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On Stationary Solutions to the Linear Boltzmann Equation with Inelastic Granular Collisions

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Abstract

This paper considers the stationary linear Boltzmann equation with inelastic (granular) collisions in the case of an interior source term together with an absorption term and general boundary reflections. First, mild L^1-solutions are constructed as limits of iterate functions. Then boundedness of all higher velocity moments are obtained.

1 Introduction

The linear Boltzmann equation is frequently used for mathematical modelling in physics, (e.g. for describing the neutron distribution in reactor physics, cf. [1]–[4]). In our earlier papers [5]–[11] we have studied the linear Bolzmann equation for a function $f(\mathbf{x}, \mathbf{v}, t)$ (in the time-dependent case), or $F(\mathbf{x}, \mathbf{v})$ in the stationary case, representing the distribution of particles with mass m colliding elastically and binary with other particles with mass m_* and with a given (known) distribution function $Y(\mathbf{x}, \mathbf{v}_*)$. In recent years a significant interest has been focused on the study of kinetic models for granular flows, see e.g. [13]–[15], whose papers study the non-linear Boltzmann equation for granular gases, (mostly in the case of hard sphere collisions). Our paper [12] considers the time-dependent linear Boltzmann equation for inelastic (granular) collisions. The purpose of this paper is to generalize our earlier results for the stationary equation, cf. [11], to the case of inelastic collision for granular gases.

So we will study collisions between particles with mass m and particles with mass m_*, such that momentum is conserved,

$$m\mathbf{v} + m_*\mathbf{v}_* = m\mathbf{v}' + m_*\mathbf{v}'_*, \tag{1.1}$$

where $\mathbf{v}, \mathbf{v}_*$ are velocities before and $\mathbf{v}', \mathbf{v}'_*$ are velocities after a collision.

In the elastic case, where also kinetic energy is conserved, one finds that the velocities after a binary collision terminate on two concentric spheres, so all velocities $\mathbf{v}'$ lie on a sphere around the center of mass, $\bar{\mathbf{v}} = (m\mathbf{v}+m_*\mathbf{v}_*)/(m+m_*)$, with radius $\frac{m_*}{m+m_*}|\mathbf{v}-\mathbf{v}_*|$, and all velocities $\mathbf{v}_*'$ lie on a sphere with the same center $\bar{\mathbf{v}}$ and with radius $\frac{m}{m+m_*}|\mathbf{v}-\mathbf{v}_*|$, cf. Figure 1 in [5].

In the granular, inelastic case we assume the following relation between the relative velocity components normal to the plane of contact of the two particles,

$$\mathbf{w}' \cdot \mathbf{u} = -a(\mathbf{w} \cdot \mathbf{u}), \tag{1.2}$$

where a is a constant, $0 < a \leq 1$, and $\mathbf{w} = \mathbf{v} - \mathbf{v}_*, \mathbf{w}' = \mathbf{v}' - \mathbf{v}_*'$ are the relative velocities before and after the collision, and $\mathbf{u}$ is a unit vector in the direction of impact, $\mathbf{u} = (\mathbf{v}-\mathbf{v}')/|\mathbf{v}-\mathbf{v}'|$. Then we find that $\mathbf{v}' = \mathbf{v}_a'$ lies on the line between $\mathbf{v}$ and $\mathbf{v}_1'$, where $\mathbf{v}_1'$ is the postvelocity in the case of elastic collision, i.e. with $a = 1$, and $\mathbf{v}_{*a}'$ lies on the (parallel) line between $\mathbf{v}_*$ and $\mathbf{v}_{*1}'$.

Now it follows that the following relations hold for the velocities in the granular, inelastic case,

$$\begin{aligned} \mathbf{v}' &= \mathbf{v} - (a+1)\frac{m_*}{m+m_*}(\mathbf{w}\cdot\mathbf{u})\cdot\mathbf{u}, \\ \mathbf{v}_*' &= \mathbf{v}_* + (a+1)\frac{m}{m+m_*}(\mathbf{w}\cdot\mathbf{u})\cdot\mathbf{u}, \end{aligned} \tag{1.3}$$

where $\mathbf{w}\cdot\mathbf{u} = w\cos\theta, w = |\mathbf{v}-\mathbf{v}_*|$, if the unit vector $\mathbf{u}$ is given in spherical coordinates,

$$\mathbf{u} = (\sin\theta\cos\zeta, \sin\theta\sin\zeta, \cos\theta), \tag{1.4}$$

$0 \leq \theta \leq \pi/2$, $0 \leq \zeta < 2\pi$.

By (1.3) we get for the relative velocity after collision, $\mathbf{w}' = \mathbf{v}' - \mathbf{v}_*'$, that

$$\mathbf{w}' = \mathbf{w} - (a+1)(\mathbf{w}\cdot\mathbf{u})\cdot\mathbf{u}, \tag{1.5}$$

and we also find (for $\mathbf{w}' = \mathbf{w}_a'$) that

$$|\mathbf{w}_a'| = |\mathbf{w}|\sqrt{\sin^2\theta + a^2\cos^2\theta}. \tag{1.6}$$

Furthermore, the change of kinetic energy ΔE in a binary granular collision can be calculated by

$$\begin{aligned} 2\Delta E &\equiv m|\mathbf{v}'|^2 + m_*|\mathbf{v}_*'|^2 - m|\mathbf{v}|^2 - m_*|\mathbf{v}_*|^2 = \\ &= -(1-a^2)\frac{mm_*}{m+m_*}w^2\cos^2\theta. \end{aligned} \tag{1.7}$$

One can also see that all velocities $\mathbf{v}'$ and $\mathbf{v}_*'$ terminate on two different spheres (with different centers, if $a \neq 1$), cf. [13]–[15],

$$\begin{aligned} \mathbf{v}' &= \bar{\mathbf{v}} + \frac{(1-a)m_*}{2(m+m_*)}\mathbf{w} + \frac{(1+a)m_*w}{2(m+m_*)}\boldsymbol{\pi}, \\ \mathbf{v}_*' &= \bar{\mathbf{v}} - \frac{(1-a)m}{2(m+m_*)}\mathbf{w} - \frac{(1+a)mw}{2(m+m_*)}\boldsymbol{\pi}, \end{aligned} \tag{1.8}$$

where $\bar{\mathbf{v}} = (m\mathbf{v} + m_*\mathbf{v}_*)/(m + m_*)$, $\mathbf{w} = \mathbf{v} - \mathbf{v}_*$, $w = |\mathbf{w}|$, and $\boldsymbol{\pi}$ is a unit vector.

Moreover, if we change notations, and let $'\mathbf{v}, '\mathbf{v}_*$ be the velocities before, and $\mathbf{v}, \mathbf{v}_*$ the velocities after a binary inelastic collision, then by (1.2) and (1.3), cf. [13]–[15],

$$\begin{aligned} '\mathbf{v} &= \mathbf{v} - \frac{(a+1)m_*}{a(m+m_*)}(\mathbf{w}\mathbf{u}) \cdot \mathbf{u}, \\ '\mathbf{v}_* &= \mathbf{v}_* + \frac{(a+1)m}{a(m+m_*)}(\mathbf{w}\mathbf{u}) \cdot \mathbf{u}. \end{aligned} \tag{1.9}$$

In the following sections of this paper, we give in Section 2 some preliminaries on the stationary, linear Boltzmann equation with an interior source term together with an absorption term and general boundary reflections. Then in Section 3 solutions are constructed as limits of iterate functions, and in Section 4 boundedness of higher velocity moments are studied.

2 Preliminaries

We consider the stationary transport equation for a distribution function $F(\mathbf{x}, \mathbf{v})$, depending on a space variable $\mathbf{x} = (x_1, x_2, x_3)$ in a bounded convex body D with (piecewise) C^1-boundary $\Gamma = \partial D$, and depending on a velocity variable $\mathbf{v} = (v_1, v_2, v_3) \in V = \mathbb{R}^3$. The stationary linear Boltzmann equation in the case of given interior source $\alpha_0 G(\mathbf{x}, \mathbf{v})$, where $\alpha_0 > 0$ is a constant and $G \geq 0$ is a given (measurable) function, together with an absorption term $\alpha F(\mathbf{x}, \mathbf{v})$, $\alpha > 0$, is in the strong form

$$\alpha F(\mathbf{x}, \mathbf{v}) + \mathbf{v}\nabla_{\mathbf{x}} F(\mathbf{x}, \mathbf{v}) = \alpha_0 G(\mathbf{x}, \mathbf{v}) + (QF)(\mathbf{x}, \mathbf{v}). \tag{2.1}$$

The collision term can, in the case of inelastic (granular) collision, be written, cf. [13]–[15], and also [1]–[12],

$$\begin{aligned} (QF)(\mathbf{x}, \mathbf{v}) = \iint_{V\Omega} &[J_a(\theta, w) Y(\mathbf{x}, '\mathbf{v}_*) F(\mathbf{x}, '\mathbf{v}) - \\ &- Y(\mathbf{x}, \mathbf{v}_*) F(\mathbf{x}, \mathbf{v})] B(\theta, w) d\mathbf{v}_* d\theta d\zeta, \end{aligned} \tag{2.2}$$

with $w = |\mathbf{v} - \mathbf{v}_*|$, where $Y \geq 0$ is a known distribution, and $B \geq 0$ is given by the collision process, and finally J_a is a factor depending on the granular process, (and giving mass-conservation, if the gain and the loss integrals converges separately). Furthermore, $'\mathbf{v}, '\mathbf{v}_*$ in (2.2) are the velocities before and $\mathbf{v}, \mathbf{v}_*$ the velocities after the binary collision, cf. equation (1.9), and $\Omega = \{(\theta, \zeta) : 0 \leq \theta < \hat{\theta}, 0 \leq \zeta < 2\pi\}$ is the impact plane.

If the collision term is written in a weak form with a testfunction $g = g(\mathbf{v})$, then we (formally) get

$$\begin{aligned}(QF, g) &\equiv \int_V (QF)(\mathbf{x}, \mathbf{v}) g(\mathbf{v}) d\mathbf{v} = \\ &= \iiint_{VV\Omega} [g(\mathbf{v}') - g(\mathbf{v})] F(\mathbf{x}, \mathbf{v}) Y(\mathbf{x}, \mathbf{v}_*) B(\theta, |\mathbf{v} - \mathbf{v}_*|) d\mathbf{v} d\mathbf{v}_* d\theta d\zeta,\end{aligned} \tag{2.3}$$

where $\mathbf{v}'$ is the velocity after collision.

In the following of this paper we will study the angular cut-off-case with $\hat{\theta} < \pi/2$. Then the gain and the loss term in (2.2) can be separated

$$(QF)(\mathbf{x}, \mathbf{v}) = (Q^+F)(\mathbf{x}, \mathbf{v}) - (Q^-F)(\mathbf{x}, \mathbf{v}), \tag{2.4}$$

where we can write

$$\begin{aligned}(Q^+F)(\mathbf{x}, \mathbf{v}) &= \iint_{V\Omega} J_a(\theta, w) Y(\mathbf{x},' \mathbf{v}_*) F(\mathbf{x},' \mathbf{v}) B(\theta, w) d\mathbf{v}_* d\theta d\zeta = \\ &= \int_V K_a(\mathbf{x},' \mathbf{v} \to \mathbf{v}) F(\mathbf{x},' \mathbf{v}) d' \mathbf{v},\end{aligned} \tag{2.5}$$

and

$$(Q^-F)(\mathbf{x}, \mathbf{v}) = L(\mathbf{x}, \mathbf{v}) F(\mathbf{x}, \mathbf{v}) \tag{2.6}$$

with the collision frequency

$$L(\mathbf{x}, \mathbf{v}) = \iint_{V\Omega} Y(\mathbf{x}, \mathbf{v}_*) B(\theta, w) d\mathbf{v}_* d\theta d\zeta, w = |\mathbf{v} - \mathbf{v}_*|. \tag{2.7}$$

In the case of a non-absorbing body we have the following relation

$$L(\mathbf{x}, \mathbf{v}) = \int_V K_a(\mathbf{x}, \mathbf{v} \to \mathbf{v}') d\mathbf{v}'. \tag{2.8}$$

For hard sphere collisions the function $B(\theta, w)$ can be written, cf. [1]–[4] and [13]–[15],

$$B(\theta, w) = \text{const} \cdot w \sin\theta \cos\theta, w = |\mathbf{v} - \mathbf{v}_*|. \tag{2.9}$$

Another physically interesting case is that with inverse k-th power collision forces

$$B(\theta, w) = b(\theta) w^\gamma, \gamma = \frac{k-5}{k-1} \tag{2.10}$$

with hard forces for $k > 5$, Maxwelllian for $k = 5$, and soft forces for $3 < k < 5$.

The factor J_a in the gain term can in the hard sphere case be calculated and found to be proportional to a^{-2}, cf. [13]–[15].

The equation (2.1) is supplemented with (general) boundary conditions

$$F_-(\mathbf{x}, \mathbf{v}) = (1 - \beta) \int_V \frac{|\mathbf{n}\bar{\mathbf{v}}|}{|\mathbf{n}\mathbf{v}|} R(\mathbf{x}, \bar{\mathbf{v}} \to \mathbf{v}) F_+(\mathbf{x}, \bar{\mathbf{v}}) d\bar{\mathbf{v}}, \tag{2.11}$$

where β is a constant, $0 \leq \beta \leq 1$. The function $R \geq 0$ satisfies

$$\int_V R(\mathbf{x}, \tilde{\mathbf{v}} \to \mathbf{v}) d\mathbf{v} \equiv 1, \tag{2.12}$$

and $\mathbf{n} = \mathbf{n}(\mathbf{x})$ is the unit outward normal at $\mathbf{x} \in \Gamma$. The functions F_- and F_+ represent the ingoing and outgoing trace functions corresponding to F. Furthermore, in the specular reflection case, the function R is represented by Dirac measure $R(\mathbf{x}, \tilde{\mathbf{v}} \to \mathbf{v}) = \delta(\mathbf{v} - \tilde{\mathbf{v}} + 2\mathbf{n}(\mathbf{n}\tilde{\mathbf{v}}))$, and in the diffuse reflection case $R(\mathbf{x}, \tilde{\mathbf{v}} \to \mathbf{v}) = |\mathbf{n}\mathbf{v}| W(\mathbf{x}, \mathbf{v})$ with some given function $W \geq 0$ (e.g. Maxwellian function).

Now using differentiation along the characteristics, the equation (2.1) can formally be written

$$\frac{d}{dt}(F(\mathbf{x} + t\mathbf{v}, \mathbf{v})) = \alpha_0 G(\mathbf{x} + t\mathbf{v}, \mathbf{v}) + \\ + \int_V K_a(\mathbf{x} + t\mathbf{v}, {}'\mathbf{v} \to \mathbf{v}) F(\mathbf{x} + t\mathbf{v}, {}'\mathbf{v}) d'\mathbf{v} - [\alpha + L(\mathbf{x} + t\mathbf{v}, \mathbf{v})] F(\mathbf{x} + t\mathbf{v}, \mathbf{v}). \tag{2.13}$$

Let

$$t_b \equiv t_b(\mathbf{x}, \mathbf{v}) = \inf_{\tau \in \mathbb{R}_+} \{\tau : \mathbf{x} - \tau\mathbf{v} \notin D\}$$

and $\mathbf{x}_b \equiv \mathbf{x}_b(\mathbf{x}, \mathbf{v}) = \mathbf{x} - t_b\mathbf{v}$. Here t_b represents the time for a particle going with velocity $\mathbf{v}$ from the boundary point $\mathbf{x}_b = \mathbf{x} - t_b\mathbf{v}$ to the point $\mathbf{x}$.

Then we have the following *mild form* of the linear Boltzmann equation

$$F(\mathbf{x}, \mathbf{v}) = F_-(\mathbf{x}_b, \mathbf{v}) + \int_0^{t_b} [(QF)(\mathbf{x} - \tau\mathbf{v}, \mathbf{v}) + \alpha_0 G(\mathbf{x} - \tau\mathbf{v}, \mathbf{v}) - \alpha F(\mathbf{x} - \tau\mathbf{v}, \mathbf{v})] d\tau \tag{2.14}$$

and the *exponential form*

$$F(\mathbf{x}, \mathbf{v}) = F_-(\mathbf{x}_b, \mathbf{v}) e^{-\int_0^{t_b} (\alpha + L(\mathbf{x} - s\mathbf{v}, \mathbf{v})) ds} + \\ + \int_0^{t_b} e^{-\int_0^{\tau} (\alpha + L(\mathbf{x} - s\mathbf{v}, \mathbf{v})) ds} [\alpha_0 G(\mathbf{x} - \tau\mathbf{v}, \mathbf{v}) + \\ + \int_V K_a(\mathbf{x} - \tau\mathbf{v}, {}'\mathbf{v} \to \mathbf{v}) F(\mathbf{x} - \tau\mathbf{v}, {}'\mathbf{v}) d'\mathbf{v}] d\tau. \tag{2.15}$$

Remark. One finds that F is a mild solution (2.14) if and only if F satisfies the exponential form (2.15) of the equation, (cf. our earlier papers and also the classical result by di Perna-Lions). For a proof we (among others) use that

$$\frac{d}{dt}(t_b(\mathbf{x} + t\mathbf{v}, \mathbf{v})) \equiv 1. \tag{2.16}$$

3 Construction of stationary solutions

We construct mild L^1-solutions to our problem as limits of iterate functions F^n, when $n \to \infty$. Let first $F^{-1}(\mathbf{x},\mathbf{v}) \equiv 0$ for all $\mathbf{x},\mathbf{v} \in \mathbb{R}^3$. Then define, for given function F^{n-1} the next iterate F^n, first at the ingoing boundary (using the appropriate boundary condition), and then inside D and at the outgoing boundary (using the exponential form of the equation);

$$F^n_-(\mathbf{x},\mathbf{v}) = (1-\beta)\int_V \frac{|\mathbf{n}\tilde{\mathbf{v}}|}{|\mathbf{n}\mathbf{v}|} R(\mathbf{x},\tilde{\mathbf{v}} \to \mathbf{v}) F^{n-1}_+(\mathbf{x},\tilde{\mathbf{v}}) d\tilde{\mathbf{v}}, \tag{3.1}$$

$$\mathbf{n}\mathbf{v} < 0, \mathbf{x} \in \Gamma = \partial D,\ \mathbf{v} \in V = \mathbb{R}^3;$$

$$F^n(\mathbf{x},\mathbf{v}) = F^n_-(\mathbf{x} - t_b\mathbf{v},\mathbf{v}) e^{-\int_0^{t_b}(\alpha + L(\mathbf{x}-s\mathbf{v},\mathbf{v}))ds} +$$
$$+ \int_0^{t_b} e^{-\int_0^{\tau}(\alpha + L(\mathbf{x}-s\mathbf{v},\mathbf{v}))ds} [\alpha_0 G(\mathbf{x} - \tau\mathbf{v},\mathbf{v}) +$$
$$+ \int_V K_a(\mathbf{x} - \tau\mathbf{v},\ '\mathbf{v} \to \mathbf{v}) F^{n-1}(\mathbf{x} - \tau\mathbf{v},\ '\mathbf{v}) d'\mathbf{v}] d\tau, \tag{3.2}$$
$$\mathbf{x} \in D \setminus \Gamma_-(\mathbf{v}),\ \mathbf{v} \in V = \mathbb{R}^3.$$

Let also $F^n(\mathbf{x},\mathbf{v}) \equiv 0$ for $\mathbf{x} \in \mathbb{R}^3 \setminus D$. Now we get a monotonicity lemma, which is essential in the following, and which can be proved by induction.

Lemma 3.1 $F^n(\mathbf{x},\mathbf{v}) \geq F^{n-1}(\mathbf{x},\mathbf{v}), \mathbf{x} \in D, \mathbf{v} \in V, \mathbf{n} \in \mathbb{N}$.

Using differentiation along the characteristics, we get by (3.2) that

$$\frac{d}{dt}(F^n(\mathbf{x}+t\mathbf{v},\mathbf{v})) = \alpha_0 G(\mathbf{x}+t\mathbf{v},\mathbf{v}) - \alpha F^n(\mathbf{x}+t\mathbf{v},\mathbf{v}) +$$
$$+ \int_V K_a(\mathbf{x}+t\mathbf{v},\ '\mathbf{v} \to \mathbf{v}) F^{n-1}(\mathbf{x}+t\mathbf{v},\ '\mathbf{v}) d'\mathbf{v} - L(\mathbf{x}+t\mathbf{v},\mathbf{v}) F^n(\mathbf{x}+t\mathbf{v},\mathbf{v}). \tag{3.3}$$

Then integrating (3.3), it follows by Green's formula that

$$\alpha \iint_{DV} F^n(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} + \iint_{\Gamma V} F^n_+(\mathbf{x},\mathbf{v}) |\mathbf{n}\mathbf{v}| d\mathbf{v} d\Gamma =$$
$$= \alpha_0 \iint_{DV} G(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} + \iint_{\Gamma V} F^n_-(\mathbf{x},\mathbf{v}) |\mathbf{n}\mathbf{v}| d\mathbf{v} d\Gamma + \tag{3.4}$$
$$+ \iint_{DV} L(\mathbf{x},\mathbf{v}) [F^{n-1}(\mathbf{x},\mathbf{v}) - F^n(\mathbf{x},\mathbf{v})] d\mathbf{x} d\mathbf{v},$$

where by (2.12) and (3.1)

$$\iint_{\Gamma V} F^n_-(\mathbf{x},\mathbf{v}) |\mathbf{n}\mathbf{v}| d\mathbf{v} d\Gamma = (1-\beta) \iint_{\Gamma V} F^{n-1}_+(\mathbf{x},\mathbf{v}) |\mathbf{n}\mathbf{v}| d\mathbf{v} d\Gamma \tag{3.5}$$

Now by Lemma 3.1 it follows that

$$\alpha \iint_{DV} F^n(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v} + \beta \iint_{\Gamma V} F_+^n(\mathbf{x},\mathbf{v})|\mathbf{n}\mathbf{v}|d\mathbf{v}d\Gamma \leq \alpha_0 \iint_{DV} G(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v} \tag{3.6}$$

So, if $G \in L^1(D \times V)$, then we have for all $\alpha > 0$ that

$$\iint_{DV} F^n(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v} \leq \frac{\alpha_0}{\alpha} \iint_{DV} G(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v} < \infty. \tag{3.7}$$

Then Levi's theorem (on monotone convergence) gives (also in the inelastic granular case) existence of a mild (defined by (2.14)) L^1-solution $F(\mathbf{x},\mathbf{v}) = \lim_{n\to\infty} F^n(\mathbf{x},\mathbf{v})$ to the stationary linear Boltzmann equation (2.1) with (2.11), and $F \equiv F_{a,\alpha,\beta,\alpha_0}$ satisfies for $0 < a \leq 1, \alpha, \alpha_0 > 0, 0 \leq \beta \leq 1$, the inequality

$$\begin{aligned} &\alpha \iint_{DV} F(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v} + \beta \iint_{\Gamma V} F_+(\mathbf{x},\mathbf{v})|\mathbf{n}\mathbf{v}|d\mathbf{v}d\Gamma \leq \\ &\leq \alpha_0 \iint_{DV} G(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v}. \end{aligned} \tag{3.8}$$

Furthermore, if $L(\mathbf{x},\mathbf{v})F(\mathbf{x},\mathbf{v}) \in L^1(D\times V)$, then we get equality in (3.8) together with uniqueness in the relevant function space, cf. [6] and also [3].

So, for instance, if $\beta = \rho \cdot \alpha, \alpha_0 = \alpha > 0, \rho \geq 0$, then

$$\iint_{DV} F(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v} + \rho \iint_{\Gamma V} F_+(\mathbf{x},\mathbf{v})|\mathbf{n}\mathbf{v}|d\mathbf{v}d\Gamma = \iint_{DV} G(\mathbf{x},\mathbf{v})d\mathbf{x}d\mathbf{v}. \tag{3.9}$$

In summary, we have the following existence theorem for solutions to our stationary linear Boltzmann equation with general boundary reflections.

Theorem 3.2 *Assume that K_a, L and R are nonnegative, measurable functions, such that (2.8) and (2.12) hold, and $L(\mathbf{x},\mathbf{v}) \in L^1_{\text{loc}}(D \times V)$. Let $\alpha, \alpha_0 > 0$ and $0 \leq \beta \leq 1$ be constants, and $G(\mathbf{x},\mathbf{v}) \in L^1(D \times V)$ with $\iint G d\mathbf{x}d\mathbf{v} > 0$.*

a) Then there exists a mild L^1-solution $F(\mathbf{x},\mathbf{v})$ to the problem (2.1), (2.4)–(2.6) with (2.11). This solution, depending on a, α, α_0 and β, satisfies the inequality (3.8).

b) Moreover, if $L(\mathbf{x},\mathbf{v})F(\mathbf{x},\mathbf{v}) \in L^1(D \times V)$, then the trace of the solution F satisfies the boundary condition (2.11) for a.e. $(\mathbf{x},\mathbf{v}) \in \Gamma\times V$. Furthermore, mass conservation, giving equality in (3.8), holds, together with uniqueness in the relevant L^1-space.

Remarks.

1) The statement in Theorem 3.2(b) on existence of traces follows e.g. from Proposition 3.3, Chapter XI in [3].

2) The assumption $LF \in L^1(D \times V)$ is for instance satisfied for the solution F in the case of inverse power collision forces, cf. (2.11), together with e.g. specular or diffuse boundary reflections. This follows from a statement on global boundedness of higher velocity moments, cf. Theorem 4.1 and Corollary 4.2 in the next section.

3) For bounded gain operators the problem of existence and uniqueness for solutions to the linear Boltzmann equation has been studied earlier by a different technique, cf. ref. [3]. But in our approach unbounded operators also are included, e.g. the case of hard inverse power collision forces.

4) An interesting problem concerns the question: what happens, when the coefficients α, α_0 and β go to zero? That problem has been studied (and partly solved) in our earlier papers, but a general existence result for the stationary linear Boltzmann equation is not yet recieved; cf. also the L^1-result in [16] for velocities bounded away from zero, and cf. [17]. But the problem on uniqueness for stationary solutions has been solved in our earlier papers by use of an H-theorem for a relative entropy functional.

4 On boundedness of higher velocity moments

In this section we first study some velocity estimates, and then use these results to prove boundedness of higher velocity moments in the case of inverse power collision forces together with e.g. specular boundary conditions. These results generalize our earlier statements to the granular inelastic case. The following proposition on the difference of the squares of velocities after and before collision is an analogue of Proposition 1.1 in [5], and can be proved in (almost) the same way.

Proposition 4.1 *Let* $\mathbf{v}$ *and* $\mathbf{v}'_a(\theta, \zeta)$ *be the velocities for a particle with mass* m *before and after a binary granular collision with a particle, having the corresponding velocities* $\mathbf{v}_*$ *and* $\mathbf{v}'_*$ *and mass* m_**, such that (1.1) and (1.2) hold. Then the following estimate holds, (for* $0 < a \leq 1, 0 \leq \theta < \pi/2, 0 \leq \zeta < 2\pi$*),*

$$\begin{aligned} &|\mathbf{v}'_a(\theta,\zeta)|^2 - |\mathbf{v}|^2 \leq \\ &\leq 2(a+1)\rho_* w \cos\theta[3|\mathbf{v}_*| - \rho|\mathbf{v}|\cos\theta], \end{aligned} \tag{4.1}$$

with $w = |\mathbf{v} - \mathbf{v}_*|, \rho_* = m_*/(m + m_*), \rho = m/(m + m_*)$.

Now we can prove that the following type of estimate holds also in the inelastic granular collision case, (analogously to the elastic case, cf. Proposition 1.2 in [5]).

Proposition 4.2 *Suppose* $\mathbf{v}$ *and* $\mathbf{v}'_a$ *are velocities as in Proposition 4.1. Then for all* $\sigma > 0$*, there are positive constants* C_1 *and* C_2 *(depending on* σ, m, m_* *and* a*), such that*

$$\begin{aligned} &(1 + |\mathbf{v}'_a(\theta)|^2)^{\sigma/2} - (1 + |\mathbf{v}|^2)^{\sigma/2} \le \\ &\le C_1(w\cos\theta)(1 + |\mathbf{v}_*|)^{\max(1,\sigma-1)}(1 + |\mathbf{v}|^2)^{(\sigma-2)/2} - \\ &- C_2(w\cos^2\theta)(1 + |\mathbf{v}|^2)^{(\sigma-1)/2}. \end{aligned} \tag{4.2}$$

In the rest of this section we assume inverse power collision forces with the function $B(\theta, w)$ given by (2.10).

To get higher velocity estimates for our solution F (given by Theorem 3.2 (a)), we start from equation (3.3), i.e. the differentiated mild form (along the characteristics) for the iterate function F^n, and multiply this equation by $(1 + v^2)^{\sigma/2}$, where $v = |\mathbf{v}|, \sigma > 0$. Then

$$\begin{aligned} &\frac{d}{dt}[(1 + v^2)^{\sigma/2} F^n(\mathbf{x} + t\mathbf{v}, \mathbf{v})] = \alpha_0(1 + v^2)^{\sigma/2} G(\mathbf{x} + t\mathbf{v}, \mathbf{v}) + \\ &+ \int_V K_a(\mathbf{x} + t\mathbf{v}, {}'\mathbf{v} \to \mathbf{v})(1 + v^2)^{\sigma/2} F^{n-1}(\mathbf{x} + t\mathbf{v}, {}'\mathbf{v}) d{}'\mathbf{v} - \\ &- [\alpha + L(\mathbf{x} + t\mathbf{v}, \mathbf{v})](1 + v^2)^{\sigma/2} F^n(\mathbf{x} + t\mathbf{v}, \mathbf{v}). \end{aligned} \tag{4.3}$$

Now integrating (4.3), it follows by Green's formula that

$$\begin{aligned} &\alpha \iint_{DV} (1 + v^2)^{\sigma/2} F^n(\mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v} + \iint_{\Gamma V} (1 + v^2)^{\sigma/2} F^n_+(\mathbf{x}, \mathbf{v}) |\mathbf{n}\mathbf{v}| d\mathbf{v} d\Gamma = \\ &= \alpha_0 \iint_{DV} (1 + v^2)^{\sigma/2} G(\mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v} + \iint_{\Gamma V} (1 + v^2)^{\sigma/2} F^n_-(\mathbf{x}, \mathbf{v}) |\mathbf{n}\mathbf{v}| d\mathbf{v} d\Gamma + \\ &+ \iiint_{DVV} [K_a(\mathbf{x}, {}'\mathbf{v} \to \mathbf{v})(1 + v^2)^{\sigma/2} F^{n-1}(\mathbf{x}, {}'\mathbf{v}) - \\ &- K_a(\mathbf{x}, \mathbf{v} \to {}'\mathbf{v})(1 + v^2)^{\sigma/2} F^n(\mathbf{x}, \mathbf{v})] d\mathbf{x} d\mathbf{v} d{}'\mathbf{v}, \end{aligned} \tag{4.4}$$

Using the collision estimate (4.2) together with the assumption on inverse power forces (2.10) and some elementary estimate, cf. [5], $w = |\mathbf{v} - \mathbf{v}_*|, -1 < \gamma \le 1$,

$$-w^{\gamma+1} \le (1 + v_*)^{\gamma+1} - 2^{-1}(1 + v^2)^{\frac{\gamma+1}{2}},$$

we find that the interior collision term in (4.4) is bounded from above by

$$\begin{aligned} &+ \tilde{C}_1 \iint_{DV} (1 + v^2)^{\frac{\sigma+\gamma-1}{2}} F^n(\mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v} + \\ &+ \tilde{C}_2 \iint_{DV} (1 + v^2)^{\frac{\sigma-1}{2}} F^n(\mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v} - \\ &- \tilde{C}_0 \iint_{DV} (1 + v^2)^{\frac{\sigma+\gamma}{2}} F^n(\mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v} \end{aligned}$$

with positive constants $\tilde{C}_0, \tilde{C}_1, \tilde{C}_2 > 0$. Here we have assumed that the function Y in (2.2) satisfies

$$\int_V (1+v_*)^{\gamma+\max(2,\sigma)} \sup_{\mathbf{x}\in D}(Y(\mathbf{x},\mathbf{v}_*))d\mathbf{v}_* < \infty, \int_V \inf_{\mathbf{x}\in D}(Y(\mathbf{x},\mathbf{v}_*))d\mathbf{v}_* > 0. \tag{4.5}$$

To handle the ingoing boundary term, $I_b^-(\sigma)$, in (4.4), we specialize in two physically interesting cases:

a) "Non-heating boundary" (e.g. specular reflection):

$$R(\mathbf{x}, \tilde{\mathbf{v}} \to \mathbf{v}) \equiv 0 \quad \text{for } |\mathbf{v}| > |\tilde{\mathbf{v}}|. \tag{4.6}$$

Here we find that

$$I_b^-(\sigma) \le (1-\beta) \iint_{\Gamma V} (1+v^2)^{\sigma/2} F_+^n(\mathbf{x},\mathbf{v}) |\mathbf{n}\mathbf{v}| d\mathbf{v} d\Gamma.$$

b) Diffuse reflection boundary:

$$R(\mathbf{x}, \tilde{\mathbf{v}} \to \mathbf{v}) = |\mathbf{n}\mathbf{v}| W(\mathbf{x},\mathbf{v}). \tag{4.7}$$

Here we get

$$I_b^-(\sigma) \le (1-\beta) C_{W,\sigma} \iint_{\Gamma V} F_+^n(\mathbf{x},\mathbf{v}) |\mathbf{n}\mathbf{v}| d\mathbf{v} d\Gamma$$

with a constant

$$C_{W,\sigma} = \int_V (1+v^2)^{\sigma/2} \sup_{\mathbf{x}\in\Gamma}(W(\mathbf{x},\mathbf{v}))d\mathbf{v} < \infty. \tag{4.8}$$

Then, the higher moment estimations follow in both the boundary cases, respectively

a) "non-heating boundary":

$$\begin{aligned}
&\alpha \iint_{DV} (1+v^2)^{\sigma/2} F^n(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} + \beta \iint_{\Gamma V} (1+v^2)^{\sigma/2} F_+^n(\mathbf{x},\mathbf{v}) |\mathbf{n}\mathbf{v}| d\mathbf{v} d\Gamma \\
&+ \tilde{C}_0 \iint_{DV} (1+v^2)^{\frac{\sigma+\gamma}{2}} F^n(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} \le \alpha_0 \iint_{DV} (1+v^2)^{\sigma/2} G(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} + \\
&+ \tilde{C}_1 \iint_{DV} (1+v^2)^{\frac{\sigma+\gamma-1}{2}} F^n(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} + \tilde{C}_2 \iint_{DV} (1+v^2)^{\frac{\sigma-1}{2}} F^n(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v},
\end{aligned} \tag{4.9}$$

b) diffuse boundary:

$$\alpha \iint_{DV} (1+v^2)^{\sigma/2} F^n(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} + \iint_{\Gamma V} (1+v^2)^{\sigma/2} F_+^n(\mathbf{x},\mathbf{v}) |\mathbf{nv}| d\mathbf{v} d\Gamma +$$
$$+ \tilde{C}_0 \iint_{DV} (1+v^2)^{\frac{\sigma+\gamma}{2}} F^n(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} \leq \alpha_0 \iint_{DV} (1+v^2)^{\sigma/2} G(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} +$$
$$+ \tilde{C}_1 \iint_{DV} (1+v^2)^{\frac{\sigma+\gamma-1}{2}} F^n(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} + \tilde{C}_2 \iint_{DV} (1+v^2)^{\frac{\sigma-1}{2}} F^n(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} +$$
$$+ (1-\beta) C_{W,\sigma} \iint_{\Gamma V} F_+^n(\mathbf{x},\mathbf{v}) |\mathbf{nv}| d\mathbf{v} d\Gamma, \tag{4.10}$$

where by (3.6)

$$\beta \iint_{\Gamma V} F_+^n(\mathbf{x},\mathbf{v}) |\mathbf{nv}| d\mathbf{v} d\Gamma \leq \alpha_0 \iint_{DV} G(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v}.$$

Letting $n \to \infty$ and using that $F^n(\mathbf{x},\mathbf{v}) \nearrow F(\mathbf{x},\mathbf{v})$, then the estimates (4.9) and (4.10) hold also for $F(\mathbf{x},\mathbf{v})$. Now by (3.8), the moment for $\sigma = 0$, i.e., the mass $\int\int F d\mathbf{x} d\mathbf{v}$, is bounded. So, successively, by (4.9), (4.10), we get boundedness of all higher velocity moments, $\sigma > 0$, for both soft and hard collision forces, $-1 < \gamma \leq 1$, i.e., $-3 < k \leq \infty$, and for both "non-heating" and diffuse boundaries. Then the following theorem for higher velocity moments holds for our solution also in the inelastic case.

Theorem 4.1 *Assume that the collision function $B(\theta, w)$ is given for inverse k-th power forces by equation (2.10) with $3 < k \leq \infty$, i.e., $-1 < \gamma \leq 1$, and suppose that the function $Y(\mathbf{x}, \mathbf{v}_*)$ satisfies (4.5). Let the boundary relation (2.11) be given by a "non-heating" boundary (4.6), (e.g. specular reflection), or by diffuse reflections (4.7) with (4.8).*

Then the higher velocity moments belonging to the mild solution F in Theorem 3.2 (a), are all bounded,

$$\iint_{DV} (1+v^2)^{\sigma/2} F(\mathbf{x},\mathbf{v}) d\mathbf{x} d\mathbf{v} < \infty;$$

i.e., for all $\sigma > 0$, $0 < a \leq 1$, $\alpha, \alpha_0 > 0$, $0 < \beta \leq 1$, and $-1 < \gamma \leq 1$, if $(1+v^2)^{\sigma/2} G(\mathbf{x},\mathbf{v}) \in L^1(D \times V)$.

We will now finish this section by proving that our solution F satisfies the assumption $LF \in L^1(D \times V)$, giving existence of traces at the boundary together with uniqueness and mass-conservation; cf. Theorem 3.2 (b) and Remark 2 in Section 3. This result holds for both hard and soft inverse collision forces.

Corollary 4.2 *The solution $F = F_a(\mathbf{x}, \mathbf{v})$ in Theorem 3.2 (a) satisfies*

$$L(\mathbf{x}, \mathbf{v})F(\mathbf{x}, \mathbf{v}) \in L^1(D \times V), \tag{4.11}$$

if a) in the hard force case $\gamma = (k-5)/(k-1) \geq 0$, the assumptions of Theorem 4.1 are satisfied together with

$$(1 + v_*)^\gamma \sup_{\mathbf{x} \in D}(Y(\mathbf{x}, \mathbf{v}_*)) \in L^1(V), \tag{4.12}$$

$$(1 + v^2)^{\gamma/2} G(\mathbf{x}, \mathbf{v}) \in L^1(D \times V), \tag{4.13}$$

and if b) in the soft force case, $-1 < \gamma = (k-5)/(k-1) < 0$,

$$\sup_{\mathbf{x} \in D}(Y(\mathbf{x}, \mathbf{v}_*)) \in L^1(V) \cap L^\infty(V). \tag{4.14}$$

Proof. We estimate the collision frequency as follows

a)

$$\begin{aligned} L(\mathbf{x}, \mathbf{v}) &= \iint_{V\Omega} b(\theta)|\mathbf{v} - \mathbf{v}_*|^\gamma Y(\mathbf{x}, \mathbf{v}_*) d\mathbf{v}_* d\theta d\zeta \leq \\ &\leq \iint b(\theta)(1 + v^2)^{\gamma/2}(1 + v_*)^\gamma Y(\mathbf{x}, \mathbf{v}_*) d\mathbf{v}_* d\theta d\zeta \leq \\ &\leq \text{const.}(1 + v^2)^{\gamma/2}, \end{aligned}$$

and then we use Theorem 4.1.

b) For soft forces, the collision frequency is bounded,

$$\begin{aligned} L(\mathbf{x}, \mathbf{v}) &= \iint_{V\Omega} b(\theta) w^\gamma Y(\mathbf{x}, \mathbf{v}_*) d\mathbf{v}_* d\theta d\zeta = \\ &= \int_\Omega b(\theta)[\int_{w<1} w^\gamma Y(\mathbf{x}, \mathbf{v}_*) d\mathbf{v}_* + \int_{w \geq 1} w^\gamma Y(\mathbf{x}, \mathbf{v}_*) d\mathbf{v}_*] d\theta d\zeta \\ &\leq 2\pi \int_0^{\pi/2} b(\theta) d\theta [\sup_{\mathbf{x}, \mathbf{v}_*}(Y(\mathbf{x}, \mathbf{v}_*)) \int_{w<1} w^\gamma dw + \int_V \sup_{\mathbf{x}}(Y(\mathbf{x}, \mathbf{v}_*)) d\mathbf{v}_*] \\ &\leq \text{constant}, \end{aligned}$$

so $\iint LF d\mathbf{x} d\mathbf{v} \leq \text{const.} \iint G d\mathbf{x} d\mathbf{v}$.

$\square$

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