PREPRINT

Sign-Graded Posets, Unimodality of
W-Polynomials and the Charney-Davis
Conjecture

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Göteborg, Sweden 2004
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Abstract. We generalize the notion of graded posets to what we call sign-graded (labeled) posets. We prove that the W-polynomial of a sign-graded poset is symmetric and unimodal. This extends a recent result of Reiner and Welker who proved it for graded posets by associating a simplicial polytope to each graded poset. So proving that the W-polynomials of sign-graded posets has the right sign at 1, we are able to prove the Charney-Davis Conjecture for these spheres (whenever they are flag).

1. Introduction and preliminaries

Recently Reiner and Welker [8] proved that the W-polynomial of a graded naturally labeled poset P has unimodal coefficients. They proved this by associating to P a simplicial polytope D_\omega(P), where D_\omega(P) is the W-polynomial of P, and invoking McMullen's g-theorem [11]. Whenever this sphere is flag, i.e., its minimal faces all have cardinality two, they noted that the Neggers-Stanley Conjecture implies the Charney-Davis Conjecture for D_\omega(P). In this paper we give a completely different proof of the unimodality of W-polynomials of graded posets, and we also prove the Charney-Davis Conjecture for D_\omega(P) (whenever they are flag). Our proof is by studying a family of labeled posets, which we call sign-graded posets, of which the class of graded naturally labeled posets is a sub-class.

In this paper all posets will be finite, for undefined terminology on posets we refer the reader to [3]. We denote the cardinality of a poset P with a small letter p. Let P be a poset and let \omega : P \to \{1, 2, \ldots, p\} be a bijection. The pair (P, \omega) is called a labeled poset. If \omega is order preserving then (P, \omega) is said to be naturally labeled. A (P, \omega)-partition is a map \sigma : P \to \{1, 2, 3, \ldots\} such that

- \sigma is order reversing, that is if x \leq y then \sigma(x) \geq \sigma(y);
- if x \leq y and \omega(x) > \omega(y) then \sigma(x) > \sigma(y).

The theory of (P, \omega)-partitions was developed by Stanley in [10]. The number of (P, \omega)-partitions \sigma : P \to \{1, 2, \ldots, n\} is a polynomial of degree \( n \) is called the order polynomial of (P, \omega) and is denoted \#(P, \omega; n). The W-polynomial of (P, \omega) is defined by

\[ W(P, \omega; t) = \sum_{n \geq 0} \#(P, \omega; n) t^n. \]

The Jordan-Hölder set \( L(P, \omega) \) of (P, \omega) is the set of permutations \( \omega(x_1), \omega(x_2), \ldots, \omega(x_p) \) where \( x_1, x_2, \ldots, x_p \) is a linear extension of P. A descent in a permutation \( \pi \) is an index \( 1 \leq i \leq p-1 \) such that \( \pi_i > \pi_{i+1} \). The number of descents of \( \pi \) is denoted \( \text{des}(\pi) \). A result of Stanley’s [10] implies that the W-polynomial can be written as

\[ W(P, \omega; t) = \sum_{\pi \in L(P, \omega)} t^{\text{des}(\pi)}, \]

The Neggers-Stanley Conjecture is the following:

Conjecture 1.1 (Neggers-Stanley). For any labeled poset (P, \omega) the polynomial W(P, \omega; t) has only real zeros.

It was first conjectured by Neggers [6] in 1978 for natural labelings and by Stanley in 1984 for arbitrary labelings. The conjecture has been proved for special cases, see [1, 2, 8, 14] for the state of the art. If a polynomial has only real non-positive zeros then its coefficients form a unimodal sequence. For the W-polynomials of graded posets unimodality was first proved by Gasharov [5] whenever the rank is at most 2, and as mentioned by Reiner and Welker for all graded posets.

For the relevant definitions concerning the topology behind the Charney-Davis Conjecture we refer the reader to [3, 8, 12].

Conjecture 1.2 (Charney-Davis, [3]). Let \( \Delta \) be a flag simplicial sphere (d = 1), sphere, where d is even. Then the h-vector, \( h(\Delta, t) \), of \( \Delta \) satisfies

\[ (-1)^k h(\Delta, -1) \geq 0, \]

where \( h_0 \) is a non-negative integer for all \( i \). This was proved by Forman and Schütte in [4] and combinatorially by Shapiro, Getu and Wean in [6]. From this expansion we immediately that \( h_{2d}(\Delta) \) is symmetric and that the coefficients in the standard basis are unimodal. It also follows that \( (-1)^d h_d(\Delta) \geq 0 \).

We will in Section 2 define a class of labeled posets whose members we call sign-graded posets. This class includes the class of naturally labeled graded posets, in Section 4 we show that the W-polynomial of a sign-graded poset (P, \omega) of rank r can be expanded, just as the Eulerian polynomial as

\[ W(P, \omega; t) = \sum_{i=0}^{r} \binom{r}{i} a_i t^i, \]

where \( a_i \) are non-negative integers. Hence, symmetry and unimodality follow, and W(P, \omega; t) has the right sign at -1. Consequently, whenever the associated sphere D_\omega(P) of a graded poset P is flag the Charney-Davis Conjecture holds for D_\omega(P). We also note that all symmetric polynomials with non-negative zeros only, admits an expansion such as (1.1), Hence, that W(P, \omega; t) has such an expansion can be seen as further evidence for the Neggers-Stanley Conjecture.

In [7] the Charney-Davis quantity of a graded naturally labeled poset (P, \omega) of rank r was defined to be \( \binom{r}{i} a_i t^i W(P, \omega; t) \). In Section 5 we give a combinatorial interpretation of the Charney-Davis quantity as counting certain reverse alternating permutations. Finally in Section 6 we give a characterization of sign-graded posets in terms of properties of order polynomials.
2. Sign-Graded Posets

Let \((P, \omega)\) be a labeled poset and let \(E = E(P) = \{(x, y) \in P \times P : x \prec y\}\) be the covering relations of \(P\). An element \(y\) covers \(x\), written \(x \prec y\), if \(x < y\) and \(x < z < y\) for no \(z \in P\). We associate a labeling \(\epsilon : E \to \{-1, 1\}\) of the Hasse diagram of \(P\) by

\[
\epsilon(x, y) = \begin{cases} 
1 & \text{if } \omega(x) < \omega(y), \\
-1 & \text{if } \omega(x) > \omega(y).
\end{cases}
\]

Note that the definition of a \((P, \omega)\)-partition only depends on the function \(\epsilon\), in what follows we will often refer to \(\epsilon\) as the labeling and write \(\Omega(P, \epsilon; t)\).

**Definition 2.1.** Let \(\epsilon : E \to \{-1, 1\}\) be a labeling of \(E\). We say that \(P\) is sign-graded with respect to \(\epsilon\) (or \(\epsilon\)-graded for short) if for every maximal chain \(x_0 < x_1 < \cdots < x_n\), the sum

\[
\epsilon(x_i, x_{i+1})
\]

is the same. The common value, \(r(\epsilon)\), of the above sum is called the rank of \(\epsilon\). The rank function, \(\rho : P \to \mathbb{Z}\), is defined by

\[
\rho(x) = \sum_{i=1}^{n} \epsilon(x_i, x_{i+1}),
\]

where \(x_0 < x_1 < \cdots < x_n = x\) is any maximal chain from a minimal element to \(x\).

See Fig. 1 for an example of a sign-graded poset. Note that if \(\epsilon\) is identically equal to 1, then a sign-graded poset with respect to \(\epsilon\) is just a graded poset. Note also that if \(P\) is \(\bar{\epsilon}\)-graded then \(P\) is also \(\epsilon\)-graded, where \(\bar{\epsilon}\) is defined by \(\bar{\epsilon}(x, y) = -\epsilon(x, y)\). It may come as a surprise to the reader that when it comes to order polynomials of sign-graded posets, the specific labeling does not matter:

**Theorem 2.2.** Let \(P\) be \(\epsilon\)-graded and \(\bar{\epsilon}\)-graded. Then

\[
\Omega(P, \epsilon; t) = \Omega(P, \bar{\epsilon}; t - r(\epsilon)) = \Omega(P, \bar{\epsilon}; t - r(\bar{\epsilon})),
\]

and proves the theorem.

**Theorem 2.3.** Let \(P\) be \(\epsilon\)-graded. Then

\[
\Omega(P, \epsilon; t) = (-1)^{r(\epsilon)} \Omega(P, \epsilon; t - r(\epsilon)),
\]

**Proof.** We have the following reciprocity for order polynomials, see [10]:

\[
\Omega(P, \epsilon; t) = (-1)^{r(\epsilon)} \Omega(P, \epsilon; t - r(\epsilon)),
\]

Note that \(r(\epsilon) = r(\bar{\epsilon})\), so by Theorem 2.2 we have:

\[
\Omega(P, \epsilon; t) = \Omega(P, \epsilon; t - r(\epsilon)),
\]

which, combined with (2.1), gives the desired result.

**Corollary 2.4.** Let \(P\) be an \(\epsilon\)-graded poset. Then \(W(P, \epsilon; t)\) is symmetric with center of symmetry \(p = r(\epsilon) + 1/2\). If \(P\) is also \(\bar{\epsilon}\)-graded then

\[
W(P, \bar{\epsilon}; t) = W(P, \epsilon; (t - r(\epsilon))/2).\]
Proof. It is known, see [10], that if \( W(P, e, t) = \sum_{t \geq b} w_t(P, e) t^b \) then \( \Omega(P, e, t) = \sum_{t \geq b} w_t(P, e) \left( \frac{t + p - 1}{p} \right) \).

Let \( r = r(e) \), Theorem 2.3 gives:

\[
\Omega(P, e, t) = \sum_{t \geq b} w_t(P, e) \left( \frac{t - p + 1}{t} \right)
- \sum_{t \geq b} w_t(P, e) \left( \frac{t + p - 1}{p} \right)
- \sum_{t \geq b} w_t(P, e) \left( \frac{t + p - 1}{t} \right),
\]

so \( w_t(P, e) = w_t r + i(P, e) \) for all \( i \), and the symmetry follows. The relationship between the \( W \)-polynomials of \( \epsilon \) and \( \mu \) follows from Theorem 2.2 and the expansion of \( \epsilon \)-polynomials in the basis \( \left( \frac{t - p + 1}{t} \right) \).

The following theorem tells us that the class of \( \epsilon \)-graded posets is considerably greater than the class of \( \epsilon \)-graded posets.

Theorem 2.5. Let \( P \) be a finite poset. Then there exists a labeling \( \epsilon : E \rightarrow \{0, 1\} \) such that \( (P, \epsilon) \) is \( \epsilon \)-graded if and only if all maximal chains in \( P \) have the same parity (cardinality modulo 2).

Moreover, the labeling \( \epsilon \) can be chosen so that the corresponding rank function has values in \( \{0, 1\} \).

Proof. It is clear that if \( P \) is \( \epsilon \)-graded then all maximal chains have the same parity. Let \( P \) be a poset whose maximal chains have the same parity. Then, for any \( e \in E \), all saturated chains starting at a minimal element and ending at \( e \) have the same length modulo 2. Hence, we may define a labeling \( \epsilon : E \rightarrow \{0, 1\} \) by \( \epsilon(x, y) = \left( \frac{|x|}{2} \right) \), where \( |x| \) is the length of any saturated chain starting at a minimal element and ending at \( x \). It follows that \( P \) is \( \epsilon \)-graded and that its rank function has values in \( \{0, 1\} \).

We say that \( \omega : P \rightarrow \{1, 2, \ldots, p\} \) is a canonical labeling if \( (P, \omega) \) has a rank function \( \rho \) with values in \( \{0, 1\} \), and \( \rho(x) < \rho(y) \) implies \( \omega(x) < \omega(y) \). By Theorem 2.5, we know that \( P \) admits a canonical labeling if \( P \) is \( \epsilon \)-graded with respect to some \( \epsilon \).

3. THE JORDAN-BÖHLER SET OF A \( \epsilon \)-GRATED POSET

Let \( (P, \omega) \) be \( \epsilon \)-graded. We may assume that \( \omega(x) < \omega(y) \) whenever \( x < y \) (\( x \) is incomparable to \( y \), assume that \( x, y \in P \) are incomparable and that \( \rho(y) = \rho(x) + 1 \). Then the Jordan-Bölder set of \( (P, \omega) \) can be partitioned into two sets: One where in all permutations \( \omega(x) \) comes before \( \omega(y) \) and one where \( \omega(y) \) comes before \( \omega(x) \). This means that

\[
E(P, \omega) = \{ x < y \} \cup \{ x < y \},
\]

where \( P \) is the transitive closure of \( E \cup \{ x < y \} \), and \( P^\epsilon \) is the transitive closure of \( E \cup \{ y < x \} \).

Lemma 3.1. With definitions as above \((P, \omega)\) and \((P^\epsilon, \omega)\) are \( \epsilon \)-graded with the same rank function as that for \((P, \omega)\).

Proof. Let \( C : x_0 < x_1 < \cdots < x_k = z \) be a saturated chain in \( P^\epsilon \), where \( x_0 \) is a minimal element in \( P^\epsilon \). Of course \( x_0 \) is also a minimal element in \( P \). We have to prove that

\[
\rho(z) = \sum_{i=0}^k \rho(x_i, x_{i+1}),
\]

where \( \epsilon \) is the "value"-labeling of \( P^\epsilon \) and \( \rho \) is the rank function of \( (P, \omega) \).

All covering relations in \( P^\epsilon \), except \( x < y \), are also covering relations in \( P \). Note that \( \rho(x, y) = 1 \). If \( y \) and \( x \) do not appear in \( C \), then \( C \) is a saturated chain in \( P \) and we have nothing to prove. Otherwise

\[
C : y_0 < \cdots < y_k = x < z,
\]

where \( y_0 < \cdots < y_k = x = x_1 < x_2 < \cdots < x_k = z \). Note that if \( y_0 < x_1 < \cdots < y_k \) is any saturated chain in \( P \) then \( \sum_{i=0}^k \rho(y_i, y_{i+1}) = \rho(y_0) = \rho(y_k) \). Since \( y_0 < \cdots < y_k = y \) and \( x = x_1 < x_2 < \cdots < x_k = z \) are saturated chains in \( P \) we have

\[
\sum_{i=0}^k \rho(x_i, x_{i+1}) = \rho(z) + \rho(z) = \rho(z) + \rho(z) = \rho(z),
\]

as was to be proved. The statement for \((P^\epsilon, \omega)\) follows similarly.

We say that a \( \epsilon \)-graded poset \((P, \omega)\) is \( \epsilon \)-graded if for all \( x, y \in P \) we have that \( x < y \) are comparable whenever \( \rho(x) = \rho(y) \), and \( P \) and \( Q \) be posets on the same set. Then \( Q \) extends \( P \) if \( x < y \) whenever \( x \in P \).
E(P ⊗ Q) by

\((e ⊗ 1)μ(x, y) = e(x, y)\) if \((x, y) ∈ E(P)\),

\((e ⊗ 1)μ(x, y) = μ(x, y)\) if \((x, y) ∈ E(Q)\) and

\((e ⊗ 1)μ(x, y) = 1\) otherwise,

\((e ⊗ 1)μ(x, y) = e(x, y)\) if \((x, y) ∈ E(P)\),

\((e ⊗ 1)μ(x, y) = μ(x, y)\) if \((x, y) ∈ E(Q)\) and

\((e ⊗ 1)μ(x, y) = 1\) otherwise.

With a slight abuse of notation we write \(P \oplus Q\) when the labellings of \(P\) and \(Q\) are understood from the context. Note that ordinal sums are associative, i.e.,

\((P \oplus Q) \oplus R = P \oplus (Q \oplus R)\), and preserve the property of being sign-
good. The following result is obtained easily by combinatorial reasoning, see [2, 14]:

**Proposition 3.3.** Let \((P, ω)\) and \((Q, v)\) be two labeled posets, Then

\[ W(P ⊗ Q, ω ⊗ v; t) = W(P, ω t)W(Q, v t) \]

and

\[ W(P \oplus Q, ω ⊗ v; t) = t W(P, ω; t)W(Q, v; t) \]

**Proposition 3.4.** Suppose that \((P, ω)\) is a naturally ordered canonically signed graded poset, Then \((P, ω)\) is the direct sum

\[ (P, ω) = A_0 \oplus A_1 \oplus A_2 \oplus A_3 \oplus \cdots \oplus A_k \]

where the \(A_i\)s are antichains.

Proof. Let \(π ∈ J(P, ω)\), then we may write \(π = ω(x_0, \ldots, x_k)\), where the \(x_i\)s are maximal words with respect to the property: If \(a \preceq b\) then \(ρ(ω(−1)(a)) = ρ(ω(−1)(b))\), \(π ∈ J(Q, ω)\)

\[ (Q, ω) = A_0 \oplus A_1 \oplus A_2 \oplus A_3 \oplus \cdots \oplus A_k \]

and \(A_0\) is the antichain consisting of the elements \(ω(−1)(a)\) where \(a\) is a letter of \(ω(−1)(a)\) \((A_0\) is an antichain, since if \(x < y\) where \(x, y \in A_0\) there would be a letter in \(π\) between \(ω(x)\) and \(ω(y)\) whose rank was different than that of \(x, y\)). Now, \((ω, ω)\)

is symmetric, so \(P = Q\). □

Note that the argument in the above proof also can be used to give a simple proof of Corollary 3.2 when \(ω\) is canonical. However, we wanted to prove Corollary 3.2 in its generality even though we only need it for canonical labellings.

4. The \(W\)-Polynomial of a Sign-Graded Poset

The space, \(S_0^d\), of symmetric polynomials in \(S(t)\) with center of symmetry \(d/2\) has a basis

\[ B_0 = \{ t^{(1 + t)d/2} \} \]

If \(h ∈ S_0^d\) has non-negative coefficients in this basis it follows immediately that the coefficients of \(h\) in the standard basis are unimodal. Let \(S_0^d\) be the nonnegative span of \(B_0\). Thus \(S_0^d\) is a cone. Another property of \(S_0^d\) is that if \(h ∈ S_0^n\) then it has the correct sign at \(1\), i.e.,

\[ (-1)^d/2 h(−1) ≥ 0, \]

**Lemma 4.1.** Let \(a, d ∈ N\). Then

\[ S_0^d ⊆ S_{d+1}^d \]

\[ S_0^d ⊆ S_{d+1}^d \]

Suppose further that \(h ∈ S_0^d\) has positive leading coefficient and that all zeros of \(h\) are real and non-positive, Then \(h ∈ S_0^d\).

Proof. The inclinations are obvious, since \(t ∈ S_0^d\) and \((1 + t) ∈ S_0^d\) we may assume that none of them divides \(h\), but then we may collect the zeros of \(h\) in pairs \(2\) and \(θ^0\),\(\theta^1\), Let \(A_0 = 2θ = \theta^0\),\(\theta^1\), then

\[ h = C \prod_{θ^0 \neq 1} \left( t^2 + A_0 t + 1 \right), \]

where \(C > 0\), since \(A_0 > 2\) we have

\[ t^2 + A_0 t + 1 = (t + 1)^2 + (A_0 - 2) t ∈ S_0^1. \]

and the lemma follows. □

We can now prove our main theorem,

**Theorem 4.2.** Suppose that \((P, ω)\) is a signed graded poset of rank \(r\), Then \(W(P, ω; t) ∈ S_0^r \).\(\tau \geq 0\),

Proof. By Corollary 2.4 and Lemma 2.5 we may assume that \((P, ω)\) is canonically labeled. By Corollary 3.2 we know that

\[ W(P, ω; t) = \sum_{Q} W(Q, ω; t) \]

where \((Q, ω)\) are saturated and signed graded with the same rank function as that of \((P, ω)\). The W-polynomials of antichains are the Eulerian polynomials, which only have real non-negative zeros. By Proposition 3.4 and Proposition 3.3 the polynomial \(W(Q, ω; t)\) has only non-negative zeros by Lemma 4.1 and Corollary 2.4 we have \(W(Q, ω; t) ∈ S_0^r \).\(\tau \geq 0\). The theorem now follows since \(S_0^r \).\(\tau \geq 0\) is a cone.

**Corollary 4.3.** Let \((P, ω)\) be a signed graded of rank \(r\) then \(W(P, ω; t)\) is symmetric and its coefficients are unimodal. Moreover, \(W(P, ω; t)\) has the correct sign at \(1\), i.e.,

\[ (-1)^{r/2} W(P, ω; t) ≥ 0, \]

**Corollary 4.4.** Let \(P\) be a (naturally labeled) graded poset. Suppose that \(Δ_D(P)\) is flat. Then the Chevalley-Eilenberg Conjecture holds for \(Δ_D(P)\).

If \(h(0)\) is any polynomial with integer coefficients and \(h(0) ∈ S_0^d\), it follows that \(h(0)\) has integer coefficients in the basis \(t^{(1 + t)d/2}\). Thus we know that if \((P, ω)\) is signed graded of rank \(r\), then

\[ W(P, ω; t) = \sum_{i=0}^{\lfloor r/2 \rfloor} a_i(P, ω) t^{(1 + t)^i} r \geq 0, \]

where \(a_i(P, ω)\) are nonnegative integers. It would be interesting to have a combinatorial interpretation of these coefficients, and thus a combinatorial proof of Theorem 4.2.
Let \((P, e)\) be a labeled poset. We say that \((P, e)\) admits a rank function if for every \(x \in P\) and saturated chain \(x_0 \prec x_1 \prec \ldots \prec x_k = x\), where \(x_0\) is a minimal element, the quantity
\[
p(x) = \sum_{i=1}^{k} e(x_i, x_{i-1})
\]
is the same. Hence, a labeled poset \((P, e)\) with a rank function is sign-graded if and only if \(p\) is constant on maximal elements.

**Theorem 4.5.** Suppose that \((P, e)\) admits a rank function with values in \([0, 1]\). Then \(W(P, e; t)\) has unimodal coefficients, proof. One may check that the proof of Lemma 3.1, Corollary 3.2 and Proposition 3.4 holds for this case too. Then
\[
W(P, e; t) = \sum_{Q} W(Q, e; t)
\]
where \(W(Q, e; t)\) is unimodal and symmetric with center of symmetry \((p = 1)/2\) or \((p = 2)/2\). The sum of such polynomials is again unimodal.

5. THE CHARNEY-DAVIS QUANTITY

In [7] Reiner, Stanton and Weigel defined the Charney-Davis quantity of a graded naturally labeled poset \((P, \omega)\) of rank \(r\) to be
\[
CD(P, \omega) = (\omega(1) + r)/2 W(P, \omega; 1).
\]
We may define it in the exact same way for sign-graded posets. Since the particular labeling does not matter we write \(CD(P)\). Let \(\pi = \pi_{1} \prec \pi_{2} \prec \cdots \pi_{r}\) be any permutation. We say that \(\pi\) is alternating if \(\pi_{1} \prec \pi_{2} \prec \cdots \) and reverse alternating if \(\pi_{1} \prec \pi_{2} \succ \cdots \). Let \((P, \omega)\) be a canonically labeled sign-graded poset. If \(\pi \in L(P, \omega)\) then we may write \(\pi = \pi_{1} \omega_{1} \cdots \omega_{u}\) where \(\omega_{1}\) are maximal words with respect to the property: If \(a\) and \(b\) are letters of \(\omega_{1}\) then \(p(\omega_{1}^{-1}(a)) = p(\omega_{1}^{-1}(b))\). The words \(\omega_{1}\) are the components of \(\pi\). The following theorem is well known, see for example [9], and gives the Charney-Davis quantity of an antichain.

**Proposition 5.1.** Let \(n \geq 0\) be an integer. Then \((\omega(1)^{n} + 1) A_{n}(\omega)\) is equal to 0 if \(n\) is even and equal to the number of (reverse) alternating permutations of the set \(\{1, 2, \ldots, n\}\) if \(n\) is odd.

**Theorem 5.2.** Let \((P, \omega)\) be a canonically labeled sign-graded poset. Then the Charney-Davis quantity, \(CD(P)\), is equal to the number of reverse alternating permutations in \(L(P, \omega)\) such that all components have odd numbers of letters.

**Proof.** It suffices to prove the theorem when \((P, \omega)\) is saturated. By Proposition 3.4 we know that
\[
(P, \omega) = A_{0} \ominus A_{1} \oplus A_{2} \ominus A_{3} \oplus \cdots \ominus A_{k} \oplus A_{k+1},
\]
where the \(A_{k}\) are antichains. This means that \(CD(P) = CD(A_{0}) CD(A_{1}) \cdots CD(A_{k})\). Let \(\pi = \omega_{1} \omega_{2} \cdots \omega_{u} \in L(P, \omega)\) where \(\omega_{1}\) is a permutation of \(A_{0}\). Then \(\pi\) is a reverse alternating such that all components have odd numbers of letters if and only if, for all \(i\), \(\omega_{i}\) is reverse alternating if \(i\) is even and alternating if \(i\) is odd. Hence, by Proposition 5.1, the number of such permutations is indeed \(CD(A_{0}) CD(A_{1}) \cdots CD(A_{k})\).

6. A CHARACTERIZATION OF SIGN-GRADED POSETS

Here we give a characterization of sign-graded posets along the lines of the characterization of graded poset given by Stanley in [10]. Let \((P, e)\) be any labeled poset. Define a function \(\delta = \Phi : P \rightarrow \mathbb{Z}\) by
\[
\delta(x) = \max \left\{ \sum_{i=1}^{k} e(x_i, x_{i-1}) \right\},
\]
where \(x = x_0 \prec x_1 \prec \cdots \prec x_k\) is any saturated chain starting at \(x_0\) and ending at a maximal element \(x_k\). Define a map \(\Phi = \Phi_{\omega} : A(\omega) \rightarrow \mathbb{Z}^P\) by
\[
\Phi_{\omega} = \sigma + \delta,
\]
where we have
\[
\delta(x) \geq \delta(y) + e(x, y),
\]
(6.1)
This means that \(\Phi_{\omega}(x) > \Phi_{\omega}(y)\) if \(e(x, y) = 0\) and \(\Phi_{\omega}(x) \geq \Phi_{\omega}(y)\) if \(e(x, y) > 0\).

**Theorem 6.1.** Let \((P, e)\) be a labeled poset. Then \(\Phi_{\omega} : A(\omega) \rightarrow A(\omega)\) is a bijection if and only if \(\sigma\) satisfies the \(\delta\)-chain condition.

**Proof.** If \(e\) satisfies the \(\delta\)-chain condition, then so does \(\sigma\) and \(\delta_{\omega}(x) = \delta_{\omega}(x)\) for all \(x \in P\). Thus the map is a bijection since the inverse of \(\Phi_{\omega}\) is \(\Phi_{\omega}^{-1}\).

For the only if direction note that \(\delta\) satisfies the \(\delta\)-chain condition if and only if for all \((x, y) \in E\) we have
\[
\delta(x) = \delta(y) + e(x, y).
\]
If \(e\) fails to satisfy the \(\delta\)-chain property we have, by (6.1), that there is a covering relation \((x, y) \in E\) such that \(\sigma(x, y) = 1\) and \(\delta(x) \geq \delta(y) + 2\) or \(e(x, y) = 1\) and \(\sigma(x, y) = 0\).

Suppose that \(e(x, y) = 1\). It is clear that there is a \(\sigma \in A(\omega)\) such that \(\sigma(x) = \sigma(y) + 1\). But then
\[
\sigma(x) = \delta(x) \leq \sigma(y) + \delta(y) = 1,
\]
so \(\sigma \notin A(\omega)\).

Similarly, if \(e(x, y) = 1\) then we can find a partition \(\sigma \in A(\omega)\) with \(\sigma(x) = \sigma(y)\), and then
\[
\sigma(x) \leq \delta(x) \leq \sigma(y) + \delta(y),
\]
so \(\sigma \notin A(\omega)\).
Define \( r(e) \) by
\[
    r(e) = \max \left\{ \sum_{i=1}^t e(x_i, x_{i+1}) : x_0 < x_1 < \cdots < x_t \text{ is maximal} \right\}.
\]
We then have:
\[
    \max \{ \Phi \sigma(x) : x \in P \} = \max \{ \sigma(x) + \delta_0(x) : x \text{ is minimal} \} \\
    \leq \max \{ \sigma(x) : x \in P \} + r(e),
\]
So if we let \( A_0(e) \) be the \((P,e)\)-partitions with largest part at \( n \) we have that \( \Phi : A_0(e) \to A_{n+1,e}(\omega) \) is an injection, A labeling \( e \) of \( P \) is said to satisfy the \( \lambda \)-chain condition if for every \( x \in P \) there is a maximal chain \( c : x_0 < x_1 < \cdots < x_t \) containing \( x \) such that \( \sum_{i=1}^t e(x_i, x_{i+1}) = r(e) \).

**Lemma 6.2.** Suppose that \( n \) is a non-negative integer such that \( \Omega(P, e; n) \neq 0 \). If \( \Omega(P, e; n + r(e)) = \Omega(P, e; n) \)

then \( e \) satisfies the \( \lambda \)-chain condition.

**Proof.** Define \( \delta^* : P \to Z \) by
\[
    \delta^*(x) = \max \left\{ e(x_i, x_{i+1}) : i = 1, \ldots, t \right\},
\]
where the maximum is taken over all maximal chains starting at a minimal element and ending at \( x \). Then
\[
    \delta(x) + \delta^*(x) \leq r(e) \quad (6.2)
\]
for all \( x \), and \( e \) satisfies the \( \lambda \)-chain condition if and only if we have equality in (6.2) for all \( x \in P \). It is easy to see that the map \( \Phi^* : A_0(e) \to A_{n+1,e}(\omega) \) defined by
\[
    \Phi^* \sigma(x) = \sigma(x) + r(e) = \delta^*(x),
\]
is well defined and is an injection. By (6.2) we have \( \Phi \sigma(x) \leq \Phi^* \sigma(x) \) for all \( \sigma \) and all \( x \in P \), with equality if and only if \( x \) is in a maximal chain of maximal weight. This means that in order for \( \Phi : A_0(e) \to A_{n+1,e}(\omega) \) to be a bijection it is necessary for \( e \) to satisfy the \( \lambda \)-chain condition. \( \square \)

**Theorem 6.3.** Let \( e \) be a labeling of \( P \). Then
\[
    \Omega(P, e; t) = \left( \prod t \right)^{\Omega(P, e; t - r(e))}
\]
if and only if \( P \) is \( e \)-graded if \( \omega \) is a root of \( r(e) \).

**Proof.** The \( \Theta \)-part is Theorem 2.3, so suppose that the equality of the theorem holds. By reciprocity we have
\[
    \left( -1 \right)^{\theta} \Omega(P, e; t - r(e)) = \Omega(P, e; t + r(e)),
\]
and since \( \Phi : A_0(e) \to A_{n+1,e}(\omega) \) is an injection it is also a bijection. By Proposition 6.1, \( e \) satisfies the \( \delta \)-chain condition, and, by Lemma 6.2, we have that all minimal elements are members of maximal chains of maximal weight. In other words \( P \) is \( e \)-graded. \( \square \)