e^{i\xi x}}{i\xi} dx \right| \\
&= \left| \int_0^{C-F} E'(x) e^{i\xi x} dx \right| \\
&\leq \frac{h^3}{24}(C-F) \sup_{x\in[0,C-F]} \left| \frac{d^4}{dx^4} \Phi\left(\frac{a_n - \log(1+C-x)}{b_n} \right) \right|
\end{aligned}$$

by partial integration and the second statement of Lemma 4. Here we have also used the fact that $x = 0$ and $x = C - F$ are points of interpolation with zero error.

□

There is an explicit formula for the fourth derivative derived in Mathematica but it is too messy to be presentable.

In the computation of the greeks we also need to compute the integral formulas for the partial derivatives $\frac{\partial \varphi_{\tilde{R}_m^t}}{\partial s}$, $\frac{\partial \varphi_{\tilde{R}_m^t}}{\partial t}$ and $\frac{\partial^2 \varphi_{\tilde{R}_m^t}}{\partial s^2}$. They are evaluated directly by the trapezoid rule.

3.3.4 Trapezoid rule for the outer integral

If we write

$$\psi(\xi) = \text{sinc}^2(\frac{\xi A}{2}) \times \text{Re} \left\{ \varphi_{\tilde{R}_m^t}(\xi) \times \prod_{n=m+1}^{N} \varphi_{\tilde{R}_n}(\xi) \right\}$$

the price formula can be written

$$V_t = e^{-\tilde{r}_t(T-t)} \left\{ F_g + \frac{A^2}{\pi} \int_0^\infty \psi(\xi) d\xi \right\}$$

The integal is then truncated at ξ_{max}, which is set according to the rule proposed in Section 3.3.2. Dividing $[0, \xi_{max}]$ into N intervals gives the well known trapezoid rule of numerical quadrature

$$\int_0^{\xi_{max}} \psi(\xi) d\xi \approx \sum_{n=0}^{N-1} \left(\frac{\psi(\xi_n) + \psi(\xi_{n+1})}{2} \right) (\xi_{n+1} - \xi_n).$$

According to [7], the quadrature error e_n is bounded by

$$|e_n| \leq \frac{(\xi_{n+1} - \xi_n)^3}{12} \sup_{\xi\in[\xi_n,\xi_{n+1}]} |\psi''(\xi)|.$$

If a tolerance level *tol* is specified, a rule for the step length can be obtained as

$$
\begin{aligned}
|tol| &= \frac{\Delta\xi_n^3}{12} \sup_{\xi\in[\xi_n,\xi_{n+1}]} |\psi''(\xi)| \\
&\Leftrightarrow \\
\Delta\xi_n &= \left(\frac{12 \cdot tol}{\sup_{\xi\in[\xi_n,\xi_{n+1}]} |\psi''(\xi)|} \right)^{1/3}
\end{aligned}
$$

The second derivative $\psi''(\xi)$ is approximated with

$$\psi''(\xi) \approx \frac{\psi(\xi + d\xi) - 2\psi(\xi) + \psi(\xi - d\xi)}{(d\xi)^2}$$

where $d\xi$ is some small number.

We also replace $\sup_{\xi\in[\xi_n \xi_{n+1}]} |\psi''(\xi)|$ by $|\psi''(\xi_n)|$, which is justified if the second derivative does not change too much over the interval $[\xi_n, \xi_{n+1}]$.

The proposed integration scheme can now be summarized in the following algorithm:

```
ξ_0 = 0, ψ_0 = 1;
integral=0;
k=0;
while (ξ_n < ξ_max)
      Compute ψ''(ξ_k) using numerical differentiation;
      Δξ_k = (12tol / |ψ''(ξ_k)|)^(1/3);
      ξ_{k+1} = ξ_k + Δξ_k;
      Compute φ_{R̃^t_m}(ξ_{k+1}) × ∏_{n=m+1}^N φ_{R̃_n}(ξ_{k+1}) with the spline algorithm;
      Compute ψ_{k+1} = ψ(ξ_{k+1}) using φ_{R̃^t_m}(ξ_{k+1}) × ∏_{n=m+1}^N φ_{R̃_n}(ξ_{k+1});
      integral=integral + (ψ_k+ψ_{k+1})/2 (ξ_{k+1} − ξ_k);
      n=n+1;
end
```

3.4 Adding a global cap C_g

It is possible to generalize the Fourier method to compute the price and greeks of a derivative with a global cap C_g added. At time T the holder of this derivative is paid

$$Y = \min(\max(\sum_{n=1}^{N} \overline{R}_n, F_g), C_g).$$

As in Section 3.1 $A = MC + z - F_g$, $\tilde{R}_n = C - \overline{R}_n$ and $\Lambda_A(x) = \max(A - |x|, 0)$. First,

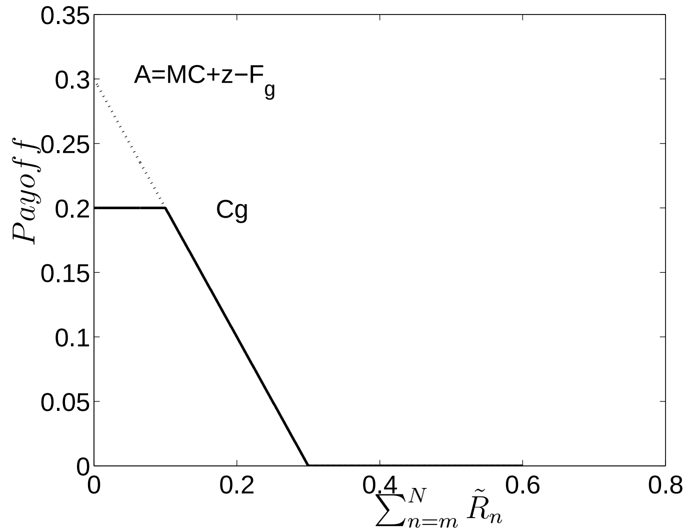

Figure 3.1: Payoff with a global cap added. Parameters in this example: $N = 12$, $m = 7$, $z = 0$, $F_g = 0$, $C_g = 0.20$

if $C_g \geq A$ the global cap cannot be attained and we are back in the setting with a global floor only presented in Section 3.1. This is illustrated in Figure 3.1 above. If the global cap can be reached, the price of the derivative is given by Proposition 11 below.

Proposition 11. *If $A > C_g - F_g > 0$, the price V_t of a cliquet with global floor and cap is given by*

$$V_t = e^{-\tilde{r}_t(T-t)}\left\{F_g + \int_{-\infty}^{\infty} H(\xi) \times \varphi_{\tilde{R}_m^t}(\xi) \times \prod_{n=m+1}^{N} \varphi_{\tilde{R}_n}(\xi)\frac{d\xi}{2\pi}\right\}$$

where

$$H(\xi) = A^2\text{sinc}^2(\frac{\xi A}{2}) - (A - C_g + F_g)^2\text{sinc}^2(\frac{\xi(A - C_g + F_g)}{2}),$$

$$\varphi_{\tilde{R}_m^t}(\xi) = e^{i\xi(C-F)} - i\xi \int_0^{C-F} \Phi\left(\frac{a_m - \log(1 + C - x)}{b_m}\right) e^{i\xi x}dx,$$

and

$$\varphi_{\tilde{R}_n}(\xi) = e^{i\xi(C-F)} - i\xi \int_0^{C-F} \Phi\left(\frac{a_n - \log(1 + C - x)}{b_n}\right) e^{i\xi x}dx.$$

PROOF: We have that

$$\begin{aligned} \min(\max(z+\sum_{n=m}^{N}\overline{R}_n, F_g), C_g) &= \\ F_g + \min(\max(MC + z - F_g - \sum_{n=m}^{N}\tilde{R}_n, 0), C_g - F_g) &= \\ F_g + \Lambda_A(\sum_{n=m}^{N}\tilde{R}_n) - \Lambda_{A-C_g+F_g}(\sum_{n=m}^{N}\tilde{R}_n). \end{aligned}$$

Repetition of the proof of Proposition 2 completes the proof.

□

The greeks are obtained by replacing $A^2\text{sinc}^2(A\xi/2)$ by $H(\xi)$ in the formulas of Propositions 3 to 5.

Chapter 4

A PDE method

In this section we derive a PDE for the cliquet with global floor. The method is inspired by Andreasen's approach for pricing of discrete Asian options presented in [1]. At the end of this section we indicate how the method could be modified to incorporate early exercise features.

4.1 Derivation of a Black-Scholes PDE

Initially we only consider the case without dividends and no global cap C_g. In Section 4.2.5 we show how to incorporate the discrete dividend yield approach used in the Fourier method of Chapter 3.

Proposition 12. *The price process $t \to V_t$ is continuous.*

PROOF: This is an immediate consequence of the martingale representation theorem. However, it could also be verified from the price formula in Proposition 2 (or Proposition 11 in the presence of a global cap C_g).

□

From Proposition 2 it is evident that the price of the derivative depends on four state variables. These are the time t, the current share price s, the share price at the previous reset day $\bar{s}$ and z. If $T_{m-1} \leq t < T_m$, we have $s = S_t$, $\bar{s} = S_{T_{m-1}}$ and $z = \sum_{n=1}^{m-1} \overline{R}_n$. Moreover, if $t = T_N$, we define $s = \bar{s} = S_{T_N}$ and $z = \sum_{n=1}^{N} \overline{R}_n$. Also note that $s = \bar{s}$ at the reset dates $t = T_n$, $0 \leq n \leq N$.

Having fixed the notation, we are ready to show that the floored cliquet satisfies the Black-Scholes equation between the reset days.

Proposition 13. *The price $V = V(t, s, \bar{s}, z)$ satisfies the partial differential equation*

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{\sigma_t^2 s^2}{2}\frac{\partial^2 V}{\partial s^2} + r_t s \frac{\partial V}{\partial s} - r_t V = 0, \quad T_{n-1} \leq t < T_n \\ V(T_N, s, \bar{s}, z) = \max(z, F_g) \\ V(T_n^-, s, \bar{s}, z) = V(T_n, s, s, z + \max(\min(s/\bar{s} - 1, C), F)), \quad 1 \leq n \leq N. \end{cases}$$

PROOF: The claim is proved by backwards induction.
Step 1: The Proposition is true for $n = N$.
If $n = N$ we have $t \in [T_{N-1}, T_N)$ and the derivative has a payoff Y of

$$Y = \max(z + \max(\min(\frac{S_{T_N}}{S_{T_{N-1}}} - 1, C), F), F_g) = f(S_T).$$

Hence we have a simple European derivative for which the Black Scholes equation holds by delta hedging.
Step 2: Induction assumption. Assume that we have computed $V(T_n, s, s, z)$, $n < N$.
Step 3: $V(t, s, \bar{s}, z)$, $t \in [T_{n-1}, T_n)$ can now be computed by the Black Scholes equation. By step 2, $V(T_n, \cdot, \cdot, \cdot)$ is known, and the continuity property of Proposition 12 relates $V(T_n, \cdot, \cdot, \cdot)$ to $V(T_n^-, \cdot, \cdot, \cdot)$ by the equation

$$\begin{aligned} V(T_n^-, s, \bar{s}, z) &= \\ V(T_n, s, s, z + \max(\min(\frac{s}{\bar{s}} - 1, C), F) &= f(S_{T_n}). \end{aligned}$$

Hence we have a simple European derivative over the period $[T_{n-1}, T_n)$ which implies that the Black Scholes equation holds in this period with $V(T_n^-, s, \bar{s}, z)$ as final condition. The principle of induction states that this holds for any $n < N$.

□

If a global cap is wanted, the end condition $\max(z, F_g)$ is changed to $\min(\max(z, F_g), C_g)$. The proofs of Propositions 12 and 13 show that delta hedging works for cliquet options with global floor.

In the integral formula in Proposition 2 s and $\bar{s}$ appear as $x = \log(s/\bar{s})$. Since the volatility in our model does not depend on s explicitly, the number of dimensions can be reduced from four to three. Another advantage of this change of variables is that the coefficients in front of the derivatives are constant over each reset period. The Black-Scholes equation now takes the following form

Proposition 14. *Let $x = \log(s/\bar{s})$. The price $V = V(t, x, z)$ satisfies the partial differential equation*

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{\sigma_t^2}{2}\frac{\partial^2 V}{\partial x^2} + (r_t - \frac{\sigma_t^2}{2})\frac{\partial V}{\partial x} - r_t V = 0, \quad T_{n-1} \le t < T_n \\ V(T_N, x, z) = \max(z, F_g) \\ V(T_n^-, x, z) = V(T_n, 0, z + \max(\min(e^x - 1), C), F), \quad 1 \le n \le N. \end{cases}$$

PROOF: First, let $y = s/\bar{s}$. For $T_{n-1} \le t < T_n$, $\bar{s}$ is constant and we have $\frac{\partial V}{\partial s} = \frac{\partial V}{\partial y} \cdot \frac{1}{\bar{s}}$, $\frac{\partial^2 V}{\partial s^2} = \frac{\partial^2 V}{\partial y^2} \cdot \frac{1}{\bar{s}^2}$. This gives

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{\sigma_t^2 y^2}{2}\frac{\partial^2 V}{\partial y^2} + r_t y\frac{\partial V}{\partial y} - r_t V = 0, \quad T_{n-1} \le t < T_n \\ V(T_N, y, z) = \max(z, F_g) \\ V(T_n^-, y, z) = V(T_n, 1, z + \max(\min(y - 1, C), F)), \quad 1 \le n \le N. \end{cases}$$

Secondly, the transformation $x = \log(y)$ gives $\frac{\partial V}{\partial y} = \frac{\partial V}{\partial x} \cdot \frac{1}{y}$ and $\frac{\partial^2 V}{\partial y^2} = (\frac{\partial^2 V}{\partial x^2} - \frac{\partial V}{\partial x}) \cdot \frac{1}{y^2}$, and the result follows by insertion.

□

One might wonder if it is possible to combine x and z into one variable analogous to the method for Asian options presented in [1]. Unfortunately, no such further reduction is possible since $z = \sum_{n=1}^{m-1} \overline{R}_n$ and $x = \log(S_t/S_{T_{m-1}})$ are independent random variables.

Although the solution depends on two spatial variables, only x occurs explicitly in the differential equation. An interpretation of this is that the diffusion takes place only in the x-direction and hence there is no flow in the z-direction, similar to the situation of laminar flow in fluid dynamics.

Mathematically this means that between the reset dates, we have a set of one dimensional PDE's (one for each fixed z) rather than a full two dimensional PDE. This observation will reduce the computational work considerably compared to if we were to solve a full two dimensional problem.

4.2 Numerical solution of the PDE

In this section an explicit finite difference scheme for the equation in Proposition 14 will be derived. The main rationales for choosing an explicit instead of an implicit or Crank-Nicholson scheme are the following:

- Only $V(T_n, 0, z)$ is needed at the beginning of each reset period in order to use the continuity condition update. If an implicit method were used, $V(T_n, x, z)$ would be obtained over an entire square at an extra computational cost but for no use.

- No need for artificial boundary conditions.

- No difficult stability conditions since the coefficients of the PDE in Proposition 14 are constant in each reset period.

- Special case of the trinomial tree method which is familiar to most practitioners.

Inspired by Proposition 14, a coarse solution algorithm is presented below.

$s = S_t, \bar{s} = S_{T_{m-1}}$;
$x_0 = \log(s/\bar{s})$, $z_0 = \sum_{n=1}^{m-1} \overline{R}_n$;
$V(T_N, 0, z) = \max(z, F_g)$;
for (n=N:-1:m);
    1.Continuity: $V(T_n^-, x, z) = V(T_n, 0, z + \max(\min(e^x - 1), C), F))$;
    2.Compute $V(T_{n-1}, 0, z)$ by solving the PDE with
    $V(T_n^-, x, z)$ as final condition;
end

$V(T_m^-, x, z) = V(T_m, 0, z + \max(\min(e^x - 1), C), F))$;
`Compute` $V(t, x_0, z_0)$ `by solving the PDE with` $V(T_m^-, x, z)$ `as`
`final condition;`

In order to make this algorithm work in practice, the following three issues need to be resolved:

- **Discretization:** The t, x, z-space need to be discretized in order to use finite differences.

- **Continuity condition:** A discrete version needed.

- **PDE solution:** Explicit method for solution over one reset period.

4.2.1 Discretization

Each reset period is divided into N_t intervals of length $\Delta t = \frac{T_n - T_{n-1}}{N_t}$. Typically, $N_t = 75$ gives accurate answers. This gives $t_{n_t} = n_t \Delta t$, $0 \leq n_t \leq N_t$.

Once the time step is selected stability issues put constraints on the space step Δx. According to [16], $\Delta x = \sigma_n \sqrt{3\Delta t}$ is a stable and good choice. Since x can move up or down in each time step, we have $x_k = k\Delta x$, $-n_t \leq k \leq n_t$, $0 \leq n_t \leq N_t$. This $x - t$ discretization is illustrated in figure 4.1.

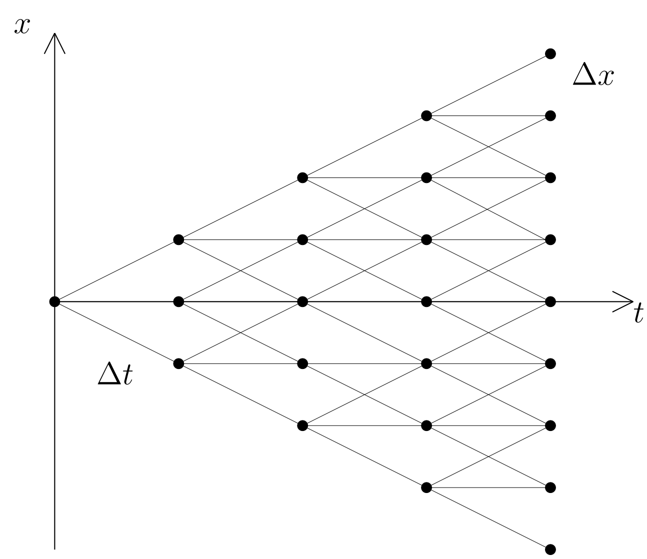

Figure 4.1: The discretization of one reset period

The presence of local floors and caps ensures that the z-variable in reset period $n \geq m$ is bounded by $z_0 + (n - m)F \leq z \leq z_0 + (n - m)C$. If we divide $C - F$ into N_z intervals of length $\Delta z = \frac{C-F}{N_z}$, we get $z_l = (n - m)F + z_0 + l\Delta z$, $0 \leq l \leq N_z(n - m)$ for period n.

In Figure 4.2, the structure of the global discretization is shown. Each triangle represents an independent trinomial tree of the type described in Figure 4.1. This is consequence of the laminar structure of the diffusion.

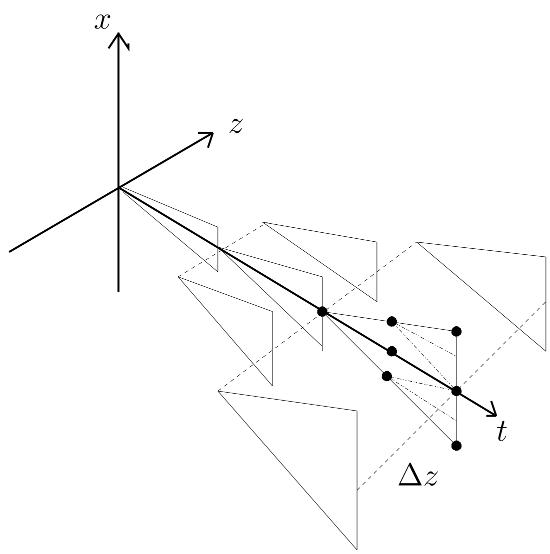

Figure 4.2: The structure of the (t, x, z)-discretisation for $N = 3$ reset periods.

4.2.2 Continuity condition

The discrete version of the continuity condition from Proposition 14 is

$$V(T_{n-1} + t_{N_t}, x_k, z_l) = V(T_n, 0, z_l + \max(\min(e^{x_k} - 1, C), F)).$$

A problem is that the solution V is known only at the mesh points we cannot be sure that $z_l + \max(\min(e^{x_k} - 1, C), F) \in \{z_{l'}\}_{l'=0}^{N_z(M-1)}$. To overcome this problem the solution is interpolated with the same kind of cubic splines used in Section 3.3.3. The rationale for doing so is that the price is convex with respect to z. This is easily proved by noting that the payoff $Y = \max(z, F_g)$ is monotone and convex and that this property is preserved when taking conditional expectations.

4.2.3 PDE solution

For the solution of the PDE within each reset period, an explicit finite difference scheme is used. It approximates the partial derivatives at the mesh points with the finite difference quotients presented below. Remember that time runs backwards.

$$
\begin{aligned}
\frac{\partial V}{\partial t}(t_{n_t}, x_k, z_l) &\approx \frac{V(t_{n_t}, x_k, z_l) - V(t_{n_t-1}, x_k, z_l)}{\Delta t} \\
\frac{\partial^2 V}{\partial x^2}(t_{n_t}, x_k, z_l) &\approx \frac{V(t_{n_t}, x_{k+1}, z_l) - 2V(t_{n_t}, x_k, z_l) + V(t_{n_t}, x_{k-1}, z_l)}{(\Delta x)^2} \\
\frac{\partial V}{\partial x}(t_{n_t}, x_k, z_l) &\approx \frac{V(t_{n_t}, x_{k+1}, z_l) - V(t_{n_t}, x_{k-1}, z_l)}{2\Delta x}
\end{aligned}
$$

Inserting this into the Black-Scholes equation for reset period m yields

$$V(t_{n_t-1}, x_k, z_l) = c_1^{(m)} V(t_{n_t}, x_{k-1}, z_l) + c_2^{(m)} V(t_{n_t}, x_k, z_l) + c_3^{(m)} V(t_{n_t}, x_{k+1}, z_l)$$

for appropriate constants $c_1^{(m)}$, $c_2^{(m)}$, and $c_2^{(m)}$.

4.2.4 Enhancements

A main driver of the computational complexity is the $1 + (n - m)N_z$ PDE's, that have to be solved in reset period n. However, it is possible to achieve a 40% reduction of this number by utilizing the result from Section 3.1 that states there is an analytical solution available for some values of z depending on the reset period n. The situation is illustrated in detail in Figure 4.2.4 below for the case $F = -C$, $F_g = 0$.

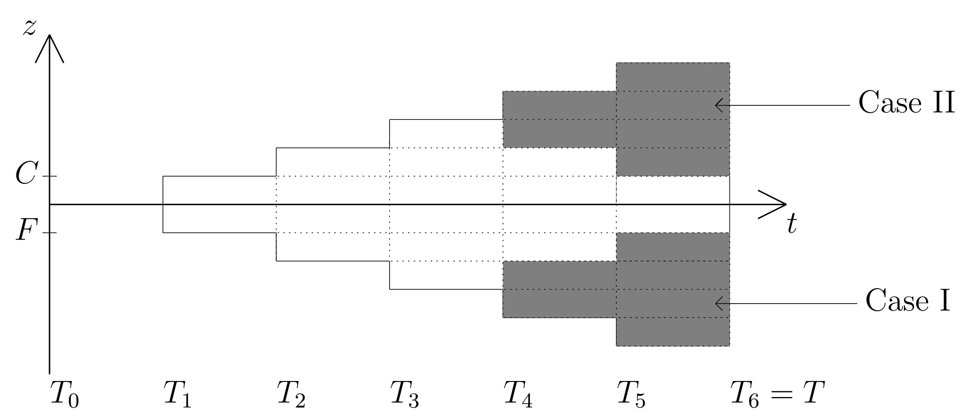

Figure 4.3: The values of z where the solution V is known. Cases I-II refer to the cases presented in Section 3.1.

With this result at hand, the algorithm is modified such that $V(T_{n-1}, 0, z_l)$ is computed by the finite difference method only if z_l is in the white region. If in the shaded region, analytical formulas are used instead.

Greeks

When using finite differences or trees, the only way to obtain the greeks is by difference quotients. To be able to so, the solution is computed back to time $t-\Delta t$ instead of t which gives the following estimates of the greeks:

$$
\begin{aligned}
\Delta &= \frac{V(t, x_0+\Delta x, z_0) - V(t, x_0-\Delta x, z_0)}{2\Delta x} \cdot \frac{1}{s} \\
\Theta &= \frac{V(t+\Delta t, x_0, z_0) - V(t-\Delta t, x_0, z_0)}{2\Delta t} \\
\Gamma &= \frac{2(r_m V(t, x_0, z_0) - r_m s\Delta - \Theta)}{\sigma_m^2 s^2}
\end{aligned}
$$

where $x_0 = \log(s/\bar{s})$ and the Γ is derived from the Black-Scholes PDE.

4.2.5 Dividends

Like for the Fourier method, we use the approximation of fixed dividends with fixed dividend yields derived in Section 2.3. This is necessary if we want to use the relative performance variable x rather than the absolute s.

The martingale representation theorem ensures that V_t is continuous over a discrete dividend yield. As described in Section 2.3, the share price drops from S_{τ^-} to $(1-\alpha)S_{\tau^-}$ each time τ a dividend yield of size α is paid to the share holder. This corresponds to a shift in the x-variable from x to $x+\log(1-\alpha)$ at time τ. When working with dividend yields, the exact time of a future dividend pay out is irrelevant, as long as it occurs in the right reset period. This observation enables us to move all remaining pay outs of a reset period to the end of that period and incorporate x-shift with the continuity condition at the reset date. This now takes the form

$$V(T_n^-, x, z) = V(T_n, 0, z + \max(\min((1-\alpha_n)e^x - 1), C), F), 1 \leq n \leq N.$$

The main benefits of this approach are that no shifts of the mesh within a reset period are needed and that the extra computational cost is negligible.

4.3 Early exercise

In this section we briefly discuss early exercise features of a cliquet with global floor. For simplicity we only consider the situation without dividends.

An *American cliquet* option with global floor enables the holder to exercise the option at any time, then paying

$$Y_A(t) = B \max(z + \min(\max(s/\bar{s} - 1, F), C), F_g).$$

Allowing exercise only immediately after each reset date gives a so called *Bermudan cliquet* option with global floor, paying

$$Y_B(t) = B \max(z, F_g)$$

at $t = T_n^+$.

Let $\Sigma_{t,T}$ be the set of all stopping times with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ attaining values in $[t, T]$. From Theorem 3.39 in [13] it then follows that the price of the American cliquet is given by

$$V_t = \sup_{\tau \in \Sigma_{t,T}} E[e^{-\int_t^\tau r_u du} Y_A(\tau)].$$

In a similar fashion the Bermudan option is the maximum over all stopping times with respect to $(\mathcal{F}_t)_{t\geq 0}$ taking values in the discrete set $\{T_m, \ldots, T_N\}$.

Although the theory for American contracts is intricate, the practical implementation is fairly straightforward provided that a PDE method like finite differences or finite elements is available for the European version of the contract. In nodes *where exercise is allowed*, one simply has to check whether it is optimal to exercise or not. For the explicit method proposed in this thesis this yields at nodes where exercise is allowed

$$V(t_{n_t}, x_k, z_l) = \max\left\{Y, V_{keep}(t_{n_t}, x_k, z_l)\right\}$$

where

$$\begin{aligned} Y &= B \max(z_l, F_g), && \text{(if Bermudan)}, \\ Y &= B \max(z_l + \min(\max(e^{x_k} - 1, F), C), F_g), && \text{(if American)}, \end{aligned}$$

and

$$V_{keep}(t_{n_t}, x_k, z_l) = c_1^{(m)} V(t_{n_t+1}, x_{k-1}, z_l) + c_2^{(m)} V(t_{n_t+1}, x_k, z_l) + c_3^{(m)} V(t_{n_t+1}, x_{k+1}, z_l).$$

Here $c_1^{(m)}$, $c_2^{(m)}$ and $c_3^{(m)}$ are as in Section 4.2.1.

Time continuity does not follow directly from the martingale representation theorem as in the European case. Instead we proceed as follows. Assume that the the American cliquet has not been exercised yet. If $T_{N-1} \leq t \leq T_N$, we have an American contract with payoff

$$Y_A(t) = \max(z + \max(\min(s/\bar{s} - 1, C), F), F_g) = f(s)$$

when exercised. This depends only on the current share price s and since f is bounded and uniformly Lipschitz, V_t is continuous in time. For details, see [14]. Once $V_{T_{N-1}}$ has been computed, it is simple and satisfies the same conditions as V_{T_N}. Hence the argument above can be repeated recursively to prove continuity back to an arbitrary time $t \geq T_0$.

In Section 5 some comparisons between European, Bermudan and American style options are presented.

Finally, it should be noted that the Fourier method cannot be extended to allow early exercise.

Chapter 5

Results

5.1 Comparison of dividend models

This section investigates the relative differences[1] of option prices from the discrete dividend yield and fixed dividend models presented in Section 2.3. Prices are computed with a Monte-Carlo algorithm for a vide range of input parameters using the same set of 10^6 random numbers for both models.

The following observations can be made from Figures 5.1 to 5.5 below.

- **Relative difference increases with dividend size:** This is obvious in all figures and what could be expected with an increasing error from the Taylor series approximation. It also seems to hold for different dividend dates (Figure 5.2), volatilities (Figure 5.3 left), F and C (Figure 5.4 left) and number of reset dates (Figure 5.5 left).

  The somewhat large difference for $D = 0.05$ EUR and $\tau = 2.5$ years is not too troublesome considering the difficulty to predict a dividend 2.5 years in the future. For most other parameters, the relative difference is of the order $0.3 - 0.6\%$.

- **Relative difference seems to increase with τ:** With exception of Figure 5.2 (right) this effect is present in all cases. For $\tau = 0.5$, relative differences are in the range of $0 - 0.2\%$ for all tested values of σ, D, F and C and N, which means that for known dividends in the near future, the fixed and yield models give very similar results. This is exactly what we required in Section 2.3.

- **Relative difference increases as F decreases and C increases:** This is obvious from Figure 5.4.

[1] Relative difference between x and $\hat{x}$ is defined as $\frac{|x-\hat{x}|}{x}$

Relative difference as a function of dividend size

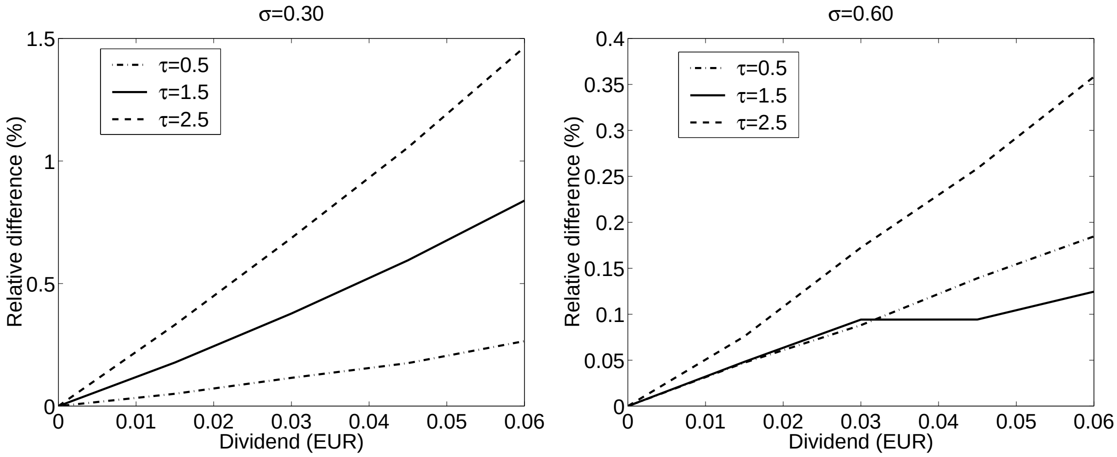

Figure 5.1: Relative difference dependence on the size a dividend paid at time τ for $\sigma = 0.30$ (left) and $\sigma = 0.60$ (right). Other parameters: Today's share price $S_0 = 1$ EUR, $T = 3$ years, $N = 12$ reset periods, $F_g = 0$, $F = -0.05$, $C = 0.05$ and $r = 0.05$.

Relative difference as a function of dividend time

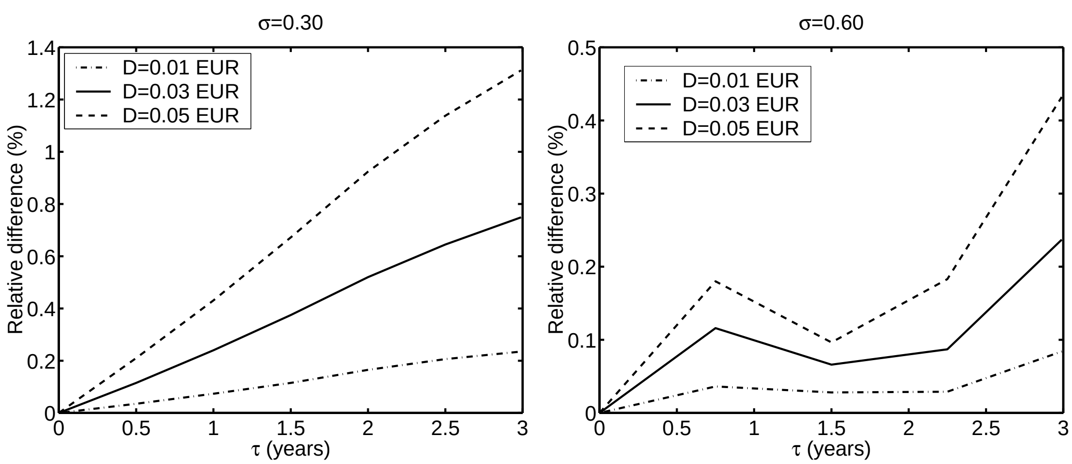

Figure 5.2: Relative difference dependence on the time τ a dividend of size D EUR is paid for $\sigma = 0.30$ (left) and $\sigma = 0.60$ (right). Other parameters: Today's share price $S_0 = 1$ EUR, $T = 3$ years, $N = 12$ reset periods, $F_g = 0$, $F = -0.05$, $C = 0.05$ and $r = 0.05$.

Relative difference as a function of volatility

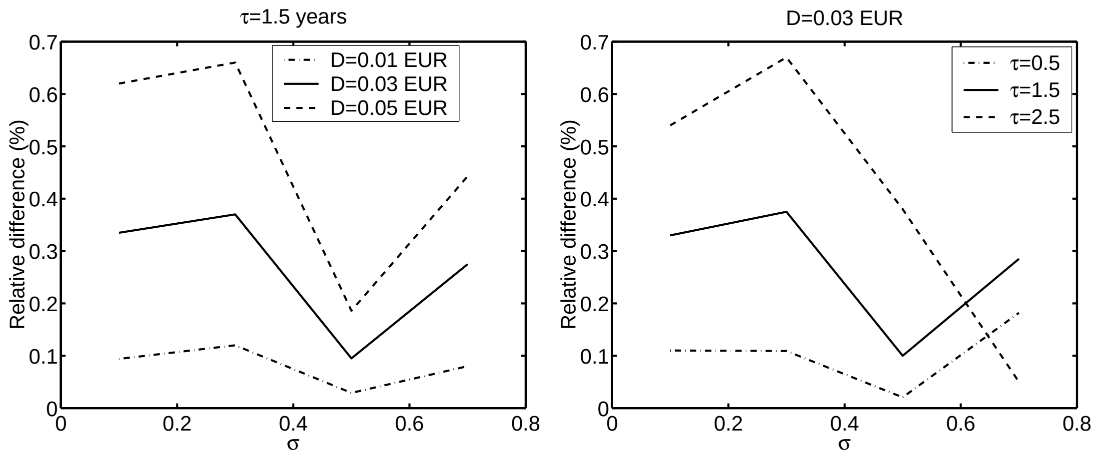

Figure 5.3: Relative difference dependence on the volatility for different dividend levels D (left) and dividend times τ (right). Other parameters: Today's share price $S_0 = 1$ EUR, $T = 3$ years, $N = 12$ reset periods, $F_g = 0$, $F = -0.05$, $C = 0.05$ and $r = 0.05$.

Relative difference as a function of F and C

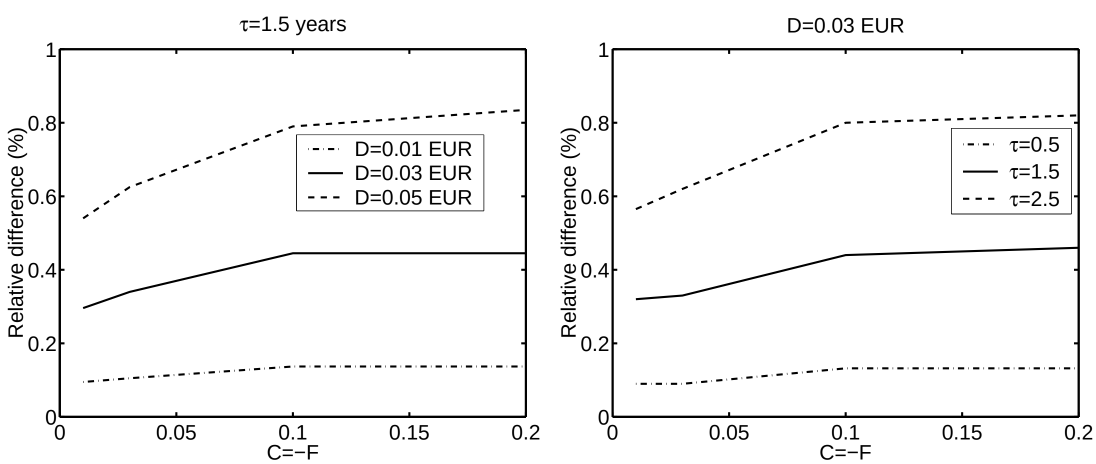

Figure 5.4: Relative difference dependence on F and C for different dividend levels D (left) and dividend times τ (right). Other parameters: Today's share price $S_0 = 1$ EUR, $T = 3$ years, $N = 12$ reset periods, $F_g = 0$, $\sigma = 0.30$ and $r = 0.05$.

Relative difference as a function of number of reset periods

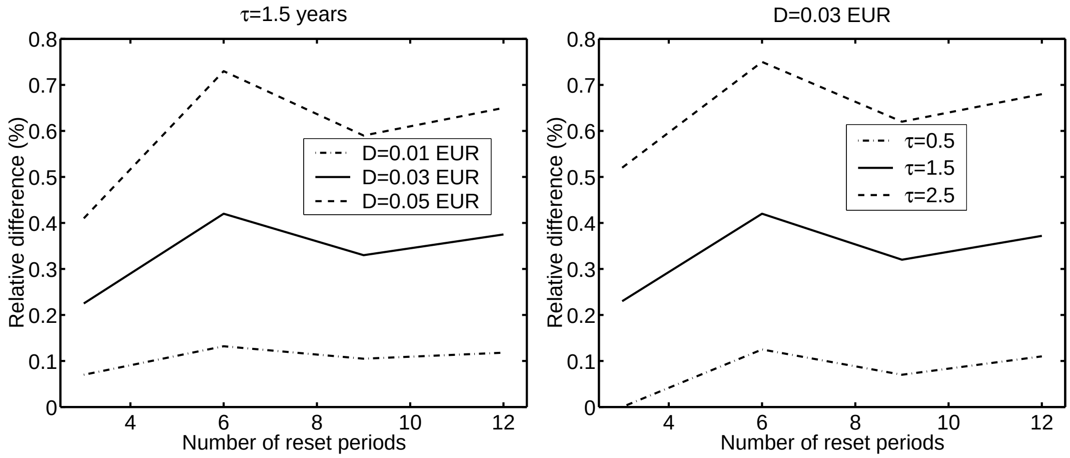

Figure 5.5: Relative difference dependence on the number of reset periods for different dividend levels D (left) and dividend times τ (right). Other parameters: Today's share price $S_0 = 1$ EUR, $T = 3$ years, $N = 12$ reset periods, $F_g = 0$, $F = -0.05$ and $C = 0.05$, $\sigma = 0.30$ and $r = 0.05$.

5.2 Accuracy versus computational effort

In order to rank the Fourier, PDE and Monte-Carlo methods, we investigate their accuracy and computational effort for the benchmark options displayed in the table below.

Option	T	N	F_g	F	C
Cliquet 1	3	6	0	-0.10	0.10
Cliquet 2	3	12	0	-0.05	0.05
Cliquet 3	3	36	0	-0.02	0.02

Each option is priced at $t = 0$ with the three methods for a flat volatility of $\sigma = 0.1$, $\sigma = 0.3$ and $\sigma = 0.5$ assuming no dividends. The reference answers in the table below are obtained using the Monte-Carlo method with 10^9 simulations to compte the price while the Fourier method with very high accuracy has been used for the greeks.

Option	σ	V	Δ	Θ	Γ
Cliquet 1	0.10	0.1180	0.5529	-0.01633	-1.076
Cliquet 1	0.30	0.0776	0.1610	0.00182	-0.134
Cliquet 1	0.50	0.0567	0.0797	0.00537	-0.0512
Cliquet 2	0.10	0.0952	0.4451	-0.01008	-1.484
Cliquet 2	0.30	0.0566	0.1154	-0.00167	-0.102
Cliquet 2	0.50	0.0426	0.0567	0.00385	-0.0366
Cliquet 3	0.10	0.0717	0.3339	-0.00602	-1.419
Cliquet 3	0.30	0.0401	0.0804	0.00138	-0.0755
Cliquet 3	0.50	0.0300	0.0398	0.00276	-0.0258

All three methods are implemented in the C programming language and compiled to a DLL file that is called from a test routine written in Python. Computations are made on a Dell Inspirion 8200 laptop with a 1.6 GHz Pentium® m4 processor and a 256 MB RAM. In Figures 5.6 to 5.8 below the relative errors are plotted against the computational time needed to compute the price, delta, theta and gamma. For the Monte-Carlo method, the standard error has been used to measure accuracy. In order to save space, only the results for the price V and gamma Γ are shown. The gamma is the most difficult of the greeks to compute, so the relative errors of delta and theta are expected to be between that of the price and gamma.

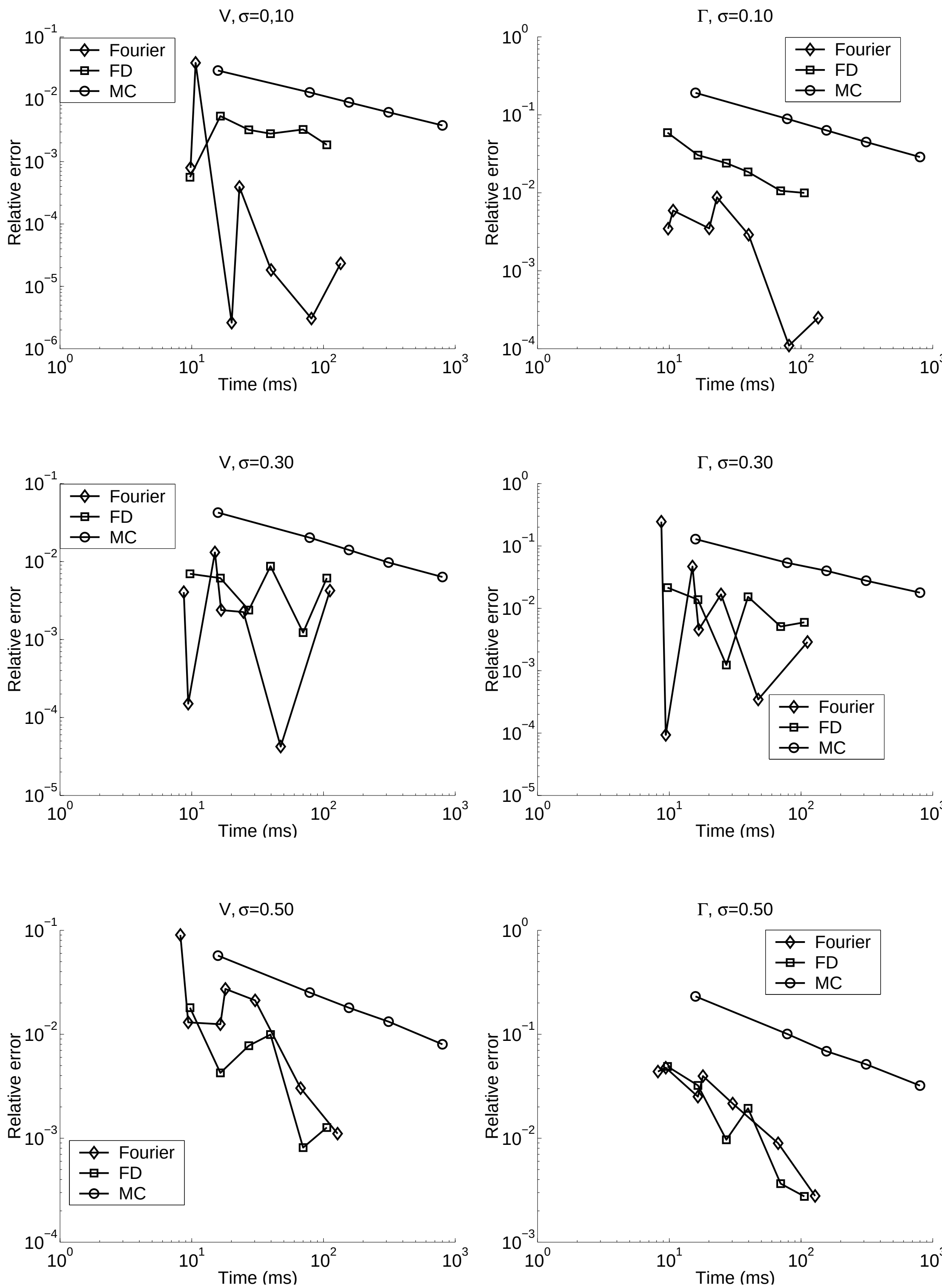

Figure 5.6: Cliquet 1: Price and gamma for the three methods for different volatilities.

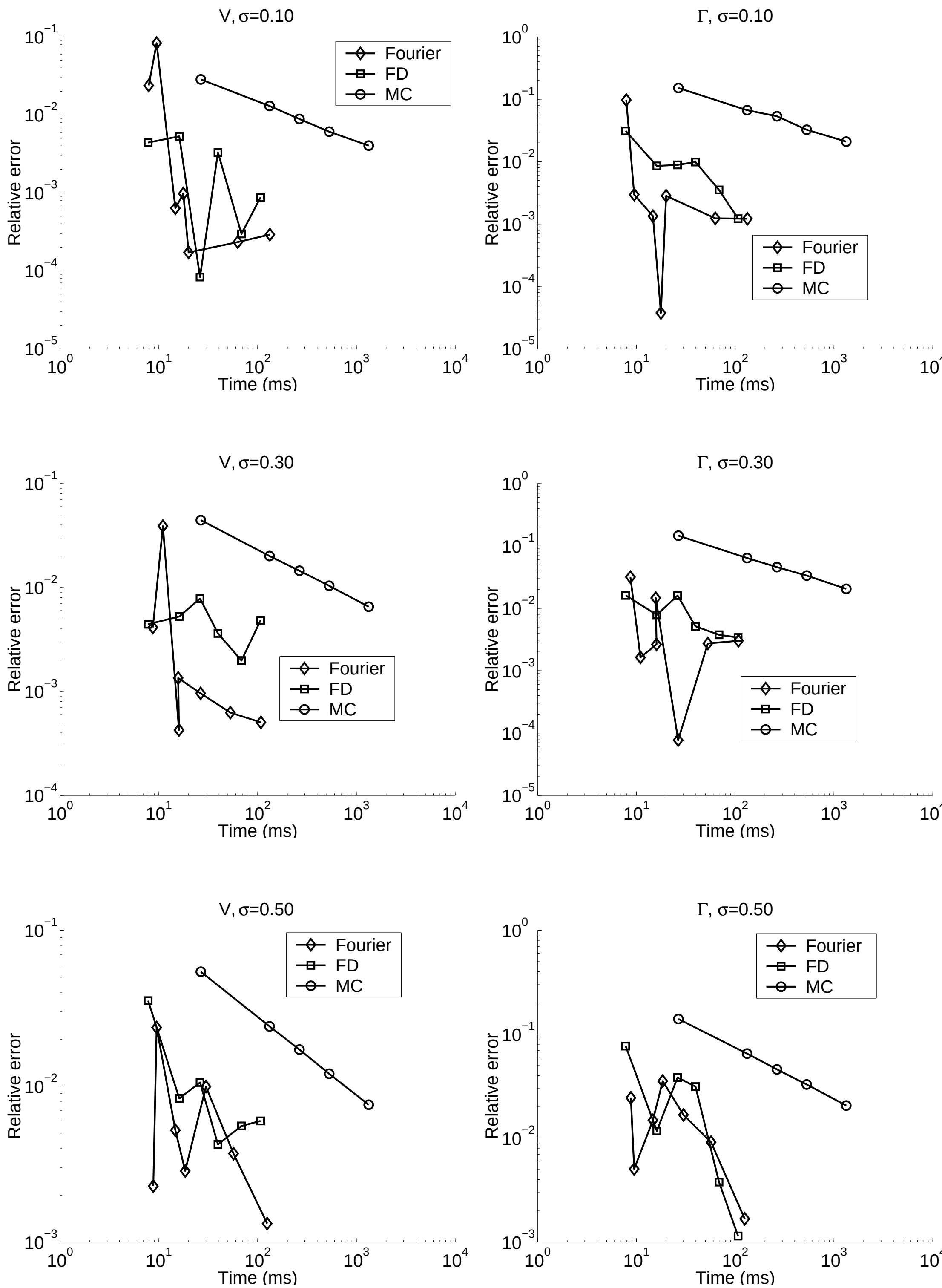

Figure 5.7: Cliquet 2: Price and gamma for the three methods for different volatilities.

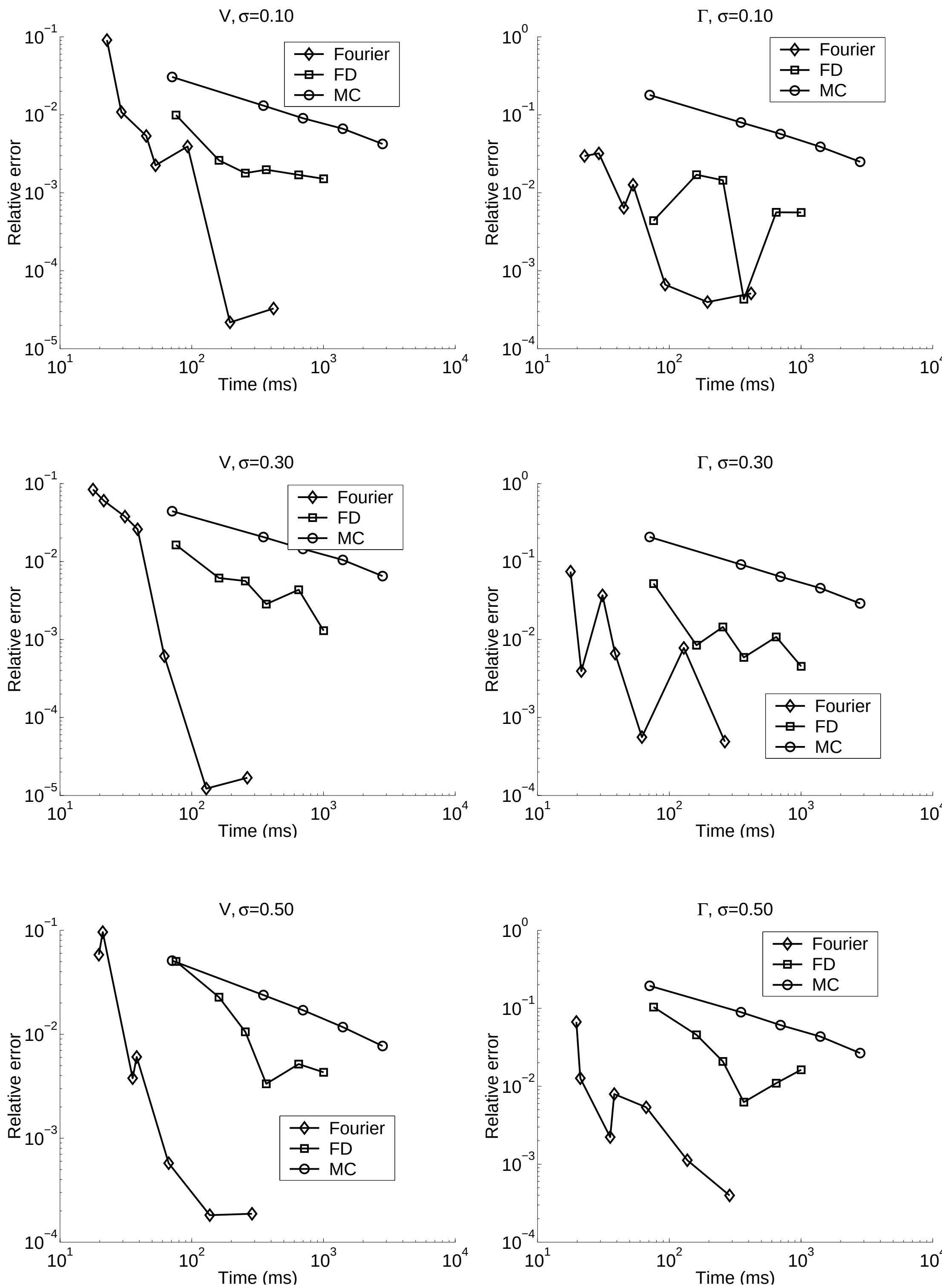

Figure 5.8: Cliquet 3: Price and gamma for the three methods for different volatilities.

It is obvious that both the Fourier and PDE method outperform the Monte-Carlo method, in particular in the computation of the gamma. The only benefit of the Monte-Carlo method is that it is extremely easy to implement.

For $N = 6$ and $N = 12$ reset periods, the Fourier method mostly gives better accuracy than the PDE method for a given computational time. This picture changes dramatically when $N = 36$ in which case the Fourier method is much faster and more accurate. One reason for this is that the PDE method has to spend a lot of computational effort on performing continuity updates in addition to solving a large number of one dimensional PDE's.

5.3 Volatility dependence

As a consequence of Proposition 9 and the results of Section 3.1.1, the price of a cliquet with global floor tends to a known limit as the volatility tends to infinity. But before eventually reaching it, the price dependence on the volatility may be very exotic as illustrated by the examples displayed in Figures 5.9 to 5.10. It should be particularly noted that in the example in Figure 5.10, the price first increases and then decreases with volatility.

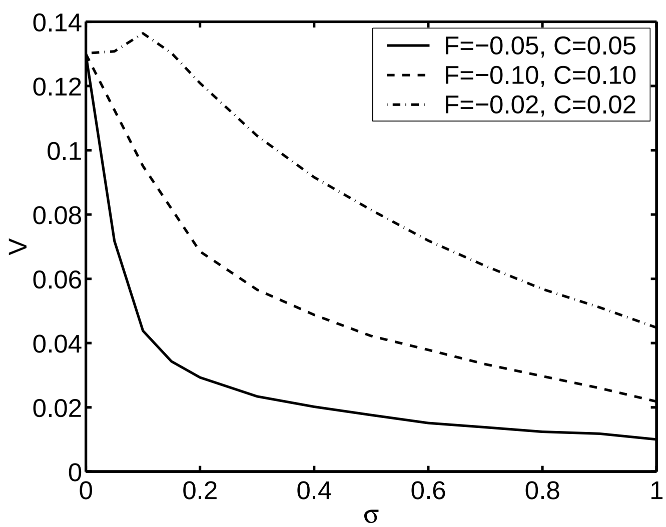

Figure 5.9: Price V as a function of volatility for different symmetric local floors and caps. Parameters: $T = 3$, $N = 12$, $F_g = 0$, $r = 0.05$

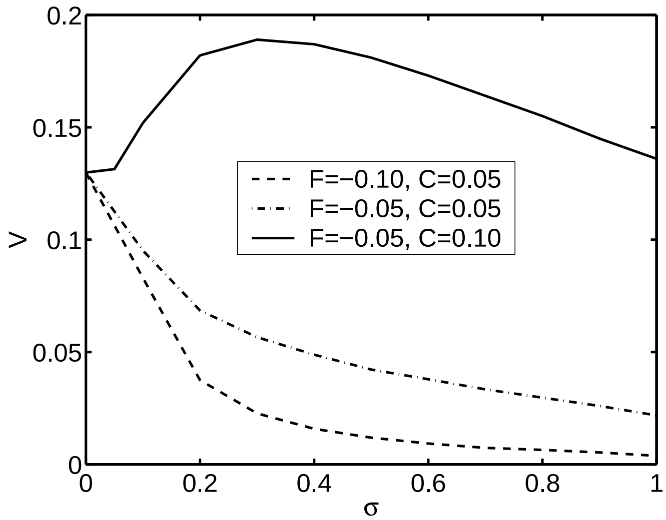

Figure 5.10: Price V as a function of volatility for different symmetric local floors and caps. Parameters: $T = 3$, $N = 12$, $F_g = 0$, $r = 0.05$

5.4 Early exercise

In this section prices of Cliquets 1 to 3 with European, Bermudan and American exercise are given. In the Bermudan case exercise is allowed directly after each reset date. The results are presented in the table below.

Option	σ	European	Bermudan	American
Cliquet 1	0.10	0.1180	0.1180	0.1215
Cliquet 1	0.30	0.0773	0.0793	0.1066
Cliquet 1	0.50	0.0587	0.0669	0.1045
Cliquet 2	0.10	0.0952	0.0952	0.0994
Cliquet 2	0.30	0.0563	0.0583	0.0757
Cliquet 2	0.50	0.0424	0.0482	0.0706
Cliquet 3	0.10	0.0713	0.0713	0.0752
Cliquet 3	0.30	0.0397	0.0411	0.0495
Cliquet 3	0.50	0.0300	0.0338	0.0434

From this we see that the Bermudan style option is not so much more expensive than its European counterpart. The American version could be up to 40% more expensive but this seems to happen for large volatilities where the option price is comparatively low anyway. This means that a Bermuda style cliquet with global floor could become popular since it offers extra flexibility at a low extra cost.

Chapter 6

Discussion

First a discrete dividend yield approximation of the fixed dividend model by Heath and Jarrow ([11]) was introduced in Section 2.3. Numerical testing in Section 5.1 suggests that it gives results very close to those of the Heath-Jarrow model for dividends occurring in the near future. For more distant dividends, the relative difference is mostly less than one percent, which is not a problem considering the fact that dividends occurring half a year or more ahead in time are unknown and thus have to be estimated. In addition to analytical tractability, the model ensures that share prices are positive.

Second, the Fourier and PDE methods were proposed for the computation of the price and greeks. Both methods are much faster than Monte-Carlo simulation for a given level of accuracy. For a moderate number of reset periods (six to twelve) the Fourier method mostly seems to be better than the PDE method if the desired relative error is to be less than 0.2%. If errors up to 1% are acceptable, the PDE method becomes much more competitive. When the number of reset periods are increased to 36, which is common in practice, the Fourier method offers an accuracy several orders of magnitude better than the PDE method.

The PDE-method could be modified to allow early exercise and some numerical simulations in Section 5.4 indicate that at least Bermudan cliquets could become an attractive product.

Chapter 7

Future research and development

In this section we discuss possible improvements of the methods as well as extensions to more sophisticated market models.

The PDE method

The existing implementation could be modified to give better greeks if the number of time steps used in the last reset period is set to be larger than in the other periods. This would not improve the price, but result in better estimation of the partial derivatives.

One way of incorporating smile and skew are the local volatility models presented in Section 2.2.2. Adapting the proof of Proposition 13 to this model gives the Black-Scholes PDE with $\sigma(t)$ replaced with $\sigma(t, s)$. Since explicit knowledge of s is needed, unless the volatility satisfies some homogeneity condition, the dimensionality reduction of Proposition 14 is not possible. Instead, the full three state variables equation of Proposition 13 would have to be solved, possibly with the change of variable $x = \log(s)$. Provided that issues of stability of the finite difference method are resolved, this task should be feasible given the low computational times observed in Section 5.2 for the two dimensional problem.

Finally, the possibility to deal with American styled cliquet options with global floor would need a much deeper treatment. This includes understanding hedging issues and theoretical investigations of free boundary value problems in the context of a set of one dimensional PDE's embedded in a two or three dimensional space.

The Fourier method

Possible immediate improvements of the Fourier method include replacing the trapezoid and Simpson quadrature schemes with some Gauss-Legendere method for the outer integrals with respect to ξ in the formulas for the price and greeks in Propositions 2 to 5. For the evaluation of the characteristic function, some optimal placement of the spline knots could replace the existing uniform spacing. This might reduce the number of spline knots needed.

Requiring independent returns, the Fourier method does not extend to the context of local or stochastic volatility market models.

One framework, in which it does work, is the *Levy-process market models* with constant process coefficients over each reset period. It has the advantage of allowing skew and fat tails, while keeping the independent increments. Reviewing the proof of Proposition 2 suggests that only the distribution function to be splined has to be changed. One potential threat to this approach is a possible absence of fast rational approximation formulas used to compute the distribution function at the spline knots.

As discussed in Section 2.2.2, the choice of measure and hedging strategy is not straightforward. One approach to hedging could be to compute the price and greeks as usual, and then use delta-hedging. This should reduce the risk *somewhat*, since $\Delta V \approx \frac{\partial V}{\partial s}\Delta s$. An interesting topic of research would be the selection of martingale measures and hedging strategies in incomplete markets given different utility functions.

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