CHALMERS | GÖTEBORGS UNIVERSITET

*PREPRINT*

# Sign-Graded Posets, Unimodality of $W$-Polynomials and the Charney-Davis Conjecture

**PETTER BRÄNDÉN**

*Department of Mathematics*
**CHALMERS UNIVERSITY OF TECHNOLOGY**
**GÖTEBORG UNIVERSITY**
Göteborg Sweden 2004

Preprint 2004:34

# Sign-Graded Posets, Unimodality of $W$-Polynomials and the Charney-Davis Conjecture

**Petter Brändén**

CHALMERS | GÖTEBORGS UNIVERSITET

Mathematics
Department of Mathematics
Chalmers University of Technology and Göteborg University
SE-412 96 Göteborg, Sweden
Göteborg, June 2004

NO 2004:34
ISSN 0347-2809

Matematiska Vetenskaper
Göteborg 2004

# SIGN-GRADED POSETS, UNIMODALITY OF $W$-POLYNOMIALS AND THE CHARNEY-DAVIS CONJECTURE

PETTER BRÄNDÉN

ABSTRACT. We generalize the notion of graded posets to what we call sign-graded (labeled) posets. We prove that the $W$-polynomial of a sign-graded poset is symmetric and unimodal. This extends a recent result of Reiner and Welker who proved it for graded posets by associating a simplicial polytopal sphere to each graded poset $P$. By proving that the $W$-polynomials of sign-graded posets has the right sign at $-1$, we are able to prove the Charney-Davis Conjecture for these spheres (whenever they are flag).

## 1. INTRODUCTION AND PRELIMINARIES

Recently Reiner and Welker [8] proved that the $W$-polynomial of a graded naturally labeled poset $P$ has unimodal coefficients. They proved this by associating to $P$ a simplicial polytopal sphere, $\Delta_{eq}(P)$, whose $h$-polynomial is the $W$-polynomial of $P$, and invoking McMullen's $g$-theorem [11]. Whenever this sphere is flag, i.e., its minimal non-faces all have cardinality two, they noted that the Neggers-Stanley Conjecture implies the Charney-Davis Conjecture for $\Delta_{eq}(P)$. In this paper we give a completely different proof of the unimodality of $W$-polynomials of graded posets, and we also prove the Charney-Davis Conjecture for $\Delta_{eq}(P)$ (whenever they are flag). Our proof is by studying a family of labeled posets, which we call sign-graded posets, of which the class of graded naturally labeled posets is a sub-class.

In this paper all posets will be finite. For undefined terminology on posets we refer the reader to [13]. We denote the cardinality of a poset $P$ with a small letter $p$. Let $P$ be a poset and let $\omega : P \to \{1, 2, \dots, p\}$ be a bijection. The pair $(P, \omega)$ is called a *labeled poset*. If $\omega$ is order-preserving then $(P, \omega)$ is said to be *naturally labeled*. A $(P, \omega)$-partition is a map $\sigma : P \to \{1, 2, 3, \dots\}$ such that

- $\sigma$ is order reversing, that is, if $x \leq y$ then $\sigma(x) \geq \sigma(y)$,
- if $x < y$ and $\omega(x) > \omega(y)$ then $\sigma(x) > \sigma(y)$.

The theory of $(P, \omega)$-partitions was developed by Stanley in [10]. The number of $(P, \omega)$-partitions $\sigma : P \to \{1, 2, \dots, n\}$ is a polynomial of degree $p$ in $n$ called the *order polynomial* of $(P, \omega)$ and is denoted $\Omega(P, \omega; n)$. The $W$-polynomial of $(P, \omega)$ is defined by

$$\sum_{n \geq 0} \Omega(P, \omega; n) t^n = \frac{tW(P, \omega; t)}{(1-t)^{p+1}}.$$

The *Jordan-Hölder set*, $\mathcal{L}(P, \omega)$, of $(P, \omega)$ is the set of permutations $\omega(x_1), \omega(x_2), \dots, \omega(x_p)$ where $x_1, x_2, \dots, x_p$ is a linear extension of $P$. A *descent* in a permutation $\pi =$

Part of this work was financed by the EC's IHRP Programme, within the Research Training Network "Algebraic Combinatorics in Europe", grant HPRN-CT-2001-00272, while the author was at Universitá di Roma "Tor Vergata", Rome, Italy.



$\pi_1 \pi_2 \cdots \pi_p$ is an index $1 \leq i \leq p-1$ such that $\pi_i > \pi_{i+1}$. The number of descents of $\pi$ is denoted $\text{des}(\pi)$. A result of Stanley's [10] implies that the $W$-polynomial can be written as

$$W(P, \omega; t) = \sum_{\pi \in \mathcal{L}(P, \omega)} t^{\text{des}(\pi)},$$

The Neggers-Stanley Conjecture is the following:

**Conjecture 1.1** (Neggers-Stanley). *For any labeled poset $(P, \omega)$ the polynomial $W(P, \omega; t)$ has only real zeros.*

It was first conjectured by Neggers [6] in 1978 for natural labelings and by Stanley in 1986 for arbitrary labelings. The conjecture has been proved for special cases, see [1, 2, 8, 14] for the state of the art. If a polynomial has only real non-positive zeros then its coefficients form a unimodal sequence. For the $W$-polynomials of graded posets unimodality was first proved by Gasharov [5] whenever the rank is at most 2, and as mentioned by Reiner and Welker for all graded posets.

For the relevant definitions concerning the topology behind the Charney-Davis Conjecture we refer the reader to [3, 8, 12].

**Conjecture 1.2** (Charney-Davis, [3]). *Let $\Delta$ be a flag simplicial homology $(d-1)$-sphere, where $d$ is even. Then the $h$-vector, $h(\Delta, t)$, of $\Delta$ satisfies*

$$(-1)^{d/2} h(\Delta, -1) \geq 0.$$

Recall that the $n$th *Eulerian polynomial*, $A_n(x)$, is the $W$-polynomial of an anti-chain of $n$ elements. The Eulerian polynomials can be written as

$$A_n(x) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} a_{n,i} x^i (1+x)^{n-1-2i},$$

where $a_{n,i}$ is a non-negative integer for all $i$. This was proved by Foata and Schützenberger in [4] and combinatorially by Shapiro, Getu and Woan in [9]. From this expansion we see immediately that $A_n(x)$ is symmetric and that the coefficients in the standard basis are unimodal. It also follows that $(-1)^{(n-1)/2} A_n(-1) \geq 0$.

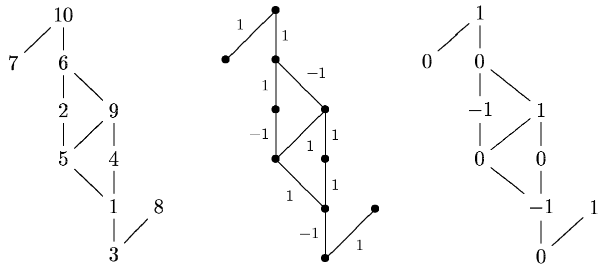
We will in Section 2 define a class of labeled poset whose members we call sign-graded posets. This class includes the class of naturally labeled graded posets. In Section 4 we show that the $W$-polynomial of a sign-graded poset $(P, \omega)$ of rank $r$ can be expanded, just as the Eulerian polynomial, as

$$W(P, \omega; t) = \sum_{i=0}^{\lfloor (p-r-1)/2 \rfloor} a_i(P, \omega) t^i (1+t)^{p-r-1-2i}, \qquad (1.1)$$

where $a_i(P, \omega)$ are non-negative integers. Hence, symmetry and unimodality follow, and $W(P, \omega; t)$ has the right sign at $-1$. Consequently, whenever the associated sphere $\Delta_{eq}(P)$ of a graded poset $P$ is flag the Charney-Davis Conjecture holds for $\Delta_{eq}(P)$. We also note that all symmetric polynomials with non-positive zeros only, admits an expansion such as (1.1). Hence, that $W(P, \omega; t)$ has such an expansion can be seen as further evidence for the Neggers-Stanley Conjecture.

In [7] the Charney-Davis quantity of a graded naturally labeled poset $(P, \omega)$ of rank $r$ was defined to be $(-1)^{(p-1-r)/2} W(P, \omega; -1)$. In Section 5 we give a combinatorial interpretation of the Charney-Davis quantity as counting certain reverse alternating permutations. Finally in Section 6 we give a characterization of sign-graded posets in terms of properties of order polynomials.

FIGURE 1. A sign-graded poset, its two labelings and the corresponding rank function.




## 2. SIGN-GRADED POSETS

Let $(P, \omega)$ be a labeled poset and let $E = E(P) = \{(x, y) \in P \times P : x \prec y\}$ be the covering relations of $P$. An element $y$ covers $x$, written $x \prec y$, if $x < y$ and $x < z < y$ for no $z \in P$. We associate a labeling $\epsilon : E \to \{-1, 1\}$ of the Hasse-diagram of $P$ by

$$\epsilon(x, y) = \begin{cases} 1 \text{ if } & \omega(x) < \omega(y), \\ -1 \text{ if } & \omega(x) > \omega(y). \end{cases}$$

Note that the definition of a $(P, \omega)$-partition only depends on the function $\epsilon$. In what follows we will often refer to $\epsilon$ as the labeling and write $\Omega(P, \epsilon; t)$.

**Definition 2.1.** Let $\epsilon : E \to \{-1, 1\}$ be a labeling of $E$. We say that $P$ is *sign-graded with respect to* $\epsilon$ (or $\epsilon$-graded for short) if for every maximal chain $x_0 \prec x_1 \prec \cdots \prec x_n$ the sum

$$\sum_{i=1}^{n} \epsilon(x_{i-1}, x_i)$$

is the same. The common value, $r(\epsilon)$, of the above sum is called the *rank* of $\epsilon$. The *rank function*, $\rho : P \to \mathbb{Z}$ is defined by

$$\rho(x) = \sum_{i=1}^{m} \epsilon(x_{i-1}, x_i),$$

where $x_0 \prec x_1 \prec \cdots \prec x_m = x$ is any saturated chain from a minimal element to $x$.

See Fig. 1 for an example of a sign-graded poset. Note that if $\epsilon$ is identically equal to 1, then a sign-graded poset with respect to $\epsilon$ is just a graded poset. Note also that if $P$ is $\epsilon$-graded then $P$ is also $-\epsilon$-graded, where $-\epsilon$ is defined by $(-\epsilon)(x, y) = -\epsilon(x, y)$. It may come as a surprise to the reader that when it comes to order-polynomials of sign-graded posets, the specific labeling does not matter:

**Theorem 2.2.** *Let $P$ be $\epsilon$-graded and $\mu$-graded. Then*

$$\Omega(P, \epsilon; t - \frac{r(\epsilon)}{2}) = \Omega(P, \mu; t - \frac{r(\mu)}{2}).$$



*Proof.* Let $\rho_\epsilon$ and $\rho_\mu$ denote the rank functions of $(P, \epsilon)$ and $(P, \mu)$ respectively, and let $\mathcal{A}(\epsilon)$ denote the set of $(P, \epsilon)$-partitions. Define a function $\xi : \mathcal{A}(\epsilon) \to \mathbb{Q}^P$ by $\xi\sigma(x) = \sigma(x) + \Delta(x)$, where

$$\Delta(x) = \frac{r(\epsilon) - \rho_\epsilon(x)}{2} - \frac{r(\mu) - \rho_\mu(x)}{2}.$$

The four possible combinations of labelings of a covering-relation $(x, y) \in E$ are

TABLE 1

| $\epsilon(x,y)$ | $\mu(x,y)$ | $\sigma$ | $\Delta$ | $\xi\sigma$ |
|---|---|---|---|---|
| 1 | 1 | $\sigma(x) \geq \sigma(y)$ | $\Delta(x) = \Delta(y)$ | $\xi\sigma(x) \geq \xi\sigma(y)$ |
| 1 | −1 | $\sigma(x) \geq \sigma(y)$ | $\Delta(x) = \Delta(y) + 1$ | $\xi\sigma(x) > \xi\sigma(y)$ |
| −1 | 1 | $\sigma(x) > \sigma(y)$ | $\Delta(x) = \Delta(y) - 1$ | $\xi\sigma(x) \geq \xi\sigma(y)$ |
| −1 | −1 | $\sigma(x) > \sigma(y)$ | $\Delta(x) = \Delta(y)$ | $\xi\sigma(x) > \xi\sigma(y)$ |

given in Table 1.

According to the table $\xi\sigma$ is a $(P, \mu)$-partition provided that $\xi\sigma(x) > 0$ for all $x \in P$. But $\xi\sigma$ is order-reversing so it attains its minima on maximal elements. If $z$ is a maximal element we have $\xi\sigma(z) = \sigma(z)$ so $\xi : \mathcal{A}(\epsilon) \to \mathcal{A}(\mu)$. By symmetry we also have a map $\eta : \mathcal{A}(\mu) \to \mathcal{A}(\epsilon)$ defined by

$$\eta\sigma(x) = \sigma(x) + \frac{r(\mu) - \rho_\mu(x)}{2} - \frac{r(\epsilon) - \rho_\epsilon(x)}{2}.$$

Hence, $\eta = \xi^{-1}$ and $\xi$ is a bijection.

Since $\sigma$ and $\xi\sigma$ are order-reversing they attain their maxima on minimal elements. But if $z$ is a minimal element then $\xi\sigma(z) = \sigma(z) + \frac{r(\epsilon) - r(\mu)}{2}$, which gives

$$\Omega(P, \mu; n) = \Omega(P, \epsilon; n + \frac{r(\mu) - r(\epsilon)}{2}),$$

and proves the theorem. □

**Theorem 2.3.** *Let $P$ be $\epsilon$-graded. Then*

$$\Omega(P, \epsilon; t) = (-1)^p \Omega(P, \epsilon; -t - r(\epsilon)).$$

*Proof.* We have the following reciprocity for order polynomials, see [10]:

$$\Omega(P, -\epsilon; t) = (-1)^p \Omega(P, \epsilon; -t). \tag{2.1}$$

Note that $r(-\epsilon) = -r(\epsilon)$, so by Theorem 2.2 we have:

$$\Omega(P, -\epsilon; t) = \Omega(P, \epsilon, t - r(\epsilon)),$$

which, combined with (2.1), gives the desired result. □

**Corollary 2.4.** *Let $P$ be an $\epsilon$-graded poset. Then $W(P, \epsilon, t)$ is symmetric with center of symmetry $(p - r(\epsilon) - 1)/2$. If $P$ is also $\mu$-graded then*

$$W(P, \mu; t) = t^{(r(\epsilon) - r(\mu))/2} W(P, \epsilon; t).$$

*Proof.* It is known, see [10], that if $W(P,\epsilon;t) = \sum_{i\geq 0} w_i(P,\epsilon)t^i$ then $\Omega(P,\epsilon;t) = \sum_{i\geq 0} w_i(P,\epsilon)\binom{t+p-1-i}{p}$. Let $r = r(\epsilon)$. Theorem 2.3 gives:

$$\begin{aligned}\Omega(P,\epsilon;t) &= \sum_{i\geq 0} w_i(P,\epsilon)(-1)^p\binom{-t-r+p-1-i}{p} \\ &= \sum_{i\geq 0} w_i(P,\epsilon)\binom{t+r+i}{p} \\ &= \sum_{i\geq 0} w_{p-r-1-i}(P,\epsilon)\binom{t+p-1-i}{p},\end{aligned}$$

so $w_i(P,\epsilon) = w_{p-r-1-i}(P,\epsilon)$ for all $i$, and the symmetry follows. The relationship between the $W$-polynomials of $\epsilon$ and $\mu$ follows from Theorem 2.2 and the expansion of order-polynomials in the basis $\binom{t+p-1-i}{p}$. □

The following theorem tells us that the class of sign-graded posets is considerably greater than the class of graded posets.

**Theorem 2.5.** *Let $P$ be a finite poset. Then there exists a labeling $\epsilon : E \to \{-1,1\}$ such that $(P,\epsilon)$ is sign-graded if and only if all maximal chains in $P$ have the same parity (cardinality modulo 2).*

*Moreover, the labeling $\epsilon$ can be chosen so that the corresponding rank function has values in $\{0,1\}$.*

*Proof.* It is clear that if $P$ is $\epsilon$-graded then all maximal chains have the same parity. Let $P$ be a poset whose maximal chains have the same parity. Then, for any $x \in P$, all saturated chains starting at a minimal element and ending at $x$ has the same length modulo 2. Hence, we may define a labeling $\epsilon : P \to \{-1,1\}$ by $\epsilon(x,y) = (-1)^{\ell(x)}$, where $\ell(x)$ is the length of any saturated chain starting at a minimal element and ending at $x$. It follows that $P$ is $\epsilon$-graded and that its rank function has values in $\{0,1\}$. □

We say that $\omega : P \to \{1,2,\dots,p\}$ is *canonical* if $(P,\omega)$ has a rank-function $\rho$ with values in $\{0,1\}$, and $\rho(x) < \rho(y)$ implies $\omega(x) < \omega(y)$. By Theorem 2.5 we know that $P$ admits a canonical labeling if $P$ is sign-graded with respect to some $\epsilon$.

## 3. The Jordan-Hölder set of a sign-graded poset

Let $(P,\omega)$ be sign-graded. We may assume that $\omega(x) < \omega(y)$ whenever $\rho(x) < \rho(y)$. Assume that $x,y \in P$ are incomparable and that $\rho(y) = \rho(x)+1$. Then the Jordan-Hölder set of $(P,\omega)$ can be partitioned into two sets: One where in all permutations $\omega(x)$ comes before $\omega(y)$ and one where $\omega(y)$ comes before $\omega(x)$. This means that

$$\mathcal{L}(P,\omega) = \mathcal{L}(P',\omega) \sqcup \mathcal{L}(P'',\omega), \tag{3.1}$$

where $P'$ is the transitive closure of $E \cup \{x \prec y\}$, and $P''$ is the transitive closure of $E \cup \{y \prec x\}$.

**Lemma 3.1.** *With definitions as above $(P',\omega)$ and $(P'',\omega)$ are sign-graded with the same rank-function as that for $(P,\omega)$.*



*Proof.* Let $C : z_0 \prec z_1 \prec \cdots \prec z_k = z$ be a saturated chain in $P''$, where $z_0$ is a minimal element in $P''$. Of course $z_0$ is also a minimal element in $P$. We have to prove that

$$\rho(z) = \sum_{i=0}^{k-1} \epsilon''(z_i, z_{i+1}),$$

where $\epsilon''$ is the "edge"-labeling of $P''$ and $\rho$ is the rank-function of $(P,\omega)$.

All covering relations in $P''$, except $y \prec x$, are also covering relations in $P$. Note that $\epsilon''(y,x) = -1$. If $y$ and $x$ do not appear in $C$, then $C$ is a saturated chain in $P$ and we have nothing to prove. Otherwise

$$C : y_0 \prec \cdots \prec y_i = y \prec x = x_{i+1} \prec x_{i+2} \prec \cdots \prec x_k = z.$$

Note that if $s_0 \prec s_1 \prec \cdots \prec s_\ell$ is any saturated chain in $P$ then $\sum_{i=0}^{\ell-1}\epsilon(s_i,s_{i+1}) = \rho(s_\ell) - \rho(s_0)$. Since $y_0 \prec \cdots \prec y_i = y$ and $x = x_{i+1} \prec x_{i+2} \prec \cdots \prec x_k = z$ are saturated chains in $P$ we have

$$\begin{aligned}\sum_{i=0}^{k-1}\epsilon''(z_i,z_{i+1}) &= \rho(y) + \epsilon''(y,x) + \rho(z) - \rho(x) \\ &= \rho(y) - 1 - \rho(x) + \rho(z) \\ &= \rho(z),\end{aligned}$$

as was to be proved. The statement for $(P',\omega)$ follows similarly. □

We say that a sign-graded poset $(P,\omega)$ is *saturated* if for all $x,y \in P$ we have that $x$ and $y$ are comparable whenever $|\rho(y) - \rho(x)| = 1$. Let $P$ and $Q$ be posets on the same set. Then $Q$ *extends* $P$ if $x <_Q y$ whenever $x <_P y$.

**Corollary 3.2.** *Let $(P,\omega)$ be a sign-graded poset. Then the Jordan-Hölder set of $(P,\omega)$ is uniquely decomposed as the disjoint union*

$$\mathcal{L}(P,\omega) = \bigsqcup_Q \mathcal{L}(Q,\omega),$$

*where the union is over all saturated sign-graded posets $(Q,\omega)$, which extend $(P,\omega)$ and has the same rank-function as $(P,\omega)$.*

*Proof.* That the union exhausts $\mathcal{L}(P,\omega)$ follows from (3.1) and Lemma 3.1. Let $(Q_1,\omega)$ and $(Q_2,\omega)$ be two different saturated sign-graded posets that extends $(P,\omega)$ and have the same rank-function as $(P,\omega)$. Then we may assume that there is a covering relation $x \prec y$ in $Q_1$ which is not a covering relation in $Q_2$. Since $|\rho(x) - \rho(y)| = 1$ we must have $y \prec x$ in $Q_2$. Thus $\omega(x)$ precedes $\omega(y)$ in any permutation in $\mathcal{L}(Q_1,\omega)$, and $\omega(y)$ precedes $\omega(x)$ in any permutation in $\mathcal{L}(Q_2,\omega)$. Hence, the union is disjoint. □

We need two operations on labeled posets: Let $(P,\epsilon)$ and $(Q,\mu)$ be two labeled posets. The *ordinal sum*, $P \oplus Q$, of two non-empty posets $P$ and $Q$ is the poset with the disjoint union of $P$ and $Q$ as underlying set and with partial order defined by $x \leq y$ if, either $x \leq_P y$ or $x \leq_Q y$, or $x \in P, y \in Q$. Define two labelings of

$E(P \oplus Q)$ by

$$\begin{aligned} (\epsilon \oplus_1 \mu)(x,y) &= \epsilon(x,y) \text{ if } (x,y) \in E(P), \\ (\epsilon \oplus_1 \mu)(x,y) &= \mu(x,y) \text{ if } (x,y) \in E(Q) \text{ and} \\ (\epsilon \oplus_1 \mu)(x,y) &= 1 \text{ otherwise.} \\ (\epsilon \oplus_{-1} \mu)(x,y) &= \epsilon(x,y) \text{ if } (x,y) \in E(P), \\ (\epsilon \oplus_{-1} \mu)(x,y) &= \mu(x,y) \text{ if } (x,y) \in E(Q) \text{ and} \\ (\epsilon \oplus_{-1} \mu)(x,y) &= -1 \text{ otherwise.} \end{aligned}$$

With a slight abuse of notation we write $P \oplus_{\pm 1} Q$ when the labelings of $P$ and $Q$ are understood from the context. Note that ordinal sums are associative, i.e., $(P \oplus_{\pm 1} Q) \oplus_{\pm 1} R = P \oplus_{\pm 1} (Q \oplus_{\pm 1} R)$, and preserve the property of being sign-graded. The following result is obtained easily by combinatorial reasoning, see [2, 14]:

**Proposition 3.3.** *Let $(P,\omega)$ and $(Q,\nu)$ be two labeled posets. Then*

$$W(P \oplus Q, \omega \oplus_1 \nu; t) = W(P,\omega;t)W(Q,\nu;t)$$

*and*

$$W(P \oplus Q, \omega \oplus_{-1} \nu; t) = tW(P,\omega;t)W(Q,\nu;t).$$

**Proposition 3.4.** *Suppose that $(P,\omega)$ is a saturated canonically labeled sign-graded poset. Then $(P,\omega)$ is the direct sum*

$$(P,\omega) = A_0 \oplus_1 A_1 \oplus_{-1} A_2 \oplus_1 A_3 \oplus_{-1} \cdots \oplus_{\pm 1} A_k,$$

*where the $A_i$s are anti-chains.*

*Proof.* Let $\pi \in \mathcal{L}(P,\omega)$. Then we may write $\pi$ as $\pi = w_0 w_1 \cdots w_k$ where the $w_i$s are maximal words with respect to the property: If $a$ and $b$ are letters of $w_i$ then $\rho(\omega^{-1}(a)) = \rho(\omega^{-1}(b))$. Then $\pi \in J(Q,\omega)$ where

$$(Q,\omega) = A_0 \oplus_1 A_1 \oplus_{-1} A_2 \oplus_1 A_3 \oplus_{-1} \cdots \oplus_{\pm 1} A_k,$$

and $A_i$ is the anti-chain consisting of the elements $\omega^{-1}(a)$, where $a$ is a letter of $w_i$ ($A_i$ is an anti-chain, since if $x < y$ where $x, y \in A_i$ there would be a letter in $\pi$ between $\omega(x)$ and $\omega(y)$ whose rank was different than that of $x, y$). Now, $(Q,\omega)$ is saturated so $P = Q$. □

Note that the argument in the above proof also can be used to give a simple proof of Corollary 3.2 when $\omega$ is canonical. However, we wanted to prove Corollary 3.2 in its generality even though we only need it for canonical labelings.

## 4. The $W$-polynomial of a sign-graded poset

The space, $S^d$, of symmetric polynomials in $\mathbb{R}[t]$ with center of symmetry $d/2$ has a basis

$$B_d = \{t^i(1+t)^{d-2i}\}_{i=0}^{\lfloor d/2 \rfloor}.$$

If $h \in S^d$ has non-negative coefficients in this basis it follows immediately that the coefficients of $h$ in the standard basis are unimodal. Let $S^d_+$ be the non-negative span of $B_d$. Thus $S^d_+$ is a cone. Another property of $S^d_+$ is that if $h \in S^d_+$ then it has the correct sign at $-1$ i.e.,

$$(-1)^{d/2}h(-1) \geq 0.$$



**Lemma 4.1.** *Let $c, d \in \mathbb{N}$. Then*

$$\begin{aligned} S^c S^d &\subset S^{c+d} \\ S^c_+ S^d_+ &\subset S^{c+d}_+. \end{aligned}$$

*Suppose further that $h \in S^d$ has positive leading coefficient and that all zeros of $h$ are real and non-positive. Then $h \in S^d_+$.*

*Proof.* The inclusions are obvious. Since $t \in S^2_+$ and $(1+t) \in S^1_+$ we may assume that none of them divides $h$. But then we may collect the zeros of $h$ in pairs $\theta$ and $\theta^{-1}$. Let $A_\theta = -\theta - \theta^{-1}$. Then

$$h = C \prod_{\theta < -1} (t^2 + A_\theta t + 1),$$

where $C > 0$. Since $A_\theta > 2$ we have

$$t^2 + A_\theta t + 1 = (t+1)^2 + (A_\theta - 2)t \in S^2_+,$$

and the lemma follows. □

We can now prove our main theorem.

**Theorem 4.2.** *Suppose that $(P,\omega)$ is a sign-graded poset of rank $r$. Then $W(P,\omega;t) \in S^{p-r-1}_+$.*

*Proof.* By Corollary 2.4 and Lemma 2.5 we may assume that $(P,\omega)$ is canonically labeled. By Corollary 3.2 we know that

$$W(P,\omega;t) = \sum_Q W(Q,\omega;t),$$

where $(Q,\omega)$ are saturated and sign-graded with the same rank function as that of $(P,\omega)$. The $W$-polynomials of anti-chains are the Eulerian polynomials, which only have real non-negative zeros. By Proposition 3.4 and Proposition 3.3 the polynomial $W(Q,\omega;t)$ has only real non-positive zeros so by Lemma 4.1 and Corollary 2.4 we have $W(Q,\omega;t) \in S^{p-r-1}_+$. The Theorem now follows since $S^{p-r-1}_+$ is a cone. □

**Corollary 4.3.** *Let $(P,\omega)$ be sign-graded of rank $r$ then $W(P,\omega;t)$ is symmetric and its coefficients are unimodal. Moreover, $W(P,\omega;t)$ has the correct sign at $-1$, i.e.,*

$$(-1)^{(p-1-r)/2} W(P,\omega;-1) \geq 0.$$

**Corollary 4.4.** *Let $P$ be a (naturally labeled) graded poset. Suppose that $\Delta_{eq}(P)$ is flag. Then the Charney-Davis Conjecture holds for $\Delta_{eq}(P)$.*

If $h(t)$ is any polynomial with integer coefficients and $h(t) \in S^d$, it follows that $h(t)$ has integer coefficients in the basis $t^i(1+t)^{d-2i}$. Thus we know that if $(P,\omega)$ is sign-graded of rank $r$, then

$$W(P,\omega;t) = \sum_{i=0}^{\lfloor (p-r-1)/2 \rfloor} a_i(P,\omega) t^i (1+t)^{p-r-1-2i},$$

where $a_i(P,\omega)$ are non-negative integers. It would be interesting to have a combinatorial interpretation of these coefficients, and thus a combinatorial proof of Theorem 4.2.

Let $(P, \epsilon)$ be a labeled poset. We say that $(P, \epsilon)$ *admits a rank function* if for every $x \in P$ and saturated chain $x_0 \prec x_1 \prec \cdots \prec x_k = x$, where $x_0$ is a minimal element, the quantity

$$\rho(x) = \sum_{i=1}^{k} \epsilon(x_{i-1}, x_i)$$

is the same. Hence, a labeled poset $(P, \epsilon)$ with a rank function is sign-graded if and only if $\rho$ is constant on maximal elements.

**Theorem 4.5.** *Suppose that $(P, \epsilon)$ admits a rank-function with values in $\{0, 1\}$. Then $W(P, \epsilon; t)$ has unimodal coefficients.*

*Proof.* One may check that the proofs of Lemma 3.1, Corollary 3.2 and Proposition 3.4 holds for this case too. But then

$$W(P, \epsilon; t) = \sum_{Q} W(Q, \epsilon; t),$$

where $W(Q, \epsilon; t)$ is unimodal and symmetric with center of symmetry $(p-1)/2$ or $(p-2)/2$. The sum of such polynomials is again unimodal. □

## 5. The Charney-Davis quantity

In [7] Reiner, Stanton and Welker defined the *Charney-Davis quantity* of a graded naturally labeled poset $(P, \omega)$ of rank $r$ to be

$$CD(P, \omega) = (-1)^{(p-1-r)/2} W(P, \omega; -1).$$

We may define it in the exact same way for sign-graded posets. Since the particular labeling does not matter we write $CD(P)$. Let $\pi = \pi_1 \pi_2 \cdots \pi_n$ be any permutation. We say that $\pi$ is *alternating* if $\pi_1 > \pi_2 < \pi_3 > \cdots$ and *reverse alternating* if $\pi_1 < \pi_2 > \pi_3 < \cdots$. Let $(P, \omega)$ be a canonically labeled sign-graded poset. If $\pi \in \mathcal{L}(P, \omega)$ then we may write $\pi$ as $\pi = w_0 w_1 \cdots w_k$ where $w_i$ are maximal words with respect to the property: If $a$ and $b$ are letters of $w_i$ then $\rho(\omega^{-1}(a)) = \rho(\omega^{-1}(b))$. The words $w_i$ are called the *components* of $\pi$. The following theorem is well known, see for example [9], and gives the Charney-Davis quantity of an anti-chain.

**Proposition 5.1.** *Let $n \geq 0$ be an integer. Then $(-1)^{(n-1)/2} A_n(-1)$ is equal to 0 if $n$ is even and equal to the number of (reverse) alternating permutations of the set $\{1, 2, \ldots, n\}$ if $n$ is odd.*

**Theorem 5.2.** *Let $(P, \omega)$ be a canonically labeled sign-graded poset. Then the Charney-Davis quantity, $CD(P)$, is equal to the number of reverse alternating permutations in $\mathcal{L}(P, \omega)$ such that all components have an odd numbers of letters.*

*Proof.* It suffices to prove the theorem when $(P, \omega)$ is saturated. By Proposition 3.4 we know that

$$(P, \omega) = A_0 \oplus_1 A_1 \oplus_{-1} A_2 \oplus_1 A_3 \oplus_{-1} \cdots \oplus_{\pm 1} A_k,$$

where the $A_i$s are anti-chains. This means that $CD(P) = CD(A_0)CD(A_1) \cdots CD(A_k)$. Let $\pi = w_0 w_1 \cdots w_k \in \mathcal{L}(P, \omega)$ where $w_i$ is a permutation of $\omega(A_i)$. Then $\pi$ is a reverse alternating such that all components have an odd numbers of letters if and only if, for all $i$, $w_i$ is reverse alternating if $i$ is even and alternating if $i$ is odd. Hence, by Proposition 5.1, the number of such permutations is indeed $CD(A_0)CD(A_1) \cdots CD(A_k)$. □



## 6. A characterization of sign-graded posets

Here we give a characterization of sign-graded posets along the lines of the characterization of graded posets given by Stanley in [10]. Let $(P, \epsilon)$ be any labeled poset. Define a function $\delta = \delta_\epsilon : P \to \mathbb{Z}$ by

$$\delta(x) = \max\{\sum_{i=1}^{\ell} \epsilon(x_{i-1}, x_i)\},$$

where $x = x_0 \prec x_1 \prec \cdots \prec x_\ell$ is any saturated chain starting at $x$ and ending at a maximal element $x_\ell$. Define a map $\Phi = \Phi_\epsilon : \mathcal{A}(\epsilon) \to \mathbb{Z}^P$ by

$$\Phi\sigma = \sigma + \delta.$$

We have

$$\delta(x) \geq \delta(y) + \epsilon(x, y). \tag{6.1}$$

This means that $\Phi\sigma(x) > \Phi\sigma(y)$ if $\epsilon(x, y) = 1$ and $\Phi\sigma(x) \geq \Phi\sigma(y)$ if $\epsilon(x, y) = -1$. Thus $\Phi\sigma$ is a $(P, -\epsilon)$-partition provided that $\Phi\sigma(x) > 0$ for all $x \in P$. But $\Phi\sigma$ is order reversing so it attains its minimum at maximal elements and for maximal elements, $z$, we have $\Phi\sigma(z) = \sigma(z)$. This shows that $\Phi : \mathcal{A}(\epsilon) \to \mathcal{A}(-\epsilon)$ is an injection.

We say that a labeling $\epsilon$ of a poset $P$ satisfies the *$\delta$-chain condition* if for every $x \in P$ and saturated chain $x = x_0 \prec x_1 \prec \cdots \prec x_\ell$, where $x_\ell$ is a maximal element, the quantity

$$\sum_{i=1}^{\ell} \epsilon(x_{i-1}, x_i)$$

is the same.

**Proposition 6.1.** *Let $(P, \epsilon)$ be labeled poset. Then $\Phi_\epsilon : \mathcal{A}(\epsilon) \to \mathcal{A}(-\epsilon)$ is a bijection if and only if $\epsilon$ satisfies the $\delta$-chain condition.*

*Proof.* If $\epsilon$ satisfies the $\delta$-chain condition, then so does $-\epsilon$ and $\delta_{-\epsilon}(x) = -\delta_\epsilon(x)$ for all $x \in P$. Thus the if part follows since the inverse of $\Phi_\epsilon$ is $\Phi_{-\epsilon}$.

For the only if direction note that $\epsilon$ satisfies the $\delta$-chain condition if and only if for all $(x, y) \in E$ we have

$$\delta(x) = \delta(y) + \epsilon(x, y)$$

If $\epsilon$ fails to satisfy the $\delta$-chain property we have, by (6.1), that there is a covering relation $(x, y) \in E$ such that either $\epsilon(x, y) = 1$ and $\delta(x) \geq \delta(y) + 2$ or $\epsilon(x, y) = -1$ and $\delta(x) \geq \delta(y)$.

Suppose that $\epsilon(x, y) = 1$. It is clear that there is a $\sigma \in \mathcal{A}(-\epsilon)$ such that $\sigma(x) = \sigma(y) + 1$. But then

$$\sigma(x) - \delta(x) \leq \sigma(y) - \delta(y) - 1,$$

so $\sigma - \delta \notin \mathcal{A}(\epsilon)$.

Similarly, if $\epsilon(x, y) = -1$ then we can find a partition $\sigma \in \mathcal{A}(-\epsilon)$ with $\sigma(x) = \sigma(y)$, and then

$$\sigma(x) - \delta(x) \leq \sigma(y) - \delta(y),$$

so $\sigma - \delta \notin \mathcal{A}(\epsilon)$. □

Define $r(\epsilon)$ by

$$r(\epsilon) = \max\{\sum_{i=1}^{\ell} \epsilon(x_{i-1}, x_i) : x_0 \prec x_1 \prec \cdots \prec x_\ell \text{ is maximal}\}.$$

We then have:

$$\begin{aligned} \max\{\Phi\sigma(x) : x \in P\} &= \max\{\sigma(x) + \delta_\epsilon(x) : x \text{ is minimal}\} \\ &\leq \max\{\sigma(x) : x \in P\} + r(\epsilon). \end{aligned}$$

So if we let $\mathcal{A}_n(\epsilon)$ be the $(P, \epsilon)$-partitions with largest part at most $n$ we have that $\Phi_\epsilon : \mathcal{A}_n(\epsilon) \to \mathcal{A}_{n+r(\epsilon)}(-\epsilon)$ is an injection. A labeling $\epsilon$ of $P$ is said to satisfy the *$\lambda$-chain condition* if for every $x \in P$ there is a maximal chain $c : x_0 \prec x_1 \prec \cdots \prec x_\ell$ containing $x$ such that $\sum_{i=1}^{\ell} \epsilon(x_{i-1}, x_i) = r(\epsilon)$.

**Lemma 6.2.** *Suppose that $n$ is a non-negative integer such that $\Omega(P, \epsilon; n) \neq 0$. If*

$$\Omega(P, -\epsilon; n + r(\epsilon)) = \Omega(P, \epsilon; n)$$

*then $\epsilon$ satisfies the $\lambda$-chain condition.*

*Proof.* Define $\delta^* : P \to \mathbb{Z}$ by

$$\delta^*(x) = \max\{\sum_{i=1}^{\ell} \epsilon(x_{i-1}, x_i)\},$$

where the maximum is taken over all maximal chains starting at a minimal element and ending at $x$. Then

$$\delta(x) + \delta^*(x) \leq r(\epsilon) \tag{6.2}$$

for all $x$, and $\epsilon$ satisfies the $\lambda$-chain condition if and only if we have equality in (6.2) for all $x \in P$. It is easy to see that the map $\Phi^* : \mathcal{A}_n(\epsilon) \to \mathcal{A}_{n+r(\epsilon)}(-\epsilon)$ defined by

$$\Phi^*\sigma(x) = \sigma(x) + r(\epsilon) - \delta^*(x),$$

is well-defined and is an injection. By (6.2) we have $\Phi\sigma(x) \leq \Phi^*\sigma(x)$ for all $\sigma$ and all $x \in P$, with equality if and only if $x$ is in a maximal chain of maximal weight. This means that in order for $\Phi : \mathcal{A}_n(\epsilon) \to \mathcal{A}_{n+r(\epsilon)}(-\epsilon)$ to be a bijection it is necessary for $\epsilon$ to satisfy the $\lambda$-chain condition. □

**Theorem 6.3.** *Let $\epsilon$ be a labeling of $P$. Then*

$$\Omega(P, \epsilon; t) = (-1)^p \Omega(P, \epsilon; -t - r(\epsilon))$$

*if and only if $P$ is $\epsilon$-graded of rank $r(\epsilon)$.*

*Proof.* The "if" part is Theorem 2.3, so suppose that the equality of the theorem holds. By reciprocity we have

$$(-1)^p \Omega(P, \epsilon; -t - r(\epsilon)) = \Omega(P, -\epsilon; t + r(\epsilon)),$$

and since $\Phi_\epsilon : \mathcal{A}_n(\epsilon) \to \mathcal{A}_{n+r(\epsilon)}(-\epsilon)$ is an injection it is also a bijection. By Proposition 6.1, $\epsilon$ satisfies the $\delta$-chain condition, and, by Lemma 6.2, we have that all minimal elements are members of maximal chains of maximal weight. In other words $P$ is $\epsilon$-graded. □



It should be noted that it is not necessary for $P$ to be $\epsilon$-graded in order for $W(P, \epsilon; t)$ to be symmetric. For example, if $(P, \epsilon)$ is any labeled poset then the $W$-polynomial of the disjoint union of $(P, \epsilon)$ and $(P, -\epsilon)$ is easily seen to be symmetric. However, we have the following:

**Corollary 6.4.** *Suppose that*

$$\Omega(P, \epsilon; t) = \Omega(P, -\epsilon; t + s),$$

*for some $s \in \mathbb{Z}$. Then $-r(-\epsilon) \leq s \leq r(\epsilon)$, with equality if and only if $P$ is $\epsilon$-graded.*

*Proof.* We have an injection $\Phi_\epsilon : \mathcal{A}_n(\epsilon) \to \mathcal{A}_{n+r(\epsilon)}(-\epsilon)$. This means that $s \leq r(\epsilon)$. The lower bound follows from the injection $\Phi_{-\epsilon}$, and the statement of equality follows from Theorem 6.3. □

## References


[1] P. Brändén. On operators on polynomials preserving real-rootedness and the Neggers-Stanley conjecture. *J.Algebraic Comb.*, to appear.

[2] F. Brenti. Unimodal, log-concave and Pólya frequency sequences in combinatorics. *Mem. Amer. Math. Soc.*, 81(413):viii+106, 1989.

[3] R. Charney and M. Davis. The Euler characteristic of a nonpositively curved, piecewise Euclidean manifold. *Pacific J. Math.*, 171(1):117–137, 1995.

[4] D. Foata and M. Schützenberger. *Théorie géométrique des polynômes eulériens.* Lecture Notes in Mathematics, Vol. 138. Springer-Verlag, Berlin, 1970.

[5] V. Gasharov. On the Neggers-Stanley conjecture and the Eulerian polynomials. *J. Combin. Theory Ser. A*, 82(2):134–146, 1998.

[6] J. Neggers. Representations of finite partially ordered sets. *J. Combin. Inform. System Sci.*, 3(3):113–133, 1978.

[7] V. Reiner, D. Stanton, and V. Welker. The Charney-Davis quantity for certain graded posets. *Sém. Lothar. Combin.*, 50:Art. B50c, 13 pp. (electronic), 2003.

[8] V. Reiner and V. Welker. On the Charney-Davis and the Neggers-Stanley conjectures. *http://www.math.umn.edu/~reiner/Papers/papers.html*, 2002.

[9] L. W. Shapiro, W. J. Woan, and S. Getu. Runs, slides and moments. *SIAM J. Algebraic Discrete Methods*, 4(4):459–466, 1983.

[10] R. P. Stanley. *Ordered structures and partitions.* American Mathematical Society, Providence, R.I., 1972. Memoirs of the American Mathematical Society, No. 119.

[11] R. P. Stanley. The number of faces of simplicial polytopes and spheres. In *Discrete geometry and convexity (New York, 1982)*, volume 440 of *Ann. New York Acad. Sci.*, pages 212–223. New York Acad. Sci., New York, 1985.

[12] R. P. Stanley. *Combinatorics and commutative algebra*, volume 41 of *Progress in Mathematics.* Birkhäuser Boston Inc., Boston, MA, second edition, 1996.

[13] R. P. Stanley. *Enumerative combinatorics. Vol. 1*, volume 49 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1997.

[14] D. G. Wagner. Enumeration of functions from posets to chains. *European J. Combin.*, 13(4):313–324, 1992.



Matematik, Chalmers tekniska högskola och Göteborgs universitet, S-412 96 Göteborg, Sweden

*E-mail address*: branden@math.chalmers.se