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PREPRINT

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MATS ANDERSSON

Department of Mathematics
CHALMERS UNIVERSITY OF TECHNOLOGY
GÖTEBORG UNIVERSITY
Göteborg Sweden 2004

Preprint 2004:4

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Mats Andersson

CHALMERS | GÖTEBORGS UNIVERSITET

Mathematics
Department of Mathematics
Chalmers University of Technology and Göteborg University
SE-412 96 Göteborg, Sweden
Göteborg, January 2004

NO 2004:4
ISSN 0347-2809

Matematiska Vetenskaper
Göteborg 2004

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Abstract. We prove the following generalization of the classical Poincaré-Lelong formula. Given a holomorphic section f with zero set Z to a Hermitian vector bundle $E \to X$, let S be the line bundle over $X \setminus Z$ spanned by f and let $Q = E/S$. We prove that the Chern form $c(D_Q)$ is locally integrable and closed in X and that there is a current W such that $dd^c W = c(D_E) - c(D_Q) - M$, where M is a current with support on Z. In particular this implies that the top Bott-Chern class is represented by a current with support on Z. We also present a variant where we have several sections f_j to E. This leads to a current representation of lower order Bott-Chern classes.

1. Introduction

Let f be a holomorphic (or meromorphic) section to the Hermitian line bundle $L \to X$, and let $[Z]$ be the current of integration over the divisor Z defined by f. Then the Poincaré-Lelong equation states that

$$dd^c \log(1/|f|) = c_1(D_L) - [Z],$$

where $c_1(D_L)$ is the first Chern form associated with the Chern connection D_L on L, i.e., $c_1(D_L) = \alpha\Theta_L$, where Θ_L is the curvature; here and throughout this paper $\alpha = i/2\pi$. Our main result is the following generalization.

Theorem 1.1. *Let f be a holomorphic section to the Hermitian vector bundle $E \to X$ of rank m. Let $Z = \{f = 0\}$, let S denote the (trivial) line bundle over $X \setminus Z$ generated by f, and let $Q = E/S$.*

(i) The Chern form $c(D_Q)$ is in $L^1_{loc}(X)$ and closed as a current in X.

(ii) There is a current W of bidegree $(*, *)$ and order zero in X which is smooth in $X \setminus Z$ and with logarithmic singularity at Z, such that

$$dd^c W = c(D_E) - C(D_Q) - M, \tag{1.1}$$

where $C(D_Q)$ denote the natural extension of $c(D_Q)$, and M is a current of order zero with support on Z. More precisely,

$$M = M_p + \cdots + M_{\min(m,n)},$$

where M_k has bidegree (k, k) and $p = \operatorname{codim} Z$. Furthermore,

$$M_p = [Z^p],$$

Date: January 15, 2004.

The author was partially supported by the Swedish Natural Science Research Council.

where $[Z^p]$ *is the current of integration over the components of* Z *(counted with multiplicities) of codimension precisely* p.

(iii) The forms $\log|f|c(D_Q)$ *and*

$$|f|^{2\lambda}\frac{\alpha\partial|f|^2\wedge\bar\partial|f|^2}{|f|^4}\wedge c(D_Q), \quad \lambda>0, \tag{1.2}$$

are locally integrable in X *and*

$$M_k=\lim_{\lambda\to 0^+}\lambda|f|^{2\lambda}\frac{\alpha\partial|f|^2\wedge\bar\partial|f|^2}{|f|^4}\wedge c_{k-1}(D_Q)=dd^c(\log|f|c_{k-1}(D_Q))\mathbf{1}_Z.$$

Here $c_k(D)$ denotes the k:th Chern form with respect to the Chern connection D associated to the Hermitian structure, i.e., $c_k(D)$ is the component of bidegree (k,k) of the total Chern form $\det(\alpha\Theta+I)$, where $\Theta=D^2$ is the curvature tensor.

For the explicit expression for W, see the proof in Section 5. If W_k denotes the component of bidegree (k,k) then (1.1) means that

$$dd^cW_{k-1}=c_k(D_E)-c_k(D_Q)-M_k. \tag{1.3}$$

Since Q has rank $m-1$, $c_m(D_Q)=0$, and therefore

$$dd^cW_{m-1}=c_m(D_E)-M_m,$$

which means that the current M_m represents the top degree Bott-Chern class $\hat{c}_m(E)$. Of course this implies in particular that M_m represents the usual Chern class $c_m(E)$; this was proved already in [1]. It also follows that the Bott-Chern class $\hat{c}_k(E)$ is equal to $\hat{c}_k(Q)$ if $k<p$.

If f is a complete intersection, i.e., $p=m$, and W_{m-1} denotes the component of bidegree $(m-1,m-1)$, then

$$dd^cW_{m-1}=c_m(D_E)-[Z]; \tag{1.4}$$

this means that W_{m-1} is a Green current for Z.

If E is a line bundle, then $W=W_0=\log(1/|f|)$ so (1.3) is the usual Poincaré-Lelong formula.

In the recent paper [7], Meo proves (1.3) for $k=p$ under an extra integrability assumption.

Clearly M_p is always a positive current. It also follows from part (iii) that M_k is positive if $c_{k-1}(D_Q)$ is a positive form. For an even more precise formula for M, see Proposition 6.5 below. Let us say that E is positive if E^* is Nakano negative. It then follows that Q is positive, and this in turn implies that $c(Q)$ is positive. In particular then M is positive. We also have

Theorem 1.2. *If* E *in Theorem 1.1 is positive, then (one can choose* W *such that)* $W\le 0$ *where* $|f|\le 1$.

As an application, assume that X is compact, and that we have sections f_j to rank m_j bundles $E_j \to X$, such that $\sum m_j = n$. If $E = \oplus E_j$ and $f = \oplus f_j$, then the intersection number ν of the varieties $Z_j = \{f_j = 0\}$ is equal to the integral of $c_n(E)$ over X. Since M_n represents the class $\hat{c}_n(M)$, we thus get the representation

$$\nu = \int_X M_n,$$

i.e., an integral over the intersection $Z = \cap Z_j$. If E is positive then M_n is positive. If Z is discrete, then the formula is just the sum of the points in Z counted with multiplicities.

We also consider the possibility to represent lower order Chern classes with residue currents. If E has a trivial subbundle of rank r, then $c_k(E)$ vanishes for $k > m - r$. Therefore it is natural to guess that these classes $c_k(E)$ could be represented by currents with support on the set where r given sections $f_1, \ldots, f_r$ are linearly dependent. In fact we have

Theorem 1.3. *Let $f = f_1, \ldots, f_r$ be holomorphic sections to the Hermitian vector bundle $E \to X$, and let*

$$Z = \{f_1 \wedge \ldots \wedge f_r = 0\}.$$

Let S be the r-bundle over $X \setminus Z$ generated by f_j, and let $Q = E/S$.

(i) The Chern form $c(D_Q)$ has an extension $C(D_Q)$ as a current across Z, defined by analytic continuation of $\lambda \mapsto |h|^{2\lambda} c(D_Q)$ to $\lambda = 0$, where $h = f_1 \wedge \ldots\ f_r$. If, in addition, locally

$$(1.5) \qquad |f_1 \wedge \ldots \wedge f_r| \geq \delta |f_1| \cdots |f_r| \quad \text{for some} \quad \delta > 0,$$

then $c(D_Q)$ is locally integrable in X and $C(D_Q)$ is its natural extension.

(ii) There is a current M of bidegree $(*,*)$ with support on Z, and a current A of bidgree $(*,*-1)$ (smooth outside Z) such that

$$(1.6) \qquad dA = \bar{\partial} A = c(D_E) - C(D_Q) - M.$$

If $k < \operatorname{codim} Z$, then $M_k = 0$. Moreover, if (1.5) holds, then M has order zero and A is locally intergrable in X.

It follows that

$$(1.7) \qquad dA_k = c_k(D_E) - M_k, \quad k > m - r,$$

and thus M_k is a current representative for the Chern class $c_k(E)$, for $k > m - r$.

The condition (1.5) holds whenever the sections f_j take values in linearly independent subbundles. In particular, it is always fulfilled when $r = 1$. It turns out that (1.5) holds if and only if, locally, there is a resolution over which S has a trivial extension across the singularity, see Section 5.

We can also extend Theorem 1.1 to several sections, provided that (1.5) holds.

Theorem 1.4. *Under the same hypotheses as in Theorem 1.3 together with* (1.5)*, there is a current* W *of bidegree* $(*,*)$ *and order zero, such that*

$$dd^cW = c(D_E) - C(D_Q) - M,$$

where M *is a current of order zero with support on* Z*.*

Let us briefly describe how Theorem 1.1 is proved. Consider the exact sequence

$$0 \to S \xrightarrow{j} E \xrightarrow{g} Q \to 0 \tag{1.8}$$

of vector bundles over $X \setminus Z$. Notice that jj^* is the orthogonal projection onto S (or rather onto jS). Following [3] we deform the Chern connection D on E to $D_b = D - D'_{\mathrm{End}E}(jj^*)$; then $c(D_b) = c(D_S)c(D_Q)$ and one can find a form v in $X \setminus Z$ such that

$$\alpha\partial v = b \quad \text{and} \quad db = c(D_E) - c(D_S)c(D_Q).$$

We introduce another deformation D_a of D such that Θ_a vanishes on S, and therefore $c(D_a) = c(D_Q)$. This makes it possible to compute an explicit formula for $c(D_Q)$. Clearly $c(D_Q)$ is smooth in $X \setminus Z$, and it turns out that in a suitable desingularization it is actually smooth across the singularity. It follows that $c(D_Q)$ is locally integrable in X.

It also turns out that v is locally integrable, actually smooth across Z in a suitable resolution, and therefore $\alpha\partial V = B$, if V and B denote the natural extension to X. By the usual Poincaré-Lelong formula, $c(D_S) - 1 = dd^c\log(1/|f|)$, and since $C(D_Q)$ is smooth in a suitable resolution, it turns out that (1.1) holds with

$$W = \log(1/|f|)C(D_Q) - V,$$

and $M = dd^c(\log|f|C(D_Q))\mathbf{1}_Z$. Theorem 1.2 then follows essentially by applying the ideas in [3].

To prove Theorem 1.3 for $r = 1$ we find a form a in $X \setminus Z$ such that $a = c(D_E) - c(D_Q)$, and we prove that a is locally integrable in X. If A denotes its extension across Z, then

$$dA = C(D_E) - C(D_Q) - M',$$

where M' is a residue current with the stated properties. We can also explicitly reveal that $b = a - \alpha\partial\log|f|^2 \wedge c(D_Q)$, and this implies that $M = M'$.

Theorems 1.3 and 1.4 are proved along the same lines, but the analysis of the singularities of the various forms is more involved. Moreover, for the proof of Theorem 1.4 we also have to check that for a suitable solution u to $dd^cu = c(D_S) - 1$ in $X \setminus Z$, the form $u \wedge c(D_Q)$ has a a current extension to X. As before we can then take $W = U \wedge C(D_Q) - V$.

The plan of this paper is as follows. We first recall the definition of Chern classes and introduce some notation and technical tools which we use to prove Theorem 1.3 in Section 3. We then recall the techniques developed by Bott and Chern in [3] to define Bott-Chern classes. In Section 4 we complete the proofs of Theorems 1.1 and 1.4. Finally, we discuss positivity of Hermitian vector bundles and the proof of Theorem 1.2.

Contents

2. Differential geometric definition of Chern classes

We first have to recall the differential geometric definition of the Chern classes. Let $E \to X$ be any differentiable complex vector bundle over a differential manifold X, with connection $D\colon \mathcal{E}_k(X,E) \to \mathcal{E}_{k+1}(X,E)$ and curvature tensor $D^2 = \Theta \in \mathcal{E}_2(X, \mathrm{End}E)$. The connection $D = D_E$ induces in a natural way a connection $D_{\mathrm{End}E}$ on the bundle $\mathrm{End}E$ by the formula $Dg \cdot \xi = D(g \cdot \xi) - g \cdot D\xi$, and in a similar way there is a natural connection D_{E^*} on the dual bundle E^*, etc. In particular we have Bianchi's identity

$$D_{\mathrm{End}E}\Theta = 0. \tag{2.1}$$

If I denotes the identity mapping on E, then $c(D) = \det(\alpha\Theta + I)$ is a welldefined differential form whose terms have even degrees, which is called the Chern form of D. It is a basic fact that $c(D)$ is a closed form. Moreover its de Rham cohomology class is independent of D and is called the (total) Chern class $c(F)$ of the bundle F.

To prove this, one can consider a smooth one-parameter family D_t of connections of F with $D_0 = D$. If E' is the pull-back of E to $X \times [0,1]$, then $D' = D_t + d_t$ is a connection on E' and its curvature tensor is

$$\Theta' = \Theta_t + dt \wedge \dot{D}_t$$

where $\dot{D}_t = dD_t/dt$. It is readily checked that it is an element in $\mathcal{E}_1(X, \mathrm{End}(F))$. Since $(d + d_t)\det(\alpha\Theta' + I) = 0$ we have that

$$d_\zeta \int_0^1 \det(\alpha\Theta' + I) = -\int_0^1 d_t \det(\alpha\Theta' + I) = c(D) - c(D_1).$$

In order to make the computation more explicit we introduce the exterior algebra bundle $\Lambda = \Lambda(T^*(X) \oplus F \oplus F^*)$. Any section $\xi \in \mathcal{E}_k(X, F)$ corresponds to a section $\tilde{\xi}$ of Λ; if $\xi = \xi_1 \otimes e_1 + \ldots + \xi_m \otimes e_m$ in a local frame e_j, then we let $\tilde{\xi} = \xi_1 \wedge e_1 + \ldots + \xi_m \wedge e_m$. In the same way, $a \in \mathcal{E}_k(X, \mathrm{End}E)$ can be identified with

$$\widetilde{a} = \sum_{jk} a_{jk} \wedge e_j \wedge e_k^*,$$

if e_j^* is the dual frame, and $a = \sum_{jk} a_{jk} \otimes e_j \otimes e_k^*$ with respect to these frames. A given connection $D = D_F$ on F extends in a unique way to a linear mapping $\mathcal{E}(X, \Lambda) \to \mathcal{E}(X, \Lambda)$ which is a an anti-derivation with respect to the wedge product in Λ, and such that it acts as the exterior differential d on the $T^*(X)$-factor. It is readily seen that

$$\widetilde{D_E \xi} = D\tilde{\xi},$$

if ξ is a form-valued section to E. In the same way we have

Lemma 2.1. *If $a \in \mathcal{E}_k(X, \mathrm{End}E)$, then*

$$\widetilde{D_{\mathrm{End}E} a} = D\widetilde{a}. \tag{2.2}$$

Proof. If $\xi \in \mathcal{E}_k(X, E)$ and $\eta \in \mathcal{E}(X, E^*)$, then

$$D_{\mathrm{End}E}(\xi \otimes \eta) = D_E \xi \otimes \eta + (-1)^k \xi \otimes D_{E^*} \eta,$$

and thus the snake of $D_{\mathrm{End}E}(\xi \otimes \eta)$ is equal to

$$\widetilde{D_E \xi} \wedge \eta + (-1)^{k+1} \tilde{\xi} \wedge \widetilde{D_{E^*} \eta} = D(\tilde{\xi} \wedge \eta)$$

as claimed. □

Since $D_{\mathrm{End}E} I = 0$, $(I = I_E)$ we have from (2.1) and Lemma 2.1 that

$$D\widetilde{\Theta} = 0 \quad \text{and} \quad D\tilde{I} = 0. \tag{2.3}$$

We let $\tilde{I}_m = \tilde{I}^m/m!$ and use the same notation for other forms in the sequel. Any form ω with values in Λ can be written $\omega = \omega' \wedge \tilde{I}_m + \omega''$ uniquely, where ω'' has lower degree in e_j, e_k^*. If we define

$$\int_e \omega = \omega',$$

then this integral is of course linear and moreover

$$d \int_e \omega = \int_e D\omega. \tag{2.4}$$

In fact, since $D\tilde{I}_m = 0$,

$$\int_e D\omega = \int_e d\omega' \wedge \tilde{I}_m + D\omega'' = d\omega' = d\int_e \omega.$$

Observe that

$$c(D) = \int_e (\alpha\widetilde{\Theta} + \tilde{I})_m = \int_e e^{\alpha\tilde{\Theta}+\tilde{I}}. \tag{2.5}$$

Lemma 2.1 and (2.3) together imply that the Chern form $c(D)$ is closed. Furthermore, following the outline above, we get the formula

$$d\int_0^1 \int_e \alpha\widetilde{\dot{D}} \wedge e^{\alpha\widetilde{\Theta}_t+\tilde{I}} = c(D_1) - c(D_0), \tag{2.6}$$

thus showing that $c(D_0)$ and $c(D_1)$ are cohomologous. For further reference we notice that

$$d\int_0^1 \int_e \alpha\widetilde{\dot{D}} \wedge e^{\alpha\tilde{\Theta}_t} \wedge \tilde{I}_{m-k} = c_k(D_1) - c_k(D_0); \tag{2.7}$$

this follows by precisely the same argument.

Recall that if the connection D is modified to $D_1 = D - \gamma$, where $\gamma \in \mathcal{E}_1(X, \mathrm{End}E))$, then $\Theta_1 = \Theta - D_{\mathrm{End}E}\gamma + \gamma \wedge \gamma$. If we form the explicit homotopy $D_t = D - t\gamma$, therefore

$$\Theta_t = \Theta - tD_{\mathrm{End}E}\gamma + t^2\gamma \wedge \gamma \tag{2.8}$$

and hence, by Lemma 2.1,

$$\widetilde{\Theta}_t = \widetilde{\Theta} - tD\tilde{\gamma} + t^2\widetilde{\gamma \wedge \gamma}. \tag{2.9}$$

3. Chern classes defined by residue currents

From now on we assume that E is a holomorphic Hermitian bundle and that D_E is the Chern connection and D'_E is its $(1,0)$-part. Then the induced connection D_{E^*} on E^* is the Chern connection on E^* etc. In particular, our mapping D on Λ is of type $(1,0)$, i.e., $D = D' + \bar{\partial}$.

Assume that S is a trivial subbundle of E, generated by the sections $f_1, \ldots, f_r$. The reader who is mainly interested in Theorem 1.1 may assume that $r = 1$ throughout this section; in this way several technicalities disappear.

Let σ_j be the unique sections to the dual bundle E^* such that σ_j vanishes on $S^\perp$ and $\sigma_j \cdot f_k = \delta_{jk}$. Consider the new connection $D_a = D - \gamma_a$ on E, where

$$\tilde{\gamma}_a = \sum_j Df_j \wedge \sigma_j. \tag{3.1}$$

Then $D_a f_j = 0$, and so $D\xi$ is in S if ξ is a section to S. Moreover, if ξ is a section to $S^\perp$, then $D_a\xi = D_E\xi$. Let $g\colon E \to Q$ be the natural projection. Then

$$g\xi \mapsto g(D_a\xi) \tag{3.2}$$

is a welldefined connection on Q, and we claim that it is actually the Chern connection D_Q. In fact, if $\eta = g\xi$, then

$$D_Q\eta = g(D_E(g^*\eta)) = g(D_a(g^*\eta)) = g(D_a\xi).$$

It follows that $\Theta_Q\eta = g(\Theta_a\xi)$, and since $\Theta_a\xi = 0$ if ξ takes values in S, we have that

$$\alpha\Theta_a \sim \begin{pmatrix} 0 & * \\ 0 & \alpha\Theta_Q \end{pmatrix}$$

with respect to the smooth isomorphism $E \simeq S \oplus Q$ defined by $j^* \oplus g$, where j^* is the orthogonal projection. Therefore,

$$\alpha\Theta_a + I_E = \begin{pmatrix} I_S & * \\ 0 & I_Q + \alpha\Theta_Q, \end{pmatrix},$$

and taking the determinant, we find that

$$c(D_Q) = c(D_a). \tag{3.3}$$

The natural conjugate–linear isometry $E \simeq E^*$, $\eta \mapsto \eta^*$, defined by

$$\eta^* \cdot \xi = \langle \xi, \eta \rangle, \quad \xi \in \mathcal{E}(X, E),$$

extends to an isometry on the space of form–valued sections.

Lemma 3.1. *There is a matrix of* $(1,0)$*-forms* ϕ_{jk} *in* $X \setminus Z$ *such that*

$$D'\sigma_j = \sum_k \phi_{jk} \wedge \sigma_k. \tag{3.4}$$

Thus $\Pi_{S^*} D_{E^*} = D_{S^*}$, and so ϕ_{jk} is just the connection matrix for D_{S^*} with respect to the frame σ_j.

Proof. We must see that if ξ is a section to S^*, then $D'_{E^*}\xi$ is still a section to E^*. By duality this is equivalent to that D_E takes sections to $S^\perp$ to sections to $S^\perp$. However, this is easily seen to hold because if η is a section to $S^\perp$ and ξ is a section to S, then $\langle \xi, D'\eta \rangle = -\langle \bar\partial\xi, \eta \rangle = 0$, since $\bar\partial\xi$ is a section to S. □

Lemma 3.2. *If* D_E *is the Chern connection on* E*, then*

$$D'_{E^*}\eta^* = (\bar\partial\eta)^*, \quad \eta \in \mathcal{E}_k(X, E).$$

We omit the simple proof. If $r = 1$, then $\sigma = f^*/|f|^2$, and by Lemma 3.2, $D'f^* = (\bar\partial f)^* = 0$, so we have

$$D'\sigma = D'(f^*/|f|^2) = \partial\frac{1}{|f|^2} \wedge f^* = -\partial \log |f|^2 \wedge \sigma,$$

so that

$$\phi = -\partial \log |f|^2.$$

In the sequel we often use the shorthand notation $D'\sigma = \phi \wedge \sigma$ etc. We also often write ξ rather than $\tilde\xi$ when ξ is a form-valued section to E or E^*; for instance we write Θf even when we mean $\widetilde{\Theta f}$.

Proposition 3.3. *If* γ_a *is defined by* (3.1)*, then*

$$-tD\tilde\gamma_a + t^2\tilde\gamma_a \wedge \tilde\gamma_a = -t(Df \wedge \bar\partial\sigma + \Theta f \wedge \sigma) + (t - t^2)Df \wedge \phi \wedge \sigma. \tag{3.5}$$

Proof. To begin with,

$$D\tilde{\gamma}_a = \sum_j \Theta f_j \wedge \sigma_j + Df_j \wedge \bar\partial\sigma_j + \sum_{jk} Df_j \wedge \phi_{jk} \wedge \sigma_k,$$

and

$$\widetilde{\gamma_a \wedge \gamma_a} = \sum_j Df_j \wedge \sigma_j \cdot Df_k \wedge \sigma_k,$$

where the dot means the natural contraction of E and E^* so that $\xi \cdot (\alpha \wedge \eta) = \alpha(\xi \cdot \eta)$ if ξ and η are sections to E and E^*, respectively, and α is a form. Since $\sigma_j \cdot Df_k = -D'\sigma_j \cdot f_k = \phi_{jk}$ we get the desired formula. □

The following formula is the key point in the analysis of the singularities of $c(D_Q)$.

Proposition 3.4. *We have the explicit formula*

$$c(D_Q) = \int_e f_1 \wedge \sigma_1 \wedge \ldots \wedge f_r \wedge \sigma_r \wedge e^{\alpha\tilde\Theta + \tilde I - \alpha \sum Df_j \wedge \bar\partial\sigma_j} \tag{3.6}$$

in $X \setminus Z$.

Proof. Since $\Theta_a = \Theta - D_{\mathrm{End}E}\gamma_a + \gamma_a \wedge \gamma_a$, cf., (2.8), we have that

$$\tilde\Theta_a = \tilde\Theta - \sum \big(\Theta f_j \wedge \sigma_j + Df_j \wedge \bar\partial\sigma_j\big).$$

By (2.5) and (3.3) we know that

$$c(D_Q) = c(D_a) = \int_e e^{\alpha\tilde\Theta_a + \tilde I}. \tag{3.7}$$

Let δ_{f_j} denote interior multiplication with f_j. Then

$$\delta_{f_j}\big(\sum_j Df_j \wedge \bar\partial\sigma_j\big) = 0, \quad \delta_{f_j}\tilde I = -f_j, \quad \delta_{f_j}\tilde\Theta = -\Theta f_j, \tag{3.8}$$

and

$$\delta_{f_j}(1 - \alpha\Theta f_k \wedge \sigma_k) = 0, \quad k \neq j. \tag{3.9}$$

For degree reasons

$$e^{-\alpha\sum \Theta f_j \wedge \sigma_j} = (1 - \alpha\Theta f_1 \wedge \sigma_1) \wedge \ldots \wedge (1 - \alpha\Theta f_r \wedge \sigma_r),$$

and therefore

$$c(D_Q) = \int_e (1 - \alpha\Theta f_1 \wedge \sigma_1) \wedge \ldots \wedge (1 - \alpha\Theta f_r \wedge \sigma_r) e^{\alpha\tilde\Theta} \wedge e^{\tilde I - Df \wedge \bar\partial\sigma}.$$

Let $L_k = 1 - \alpha\Theta f_k \wedge \sigma_k$. By (3.8) and (3.9) we have

$$\int_e (1 - \alpha\Theta f_j \wedge \sigma_j) \wedge e^{\alpha\tilde\Theta} \wedge L_2 \wedge \ldots \wedge L_r \wedge e^{\tilde I - Df \wedge \bar\partial\sigma}$$
$$= \int_e \Big(e^{\alpha\tilde\Theta} - \delta_{f_1} e^{\alpha\tilde\Theta} \wedge \sigma_1\Big) \wedge L_2 \wedge \ldots \wedge L_r \wedge e^{\tilde I - Df \wedge \bar\partial\sigma}$$
$$= -\int_e \sigma_1 \wedge e^{\alpha\tilde\Theta} \wedge L_2 \wedge \ldots \wedge L_r \wedge e^{\tilde I - Df \wedge \bar\partial\sigma} \wedge f_1,$$

where the last equality follows from an integration by parts. Repeating this procedure for each of the factors $L_2, \ldots, L_r$ we end up with (3.6). □

Proposition 3.5. *If*

$$a = \int_e \alpha Df \wedge \sigma \wedge e^{\tilde{I}+\alpha\tilde{\Theta}} \wedge \sum_{\ell=0}^{\infty} \frac{(-1)^\ell(\alpha\Theta f \wedge \sigma + \alpha Df \wedge \bar{\partial}\sigma))^\ell}{(\ell+1)!} \tag{3.10}$$

then

$$da = c(D_E) - c(D_Q) \tag{3.11}$$

in $X \setminus Z$.

Proof. We choose the homotopy $D_t = D - t\gamma_a$ between $D = D_0$ and $D_1 = D_a$. In view of (2.6) and Proposition 3.3 we have that

$$\hat{a} = \int_e \int_0^1 \alpha Df \wedge \sigma \wedge e^{\tilde{I}+\alpha\tilde{\Theta}-t\alpha(\Theta f\wedge\sigma+Df\wedge\bar{\partial}\sigma)-(t-t^2)Df\wedge\phi\wedge\sigma} dt$$

satisfies (3.11) in $X \setminus Z$. We claim that actually the component of bidegree $(*, * - 1)$,

$$a = -\int_e \int_0^1 \alpha Df \wedge \sigma \wedge e^{\tilde{I}+\alpha\tilde{\Theta}-t\alpha(\Theta f\wedge\sigma+Df\wedge\bar{\partial}\sigma)} dt,$$

satisfies (3.11). For bidegree reasons it immediately follows that

$$da_{m,m-1} = \bar{\partial}a_{m,m-1} = c_m(D_E) - c_m(D_Q),$$

since $\partial a_{m,m-1} = \bar{\partial}_{m+1,m-2} = 0$. For any k we have, cf., (2.7), that

$$d\int_0^1 \int_e \alpha Df \wedge \sigma \wedge e^{\alpha\tilde{\Theta}-t\alpha(\Theta f\wedge\sigma+Df\wedge\bar{\partial}\sigma)-(t-t^2)Df\wedge\phi\wedge\sigma} \wedge \tilde{I}_{m-k} = c_k(D_E) - c_k(D_Q),$$

and for the same reason this must hold even if the $(2,0)$-form $(t - t^2)Df \wedge \phi \wedge \sigma$ is deleted in the exponent. Computing the t-integral we get (3.10). □

Notice that the natural conjugate linear isometry $E \simeq E^*$, $\eta \mapsto \eta^*$, extends to a conjugate linear isometry

$$\Lambda E \simeq \Lambda E^*, \quad f \wedge g \mapsto (f \wedge g)^* = f^* \wedge g^*.$$

Lemma 3.6. *Let* $f_1, \ldots, f_r$ *be given sections to* E *and define* $s_\ell \in E^*$ *by*

$$s_\ell \cdot \xi = \int_e f_r \wedge \ldots \wedge \xi \wedge \ldots \wedge f_1 \wedge f_1^* \wedge f_2^* \wedge \ldots \wedge f_r^* \wedge \tilde{I}_{m-r},$$

where ξ *is put on place* ℓ. *Then*

$$s_\ell f_\ell = |f_1 \wedge \ldots \wedge f_r|^2, \tag{3.12}$$

$$s_\ell \cdot f_k = 0, \quad k \neq \ell, \tag{3.13}$$

and

$$s_\ell \cdot \xi = 0, \quad \xi \in S^\perp, \tag{3.14}$$

if S *is the subspace (subbundle) spanned by* $f_1, \ldots, f_r$.

Proof. Let $e_1, \dots, e_m$ be an ON-basis (frame) for E such that $e_1, \dots, e_r$ is a basis for S. Then, if $\xi \in S^\perp$, $\xi = \xi_{r+1} e_{r+1} \wedge \dots \wedge \xi_m e_m$, and thus (3.14) holds. Moreover, (3.13) is obvious, and (3.12) follows easily by expressing the f_j in the frame e_j. □

It follows that

$$\sigma_j = s_j / |h|^2, \tag{3.15}$$

where h is the section $h = f_1 \wedge \dots \wedge f_r$ to $\Lambda^r E$.

Proof of Theorem 1.3. Consider the formula (3.10) for a. There is a certain homogeneity property in σ_j. Suppose that $\sigma_j = \phi \sigma_j'$ for some scalar function ϕ. Since $\sum_j Df_j \wedge \sigma_j$ has odd degree, we can replace $\sum_j Df_j \wedge \bar\partial \sigma_j$ by $\phi \sum_j Df_j \wedge \bar\partial \sigma_j'$, since $\bar\partial\phi \wedge \sum_j Df_j \wedge \sigma_j'$ will be cancelled. Clearly $|h|^{2\lambda} a$ is a welldefined current across Z if $\operatorname{Re}\lambda$ is large enough. We are going to show that it has an analytic continuation to $\operatorname{Re}\lambda > -\epsilon$, and we will define the current A as

$$A = |h|^{2\lambda} a|_{\lambda=0};$$

it will then coincide with a outside Z.

Notice that $h = f_1 \wedge \dots \wedge f_r$ is a holomorphic section to the bundle $\Lambda^r E$ over X. To obtain the analytic continuation we will rely on Hironaka's theorem. It is clearly a local question so we may assume that $h = h_1 \epsilon_1 + \dots + h_\nu \epsilon_\nu$ for some local holomorphic frame ϵ_j to $\Lambda^r E$. In a small neighborhood U of a given point in X, Hironaka's theorem provides an n-dimensional complex manifold $\widetilde U$ and a proper mapping $\Pi\colon \widetilde U \to U$ which is a biholomorphism outside $\Pi^{-1}(\{h_1 \cdots h_\nu = 0\})$, and such that locally on $\widetilde U$ there are holomorphic coordinates τ such that $\Pi^* h_j = u^j \tau_1^{\alpha-1} \cdots \tau_n^{\alpha_n}$, where u_j nonvanishing; i.e., roughly speaking $\Pi^* h_j$ are monomials. Since this set has Lebesgue measure zero it is enough to find the analytic continuation of $\lambda \mapsto \Pi^*(|h|^{2\lambda} a)$. One then define $|h|^{2\lambda} a$ for small $\operatorname{Re}\lambda$ as the push-forward of $\Pi^*(|h|^{2\lambda} a)$.

However, the analytic continuation of $\Pi^*(|h|^{2\lambda} a)$ is a local question in $\widetilde U$, and locally, by a resolution over a suitable toric manifold, see, e.g., [8], we may assume in the same way that one of the functions so obtained divides the other ones. For simplicity we will make a slight abuse of notation and suppress all occurring Π^* and thus denote these functions by h_j as well. We may therefore assume that $h = h^0 h'$ where h^0 is a holomorphic function and h' is nonvanishing. Observe that then $s_j = \bar h^0 s_j'$ and $|h| = |h^0||h'|$, and therefore

$$\sigma_j = \sigma' / h^0 \tag{3.16}$$

where σ' is smooth. In view of the homogeneity property decribed above, we get that $|h|^{2\lambda} a$ is a sum of terms like

$$(|h^0|^2 v)^\lambda \frac{\alpha_\ell}{(h^0)^\ell},$$

where v is a strictly positive function and α_ℓ is a smooth form. It is wellknown that such a form has the desired analytic continuation to

$\operatorname{Re}\lambda > -\epsilon$, see, e.g., [1]. At $\lambda = 0$ it is just the well-known principal value residue current

$$\Big[\frac{1}{(h^0)^\ell}\Big]\alpha_\ell.$$

In view of the discussion above, this proves that the current A is welldefined. In the same way, in view of formula (3.6) for $c(D_Q)$ one finds that $|h|^{2\lambda}c(D_Q)$ can be analytically continued to $\lambda = 0$. Since

$$d(|h|^{2\lambda}a) = |h|^{2\lambda}c(D_E) - |h|^{2\lambda}c(D_Q) + d|h|^{2\lambda}\wedge a$$

by Proposition 3.5, we get formula (1.6), with

$$M = -d|h|^{2\lambda}\wedge a|_{\lambda=0}.$$

Moreover, $M_k = 0$ if $k < \operatorname{codim} Z$; this follows with precisely the same argument as the corresponding statement for R^f in [1] (Theorem 1.1).

For further reference we also notice that actually

$$M = -\bar\partial|h|^{2\lambda}\wedge a|_{\lambda=0}; \tag{3.17}$$

this holds because

$$\partial|h^0|^{2\lambda}\frac{\alpha}{(h^0)^\ell}\Big|_{\lambda=0} = 0.$$

Let us now assume that in addition (1.5) holds. Notice that a factorization like $h = h^0h'$ is preserved under further desingularizations. Therefore, by a series of such desingularizations we may assume that also $f_k = f_k^0 f_k'$ for $1 \le k \le r$, where each f_k^0 is a holomorphic function and f_k' is nonvanishing. Moreover, we may assume that $h_1, \ldots, h_\nu$ are almost monomials. In particular, we may assume that h^0 is a monomial. The condition (1.5) now means that

$$|f_1^0\cdots f_r^0| \le C|h^0|,$$

and since h^0 is a monomial this implies that $f_1^0\cdots f_r^0 = gh^0$ for a necessarily nonvanishing holomorphic g. By redefining h^0 we may therefore assume that

$$f_1^0\cdots f_r^0 = h^0. \tag{3.18}$$

Recall that $\sigma_\ell = (s_\ell'/|h'|^2)/h^0$ and notice that we also have

$$s_\ell' = f_1^0\cdots\widehat{f_\ell^0}\cdots f_r^0 s_\ell''.$$

Under the assumption (1.5) we therefore have

$$\sigma_j = \sigma_j''/f_j^0 \tag{3.19}$$

where σ_j'' is smooth. Moreover,

$$Df_\ell = D(f_\ell^0 f_\ell') = f_\ell^0 Df_\ell' + df_\ell^0\wedge f_\ell'.$$

In view of (3.6), all singularities $1/f_j$ are cancelled, so $c(D_Q)$ is smooth in this resolution. Thus $c(D_Q)$ is locally integrable on the original manifold and $C(D_Q) = |h|^{2\lambda}c(D_Q)|_{\lambda=0}$ is the trivial extension. From (3.10) we know that a is a sum of terms like

$$\sum Df_j\wedge\sigma_j\wedge(\sum Df_i\wedge\bar\partial\sigma_i + \Theta f_i\wedge\sigma_i)^\ell$$

and so it is easily seen that any factor $1/f_j^0$ from σ_j occurs together with a factor f_j^0 or df_j^0 from Df_j. Therefore, a is a sum of terms like

$$\frac{df^0_{\ell_1}\wedge\ldots\wedge df^0_{\ell_k}\wedge\omega}{f^0_{\ell_1}\cdots f^0_{\ell_k}},$$

where ω is smooth. Since we may assume that each f_k^0 is a monomial, it is easy to see that a is locally integrable, and that $\bar\partial|h^0|^{2\lambda}\wedge a|_{\lambda=0}$ has measure coefficients. Therefore this holds also for A and M, and in particular, $A=|h|^{2\lambda}a|_{\lambda=0}$. Thus the proof is complete. $\square$

If $r=1$ we get the simpler expression

$$a=\int_e \alpha Df\wedge\sigma\wedge e^{\tilde I+\alpha\tilde\Theta}\wedge\sum_{\ell=0}^{\infty}\frac{(-\alpha Df\wedge\bar\partial\sigma)^\ell}{(\ell+1)!}.$$

Since each term in $\exp(\tilde I+\alpha\tilde\Theta)$ has the same degree in e_j and e_k^* it must be multiplied with terms with the same property in order to get a product with full degree. Therefore we get

$$a=-\int_e e^{\tilde I+\alpha\tilde\Theta-\alpha Df}\wedge\sum_0^\infty\sigma\wedge(\bar\partial\sigma)^\ell. \tag{3.20}$$

In [1] we introduced the currents

$$U=|f|^{2\lambda}\frac{\sigma}{1-\bar\partial\sigma}\Big|_{\lambda=0}=|f|^{2\lambda}\wedge\sigma\wedge\sum_\ell(\bar\partial\sigma)^{\ell-1}\Big|_{\lambda=0}$$

and

$$R=\bar\partial|f|^{2\lambda}\wedge\frac{\sigma}{1-\bar\partial\sigma}\Big|_{\lambda=0}=\bar\partial|f|^{2\lambda}\wedge\sigma\wedge\sum_\ell(\bar\partial\sigma)^{\ell-1}\Big|_{\lambda=0}.$$

The current R is supported on Z and

$$(\delta_f-\bar\partial)U=1-R. \tag{3.21}$$

In view of (3.20) we therefore have that

$$M=\int_e e^{\alpha\tilde\Theta+\tilde I-\alpha Df}\wedge R, \tag{3.22}$$

$$A=-\int_e e^{\alpha\tilde\Theta+\tilde I-\alpha Df}\wedge U, \tag{3.23}$$

and moreover, cf. (3.6),

$$C(D_Q)=\int_e e^{\alpha\tilde\Theta+\tilde I-\alpha Df}\wedge f\wedge U. \tag{3.24}$$

One can say more about the (p,p)-current M_p, where as before $p=\operatorname{codim} Z$. Let Z_j^p be the irreducible components of the variety Z of codimension p. From [2] we have

Theorem 3.7. *Suppose that $r=1$. If $p=\operatorname{codim}(Z)$, then $M_k=0$ for $k<p$ and*

$$M_p=\sum\alpha_j[Z_j^p],$$

where α_j are nonnegative integers, the multiplicities of f.

Let $E \to X$ be a Hermitian vector bundle with Chern connection D_E. The Bott-Chern class $\hat{c}(E)$ is the equivalence class of the Chern form $c(D_E)$ in

$$\oplus_k \mathcal{E}_{k,k}(X) \cap \mathrm{Ker}\, d / dd^c \oplus_k \mathcal{E}_{k,k}(X).$$

To begin with we recall the construction in [3] that proves that $\hat{c}(E)$ is independent of the Hermitian structure on E.

Lemma 4.1. *Let D be a connection depending smoothly on a real parameter t. Moreover, assume that $L \in \mathcal{E}(X, \mathrm{End}(E))$ depends smoothly on t and that*

$$D'_{\mathrm{End}E} L = \dot{D}. \tag{4.1}$$

Also assume that Θ_t has bidegree $(1,1)$ for all t. If

$$v = \int_0^1 \int_e \tilde{L}_t \wedge e^{\alpha\tilde{\Theta}_t + \tilde{I}} dt,$$

then $\alpha\partial v = b$, where

$$b = \int_0^1 \int_e \alpha \widetilde{\dot{D}_t} \wedge e^{\alpha\tilde{\Theta}_t + \tilde{I}} dt.$$

Proof. In view of (2.4) we have that (suppressing the index t)

$$d \int_e \tilde{L} \wedge e^{\alpha\tilde{\Theta} + \tilde{I}} = \int_e D\tilde{L} \wedge e^{\alpha\tilde{\Theta} + \tilde{I}},$$

and by identifying bidegrees we get that

$$\partial \int_e \tilde{L} \wedge e^{\alpha\tilde{\Theta} + \tilde{I}} = \int_e D'\tilde{L} \wedge e^{\alpha\tilde{\Theta} + \tilde{I}} = \int_e \widetilde{\dot{D}} \wedge e^{\alpha\tilde{\Theta} + \tilde{I}}.$$

□

Since $db = c(D_1) - c(D_0)$, cf., (2.6), and $dd^c = -\alpha\bar{\partial}\partial$, we thus have

$$-dd^c v = c(D_1) - c(D_0). \tag{4.2}$$

To see an example of such an L, let us assume that E is equipped with a metric that varies smoothly with t, and let $D = D_t$ be the Chern connection. Then we can define $L \in \mathcal{E}(X, \mathrm{End}E)$ by

$$\frac{d}{dt} \langle \xi, \eta \rangle = \langle L\xi, \eta \rangle, \quad \xi, \eta \in \mathcal{E}(X, E).$$

In fact, for fixed ξ, $\eta \mapsto d\langle \xi, \eta \rangle / dt$ is conjugate linear in η, and thus the element $L\xi$ must exist. We claim that

$$D_{\mathrm{End}E} L = \dot{D} + \dot{D}^*, \tag{4.3}$$

which of course implies (4.1). In fact, since $d\langle \xi, \eta \rangle = \langle D\xi, \eta \rangle + \langle \xi, D\eta \rangle$, differentiation with respect to t gives

$$d\langle L\xi, \eta \rangle = \langle LD\xi, \eta \rangle + \langle L\xi, D\eta \rangle + \langle \dot{D}\xi, \eta \rangle + \langle \xi, \dot{D}\eta \rangle.$$

We also have that $d\langle L\xi, \eta \rangle = \langle D(L\xi), \eta \rangle + \langle L\xi, D\eta \rangle$, and comparing, we get (4.3). Since any two given Hermitian metrics are connected by such a homotopy, it follows, cf., (4.2), that the Bott-Chern class is independent of the metric.

If the metric is given by the matrix τ is a given local frame for E, then

$$L = \tau^{-1}\dot{\tau} \tag{4.4}$$

in this frame. In fact, $\langle \xi, \eta \rangle = \eta^* \tau \xi$ so

$$\frac{d}{dt}\langle \xi, \eta \rangle = \eta^* \dot{\tau} \xi = \eta^* \tau \tau^{-1} \dot{\tau} = \langle \tau^{-1}\dot{\tau}\xi, \eta \rangle.$$

We now consider another situation. Assume that we have the short exact sequence of Hermitian vector bundles $0 \to S \xrightarrow{j} E \xrightarrow{g} Q \to 0$, where Q and S have the metrics induced by the Hermitian metric on E. Then

$$j^* \oplus g \colon E \to S \oplus Q \tag{4.5}$$

is a smooth vector bundle isomorphism. If D_S and D_Q are the Chern connections on S and Q respectivey, then

$$D_E \sim \begin{pmatrix} D_S & -\beta^* \\ \beta & D_Q \end{pmatrix}$$

with respect to the isomorphism (4.5). We shall now modify the connection D to $D_b = D - \gamma$, where $\gamma = D'_{\mathrm{End}E} jj^*$. It turns out that $\gamma = g^* \circ \beta \circ j^*$, thus $\gamma \wedge \gamma = 0$, and that $D_{\mathrm{End}E}\gamma = \bar\partial\gamma$. Moreover, it follows that

$$D_{E,b} \sim \begin{pmatrix} D_S & * \\ 0 & D_Q \end{pmatrix}$$

and hence

$$\Theta_b \sim \begin{pmatrix} \Theta_S & * \\ 0 & \Theta_Q \end{pmatrix},$$

so that $c(D_b) = c(D_S)c(D_Q)$. If $D_{E,t} = D_E - t\gamma$ we have $\Theta_t = \Theta - t\bar\partial\gamma$; thus it has bidegree $(1,1)$. If we let

$$b = \int_0^1 \int_e \alpha\tilde\gamma \wedge e^{\tilde I + \alpha\tilde\Theta - t\alpha\bar\partial\tilde\gamma} = \sum_{\ell \ge 0} \int_e \alpha\tilde\gamma \wedge e^{\tilde I + \alpha\tilde\Theta} \wedge \frac{1}{(\ell+1)!}(-\alpha\bar\partial\tilde\gamma)^\ell \tag{4.6}$$

it follows from (2.6) that $db = c(D_E) - c(D_S)c(D_Q)$. Moreover, if $L = jj^*/(1-t)$, then (4.1) holds. In fact, $\dot D = -\gamma$, and $[jj^*, g^* \circ \beta \circ j^*] = g^* \circ \beta \circ j^*$, (notice that $[a,b] = ab + ba$ if 1-forms) so that

$$D'_{\mathrm{End}E,t} L = D'_{\mathrm{End}E} L - t[\gamma, L] = \frac{1}{1-t}\gamma - \frac{t}{1-t}\gamma = \gamma. \tag{4.7}$$

Proposition 4.2. *If*

$$v = \sum_{\ell=1}^{m-1} \frac{(-1)^\ell}{\ell} \int_e \widetilde{jj^*} \wedge (\tilde I + \alpha\tilde\Theta - \alpha\bar\partial\tilde\gamma)_{m-\ell-1} \wedge (-\alpha\bar\partial\tilde\gamma)_\ell, \tag{4.8}$$

then $\alpha\partial v = b$.

Proof. Observe that

$$\partial \int_0^{1-\epsilon} \int_e \frac{\widetilde{jj^*}}{1-t} \wedge e^{\tilde{I}+\alpha\Theta_1} dt = \int_0^{1-\epsilon} \int_e \frac{D_1 \widetilde{jj^*}}{1-t} \wedge e^{\tilde{I}+\alpha\Theta_1} dt = 0,$$

since $D_1\widetilde{jj^*} = 0$ in view of (4.7). Therefore,

$$\alpha\partial \int_0^{1-\epsilon} \int_e \widetilde{jj^*} \wedge \frac{e^{\tilde{I}+\alpha\tilde{\Theta}-t\alpha\bar{\partial}\tilde{\gamma}} - e^{\tilde{I}+\alpha\tilde{\Theta}-\alpha\bar{\partial}\tilde{\gamma}}}{1-t} dt = \int_0^{1-\epsilon} \int_e \alpha\tilde{\gamma} \wedge e^{\tilde{I}+\alpha\tilde{\Theta}-t\alpha\bar{\partial}\tilde{\gamma}}.$$

The proposition now follows by letting $\epsilon \to 0$ and computing the t-integral on the left hand side. □

Altogether we therefore have that $-dd^c v = c(D_E) - c(D_S)c(D_Q)$ and thus $\hat{c}(E) = \hat{c}(S)\hat{c}(Q)$. The formula for v is precisely the same as in [3] but expressed in our notation.

5. Bott-Chern classes defined by residue currents

Let S be the bundle over $X \setminus Z$ generated by the holomorphic sections $f_1, \ldots, f_r$ to E. If σ_j are the sections to E^* as in Section 3, then clearly

$$\widetilde{jj^*} = \sum_j f_j \wedge \sigma_j.$$

Moreover, if we let $\gamma_b = D'_{\text{End}E}(jj^*)$, then with the short-hand notation from Proposition 3.3 we have

$$\tilde{\gamma}_b = (Df - f \wedge \phi) \wedge \sigma, \tag{5.1}$$

and

$$\bar{\partial}\tilde{\gamma}_b = (Df - f \wedge \phi) \wedge \bar{\partial}\sigma + (\Theta f + f \wedge \bar{\partial}\phi) \wedge \sigma. \tag{5.2}$$

Proposition 5.1. *With the notation above we have:*

(i) The forms v, b, and $c(D_S) \wedge c(D_Q)$ are locally integrable in X.

(ii) If the natural extensions are denoted by capitals, then

$$\partial V = B, \tag{5.3}$$

and

$$-dd^c V = c(D_E) - C(D_S)C(D_Q). \tag{5.4}$$

Proof. We use the same notation as in the proof of Theorem 1.3. After a suitable desingularization we may assume that $f = f^0 f'$ and $\sigma = \sigma'/f^0$. Therefore,

$$f \wedge \sigma = \sum f_j \wedge \sigma_j = \sum f'_j \wedge \sigma'_j,$$

and thus it is smooth. It follows that also $\tilde{\gamma}_b = D'(f \wedge \sigma)$ and $\bar{\partial}\tilde{\gamma}_b$ are smooth. Since (5.4) holds in $X \setminus Z$ and $c(D_E)$ is smooth it follows that also $c(D_S) \wedge c(D_Q)$ is smooth in the resolution. We can conclude that all the forms are locally integrable in X and that (5.3) and (5.4) holds. □

When $r=1$, the presence of the factor $\widetilde{jj^*} = f\wedge\sigma$ implies the simpler expressions, cf., (4.6) and (4.8),

$$(5.5)\quad v = \sum_{\ell=1}^{m-1} \frac{(-1)^\ell}{\ell} \int_e f\wedge\sigma\wedge(\tilde{I}+\alpha\tilde{\Theta}-\alpha Df\wedge\bar\partial\sigma)_{m-1-\ell}\wedge(-\alpha Df\wedge\bar\partial\sigma)_\ell$$

and

(5.6)

$$b = \int_e \alpha(Df - f\wedge\phi)\wedge\sigma\wedge e^{\tilde{I}+\alpha\tilde{\Theta}}\wedge\sum_{\ell=0}^{\infty}\frac{(-\alpha Df+\alpha f\wedge\phi)^\ell}{(1+\ell)!}\wedge(\bar\partial\sigma)^\ell.$$

We also have

Proposition 5.2. *If $r=1$ we have that*

$$(5.7)\qquad b = a - \alpha\partial\log|f|^2\wedge c(D_Q)$$

in $X\setminus Z$.

Proof. To begin with we rewrite (5.6) as

$$(5.8)\quad b = -\int_e e^{\tilde{I}+\alpha\tilde{\Theta}}\wedge\sum_{\ell=0}^{\infty}\frac{(-\alpha Df+\alpha f\wedge\phi)^{\ell+1}}{(\ell+1)!}\wedge\sigma\wedge(\bar\partial\sigma)^\ell = \\ -\int_e e^{\tilde{I}+\alpha\tilde{\Theta}-\alpha Df+\alpha f\wedge\phi}\wedge\sum_{\ell=0}^{\infty}\sigma\wedge(\bar\partial\sigma)^\ell.$$

From (5.8) we have, cf., (3.20) and recall that $\phi = -\partial\log|f|^2$,

$$b = \int_e e^{\tilde{I}+\alpha\tilde{\Theta}-\alpha Df}\wedge(1+\alpha f\wedge\phi)\wedge\sum_\ell\sigma\wedge(\bar\partial\sigma)^\ell = \\ a + \alpha\partial\log|f|^2\wedge c(D_Q).$$

□

It is now a simple matter to conclude the proof of Theorem 1.1.

Proof of Theorem 1.1. Since $c(D_Q)$ is smooth in a suitable resolution, it follows that $\log|f|c(D_Q)$ is locally integrable there. Thus it is locally integrable in X and since $c(D_S)-1 = dd^c\log(1/|f|)$ we have

$$dd^c(\log(1/|f|)c(D_Q)) = C(D_S)\wedge C(D_Q) - C(D_Q) - M$$

with

$$(5.9)\qquad M = dd^c(\log|f|c(D_Q))\mathbf{1}_Z.$$

If we define $W = \log(1/|f|)c(D_Q) - V$, we therefore have, cf., Proposition 5.1, that $dd^cW = c(D_E) - C(D_Q) - M$ as wanted. We now claim that the current M so defined coincides with the residue current in Theorem 3.7 (and in Theorem 1.3). In fact, this current is equal to $-dA\mathbf{1}_Z$ and since dB is locally integrable, we have in view of Proposition 5.2 that

$$-dA\mathbf{1}_Z = -d(\alpha\partial\log|f|^2c(D_Q))\mathbf{1}|_Z = dd^c(\log|f|c(D_Q))\mathbf{1}_Z = M.$$

To complete the proof of part (iii), take a resolution as before, so that $f = f^0 f'$. Then it is easily seen that (1.2) is locally integrable for $\lambda > 0$, and that the limit when $\lambda \to 0^+$ is equal to M. One just has to notice that $\log|f| = \log|f^0| + \log|f'|$, and hence

$$M = dd^c(\log|f|c(D_Q)) = dd^c \log|f^0| \wedge c(D_Q) = [f^0 = 0] \wedge c(D_Q),$$

where $[f^0 = 0]$ is the current of integration over the divisior defined by f^0. Thus the theorem is proved. □

Remark 1. Notice that when $m = 1$ then $W = \log(1/|f|)$ and therefore (1.1) is indeed the classical Poincaré-Lelong formula in this case. □

It remains to prove Theorem 1.4.

Proof of Theorem 1.4. Assume that we have r sections f_j and that (1.5) holds. We first have to check that $c(D_S)$ is locally integrable in X and find a locally integrable form u such that $dd^cU = C(D_S) - 1$, where as before the capitals denote the natural extensions to X.

Let τ be the metric matrix with respect to the frame f_j for $S \to X \setminus Z$; i.e., $\tau_{jk} = \langle f_j, f_k \rangle$. In $X \setminus Z$ we make the deformation

$$\tau^t_{jk} = (1-t)\tau_{jk} + t|f_j|^2\delta_{jk}. \tag{5.10}$$

Notice that when $t = 1$, the corresponding Chern class is

$$c(D_{S,1}) = \bigwedge_1^r (1 - dd^c \log|f_j|).$$

In particular, it follows that it is locally integrable in X, and we let $C(D_{S,1})$ denote the natural extension.

Proposition 5.3. *The Chern forms $c(D_S)$ and $c(D_{S,1})$ are locally integrable in X. With $L = L_t$ etc as in Section 4 we have that*

$$u_1 = \int_0^1 \int_e \tilde{L}_t \wedge e^{\tilde{\Theta}_t} dt$$

is locally integrable in X, and

$$dd^c U_1 = C(D_S) - C(D_{S,1}). \tag{5.11}$$

Proof. With the notation in the proof of Theorem 1.3 we have in a suitable desingularization that

$$f_1 \wedge \ldots \wedge f_r = (f_1^0 \cdots f_r^0) f_1' \wedge \ldots \wedge f_r',$$

and from (3.18) we conclude that

$$h' = f_1' \wedge \ldots \wedge f_r'.$$

In particular, $f_1' \wedge \ldots \wedge f_r' \neq 0$ so f_i' is a local frame for S (or rather the pullback of S to the resolution manifold). Let

$$\tau'_{jk} = \langle f_j', f_k' \rangle$$

and

$$(\tau_t')_{jk} = (1-t)\tau'_{jk} + t|f_j'|^2\delta_{jk}.$$

Then

$$\tau_{jk} = f_j^0 \overline{f_k^0} \tau'_{jk},$$

and hence

$$\tau_t = F^* \tau_t' F,$$

where $F_{jk} = f_j^0 \delta_{jk}$. The matrix F corresponds to a change of frame outside the singularity, and since the definition of u_1 is invariant and just depends on the given deformation (5.10), we can compute u_1 with respect to the new frame f_j' instead. However, τ_t' is clearly invertible, and therefore, cf., (4.4), $L = (\tau_t')^{-1}\dot{\tau}_t'$ and the corresponding Θ_t' are smooth, and so are also $c(D_S)$ and $c(D_{S,1})$. Since $dd^c u_1 = c(D_S) - c(D_{S,1})$ in this resolution, (5.11) holds on X. $\square$

It is a simple matter to find a current u_2 in $X \setminus Z$ such that

$$dd^c u_2 = \bigwedge_1^r (1 - dd^c \log|f_j|) - 1 = c(D_{S,1}) - 1.$$

For instance we can take

$$\xi^1 = -\log|f_1|, \quad \xi^{k+1} = \xi^k - \log|f_{k+1}| \bigwedge_1^k (1 - dd^c \log|f_j|), \quad u_2 = \xi^r.$$

Summing upp we thus have that

$$dd^c(u_1 + u_2) = c(D_S) - 1.$$

Since $c(D_Q)$ and u_1 are smooth in a suitable resolution, and u_2 has just logarithmic singularites, it follows that $u = u_1 + u_2$ has logarithmic singularities. Since $c(D_Q)$ is smooth in the resolution $(u_1 + u_2) \wedge c(D_Q)$ has logarithmic singularities, and

$$dd^c u \wedge c(D_Q) = (c(D_S) - 1) \wedge c(D_Q)$$

in $X \setminus Z$. We can conclude that

$$dd^c(U \wedge C(D_Q)) = C(D_S) \wedge C(D_Q) - C(D_Q) - M,$$

where M is a current of bidegree $(*, *)$ with support on M. Therefore, cf., Proposition 5.1, finally $dd^c W = c(D_E) - C(D_Q) - M$, with $W = U \wedge C(D_Q) - V$. Thus Theorem 1.4 is proved. $\square$

6. Positivity

Let $E \to X$ be a Hermitian holomorphic bundle as before and let e_j be an ortonormal local frame. A section

$$A = i \sum_{jk} A_{jk} \otimes e_j \otimes e_k^*$$

to $T_{1,1}^*(X) \otimes \mathrm{End}(E)$ is Hermitian if $A_{jk} = -\overline{A_{kj}}$. It then induces a Hermitian form a on $T^{1,0}(X) \otimes E^*$ by

$$a(\xi \otimes e_j^*, \eta \otimes e_k^*) = A_{jk}(\xi, \bar{\eta}),$$

if ξ, η are $(1,0)$-vectors. We say that A is (Bott-Chern) positive, $A \geq_B 0$ if the form a is positively semi-definite. In the same way any Hermitian A induces a Hermitian form a' on $T^{1,0}(X) \otimes E$ and it is called Nakano positive, $A \geq_N 0$, if a' is positively semi-definite.

Notice that $\alpha\Theta$ is Hermitian; it is said to be Nakano positive if $\alpha\Theta \geq_N 0$. Analogously we say that E is positive, $E \geq_B 0$, if $\alpha\Theta \geq_B 0$. Neither of these positivity concepts implies the other one unless $m = 1$.

Since $\Theta_{jk}(E^*) = -\Theta_{jk}(E)$ it follows that E is positive in our sense if and only if E^* is Nakano negative. The next proposition explains the interest of Bott-Chern positivity in this context.

Proposition 6.1. *Let*

$$0 \to S \to E \to Q \to 0 \tag{6.1}$$

be a short exact sequence of Hermitian holomorphic vector bundles. Then $E \geq_B 0$ implies that $Q \geq_B 0$.

Proof. It is well-known, see for instance [5], that $E \leq_N 0$ implies that $S \leq_N 0$. From the sequence (6.1) above we get the exact sequence $0 \to Q^* \to E^* \to S^* \to 0$. Since $E^* \leq_N 0$ implies $Q^* \leq_N 0$, it follows that $E \geq_B 0$ implies $Q \geq_B 0$. □

The next simple lemma reveals that our definition of Bott-Chern positivity coincides with the one used in [3].

Lemma 6.2. $A \geq_B 0$ *if and only if there are sections f_ℓ to $T^*_{1,0}(X) \otimes E$ such that*

$$A = i \sum_\ell f_\ell \otimes f^*_\ell. \tag{6.2}$$

Observe that if $f_\ell = \sum f^\ell_j \otimes e_j$, then $f^*_\ell = \sum \bar{f}^\ell_j \otimes e^*_j$ since e_j is ortonormal.

Proof. If (6.2) holds, then

$$a(\xi, \xi) = \sum_\ell f_\ell(\xi) f^*_\ell(\xi^*) = \sum |f_\ell(\xi)|^2 \geq 0$$

for all ξ in $T^{1,0} \otimes E^*$. Conversely, if a is positive, it is diagonalizable, and so there is a basis f_ℓ for $T^*_{1,0} \otimes E$ such that (6.2) holds. □

If we identify f_ℓ with $\sum f^\ell_j \wedge e_j$ as before, then (6.2) means that

$$\tilde{A} = -i \sum_\ell f_\ell \wedge f^*_\ell. \tag{6.3}$$

If $B = \sum B_{jk} e_j \otimes e^*_j$ is a scalar-valued section to End$E$, then it is Hermitian if and only if $B_{jk} = \bar{B}_{kj}$ and it is positively semi-definite if and only if

$$B = \sum_\ell g_\ell \otimes g^*_\ell$$

for some sections g_ℓ to E; or equivalently,

$$\tilde{B} = \sum_\ell g_\ell \wedge g^*_\ell. \tag{6.4}$$

Proposition 6.3. *Assume that A_j are $(1,1)$-form-valued Hermitian sections to E and B_k scalarvalued sections, such that $A_j \geq_B 0$ and $B_k \geq 0$. Then*

$$\int_e \tilde{A}_1 \wedge \ldots \wedge \tilde{A}_r \wedge \tilde{B}_{r+1} \wedge \ldots \wedge \tilde{B}_m \tag{6.5}$$

is a positive (r,r)-form.

Proof. In view of (6.3) and (6.4), we see that (6.5) is a sum of terms like

$$\int_e (-i)^r f_1 \wedge f_1^* \wedge \ldots \wedge f_r \wedge f_r^* \wedge g_{r+1} \wedge g_{r+1}^* \wedge \ldots \wedge g_m \wedge g_m^* =$$
$$(-i)^r c_{m-r} \int_e f_1 \wedge \ldots f_r \wedge \ldots g_m \wedge f_1^* \wedge \ldots \wedge f_r^* \wedge \ldots g_m^* =$$
$$(-i)^r c_{m-r} \int_e \omega \wedge e_1 \wedge \ldots \wedge e_m \wedge \bar{\omega} \wedge e_1^* \wedge \ldots \wedge e_m^*,$$

where ω is an $(r,0)$-form and $c_p = (-1)^{p(p-1)/2} = i^{p(p-1)}$. By further simple computations,

$$(-i)^r c_{m-r} (-1)^{mr} \int_e \omega \wedge \bar{\omega} \wedge e_1 \wedge \ldots \wedge e_m \wedge e_1^* \wedge \ldots \wedge e_m^* =$$
$$(-i)^r c_{m-r} (-1)^{mr} c_m \omega \wedge \bar{\omega} = i^{r^2} \omega \wedge \bar{\omega}$$

the proposition follows, since the last form is positive. □

Proposition 6.4. *If $E \geq_B 0$ (or $E \geq_N 0$), then the Chern forms $c_k(D_E)$ are positive for all k.*

Proof. Since $\alpha\Theta \geq_B 0$ by assumption, and clearly $I \geq 0$, it follows from Proposition 6.3 that

$$c_k(D_E) = \int_e (\widetilde{\alpha\Theta})_k \wedge \tilde{I}_{m-k}$$

is positive. □

We have already noticed, cf., Theorem 1.1 (iii), that the current M_k is positive if $c_{k-1}(D_Q) \geq_B 0$. From (3.6) we have that $(r=1)$

$$c_{k-1}(D_Q) = \int_e f \wedge \sigma \wedge (\widetilde{\alpha\Theta} - \alpha D f \wedge \bar{\partial}\sigma)_{k-1} \wedge \tilde{I}_{m-k} = \tag{6.6}$$
$$\sum_{j=1}^{k-1} \int_e f \wedge \sigma \wedge (\widetilde{\alpha\Theta})_{k-1-j} \wedge (-\alpha D f \wedge \bar{\partial}\sigma)_j \wedge \tilde{I}_{m-k}.$$

If $s = f^*$ as before, then $\sigma = s/|f|^2$, and therefore we have

$$c_{k-1}(D_Q) = \sum_{j=1}^{k-1} \int_e \frac{f \wedge s}{|f|^2} \wedge \Big(\frac{-\alpha D f \wedge \bar{\partial} s}{|f|^2}\Big)_j \wedge (\widetilde{\alpha\Theta})_{k-1-j} \wedge \tilde{I}_{m-k}. \tag{6.7}$$

Since $\bar{\partial} s = (Df)^*$ it now follows immediately from Proposition 6.3 that $c_k(D_Q)$ is positive if $\alpha\Theta \geq_B 0$.

One can prove that, if we multiply with $\lambda\partial|f|^2 \wedge \bar\partial|f|^2/|f|^2$ and let $\lambda \to 0^+$, then all terms where the power of $\bar\partial|f|^2$ is less than p will disappear; see for instance the proof of Theorem 1.1 in [1]. We thus have

Proposition 6.5. *If* $p = \operatorname{codim}\{f = 0\}$, *then*

$$M_k = \lim_{\lambda\to 0^+} \lambda|f|^{2\lambda}\alpha\frac{\partial|f|^2 \wedge \bar\partial|f|^2}{|f|^2}\wedge \sum_{j=p-1}^{k-1}\int_e \frac{f\wedge s}{|f|^2}\wedge\Big(\frac{-\alpha Df\wedge\bar\partial s}{|f|^2}\Big)_j \wedge(\alpha\widetilde\Theta)_{k-1-j}\wedge\tilde I_{m-k}.$$

From this formula it is apparant that M_k vanishes if $k < p$, and that M_p is positive, regardless of $\alpha\Theta$.

Notice that if some of the A_j in (6.5) are replaced by $A'_j \geq_B A_j$, then the resulting form will be larger. This follows immediately from proof, It is now easy to prove Theorem 1.2.

Proof of Theorem 1.2. Assume that $E \geq_B 0$. Then $\log(1/|f|)c(D_Q)$ is positive when $|f| < 1$. From (5.5) we have that

$$v_k = \sum_{\ell+1}^{k}\int_e f\wedge\sigma\wedge(\alpha\widetilde\Theta - \alpha Df\wedge\bar\partial\sigma)_{k-\ell}\wedge(-\alpha Df\wedge\bar\partial\sigma)_\ell\wedge\tilde I_{m-k-1}.$$

Thus it is an alternating sum of positive terms, and therefore it has no sign. If we replace each factor $-\alpha Df\wedge\bar\partial\sigma$ by $\alpha\widetilde\Theta - \alpha Df\wedge\bar\partial\sigma$, then we get a larger form which in addition is closed, since it is just a certain constant times $c_k(D_Q)$, cf., (6.6). Therefore, for a suitable constant ν_k $v'_k = v_k + \nu_k c_k(D_Q)$ is a positive form that is cohomologous with v_k. Thus the current

$$W'_k = V_k + \nu_k C_k(D_Q) + \log(1/|f|)C_k(D_Q)$$

will have the stated property. □

The modification of v in the proof is precisely as in [3] but with our notation, and for an arbitrary k rather than just $k = m-1$. It is not necessary to consider each v_k separately. By the same argument one can see directly that $v' = v + \nu c(D_Q)$ is positive and cohomologous with v, if ν is appropriately chosen.

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Department of Mathematics, Chalmers University of Technology and the University of Göteborg, S-412 96 GÖTEBORG, SWEDEN

E-mail address: matsa@math.chalmers.se