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Branching of some holomorphic representations of SO(2,n)

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Abstract

In this licentiate thesis we consider a family of Hilbert spaces of holomorphic functions on a bounded symmetric domain $\mathscr{D} = SO(2,n)/S(O(2) \times O(n))$ and corresponding holomorphic unitary (projective)representations of $SO(2,n)$ on these spaces. These representations are known to be irreducible. Our aim is to decompose them under the subgroup $SO(1,n)$ which acts as the isometry group of a totally real submanifold X of $\mathscr{D}$. We give a proof of a general decomposition theorem for certain unitary representations of semisimple Lie groups. In the particular case we are concerned with, we find an explicit formula for the Plancherel measure of the decomposition as the orthogonalising measure for certain hypergeometric polynomials. Moreover, we construct an explicit generalised Fourier transform that plays the role of the intertwining operator for the decomposition. We prove an inversion formula and a Plancherel formula for this transform. Finally we construct explicit realisations of the singular part appearing in the decomposition and also for the minimal representation corresponding to the given family of Hilbert spaces.

Keywords: Bounded symmetric domains, Lie groups, Lie algebras, unitary representations, spherical functions, hypergeometric functions, Fourier-Helgason transform.

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Contents

1 Introduction and background

One of the main problems in representations of Lie groups and harmonic analysis on Lie groups is to decompose some interesting representation of a Lie group G under a subgroup $H \subset G$. Among other things, this has lead to the discovery of some new representations. This decomposition is also called the *branching rule.*

Since the work by R. Howe, M. Kashiwara and M. Vergne (cf [8]), it has turned out to be fruitful to study the branching of singular and minimal holomorphic representations of a Lie group acting on a function space of holomorphic functions on a bounded symmetric domain.

In this licentiate thesis we will study the branching of a holomorphic representation of $SO(2,n)$ under the subgroup $H = SO_0(1,n)$. The subgroup H here is realised as the isometry group of a totally real manifold of the Lie ball $SO(2,n)/S(O(2)\times O(n))$. The branching for a general Lie group G of Hermitian type under a symmetric subgroup H has been studied recently by Neretin ([14]) and Zhang ([21], [23],[22]). The branching rule for regular parameter and for some minimal representations is now well understood. However there is still no complete theory for the general case.

We find in this thesis the branching for arbitrary scalar parameter v in the Wallach set and we give an explicit realisation of the discrete part appearing in the decomposition.

2 The Lie ball as a symmetric space $SO(2,n)/S(O(2)\times O(n))$

In this thesis we study representations on function spaces on the domain

$$\mathscr{D} = \{z \in \mathbb{C}^n \,|1 - 2\langle z, z\rangle + |zz^t|^2 > 0, |z| < 1\}. \tag{1}$$

We will only be concerned with the case $n > 2$. (If $n = 1$ it is the unit disk, U, and if $n = 2$, $\mathscr{D} \cong U \times U$). In this section we prove that $\mathscr{D}$ is the quotient of $SO(2,n)$ by $S(O(2)\times O(n))$ by studying a holomorphically equivalent model on which we have a natural group action induced by the linear action on a Grassmanian manifold.

Consider $\mathbb{R}^{n+2} \cong \mathbb{R}^2 \oplus \mathbb{R}^n$ equipped with the non-degenerate bilinear form

$$(x|y) := x_1y_1 + x_2y_2 - x_3y_3 - \ldots - x_{n+2}y_{n+2},$$

where the coordinates are with respect to the standard basis $e_1, \cdots, e_{n+2}$. Let $SO(2,n)$ be the group of all linear transformations on $\mathbb{R}^{n+2}$ that preserve this form and have determinant 1, i.e.,

$$SO(2,n) = \{g \in GL(2+n,\mathbb{R})|(gx|gy) = (x|y), x, y \in \mathbb{R}^{2+n}, \det g = 1\} \tag{2}$$

Let $\mathcal{G}^+_{(2,n)}$ denote the set of all two-dimensional subspaces of $\mathbb{R}^2 \oplus \mathbb{R}^n$ on which $(\cdot|\cdot)$ is positive definite. Clearly $\mathbb{R}^2 \oplus \{0\}$ is one of these subspaces. It will be the reference point in $\mathcal{G}^+_{(2,n)}$ and we will denote it by V_0. The group $SO(2,n)$ acts linearly on this set and we shall see that this action is actually transitive. Let therefore $V \in \mathcal{G}^+_{(2,n)}$ and let $V^\perp$denote the set of vectors that are orthogonal to V in the sense of $(\cdot|\cdot)$. Since the form $(\cdot|\cdot)$ is positive definite, we can construct an ordered orthonormal basis, $\{v_1, v_2\}$, for V. Given $x \in V$, $x - (x|v_1)v_1 - (x|v_2)v_2$ is in $V^\perp$ and since $V \cap V^\perp = \{0\}$ we have $\mathbb{R}^{n+2} = V \oplus V^\perp$. It is now sufficient to prove that the form is strictly negative on $V^\perp$, since in that case we can construct an ordered orthonormal basis, $\{w_1, \ldots, w_n\}$, for $V^\perp$ and by defining a linear mapping that maps the standard basis onto $\{v_1, v_2, w_1, \ldots, w_n\}$, we have found an element, g in $SO(2,n)$ that maps V_0 to V.

Now we decompose the basis vectors for V as $v_1 = \xi_1 + \eta_1$, $v_2 = \xi_2 + \eta_2$ with $\xi_i \in \mathbb{R}^2, \eta_i \in \mathbb{R}^n$ and $\|\xi_i\| > \|\eta_i\|, i \in \{1,2\}$. A vector $u + \tilde{u}$ in $V^\perp$ satisfies $(u + \tilde{u} \mid \alpha v_1 + \beta v_2) = 0$, i.e. $\langle u, \alpha\xi_1 + \beta\xi_2\rangle = \langle \tilde{u}, \alpha\eta_1 + \beta\eta_2\rangle$. Since $\{\xi_1, \xi_2\}$ forms a basis for $\mathbb{R}^2$, we have

$$\|u\| = \sup_{\substack{\alpha,\beta\ \in\mathbb{R}\\ \|\alpha\xi_1+\beta\xi_2\|=1}} |\langle u, \alpha\xi_1 + \beta\xi_2\rangle| = \sup_{\substack{\alpha,\beta\ \in\mathbb{R}\\ \|\alpha\xi_1+\beta\xi_2\|=1}} |\langle \tilde{u}, \alpha\eta_1 + \beta\eta_2\rangle \leq$$

$$\sup_{\substack{\alpha,\beta\ \in\mathbb{R}\\ \|\alpha\xi_1+\beta\xi_2\|=1}} \|\tilde{u}\|\|\alpha\eta_1 + \beta\eta_2\| < \sup_{\substack{\alpha,\beta\ \in\mathbb{R}\\ \|\alpha\xi_1+\beta\xi_2\|=1}} \|\tilde{u}\|\|\alpha\xi_1 + \beta\xi_2\| \leq \|\tilde{u}\|$$

i.e. $(\cdot \mid \cdot)$ is negative definite on $V^\perp$.

The element g constructed above is not unique. Indeed, g and h map V_0 to the same element precisely when gh^{-1} fixes V_0. We denote by K the stabilizer subroup of V_0, i.e.,

$$K = \{g \in SO(2,n) | g(V_0) = V_0\}. \tag{3}$$

Any element $g \in SO(2,n)$ can be identified with a $(2+n) \times (2+n)$- matrix of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{4}$$

where A is a 2×2-matrix. With this identification, K clearly corresponds to the matrices

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix},$$

where A and D are orthogonal 2×2- and $n \times n$-matrices respectively, i.e., $K \cong S(O(2) \times O(n))$. The space $\mathcal{G}^+_{(2,n)}$ can be realised as the unit ball in $M_{n2}(\mathbb{R})$ with the operator norm. Indeed, let $V \in \mathcal{G}^+_{(2,n)}$. If $v = v_1 + v_2 \in V$, then $v_1 = 0$ implies that $v_2 = 0$, i.e., projection onto the $\mathbb{R}^2$-component is an injective mapping. This means that there is a real $n \times 2$ matrix Z with $Z^t Z < I_2$, such that

$$V = \{(v \oplus Zv) | v \in \mathbb{R}^2\}. \tag{5}$$

Conversely, if $Z \in M_{n2}(\mathbb{R})$ satisfies $Z^t Z < I_2$, then (5) defines an element in $\mathcal{G}^+_{(2,n)}$. Using (2) to identify g with a matrix and letting V correspond to the matrix Z, then clearly

$$\begin{aligned} gV &= \{(Av + BZv \oplus Cv + DZv) | v \in \mathbb{R}^2)\} \\ &= \{v \oplus (C + DZ)(A + BZ)^{-1} v) | v \in \mathbb{R}^2)\}. \end{aligned}$$

In other words, we have an $SO(2,n)$-action on the set

$$M = \{Z \in M_{n2}(\mathbb{R}) | Z^t Z < I_2\}$$

given by

$$Z \mapsto (C + DZ)(A + BZ)^{-1}.$$

Since the elements of M correspond to the cosets of $SO(2,n)$ modulo $SO(2) \times SO(n)$, a smooth manifold structure is induced on M by this identification. The matrix $Z = (XY)$ can be identified with the vector $X + iY$ in $\mathbb{C}^n$. Thus we have the almost complex structure $J(XY) = (-YX)$ which is equivalent with multiplication with i. We now have an action of $SO(2,n)$ on the manifold $M = \{Z \in M_{n2}(\mathbb{R}) | Z^t Z < I_2\}$ (viewed as an open submanifold of $\mathbb{R}^{2n}$). In fact, the action is holomorphic with respect to J. Moreover we have the following result by Hua (see [7]):

Theorem 1. *The mapping*

$$z \mapsto Z = 2 \left(\begin{pmatrix} zz^t + 1 & i(zz^t - 1) \\ \overline{zz}^t + 1 & -i(\overline{zz}^t - 1) \end{pmatrix}^{-1} \begin{pmatrix} z \\ \overline{z} \end{pmatrix} \right)^t$$

is a holomorphic diffeomorphism of the bounded manifold

$$\mathscr{D} = \{z \in \mathbb{C}^n | 1 - 2\langle z, z\rangle + |zz^t|^2 > 0, |z| < 1\}$$

onto M.

We will call this mapping the *Hua transform* and denote it by $\mathcal{H}$. It allows us to describe $\mathscr{D}$ as a symmetric space

$$\mathscr{D} \cong SO(2,n)/S(O(2) \times O(n)).$$

Remark. SO(2,n) has two connected components (cf [6] 10.2) and one can in fact easily prove that the identity component $SO_0(2,n)$ acts transitively on $\mathscr{D}$ (cf the proof of Prop. (5)). Therefore we can also desribe $\mathscr{D}$ as the symmetric space $SO_0(2,n)/(SO(2) \times SO(n))$.

3 Bounded symmetric domains and Jordan pairs

In this section we briefly review some general theory on bounded symmetric domains and Jordan pairs. All proofs are omitted. For a more detailed account we refer to Loos ([11]).

Let $\mathcal{D}$ be a bounded open domain in $\mathbb{C}^n$ and $\mathcal{H}^2(\mathcal{D})$ be the Hilbert space of all square integrable holomorphic functions on $\mathcal{D}$,

$$\mathcal{H}^2(\mathcal{D}) = \{f, f \,\text{holomorphic on } \mathcal{D} | \int_{\mathcal{D}} |f(z)|^2 dm(z) < \infty\},$$

where m is $2n$-dimensional Lebesgue measure. It is a closed subspace of $L^2(\mathcal{D})$. For every $w \in \mathcal{D}$, the evaluation functional $f \mapsto f(w)$ is continuous, hence $\mathcal{H}^2(\mathcal{D})$ has a reproducing kernel $K(z, w)$, holomorphic in z and antiholomorphic in w such that

$$f(w) = \int_{\mathcal{D}} f(z)\overline{K(z, w)} dm(z).$$

$K(z, w)$ is called the Bergman kernel. It has the transformation property

$$K(\varphi(z), \varphi(w)) = J_\varphi(z) K(z, w) \overline{J_\varphi(w)}, \tag{6}$$

for any biholomorphic mapping φ on $\mathcal{D}$ with complex Jacobian $J_\varphi(z) = \det d\varphi(z)$. Hereafter biholomorphic mappings will be referred to as automorphisms. The formula

$$h_z(u, v) = \partial_u \partial_{\overline{v}} \log K(z, z) \tag{7}$$

defines a Hermitian metric, called the Bergman metric. It is invariant under automorphisms and its real part is a Riemannian metric on $\mathcal{D}$.

A bounded domain $\mathcal{D}$ is called *symmetric* if, for each $z \in \mathcal{D}$ there is an involutive automorphism s_z with z as an isolated fixed point. Since the group of automorphisms, $Aut(\mathcal{D})$ preserves the Bergman metric, s_z coincides with the local geodesic symmetry around z. Hence $\mathcal{D}$ is a Hermitian symmetric space.

A domain $\mathcal{D}$ is called *circled* (with respect to 0) if $0 \in \mathcal{D}$ and $e^{it} z \in \mathcal{D}$ for every $z \in \mathcal{D}$ and real t.

Theorem 2. *Every bounded symmetric domain is holomorphically isomorphic with a bounded symmetric and circled domain. It is unique up to linear isomorphisms.*

From now on $\mathcal{D}$ denotes a circled bounded symmetric domain. G is the identity component of $Aut(\mathcal{D})$, K is the isotropy group of 0 in G. The Lie algebra $\mathfrak{g}$ will be considered as a Lie algebra of holomorphic vector fields on $\mathcal{D}$, i.e., vector fields X on $\mathcal{D}$ such that Xf is holomorphic if f is.

The symmetry $s, z \mapsto -z$ around the origin induces an invoulution on G by $g \mapsto sgs^{-1}$ and, by differentiating, an involution $Ad(s)$ of $\mathfrak{g}$. We have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

into the ± 1-eigenspaces.

For every $v \in \mathbb{C}^n$, let ξ_v be the unique vector field in $\mathfrak{p}$ that takes the value v at the origin. Then

$$\xi_v(z) = v - Q(z)\overline{v}$$

where $Q(z) : \overline{V} \to V$ is a complex linear mapping and $Q : V \to \mathrm{Hom}(\overline{V}, V)$ is a homogeneous quadratic polynomial. Hence $Q(x, z) = Q(x + z) - Q(x) - Q(z) : \overline{V} \to V$ is bilinear and symmetric in x and z. For $x, y, z \in V$, we define

$$\{x\overline{y}z\} = D(x, \overline{y})z = Q(x, z)\overline{y} \tag{8}$$

Thus $\{x\overline{y}z\}$ is complex bilinear and symmetric in x and z and complex antilinear in y, and $D(x, \overline{y})$ is the endomorphism $z \mapsto \{x, \overline{y}z\}$ of V. The pair $(V, \{\ \})$ is called a *Jordan triple system.* This Jordan triple system is positive in the sense that if $v \in V, v \neq 0$ and $Q(v)\overline{v} = \lambda v$ for some $\lambda \in \mathbb{C}$, then λ is positive.

We introduce the endomorphisms

$$B(x, y) = I - D(x, \overline{y}) + Q(x)\overline{Q}(\overline{y}) \tag{9}$$

of V for $x, y \in V$, where $\overline{Q}(\overline{y})x = \overline{Q(y)\overline{x}}$.

We summarise some results in the following

Proposition 3. *a) The Lie algebra* $\mathfrak{g}$ *satisfies the relations*

$$\begin{aligned} [\xi_u, \xi_v] &= D(u, \overline{v}) - D(v, \overline{u}) && (10)\\ [l, \xi_u] &= \xi_{lu} && (11) \end{aligned}$$

for $u, v \in V$ *and* $l \in \mathfrak{k}$
b) The Bergman kernel $k(x, y)$ *of* $\mathcal{D}$ *is*

$$m(\mathcal{D})^{-1} \det B(x, y)^{-1} \tag{12}$$

c) The Bergman metric at 0 *is*

$$h_0(u, v) = trD(u, \overline{v}), \tag{13}$$

and at an arbitrary point $z \in \mathcal{D}$

$$h_z(u, v) = h_0(B(z, z)^{-1}u, v) \tag{14}$$

d) The triple product $\{\ \}$ *is given by*

$$h_0(\{u\overline{v}w\}, y) = \partial_u \partial_{\overline{v}} \partial_x \partial_{\overline{y}} \log K(z, z)|_{z=0} \tag{15}$$

We define odd powers of an element $x \in V$ by

$$x^1 = x,\ x^3 = Q(x)\overline{x}, \cdots, x^{2n+1} = Q(x)\overline{x^{2n-1}}. \tag{16}$$

An element $x \in V$ is said to be *tripotent* if $x^3 = x$, i.e., if $\{x\overline{x}x\} = 2x$. Two tripotents c and e are called orthogonal if $D(c, \overline{e}) = 0$ In this case $D(c, \overline{c})$ and $D(e, \overline{e})$ commute and $e + c$ is a tripotent.

Every $x \in V$ can be written uniquely

$$x = \lambda_1 c_1 + \cdots + \lambda_n c_n,$$

where the c_i are pairwise orthogonal nonzero tripotents which are real linear combinations of odd powers of x, and the λ_i satisfy

$$0 < \lambda_1 < \cdots < \lambda_n.$$

This expression for x is called its *spectral decomposition* and the λ_i the eigenvalues of x. Moreover, the domain $\mathcal{D}$ can be realised as the unit ball in V with the spectral norm

$$||x|| = \max |\lambda_i|,$$

where the λ_i are the eigenvalues of x, i.e.,

$$\mathcal{D} = \{x \in V | ||x|| < 1\}.$$

Let $f(t)$ be an odd complex valued function of the real variable t, defined for $|t| < \rho$. For every $x \in V$ with $|x| < \rho$ we define $f(x) \in V$ by

$$f(x) = f(\lambda_1)c_1 + \cdots + f(\lambda_n)c_n, \tag{17}$$

where $x = \lambda_1 c_1 + \cdots + \lambda_n c_n$ is the spectral resolution of x. This functional calculus is used in expressing the action on $\mathcal{D}$ of the elements $\exp \xi_v$ in G:

$$\exp \xi_v(z) = u + B(u, u)^{1/2} B(z, -u)^{-1}(z + Q(z)\overline{u}) \tag{18}$$

and

$$d(\exp \xi_v)(z) = B(u, u)^{1/2} B(z, -u)^{-1}, \tag{19}$$

where $u = \tanh v$, for $v \in \mathbb{C}^n$ and $z \in \mathcal{D}$.

4 The real part of the Lie ball

We consider the non-degenerate quadratic form

$$q(z) = z_1^2 + \cdots z_n^2 \tag{20}$$

on $V = \mathbb{C}^n$. In the following we will often denote $q(z,w)$ by (z,w). Defining $Q(x)y = q(x,y)x - q(x)y$, where $q(x,y) = q(x+y) - q(x) - q(y)$, we get a Jordan triple system. The Lie ball $\mathscr{D} = \{z \in \mathbb{C}^n | 1 - 2\langle z,z\rangle + |zz^t|^2 > 0, |z| < 1\}$ is now unit open ball in this Jordan triple system. An easy computation shows the following identity.

$$D(x,\overline{y})z = 2(\sum_{k=1}^{n} x_k\overline{y_k})z + 2(\sum_{k=1}^{n} z_k\overline{y_k})x - 2(\sum_{k=1}^{n} x_k z_k)\overline{y}$$

Recalling that $B(x,y) = I - D(x,\overline{y}) + Q(x)\overline{Q}(\overline{y})$. The Bergman kernel of $\mathscr{D}$ is

$$K(z,w) = (1 - 2\langle z,w\rangle + |zz^t|^2|\overline{ww^t}|^2)^{-n}. \tag{21}$$

We will hereafter denote it by $h(z,w)^{-n}$.

Consider the real from $\mathbb{R}^n$ in $\mathbb{C}^n$. Observe that $\mathscr{D} \bigcap \mathbb{R}^n$ is the unit ball of $\mathbb{R}^n$. We will hereafter often denote it by X. On X we have a simple expression for the Bergman metric:

$$B(x,x) = (1 - |x|^2)^{-2}I, x \in X. \tag{22}$$

X is a totally real submanifold of $\mathscr{D}$ in the sense that

$$T_x(X) + iT_x(X) = T_x(\mathscr{D}),\ T_x(X) \bigcap iT_x(X) = \{0\}$$

This implies that every holomorphic function on $\mathscr{D}$ that vanishes on X is identically zero. In fact, more is true

Lemma 4. *X is a totally geodesic submanifold of $\mathscr{D}$, i.e., for every $p \in X$ and $v \in T_p(X)$, the geodesic $t \mapsto Exp_p(tv)$ in $\mathscr{D}$ lies in X for all real t.*

Proof. First we prove that complex conjugation is an isometry of the Riemannian metric on $\mathscr{D}$. We let $\psi(z) = \overline{z}$. Clearly it is a smooth function with real differential $d\psi(z)v = \overline{v}$. Observe that $h(z,z) = h(\overline{z},\overline{z})$. Hence

$$\begin{aligned} h_{\psi(z)}(d\psi(z)u, d\psi(z)v) &= \partial_{\overline{u}}\partial_v h(\psi(z),\psi(z))^{-n} \\ &= \partial_u\partial_{\overline{v}} h(z,z)^{-n} \end{aligned}$$

Clearly

$$\Re(\partial_u\partial_{\overline{v}} h(z,z)^{-n}) = \Re(\partial_{\overline{u}}\partial_v h(z,z)^{-n}),$$

i.e., the real part of the Bergman metric is preserved.

Let $p \in X$ and choose neighbourhoods $V \subseteq T_p(\mathscr{D}) = \mathbb{C}^n, U \subseteq \mathscr{D}$ of 0 and p such that

$$\mathrm{Exp}_p|_V : V \to U$$

is a diffeomorphism. Suppose that $v \in \mathrm{Exp}_p^{-1}(U \bigcap X)$ and that $\mathrm{Exp}_p(v) = x$. Since complex conjugation is an isometry, and fixes p and x, it also fixes the geodesic from p to x, i.e., the geodesic is in X. This proves that $\mathrm{Exp}_p^{-1}(U \bigcap X)$ is an n-dimensional submanifold of $V \bigcap \mathbb{R}^n$. Hence it equals $V \bigcap \mathbb{R}^n$. □

We let G denote the identity component of $\mathrm{Aut}(\mathscr{D})$, K the isotropy group of 0 in G. We define the subgroup H as the identity component of

$$\{h \in G | h(x) \in X \,\mathrm{if}\, x \in X\}$$

We will denote $H \bigcap K$ by L.

Proposition 5. *H acts transitively on X and $X \cong H/L$ as a symmetric space.*

Proof. X is a Riemannian manifold with the Riemannian structure given by the restriction to X of the Bergman metric. Since the geodesics of X are geodesics in $\mathscr{D}$, X is geodesically complete. Hence for any $p, q \in X$, there is a $w \in T_p(X)$ such that

$$\mathrm{Exp}_p(tw) = q. \tag{23}$$

Writing $w = \xi_v(p)$ with $v \in \mathbb{R}^n$, (23) can be stated as

$$\exp \xi_v(p) = q,$$

and hence the group action is transitive. Moreover the local symmetries s_p for $p \in X$ preserve X. Indeed, $s_o(x) = -x$ and clearly preserves X. The symmetry s_p at p is given by hs_0h^{-1} where $h \in H$ is chosen so that $h(0) = p$. Thus we have proved the claim. □

We now study the image of X in the $M_{n2}(\mathbb{R})$- model of the Lie ball. For computational convenience, we now work with the transposes of these matrices. Note that the letter X is used to denote both the real manifold in $\mathscr{D}$ and the first column in the $n \times 2$ matrix Z. In this section we will therefore temporarily denote the real part of $\mathscr{D}$ by X_D.

The defining equation of the Hua-transform can be written as

$$\frac{1}{2}\begin{pmatrix} zz^t + 1 & i(zz^t - 1) \\ \overline{zz}^t + 1 & -i(\overline{zz}^t - 1) \end{pmatrix} Z = \begin{pmatrix} z \\ \overline{z} \end{pmatrix} \tag{24}$$

In coordinates, this identity takes the form

$$z_k = \frac{1}{2}(\ (zz^t + 1)x_k + i(zz^t - 1)y_k). \tag{25}$$

This gives

$$4zz^t = (zz^t)^2(X + iY)(X + iY)^t + 2(XX^t + YX^t)zz^t \tag{26}$$

$$+(X - iY)(X - iY)^t, \tag{27}$$

which is a quadratic equation in zz^t with unique solution

$$zz^t = \frac{2-(XX^t+YY^t)-2\sqrt{(1-XX^t)(1-YY^t)-(YX^t)^2}}{(X+iY)(X+iY)^t}. \tag{28}$$

From (25) we see that if z is real, then $y_k = 0$ for all k. On the other hand, if $Y = 0$, then (28) shows that zz^t is real and therefore z is real by (25). Hence the image of $X \in \mathscr{D}$ under the Hua-transform is the set

$$\mathcal{H}(X_D) = \{Z = (X\,0)\,|X \in M_{n1}(\mathbb{R})\}. \tag{29}$$

For an element $Z = (X\,0)$, the condition that $Z^*Z < I_2$ is clearly equivalent with $|X| < 1$.

Recall that the real n-dimensional unit ball can be desribed as a symmetric space $SO_0(1,n)/SO(n)$ by a procedure analogous to the one in the first section. One first considers all lines in $\mathbb{R}^{1+n}$ on which the quadratic form $x_1^2 - x_2^2 - \cdots x_{n+1}^2$ is positive definite and identifies these lines with all real $n \times 1$-matrices with norm less than or equal to one. If we write elements $g \in SO(1,n)$ as matrices of the form

$$g = \begin{pmatrix} a & - & b & - \\ | & & & \\ c & & D & \\ | & & & \end{pmatrix}, \tag{30}$$

the action is given by

$$X \mapsto (c+DX)(a+bX)^{-1}. \tag{31}$$

The group $SO(1,n)$ can be embedded into $SO(2,n)$. Indeed, the equality

$$\begin{pmatrix} 1 & 0 & - & 0 & - \\ 0 & a & - & b & - \\ | & | & & & \\ c & 0 & & D & \\ | & | & & & \end{pmatrix} \begin{pmatrix} 1 & 0 & - & 0 & - \\ 0 & a' & - & b' & - \\ | & | & & & \\ c' & 0 & & D' & \\ | & | & & & \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & - & 0 & - \\ 0 & aa' & - & ab'+bD' & - \\ | & | & & & \\ c+Dc' & 0 & & DD' & \\ | & | & & & \end{pmatrix}$$

shows that we can define an injective homomorphism $\theta : SO(1,n) \to SO(2,n)$ by

$$\begin{pmatrix} a & - & b & - \\ | & & & \\ c & & D & \\ | & & & \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & - & 0 & - \\ 0 & a & - & b & - \\ | & | & & & \\ c & 0 & & D & \\ | & | & & & \end{pmatrix}.$$

This subgroup acts on $\mathcal{H}(X_D)$ as

$$(X\,0) \mapsto \big((c+DX)(a+bX)^{-1}\ 0\big)$$

and the action is transitive. Suppose now that $h \in SO(2,n)$ preserves $H(X_D)$. Let $p = h(0)$. We can choose a $g \in SO_0(1,n)$ such that $g(0) = p$ (here we identify g with $\theta(g)$). Then $g^{-1}h(0) = 0$ and hence we can write it in block form as

$$g^{-1}h = \begin{pmatrix} I_2 & 0 \\ 0 & D \end{pmatrix},$$

with $D \in SO(n)$. This is an element in $\theta(SO(1,n))$ and hence $h \in \theta(SO(1,n))$. We have now proved the following theorem

Theorem 6. *The Hua transform $\mathcal{H} : \mathscr{D} \to M$ maps the real part X_D diffeomorphically onto*

$$\mathcal{H}(X_D) = \{Z = (X\,0)\,|\,X \in M_{n1}(\mathbb{R})\} \tag{32}$$

by $x \mapsto \frac{2x}{1+|x|^2}$. Moreover, the induced group homomorphism $h \mapsto \mathcal{H}\,h\,\mathcal{H}^{-1}$ is an isomorphism between the groups H and $SO_0(1,n)$

Remark. The model $\mathcal{H}(X)$ of $SO_0(1,n)/SO(n)$ is the real part of the complex n-dimensional unit ball $SU(1,n)/SU(n)$ with fractional linear group action. It is therefore equipped with a Riemannian metric given by the restriction of the Bergman metric of the complex unit ball. If $x \in \mathcal{H}(X), x \neq 0$, we decompose $\mathbb{R}^n = \mathbb{R}x \oplus (\mathbb{R}x)^\perp$. We let $v = v_x + v_{x^\perp}$ be the corresponding decomposition of a tangent vector v at x. In this model, the Riemannian metric at x is (cf [15])

$$g_x(v,v) = \frac{|v_x|^2}{(1-|x|^2)^2} + \frac{|v_{x^\perp}|^2}{(1-|x|^2)}.$$

We recall from equation (22) that if $x \in X$, then the Riemannian metric at x is

$$h_x(v,v) = \frac{1}{2n}\frac{|v|^2}{(1-|x|^2)^2}.$$

The Hua transform thus induces an isometry (up to a constant) of the real n-dimensional unit ball equipped with two different Riemannian structures.

4.1 Iwasawa decomposition of $\mathfrak{h}$

The Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ induces a decomposition $\mathfrak{h} = \mathfrak{l} \oplus \mathfrak{q}$.

Proposition 7. *$\mathfrak{h}$ has rank one.*

Proof. Take u and v in $\mathbb{R}^n$ and assume that $[\xi_u, \xi_v] = 0$. Then, for any $x \in \mathbb{R}^n$ we have

$$D(u,v)x = D(v,u)x.$$

A simple calculation shows that this amounts to

$$(u,x)v = (v,x)u,$$

which can only hold for all real x if $u = v$. □

We can thus choose a maximal abelian subalgebra $\mathfrak{a} = \mathbb{R}\xi_{e_1}$ in $\mathfrak{q}$. (Henceforth we will denote the vector e_1 by e.) The vector e is a maximal tripotent in the Jordan triple system corresponding to $\mathscr{D}$.

Proposition 8. *The roots with respect to $\mathfrak{a} = \mathbb{R}\xi_e$ are $\{\alpha, -\alpha\}$, where $\alpha(\xi_{e1}) = 2$. The corresponding posive root space is*

$$\mathfrak{q}_\alpha = \{\xi_v + \frac{1}{2}(D(e,v) - D(v,e)) | v \in \mathbb{R}e_2 \oplus \cdots \oplus \mathbb{R}e_n\}$$

Proof. Suppose that $[\xi_e, \xi_v + l] = \alpha(\xi_e)(\xi_v + l)$. Identifying the q- and l-components yields

$$D(e,v) - D(v,e) = \alpha(\xi_e)l \tag{33}$$

$$-\xi_{le} = \alpha(\xi_e)\xi_v \tag{34}$$

From (34) it follows that $le = -\alpha(\xi_e)v$ and, thus, applying both sides of (33) to e gives

$$D(e,v)e - D(v,e)e = -\alpha(\xi_e)^2 v,$$

i.e.,

$$D(e,e)v - D(e,v)e = \alpha(\xi_e)^2 v,$$

An easy computation gives

$$4v - 4(e,v)e = \alpha(\xi_e)^2 v.$$

Hence e is orthogonal to v and $\alpha(\xi_e)^2 = 4$. □

We shall fix the positive root α. Elements in $\mathfrak{a}_{\mathbb{C}}^*$ are of the form $\lambda\alpha$ and will hereafter be identified by the complex numbers λ. In particular, the mean value of the positive roots (with multiplicities), ρ, will be identified with the number $(n-1)/2$.

4.2 The Cayley transform

The Cayley transform is a biholomorphic mapping from a bounded symmetric domain onto a *Siegel domain.* We describe it for the domain $\mathscr{D}$ and use it to express the spherical functions on $\mathscr{D}$ in terms of the spherical functions on the unbounded domain.

We fix the maximal tripotent e. Then $\mathbb{C}^n$ equipped with the bilininear mapping

$$(z,w) \mapsto z \circ w = \frac{1}{2}\{zew\} \tag{35}$$

is a complex Jordan algebra. Observe, that since e is a tripotent, it is a unity for this multiplication. The Cayley transform is the mapping $c : \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$c(z) = (e+z) \circ (e-z)^{-1}, \tag{36}$$

where $(e-z)^{-1}$ denotes the inverse of $(e-z)$ with respect to the Jordan product.

Proposition 9. *The Cayley transform is given by the formula*

$$c(z) = \frac{1 - zz^t}{1 - 2z_1 + (zz^t)^2}e + \frac{2z'}{1 - 2z_1 + (zz^t)^2}, \tag{37}$$

for $z = z_1 e + z' \in \mathscr{D}$. Moreover, it maps X onto the halfspace

$$\{(x_1, \cdots, x_n\} \in \mathbb{R}^n | x_1 > 0\}.$$

Proof. We first find the inverse for an element x. Suppose therefore that $e = \frac{1}{2}\{xez\} = \frac{1}{2}D(x,e)z$, i.e.,

$$e = (x,e)z + (z,e)x - (x,z)e = x_1 z + z_1 x - (x,z)e$$

Identifying coordinates gives

$$\begin{aligned} 1 &= 2x_1 z_1 - (x,z) \\ 0 &= x_1 z' + z_1 x' \end{aligned}$$

These equations have the solution

$$\begin{aligned} z_1 &= x_1/(x,x) \\ z' &= -x'/(x,x). \end{aligned}$$

If we apply this to the expression $(e-z)^{-1}$ in the definition of c, we get

$$(e-z)^{-1} = \frac{1 - z_1}{(1-z_1)^2 + (z',z')}e + \frac{z'}{(1-z_1)^2 + (z',z')}.$$

Now the formula (37) follows by an easy computation. Moreover, we observe that the inverse transform is given by

$$w \mapsto (w-e) \circ (w+e)^{-1} = -c(-w).$$

Hence both c and c^{-1} preserve $\mathbb{R}^n$ and therefore

$$c(X) = c(\mathscr{D}) \bigcap \mathbb{R}^n.$$

We now determine this set. From ([11]) we know that (since e is a maximal tripotent)

$$c(\mathscr{D}) = \{u + iv | u \in A^+, v \in A\}, \tag{38}$$

where A is the real Jordan algebra

$$\{z \in V | Q(e)\overline{z} = z\}$$

and A^+ is the positive cone $\{z \circ z | z \in A\}$ in A. By a simple computation we see that

$$A = \mathbb{R}e \oplus \mathbb{R}ie_2 \oplus \cdots \oplus \mathbb{R}ie_n.$$

Since we have the identities

$$z + Q(e)\overline{z} = 2u,$$
$$z - Q(e)\overline{z} = 2iv$$

and

$$Q(e)\overline{z} = 2\overline{z_1} - \overline{z},$$

we get expressions for u and v:

$$2u = (z_1 + \overline{z_1}, z_2 - \overline{z_2}, \cdots, z_n - \overline{z_n})$$

$$2iv = (z_1 - \overline{z_1}, z_2 + \overline{z_2}, \cdots, z_n + \overline{z_n})$$

The condition that $x = u + iv$ is real thus implies that

$$u = (x_1, 0, \cdots, 0),$$

$$iv = (0, x_2, \cdots, x_n).$$

Moreover we require that

$$u = w \circ w = 2w_1 w - (w, w)e,$$

for some

$$w = c_1 e + c_2 ie_2 + \cdots c_n ie_n.$$

This yields

$$(x_1, \cdots, 0) = (c_1^2 + \cdots + c_n^2, ic_1c_2, \cdots, ic_1c_n).$$

Hence

$$c_1^2 = x_1, c_2 = \cdots = c_n = 0,$$

and thus

$$u + iv = (c_1^2, x_2, \cdots, x_n).$$

This proves the claim. $\square$

Recall the expression for the spherical functions on a symmetric space of non-compact type (cf [5] Thm 4.3)

$$\varphi_\lambda(h) = \int_L e^{(i\lambda+\rho)A(lh)} dl,$$

where $A(lh)$ is the (logarithm) of the A part of lh in the Iwasawa decomposition $H = NAL$. The integrand in this formula is called the *Harish-Chandra e-function.* For the above Siegel domain it has the form $e_\lambda(w) = (w_1)^{i\lambda+\rho}$ (cf [17]). Hence we have the following corollary.

Corollary 10. *The spherical function φ_λ on $X = H/L$ is*

$$\varphi_\lambda(x) = \int_{S^{n-1}} \left(\frac{1-|x|^2}{1-2(x,\zeta)+xx^t}\right)^{i\lambda+\rho} d\sigma(\zeta). \tag{39}$$

where σ is the $O(n)$-invariant probability measure on S^{n-1},

5 A family of unitary representations of G

5.1 The function spaces $\mathscr{H}_\nu$

The Bergman space $\mathcal{H}^2(\mathscr{D})$ has the reproducing kernel $h(z,w)^{-n}$. This means in particular that the function $h(z,w)^{-n}$ is positive definite in the sense that

$$\sum_{i,j=1}^{m} \alpha_i \overline{\alpha_j} h(z_i, z_j)^{-n} \geq 0,$$

for all $z_1, \cdots z_n \in \mathscr{D}$ and $\alpha_1, \cdots, \alpha_n \in \mathbb{C}$. It makes sense to ask for which real numbers ν the function $h(z,w)^{-\nu}$ is positive definite. It has been proved by Wallach ([19]) and Rossi-Vergne ([18]) that $h(z,w)^{-\nu}$ is positive definite precisely when ν in the set

$$\{0, \frac{n-2}{2}\} \bigcup (\frac{n-2}{2}, \infty)$$

This set will also be referred as the *Wallach set* (cf [3]). For ν in the Wallach set above, $h(z,w)^{-\nu}$ is the reproducing kernel of a Hilbert space of holomorphic functions on $\mathscr{D}$. We will call this space $\mathscr{H}_\nu$ and the reproducing kernel $K_\nu(z,w)$.

Proposition 11. *The mapping $g \mapsto \pi_\nu(g)$, where*

$$\pi_\nu(g)f(z) = J_{g^{-1}}(z)^{\frac{\nu}{n}} f(g^{-1}z)$$

defines a unitary projective representation of G on $\mathscr{H}_\nu$.

Proof. Comparison with the Bergman kernel shows that $h(z,w)^{-\nu}$ transforms under automorphisms according to the rule

$$h(gz, gw)^{-\nu} = J_g(z)^{-\frac{\nu}{n}} h(z,w)^{-\nu} \overline{J_g(w)}^{-\frac{\nu}{n}} \tag{40}$$

Recall that for functions f_1 and f_2 of the form

$$f_1(z) = \sum_{k=1}^{l} \alpha_k K_\nu(z, w_k), \;\; f_2(z) = \sum_{k=1}^{m} \beta_k K_\nu(z, w'_k),$$

the inner product is defined as

$$\langle f_1, f_2 \rangle_\nu = \sum_{i,j} \alpha_i \overline{\beta_j} K_\nu(w_i, w'_j) \tag{41}$$

Equation (40) implies that

$$K_\nu(g^{-1}z, w) = J_{g^{-1}}(z)^{-\frac{\nu}{n}} K_\nu(z, gw) \overline{J_{g^{-1}}(w)}^{-\frac{\nu}{n}} \tag{42}$$

Hence

$$\pi_\nu(g) f_1(z) = \sum_{k=1}^{l} \alpha_k \overline{J_{g^{-1}}(w_k)}^{-\frac{\nu}{n}} K_\nu(z, g w_k)$$
$$\pi_\nu(g) f_2(z) = \sum_{k=1}^{m} \beta_k \overline{J_{g^{-1}}(w'_k)}^{-\frac{\nu}{n}} K_\nu(z, g w'_k)$$

The unitarity

$$\langle \pi_\nu(g) f_1, \pi_\nu(g) f_2 \rangle_\nu = \langle f_1, f_2 \rangle_\nu$$

now follows by an application of the transformation rule (40) in the definition (41). Since functions of the form above are dense in $\mathscr{H}_\nu$, it follows that each $\pi_\nu(g)$ is a unitary operator and it is easy to see that $g \mapsto \pi_\nu(g)$ is a projective homomorphism of groups. □

In fact, π_ν is an irreducible projective representation, see [2]

5.2 Fock-Fischer spaces

It can be shown that for $\nu > (n-2)/2$ all holomorphic polynomials are in $\mathscr{H}_\nu$ and that polynomials of different homogeneous degree are orthogonal. In this context, the spaces $\mathscr{H}_\nu$ are closely linked with the *Fock-Fischer space*, $\mathscr{F}$, which we will now desribe.

The basis vector e_1 is a maximal tripotent which is decomposed into minimal tripotents as $e_1 = \frac{1}{2}(1, i, 0, \cdots, 0) + \frac{1}{2}(1, -i, 0, \cdots, 0)$. (We omit the easy computations.) In order to expand the reproducing kernels K_ν into a power series consistent with the treatment in ([2]), we need to introduce a new norm on $\mathbb{C}^n$ so that the minimal tripotents have norm 1, i.e., the Euclidean norm multiplied with $\sqrt{2}$. Then

$$\{f_1, \cdots, f_n\} := \{\frac{1}{\sqrt{2}} e_1, \cdots, \frac{1}{\sqrt{2}} e_n\}$$

is an orthonormal basis with respect to this new norm. We write points $z \in \mathscr{D}$ as $z = w_1 f_1 + \cdots + w_n f_n\}$. For polynomials $p(w) = \sum_\alpha a_\alpha w^\alpha$, we define

$$p^*(w) = \sum_\alpha \overline{a_\alpha} w^\alpha.$$

The Fock-Fischer inner product is now defined as

$$\langle p, q\rangle_{\mathscr{F}} = p(\partial)(q^*)|_{w=0},$$

where $p(\partial)$ is the differential operator $\sum_\alpha a_\alpha \frac{\partial^\alpha}{\partial w^\alpha}$, for p as above. The space $\mathscr{F}$ is the completion of the space of polynomials with respect to this norm. It is easy to see that polynomials of different homogeneous degree are orthogonal in $\mathscr{F}$. Moreover, the representation of $SO(n)$ on $\mathcal{P}^m$, the polynomials of homogeneous degree m can be decomposed into irreducible subspaces as (cf [16])

$$\mathcal{P}^m = \bigoplus_{m-2k\geq 0} E_{m-2k} \otimes \mathbb{C}(ww^t)^k, \tag{43}$$

where E_i are the spherical harmonic polynomials of degree i (cf [16]. The following relation holds between the Fock-Fischer - and the $\mathscr{H}_\nu$-norm on the space $E_{m-2k} \otimes \mathbb{C}(ww^t)^k$ (cf [2]).

$$\|p\|_\nu^2 = \frac{\|p\|_{\mathscr{F}}^2}{(\nu)_{m-k}(\nu - \frac{n-2}{2})_k}, \tag{44}$$

for $p \in E_{m-2k} \otimes \mathbb{C}(ww^t)^k$. We have the following decomposition of $\mathscr{H}_\nu$ under K:

Proposition 12. *(Faraut-Korànyi, [2]) a) If $\nu > \frac{n-2}{2}$, then*

$$\mathscr{H}_\nu|_K = \sum_{m-2k\geq 0} E_{m-2k} \otimes (zz^t)^k, \tag{45}$$

where E_{m-2k} are the spherical harmonic polynomials of degree $m-2k$. Moreover, we have the following expansion of the kernel function:

$$h(z,w)^{-\nu} = \sum_{m-2k\geq 0} (\nu)_{m-k} \left(\nu - \frac{n-2}{2}\right)_k K_{(m-k,k)}(z,w), \tag{46}$$

where $K_{(m-k,k)}$ is the reproducing kernel for the subspace $E_{m-2k} \otimes \mathbb{C}(zz^t)^k$ with the Fock-Fischer norm. The series converges in norm and uniformly on compact sets.
b) If $\nu = \frac{n-2}{2}$, then

$$\mathscr{H}_\nu|_K = \sum_m E_m \tag{47}$$

We will later need the norm of $(zz^t)^k$ in $\mathscr{H}_\nu$.

Proposition 13.

$$\|(zz^t)^k\|_\nu^2 = \frac{k!\left(\frac{n}{2}\right)_k}{(\nu)_k\left(\nu - \frac{n-2}{2}\right)_k} \tag{48}$$

Proof. A straightforward computation shows that

$$(\frac{\partial^2}{\partial z_1^2} + \cdots + \frac{\partial^2}{\partial z_n^2})(z_1^2 + \cdots + z_n^2)^k = (2^2k(k-1) + n2k)(z_1^2 + \cdots + z_n^2)^{k-1}$$

Proceeding inductively, we obtain

$$\begin{aligned}\left(\frac{\partial^2}{\partial z_1^2} + \cdots + \frac{\partial^2}{\partial z_n^2}\right)^k (z_1^2 + \cdots + z_n^2)^k &= \prod_{j=1}^k 2j(2(j-1)+n) \\ &= 4^k k!\left(\frac{n}{2}\right)_k\end{aligned}$$

The Fock-Fischer norm is computed in the w-coordinates $w_i = \sqrt{2}z_i$, so

$$(zz^t)^k = 2^{-k}(ww^t)^k$$

and

$$\left(\frac{\partial^2}{\partial z_1^2} + \cdots + \frac{\partial^2}{\partial z_n^2}\right)^k = 2^{-k}\left(\frac{\partial^2}{\partial w_1^2} + \cdots + \frac{\partial^2}{\partial w_n^2}\right)^k.$$

Hence

$$\|(zz^t)^k\|_{\mathscr{F}}^2 = k!\left(\frac{n}{2}\right)_k$$

and an application of Prop (44) gives the result. $\square$

6 Branching of π_ν under the subgroup H

6.1 A decomposition theorem

Recall the irreducible (projective)representations π_ν from the previous section. Our main objective isto decompose these into irreducible representation under the subgroup H . We saw that X is a totally real submanifold and this fact is reflected in the restrictions of the representations π_ν to H.

Proposition 14. *The constant function* 1 *is in* $\mathscr{H}_\nu$ *and is an* L*-invariant cyclic vector for the representation* $\pi_\nu : H \to \mathscr{U}(\mathscr{H}_\nu)$.

Proof. First note that

$$\begin{aligned} K_\nu(z,h0) &= J_h(h^{-1}z)^{-\nu/n} K_\nu(h^{-1}z,0)\overline{J_h(0)^{-\nu/n}} \\ &= \overline{J_h(0)^{-\nu/n}} J_{h^{-1}}(z)^{\nu/n} K_\nu(z,0) \\ &= \overline{J_h(0)^{-\nu/n}} \pi_\nu(h)1(z) \end{aligned}$$

Suppose now that the function $f \in \mathscr{H}_\nu$ is orthogonal to the linear span of the elements $\pi_\nu(h)1, h \in H$. By the above identity we have

$$\begin{aligned} f(h0) &= \langle f, K_\nu(\cdot, h0)\rangle_\nu \\ &= 0. \end{aligned}$$

Since H acts transitively on X, f is zero on X. Hence it is identically zero. $\square$

We want decompose the representation of H into a *direct integral* of irreducible representations. For the definition of a direct integral over a measurable field of Hilbert spaces we refer to Naimark ([13]). The following general decomposition theorem is stated in several references, but the author has not been able to find a proof of it in the literature. The proof we present below is based on Gelfand-Naimark representation theory for C^*-algebras.

Theorem 15. *Let π be a unitary representation of the semisimple Lie group, H, on a Hilbert space, $\mathscr{H}$. Suppose further that L is a maximal compact subgroup and that the representation has a cyclic L invariant vector. Then π can be decomposed as a direct integral of irreducible representations,*

$$\pi \cong \int_\Lambda \pi_\lambda \, d\mu(\lambda), \tag{49}$$

where Λ is a subset of the set of positive definite spherical functions on H and for $\lambda \in \Lambda, \pi_\lambda$ is the corresponding unitary spherical representation.

Proof. Recall that the representation π extends to a representation of the Banach algebra $L^1(H)$ by

$$f \mapsto \int_H f(x)\pi(x)dx$$

We will also denote this mapping of $L^1(H)$ into $\mathscr{B}(\mathscr{H})$ by π. This representation will also be cyclic. Indeed, if ξ is the L-invariant cyclic vector for H, vectors of the form

$$\pi(f)(\pi(h_1)\xi + \cdots + \pi(h_n))\xi$$

are dense in $\mathscr{H}$ and since

$$\pi(f)(\pi(h_1)\xi + \cdots + \pi(h_n)\xi) = \pi((L_{h_1} + \cdots + L_{h_n})f)\xi,$$

we see that the vectors $\pi(f)\xi$ form a dense subset in $\mathscr{H}$

Consider now the subalgebra, $L^1(H)^{\#}$, consisting of all L^1-functions that are left- and right L-invariant, i.e.,

$$L_l f = R_l f = f,$$

for all l in L. If we denote by $\mathscr{H}^L$, the subspace of L-invariant vectors in $\mathscr{H}$, the following calculation, in which f is in $L^1(H)^{\#}$

$$\pi(l)\pi(f)v = \int_H f(x)\pi(lx)v dx = \int_H f(x)\pi(x)v dx$$

shows that $\mathscr{H}^L$is invariant under all the operators $\pi(f), f \in L^1(H)^{\#}$. We claim that the subrepresentation of $L^1(H)^{\#}$ is also cyclic. To see this, suppose that $v \in \mathscr{H}^L$ is orthogonal to all $\pi(f)\xi, f \in L^1(H)^{\#}$. Then we have for an arbitrary f in $L^1(H)$

$$\begin{aligned}
\langle \pi(f)\xi, v\rangle &= \int_H f(x)\langle \pi(x)\xi, v\rangle dx \\
&= \int_H \int_L \int_L f(l_1 x l_2) dl_1 dl_2 \langle \pi(x)\xi, v\rangle dx = 0,
\end{aligned}$$

since $f^{\#} := \int_L \int_L f(l_1 \cdot l_2) dl_1 dl_2 \in L^1(H)^{\#}$. Hence $v = 0$.

The function φ defined as

$$\pi(f) \mapsto \langle \pi(f)\xi, \xi\rangle \tag{50}$$

is a state on the commutative C^*-algebra generated by $\pi(L^1(H)^{\#})$ and the identity operator (we may assume that ξ is a unit vector). It is a well known fact from the theory of C^*-algebras that the norm-decreasing positive functionals form a convex and weak* compact set (cf [12]). For a C^*-algebra with identity, its extreme points are the pure states. In the case when the algebra is commutative, these coincide with the characters. Therefore, φ can be expressed as

$$\varphi = \int_X \varphi_x d\mu, \tag{51}$$

where X is the set of characters and μ is a regular Borel measure on X. We recall the Gelfand-Naimark-Segal construction of a cyclic representation of a C^*-algebra associated with a given state. In this duality, the irreducible representations correspond to the pure states. So each φ_x in (51) parametrises an irreducible representation of $\pi(L^1(H)^{\#})$ on some Hilbert space E_x. On the other hand, we know that $\varphi_x \circ \pi : L^1(H)^{\#} \to \mathbb{C}$ is a homomorphism of algebras and is therefore of the form (cf [5] ch.4)

$$f \mapsto \int_H f(h)\phi_x(h)dh, \tag{52}$$

where ϕ_x is a bounded spherical function. Since this mapping also preserves the involution $*$, it is a positive linear functional, i.e.,

$$\int_H f(h)\varphi(h)dx \geq 0, \tag{53}$$

for every $f \in L^1(H)^{\#}$, such that $f = g * g^*$, for some $g \in L^1(H)^{\#}$. In fact, (53) will still hold if we allow f and g to be in $L^1(H)$. Indeed, if $f = g * g^*$, then a simple computation shows that $f^{\#} = g_1 * g_1^*$, where $g_1(x) = \int_L g(lx)dl$. This function is not in $L^1(H)^{\#}$, but considered as operators on $\mathscr{H}^L$, $\pi(f^{\#})$ and $\pi((g_1^{\#})^*(g_1^{\#})^*)$ coincide, as can be seen from the following calculation, where u and v are vectors in $\mathscr{H}^L$:

$$\begin{aligned}
\langle \pi(f^{\#})u, v\rangle &= \langle \pi(g_1)\pi(g_1)^*u, v\rangle \\
&= \langle \pi(g_1)\pi(g_1)^*u, v\rangle \\
&= \langle \pi(g_1)^*u, \pi(g_1)^*v\rangle \\
&= \langle \pi((g_1^*)^{\#})u, \pi((g_1^*)^{\#})v\rangle \\
&= \langle \pi((g_1^{\#})^*)u, \pi((g_1^{\#})^*)v\rangle \\
&= \langle \pi((g_1^{\#}) * (g_1^{\#})^*)u, v\rangle
\end{aligned}$$

Since $\pi(f^{\#})$ is positive, we have that $\varphi_x \circ \pi(f^{\#}) \geq 0$, i.e.,

$$\int_H f(h)\phi_x(h)dh = \int_H f^{\#}(h)\phi_x(h)dh \geq 0 \tag{54}$$

Lemma 16. *Suppose that φ is a bounded spherical function such that $\int_H f(h)\varphi(h)dh \geq 0$ for all $f \in L^1(H)$ of the form $f = g*g^*$ for some $g \in L^1(H)$. Then φ is positive definite.*

Proof. For any $f = g * g^*$ as in the statement, we have

$$\begin{aligned}
\int_H f(h)\varphi(h)dh &= \int_H \int_H g(y)\overline{g(h^{-1}y)}dy\varphi(h)dh \\
&= \int_H g(y) \int_H \overline{g(h^{-1}y)}\varphi(h)dhdy \\
&= \int_H \int_H g(y)\overline{g(z)}\varphi(yz^{-1})dzdy
\end{aligned}$$

Pick any complex numbers $c_1, \ldots, c_n$ and elements $x_1, \ldots, x_n$ in H and fix $\epsilon > 0$. We can choose a compact set $K \subset H$, containing all x_i in its interior, and a neighbourhood U of the identity such that

$$|\varphi(xy^{-1}) - \varphi(x'y'^{-1})| < \epsilon \tag{55}$$

for all (x, y) and (x', y') in $K \times K$ such that $(xx'^{-1}, yy'^{-1}) \in U \times U$. We now choose disjoint neighbourhoods E_i of x_i such that $E_i \subset K$ and $x_i^{-1} E_i \subset U$ for all i. Next, we choose an L^1-function g in such a way that its support lies in $\bigcup E_i$ and g has the constant value $c_i/|E_i|$ on E_i, where

$$|E_i| = \int_{E_i} dh.$$

Now we have

$$\begin{aligned}\int_H \int_H g(y)\overline{g(z)}\varphi(yz^{-1})dzdy &= \sum_{i,j} \int_{E_i} \int_{E_j} g(y)\overline{g(z)}\varphi(yz^{-1})dzdy \\ &= \sum_{i,j} g(y_i)\overline{g(y_j)}\varphi(y_i y_j^{-1})dh(E_i)dh(E_j),\end{aligned} \tag{56}$$

for some y_i in E_i. The choice of g implies that the sum in (56) equals

$$\sum_{i,j} c_i \overline{c_j} \varphi(y_i y_j^{-1}).$$

This yields

$$\begin{aligned}&|\int_H \int_H g(y)\overline{g(z)}\varphi(yz^{-1})dzdy - \sum_{i,j} c_i c_j \varphi(x_i x_j^{-1})| \\ &< \sum_{i,j} |c_i||c_j||\varphi(y_i y_j^{-1}) - \varphi(x_i x_j^{-1})| < n^2 \sup_i |c_i| \epsilon.\end{aligned}$$

This shows that

$$\sum_{i,j} c_i \overline{c_j} \varphi(x_i x_j^{-1}) \geq 0,$$

and hence φ is positive definite. □

Since every positive definite spherical function defines an irreducible, unitary, spherical representation of H, it also gives rise to a representation $L^1(H)^{\#}$. Its restriction to the subspace of L-invariant vectors will be irreducible and one-dimensional (cf [5], ch. 4). If the character φ_x corresponds to the spherical function Φ_x, we denote by (π_x, E_x) both the representations of H and of $L^1(H)^{\#}$ that it induces. Corresponding to this cyclic representation of $L^1(H)^{\#}$ with cyclic unit vector ϕ_x, we have the state $f \mapsto \langle \pi_x(f)\phi_x, \phi_x \rangle_x$

$$\begin{aligned}
\langle \pi_x(f)\phi_x, \phi_x\rangle_x &= \int_H f(h)\langle \pi_x(h)\phi_x, \phi_x\rangle_x dh \\
&= \int_H f(h)\langle L_h\phi_x, \phi_x\rangle_x dh \\
&= \int_H f(h)\phi_x(h^{-1})dh \\
&= \int_H f(h)\overline{\phi_x(h)}dh
\end{aligned}$$

Therefore this representation of $L^1(H)^\#$ is unitarily equivalent to the one given by the Gelfand-Naimark-Segal correspondence, i.e., we can regard the representation as coming from a representation of the group H. We now want to define a unitary operator $T : \mathscr{H} \to \int_X E_x d\mu$ that intertwines the actions of $L^1(H)^\#$. We let

$$\pi(f)\xi \mapsto \{\pi_x(f)\Phi_x\}, f \in L^1(H)^\#. \tag{57}$$

To see that this is well-defined, suppose that $\pi(f)\xi = 0$. Then we have

$$\langle \pi(f)\xi, \pi(f)\xi\rangle = \langle \pi(f^* * f)\xi, \xi\rangle = 0 \tag{58}$$

i.e.,

$$\phi(f^* * f) = 0 \tag{59}$$

By (51) we have

$$\phi(f^* * f) = \int_H \langle \pi_x(f^* * f)\phi_x, \xi\rangle_x d\mu = 0. \tag{60}$$

Therefore $\pi_x(f)\phi_x = 0$ for almost every x and hence T is well defined on a dense set of vectors. Note that (60) also shows that T is isometric on this set and it therefore extends to an isometry of $\mathscr{H}^L$ into $\int_x E_x d\mu$. We now prove that it is onto. Suppose that the vector $c = \{c_x\}$ is orthogonal to the image of T. Since each E_x is one-dimensional, $c = \{c(x)\phi_x\}$, where $c(x)$ is a complex valued function. We have now

$$\int_X \langle \pi_x(f)\phi_x, c(x)\phi_x\rangle_x d\mu = 0 \tag{61}$$

This can be stated as

$$\int_x \widetilde{\pi(f)(x)}\overline{c(x)}d\mu = 0,$$

where $\widetilde{\pi(f)}$ is the Gelfand transform of $\pi(f)$, which is a continuous function on X, defined by $\pi(f)(x) = \varphi_x(\pi(f))$. Recalling that this is an isomorphism of the involutive Banach algebras $\pi(L^1(H)^{\#}$ and $\mathcal{C}(X)$, we have that the function $c(x)$ is orthogonal to all the continuous functions in $L^2(X,\mu)$ and therefore it must be equal to zero. This establishes the claim that T is a unitary intertwining operator. Each E_x is the space of L-invariant vectors of a Hilbert space H_x on which H and $L^1(H)$ act. We would like to extend T to a unitary operator from $\mathscr{H}$ onto $\int_X H_x d\mu$ that intertwines these actions. The natural attempt is

$$\pi(f)\xi \mapsto \{\pi_x(f)\phi_x\}, f \in L^1(H). \tag{62}$$

To see that is well-defined, suppose that $\pi(f)\xi = 0$ and observe that we can assume that f is in $L^1(H)^{\#}$ and use the earlier argument. This also proves that T is isometric on a dense subspace. To prove surjectivity, suppose that $c = \{c_x\}$ is orthogonal to $T(\pi(L^1(H))$. The following equality

$$\int_x \langle \pi_x(f), \pi_x(l)c_x\rangle_x d\mu = \int_x \langle \pi_x(L_{l^{-1}}f)\phi_x, c_x^1\rangle_x d\mu$$

shows that c is L-invariant, and hence it equals zero. Thus T is a unitary isomorhism and intertwines the $L^1(H)$ actions. Thefore it also intertwines the group action of H. □

Remark. The measure μ in the above theorem is called the *Plancherel measure* for the representation π.

6.2 The Plancherel measure

In this section we find an explicit formula for the Plancherel measure μ for the representations π_ν when $\nu > (n-2)/2$. We express it as a orthogonalising measure for some hypergeometric polynomials. In this context, we will use the mapping $R : \mathscr{H}_\nu \to C^\infty(X)$ defined by

$$(Rf)(x) = h(x,x)^{\nu/2} f(x), x \in X$$

(see [21]). When $\nu > n-1$, R is in fact an H-intertwining operator onto a dense subspace of $L^2(X, d\iota)$ and the principal series representation gives the desired decomposition. This is a heuristic motivation for studying the functions $R^{-1}\varphi_\lambda$, where φ_λ is a spherical function on X.

Theorem 17. *The function $R^{-1}\varphi_\lambda(z)$ is holomorphic on $\mathscr{D}$ and has the power series expansion*

$$R^{-1}\varphi_\lambda(z) = \sum_k p_k(\lambda) \frac{(zz^t)^k}{\|(zz^t)^k\|_\nu},$$

where the coefficients $p_k(\lambda)$ is a polynomials of degree $2k$ of λ and satisfy the orthogonality relation *a) If $\nu \geq \frac{n-1}{2}$, then*

$$\frac{1}{2\pi}\int_0^\infty \left|\frac{\Gamma(\frac{1}{2}+i\lambda)\Gamma(\frac{n-1}{2}+i\lambda)\Gamma(\nu-\frac{n-1}{2}+i\lambda)}{\Gamma(2i\lambda)}\right|^2 p_{\nu,k}(\lambda)p_{\nu,l}(\lambda)d\lambda$$
$$= \Gamma\left(\frac{n}{2}\right)\Gamma\left(\nu-\frac{n-2}{2}\right)\Gamma(\nu)\delta_{kl}$$

b) If $\nu < \frac{n-1}{2}$, then

$$\frac{1}{2\pi}\int_0^\infty \left|\frac{\Gamma(\frac{1}{2}+i\lambda)\Gamma(\frac{n-1}{2}+i\lambda)\Gamma(\nu-\frac{n-1}{2}+i\lambda)}{\Gamma(2i\lambda)}\right|^2 p_{\nu,k}(\lambda)p_{\nu,l}(\lambda)d\lambda$$
$$+\frac{\Gamma(\nu)\Gamma(\nu-\frac{n-2}{2})\Gamma(n-1-\nu)\Gamma(\frac{n}{2}-\nu)}{\Gamma(n-1-2\nu)}$$
$$\times p_{\nu,k}\left(i^2\left(\nu-\frac{n-1}{2}\right)^2\right)p_{\nu,l}\left(i^2\left(\nu-\frac{n-1}{2}\right)^2\right) = \Gamma\left(\frac{n}{2}\right)\Gamma\left(\nu-\frac{n-2}{2}\right)\Gamma(\nu)\delta_{kl}$$

Proof. Recall the root space decomposition for $\mathfrak{h}$. Let $\langle\,,\,\rangle$ denote the inner product on $\mathfrak{a}_{\mathbb{C}}$ that is dual to the restriction of the Killing form to $\mathfrak{a}$. Let α_0 denote $\alpha/\langle\alpha,\alpha\rangle$. In this setting the spherical function φ_λ is determined by the formula (cf [5], ch. 4, exercise 8)

$$\varphi_\lambda(\exp(t\xi_e)0) = {}_2F_1(a',b',c';-\sinh(\alpha(t\xi_e))^2), \tag{63}$$

where

$$\begin{aligned}
a' &= \frac{1}{2}\left(\frac{1}{2}m_\alpha+m_{2\alpha}+\langle i\lambda,\alpha_0\rangle\right) = \frac{1}{2}\left(\frac{n-1}{2}+i\lambda\right),\\
b' &= \frac{1}{2}\left(\frac{1}{2}m_\alpha+m_{2\alpha}-\langle i\lambda,\alpha_0\rangle\right) = \frac{1}{2}\left(\frac{n-1}{2}-i\lambda\right),\\
c' &= \frac{1}{2}\left(\frac{1}{2}m_\alpha+m_{2\alpha}+1\right) = \frac{1}{2}\left(\frac{n+1}{2}\right).
\end{aligned}$$

Letting $x = \exp(t\xi_e)0 = \tanh t$, (63) takes the form

$$\varphi_\lambda(x) = {}_2F_1(a',b',c';\frac{xx^t}{1-xx^t}) \tag{64}$$

By Euler's formula (cf [4]) we have

$$\varphi_\lambda(x) = {}_2F_1(a',b',c';\frac{xx^t}{1-xx^t}) = (1-xx^t)^{a'}\,{}_2F_1(a',c'-b',c';xx^t) \tag{65}$$

For the function $R^{-1}\varphi_\lambda$ we thus get the expression

$$R^{-1}\varphi_\lambda(z) = (1-zz^t)^{-\nu+a'}\,{}_2F_1(a',c'-b',c;zz^t) \tag{66}$$

Expanding (66) into a power series yields

$$R^{-1}\varphi_\lambda(z) = \sum_{m=0}^{\infty}\sum_{l=0}^{m}\frac{(\nu-a')_{m-l}(a')_l(c'-b')_l}{(m-l)!l!(c')_l}(zz^t)^m. \tag{67}$$

Next, we use the following simple identities:

$$\begin{aligned}(\nu-a')_{m-l} &= \frac{(\nu-a')_m}{(\nu-a'+(m-l))_l} = \frac{(\nu-a')_m}{(-1)^l(-(\nu-a'+m-1))_l} && (68)\\ (m-l)! &= \frac{m!}{(m-l+1)_l}. && (69)\end{aligned}$$

Substitution of these in (67) yields

$$R^{-1}\varphi_\lambda(z) = \sum_{m=0}^{\infty}\frac{(\nu-a')_m}{m!}\sum_{l=0}^{m}\frac{(a')_l(c'-b')_l(-m)_l}{(c')_l(-(\nu-a'+m-1))_l}(zz^t)^m. \tag{70}$$

The inner sum in (70) can be recognised as a hypergeometric function, i.e., we have

$$\sum_{l=0}^{m}\frac{(a')_l(c'-b')_l(-m)_l}{(c')_l(-(\nu-a'+m-1))_l} = {}_3F_2(a',c'-b',-m;c',-(\nu-a'+m-1);1).$$

Now we use Thomae's transformation rule (cf [4]) for the function ${}_3F_2$:

$$\begin{aligned}&{}_3F_2(a',c'-b',-m;c',-(\nu-a'+m-1);1)\\ &= \frac{(-(\nu-a'+m-1)-(c'-b'))_m}{(-(\nu-a'+m-1))_m}\\ &\times\ {}_3F_2(c'-a',c'-b',-m;1+(c'-b')+(\nu-a'+m-1)-m;1)\end{aligned}$$

We finally obtain the following expression:

$$R^{-1}\varphi_\lambda(z) = \sum_{k=0}^{\infty}c_{n,\nu,k}(\lambda)(zz^t)^k,$$

where

$$c_{n,\nu,k}(\lambda) = \frac{(\nu-\frac{n-2}{2})_k}{k!}{}_3F_2(-k,\frac{1+i\lambda}{2},\frac{1-i\lambda}{2};\frac{n}{2},\nu-\frac{n-2}{2};1)$$

Recall the continuous dual Hahn polynomials (cf [20])

$$\begin{aligned}S_k(x^2;a,b,c) &= (a+b)_k(a+c)_k\\ &\times{}_3F_2(-k,a+ix,a-ix;a+b,a+c;1)\end{aligned} \tag{71}$$

We can thus write

$$\begin{aligned}
R^{-1}\varphi_\lambda(z) &= \sum_{k=0}^{\infty} \frac{(\nu-\frac{n-2}{2})_k}{(\frac{n}{2})_k(\nu-\frac{n-2}{2})_k k!} S_k\left((\frac{\lambda}{2})^2; \frac{1}{2}, \frac{n-1}{2}, \nu-\frac{n-2}{2}\right)(zz^t)^k \\
&= \sum_{k=0}^{\infty} p_{\nu,k}(\lambda)\frac{(zz^t)^k}{\|(zz^t)^k\|_\nu}.
\end{aligned}$$

For the orthogonality relation in the claim, we refer to [20]. □

6.3 The principal series representations π_λ on $L^2(S^{n-1})$

We will now construct an explicit realisation of the decomposition in Thereom 15. For this purpose, we consider a canonical realisation of the irreducible, unitary spherical representations of H as representations. on $L^2(S^{n-1})$.

Lemma 18. *If $g \in H$, then g transforms the surface measure, σ, on S^{n-1} as*

$$d\sigma(g\zeta) = J_g(\zeta)^{\frac{n-1}{n}} d\sigma(\eta)$$

Proof. Clearly it suffices to prove the statement for automorphisms of the form $\exp\ \xi_v, v \in \mathbb{R}^n$. Moreover we can assume that $\zeta = e_1$, since any $\zeta \in S^{n-1}$ can be written as le_1, where $l \in L$, and

$$\begin{aligned}
\exp\ \xi_v(le_1) = (\exp\ \xi_v l)(e_1) = (ll^{-1}\exp\ \xi_v l)(e_1) = (l\sigma_{l^{-1}}(\exp\ \xi_v))(e_1) \\
= l\exp\ (Ad(l^{-1})\xi_v)(e_1) = l\exp\ \xi_{\,l^{-1}v}(e_1)
\end{aligned}$$

Consider now the tangent space of $\mathbb{R}^n$ at e_1. We have an orthogonal decomposition

$$T_{e_1}(\mathbb{R}^n) = T_{e_1}(S^{n-1}) \oplus \mathbb{R}e_1.$$

At ge_1 we have the corresponding decomposition

$$T_{ge_1}(\mathbb{R}^n) = T_{ge_1}(S^{n-1}) \oplus \mathbb{R}he_1$$

Since H preserves S^{n-1}, $dg(e_1)T_{e_1}(S^{n-1}) = T_{ge_1}(S^{n-1})$, and by completing e_1 and he_1 to orthormal bases for their respective tangent spaces, $dg(e_1)$ corresponds to a matrix of the form

$$\begin{pmatrix} c & & 0 & \\ | & * & * & * \\ v & * & * & * \\ | & * & * & * \end{pmatrix}$$

Hence $J_{\mathbb{R}^n}(e_1) = cJ_{S^{n-1}}(e_1)$, where $c = (dg(e_1)e_1, ge_1)$. Now $(dg(e_1)e_1, he_1) = \lim_{r\to 1}(dg(re_1)re_1, gre_1)$. For fixed $r < 1$ we have

$$\exp\ \xi_v(re_1) = u + B(u,u)^{1/2}B(re_1,-u)^{-1}(re_1 + Q(re_1)u) = u + dh(re_1)(re_1 + Q(re_1)u),$$

where $u = \tanh v$. Since $Q(re_1)u = 2(u, re_1)re_1 - u$, we get

$$\begin{aligned}
&(dg(re_1)re_1, g(re_1)) \\
&= \quad (1 + 2(u, re_1))|dg(re_1)re_1|^2 + (dg(re_1)re_1, u - dg(re_1)u)
\end{aligned} \tag{72}$$

For any $z \in \mathscr{D} \bigcap \mathbb{R}^n$ and $v, w \in \mathbb{R}^n$, we have

$$\begin{aligned}
(dg(z)v, w) &= (B(gz, gz)^{-1}B(gz, gz)dg(z)v, w) \\
&= \frac{1}{2n}h_{gz}(B(gz, gz)dg(z)v, w)) \\
&= h(gz, gz)\frac{1}{2n}h_{gz}(dg(z)v, dg(z)dg(z)^{-1}w)) \\
&= h(gz, gz)\frac{1}{2n}h_z(v, dg(z)^{-1}w)) \\
&= h(gz, gz)(B(z, z)^{-1}v, dg(z)^{-1}w)) \\
&= \frac{h(gz, gz)}{h(z, z)}(v, dg(z)^{-1}w)
\end{aligned} \tag{73}$$

Applying (73) in the cases $z = re_1$, $v = re_1$, and $w = dg(re_1)re_1$ and $w = u - dg(re_1)u$, repectively, yields

$$(dg(re_1)re_1, dg(re_1)re_1) = \frac{h(g(re_1), g(re_1))}{h(re_1, re_1)}r^2 \tag{74}$$

and

$$(dg(re_1)re_1, u - dg(re_1)u) = \frac{h(g(re_1), g(re_1))}{h(re_1, re_1)}(re_1, dg(re_1)^{-1}u - u) \tag{75}$$

Next we find an expression for $dg(re_1)^{-1}$:

$$\begin{aligned}
dg(re_1)^{-1} &= (B(u, u)^{1/2}B(re_1, -u)^{-1})^{-1} = B(re_1, -u)B(u, u)^{-1/2} \\
&= \frac{1}{1 - |u|^2}B(re_1, -u)
\end{aligned}$$

An elementary computation shows that

$$B(re_1, -u)u = (1 - |u|^2)u + (2|u|^2|u|^2(u, re_1))re_1.$$

Therefore (75) can be written as

$$\begin{aligned}
&(dg(re_1)re_1, u - dg(re_1)u) = \\
&\qquad \frac{h(g(re_1), g(re_1))}{h(re_1, re_1)}\frac{1}{1 - |u|^2}(re_1, (2|u|^2 + 2|u|^2(u, re_1))re_1)
\end{aligned} \tag{76}$$

Comparing (72) and (76)gives us

$$\begin{aligned}(dg(re_1)re_1, g(re_1)) &= \frac{h(g(re_1), g(re_1))}{h(re_1, re_1)} r^2 \,(1 + 2(u, re_1) \\ &+ \frac{1}{1-|u|^2}(2|u|^2 + 2|u|^2(u, re_1))) \\ &= \frac{h(g(re_1), g(re_1))}{h(re_1, re_1)}\, r^2 \,\frac{1 + 2(u, re_1) + |u|^2}{1-|u|^2} \qquad (77)\end{aligned}$$

By the transformation rule for the Bergman kernel

$$h(g(re_1), g(re_1)) = |J_g(re_1)|^{2/n} h(re_1, re_1).$$

Moreover,

$$\begin{aligned}&\lim_{r\to 1} r^2 \,\frac{1 + 2(u, re_1) + |u|^2}{1-|u|^2} \\ &= \frac{1 + 2(u, e_1) + |u|^2}{1-|u|^2} \\ &= \frac{h(e_1, -u)}{h(u,u)^{1/2}} \\ &= J_h(e_1)^{-1/n}\end{aligned}$$

Letting $r \to 1$ in (77) gives

$$(dg(e_1)e_1, g(e_1)) = J_g(e_1)^{\frac{1}{n}},$$

and this proves the claim. □

Proposition 19. *For any real number λ, the map $h \mapsto \pi_\lambda(h)$, where*

$$\pi_\lambda(h)f(b) = J_{h^{-1}}(b)^{\frac{i\lambda+\rho}{n}} f(h^{-1}b)$$

defines a unitary representation of H on $L^2(B)$.

Proof. We have

$$\begin{aligned}\int_B |J_{h^{-1}}(b)^{\frac{i\lambda+\rho}{n}}|^2 |f(h^{-1}b)|^2 db &= \int_B J_{h^{-1}}(hb)^{\frac{2\rho}{n}} |f(b)|^2 d(hb) \\ &= \int_B J_h(b)^{-\frac{2\rho}{n}} |f(b)|^2 J_h(b)^{\frac{n-1}{n}} db \\ &= \int_B |f(b)|^2 db,\end{aligned}$$

where the last equality follows by lemma 18. □

It is well known that the representations π_λ above are unitarily equivalent to the canonical spherical representations associated with the corresponding functionals λ on $\mathfrak{a}_{\mathbb{C}}$ (cf [9], ch. 7).

We will now consider a family of Hilbert spaces indexed by parameter $\lambda \in \Lambda$. Each of these will be $L^2(S^{n-1})$, but we equip them with the inequivalent representations π_λ. Moreover we will follow Helgason and in this context denote S^{n-1} by B. The measure σ will be denoted by db. We thus have a family of spaces $\{L^2(B)_\lambda | \lambda \in \Lambda\}$. Recall the expression (10) for the spherical functions. In this setting we write it as

$$\varphi_\lambda(x) = \int_B e_{\lambda,b}(x)db,$$

where

$$e_{\lambda,b}(x) = \frac{h(x,x)^{1/2}}{h(x,b)}$$

by Cor 10. For fixed $z \in \mathscr{D}$ and $\lambda \in \mathbb{R}$, $R^{-1}e_{\lambda,b}(z)$ is a function in $L^2(B)$. Moreover, $\pi_\nu(H)$ makes sense as a group of mappings on $\mathcal{O}(\mathscr{D})$, the set of holomorphic functions on $\mathscr{D}$. We have a relationship between these representations.

Lemma 20. *For every $g \in H$,*

$$\pi_\nu(g)\pi_\lambda(g)R^{-1}e_{\lambda,b}(z) = R^{-1}e_{\lambda,b}(z). \tag{78}$$

On the Lie algebra-level, we have the relation

$$\pi_\nu(X)R^{-1}e_{\lambda,b}(z) = -\pi_\lambda(X)R^{-1}e_{\lambda,b}(z), \tag{79}$$

for $X \in \mathfrak{h}$

The proof is straightforward by applying the transformation rules for the function $h(z,w)$.

6.4 The Fourier-Helgason transform

In this section we let μ ($=\mu_\nu$) be the finite measure on the real line that orthogonalises the coefficients $p_k(\lambda)$ in (63). Let Λ_ν be its support. As we saw above, μ can, depending on the value of ν, either be absolutely continuous with respect to Lebesgue measure or have a point mass at $\lambda = i(\nu - (n-1)/2)$, i.e., we either have

$$\Lambda_\nu = (0,\infty)\bigcup\{i(\nu - (n-1)/2)\},\ \nu \geq (n-1)/2$$

or

$$\Lambda_\nu = (0,\infty),\ \nu < (n-1)/2.$$

The results in this section apply to both cases and we will therefore suppress the index ν and simply denote the support of μ by Λ.

Any holomorphic function, f, on $\mathscr{D}$ has a power series expansion

$$f(z) = \sum_{\alpha} f_\alpha z^\alpha, \tag{80}$$

where $f_\alpha = \frac{\partial^\alpha f}{\alpha! \partial z^\alpha}(0)$. We can collect the powers of equal homegeneous degree together and write

$$f(z) = \sum_{k} f_k(z), \tag{81}$$

where f_k is of homogeneous degree k.

We now consider the mapping $(\cdot,\cdot)_\nu : \mathcal{P} \times \mathcal{O}(\mathscr{D}) \to \mathbb{C}$ defined as

$$(f, g)_\nu = \sum_{k} \langle f, g_k \rangle_\nu. \tag{82}$$

Observe that the definition makes sense since every polynomial is orthogonal to all but finitely many g_k

Definition 21. If f is a polynomial in $\mathscr{H}_\nu$, its generalised Fourier-Helgason transform is the function $\tilde{f}$ on $\Lambda \times B$ defined by

$$\tilde{f}(\lambda, b) = (f, R^{-1} e_{\lambda,b})_\nu \tag{83}$$

Proposition 22. *If the polynomial f is in $\mathscr{H}_\nu^L$, then $\tilde{f}$ is L-invariant and*

$$\|f\|_\nu = \|\tilde{f}\|_{L^2(\Lambda \times B, d\mu \times db)}$$

Moreover, the Fourier-Helgason transform extends to an isometry from $\mathscr{H}_\nu^L$ into $L^2(\Lambda, d\mu)$, and the inversion formula

$$f(z) = \int_\Lambda \tilde{f}(\lambda) R^{-1} \varphi_\lambda(z) d\mu(\lambda) \tag{84}$$

holds.

Proof. Writing

$$R^{-1} e_{\lambda,b} = \sum_{\alpha} c_\alpha(\lambda, b) z^\alpha = \sum_{k} e_{\lambda,b,k}$$

and

$$R^{-1} \varphi_\lambda(z) = \sum_{\alpha} c_\alpha(\lambda) z^\alpha = \sum_{k} p_k(\lambda) \frac{(zz^t)^k}{\|(zz^t)^k\|_\nu}$$

we see that the coefficients and polynomials of homogeneous degree k are related by

$$c_{\lambda,\alpha} = \int_B c_\alpha(\lambda, b)db \tag{85}$$

and

$$p_k(\lambda)\frac{(zz^t)^k}{\|(zz^t)^k\|_\nu} = \int_B e_{\lambda,b,k}(z)db \tag{86}$$

respectively. Therefore we have

$$\begin{aligned}
\tilde{f}(\lambda, b) &= \sum_k \langle f, e_{\lambda,b,k}\rangle_\nu \\
&= \sum_k \langle \int_L \pi_\nu(l) f dl, e_{\lambda,b,k}\rangle_\nu \\
&= \sum_k \langle f, \int_L \pi_\nu(l^{-1}) e_{\lambda,b,k} dl\rangle_\nu \\
&= \sum_k \langle f, \int_L \pi_\lambda(l) e_{\lambda,b,k} dl\rangle_\nu \\
&= (f, R^{-1}\varphi_\lambda)_\nu
\end{aligned}$$

This proves the L-invariance. Moreover, we have

$$(f, R^{-1}\varphi_\lambda)_\nu = \sum_k \overline{p_k(\lambda)}\langle f, e_k\rangle_\nu.$$

Hence

$$\|\tilde{f}\|^2_{L^2(\Lambda\times B, d\mu\times db)} = \sum_k |\langle f, e_k\rangle_\nu|^2 = \|f\|^2_\nu$$

This proves the first part of the claim.

To prove the inversion formula, we now let f be an L-invariant polynomial and x be a point in $\mathscr{D}\bigcap\mathbb{R}^n$. Since we have the estimate

$$|R^{-1}\varphi_\lambda(x)| \le (1 - |x|^2)^{-\frac{\nu}{2}} \tag{87}$$

independently of λ, the integral

$$\int_\Lambda \tilde{f}(\lambda) R^{-1}\varphi_\lambda(x) d\mu(\lambda)$$

makes sense. In the following we let e_k denote the unit basis vector $\frac{(zz^t)^k}{\|(zz^t)^k\|_\nu}$ in $\mathscr{H}_\nu^L$. We then have

$$\begin{aligned}
\int_\Lambda \tilde{f}(\lambda)R^{-1}\varphi_\lambda(x)d\mu(\lambda) &= \sum_k \int_\Lambda \langle f, e_k\rangle_\nu \overline{p_k(\lambda)}R^{-1}\varphi_\lambda(x)d\mu(\lambda) \\
&= \sum_k \langle f, e_k\rangle_\nu \int_\Lambda \sum_j \overline{p_k(\lambda)}p_j(\lambda)e_j(x)d\mu(\lambda) \\
&= f(x)
\end{aligned}$$

Now let $f \in \mathscr{H}_\nu^L$ be arbitrary. We choose a sequence of polynomials $f_n \in \mathscr{H}_\nu^L$ such that

$$f = \lim f_n.$$

Since the evaluation functionals are continuous, we have

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \int_\Lambda \tilde{f}_n(\lambda)R^{-1}\varphi_\lambda(x)d\mu(\lambda)$$

for every real point x. By Jensen's inequality and (87)

$$\left|\int_\Lambda (\tilde{f}(\lambda) - \tilde{f}_n(\lambda))R^{-1}\varphi_\lambda(x)d\mu(\lambda)\right|^2 \leq \mu(\Lambda)\int_\Lambda |\tilde{f}(\lambda) - \tilde{f}_n(\lambda)|^2(1-|x|^2)^{-\nu}d\mu(\lambda).$$

Hence

$$f(x) = \int_\Lambda \tilde{f}(\lambda)R^{-1}\varphi_\lambda(x)d\mu(\lambda).$$

Thus the inversion formula holds for real points, x. To see that the formula holds for arbitrary points, we note that both the left hand- and the right hand side of the formula define holomorphic functions on $\mathscr{D}$. Since they agree on the totally real submanifold X, they are equal. $\square$

Proposition 23. *The Fourier-Helgason transform intertwines the action of the group H, i.e.,*

$$\widetilde{\pi_\nu(h)f}(\lambda, b) = \pi_\lambda(h)\widetilde{f}(\lambda, b),\ h \in H. \tag{88}$$

Proof. We consider the corresponding representations of the Lie algebra $\mathfrak{h}$. These will also be denoted by π_ν and π_λ respectively. Since the representations are unitary, we have the following equalities (on the respective dense spaces of analytic vectors):

$$\pi_\nu(\exp(X)) = e^{\pi_\nu(X)} \tag{89}$$

$$\pi_\lambda(\exp(X)) = e^{\pi_\lambda(X)}, \tag{90}$$

for $X \in \mathfrak{h}$. It therefore suffices to prove that the action of $\mathfrak{h}$ is preserved. Let $X \in \mathfrak{h}$. If f is a polynomial in $\mathscr{H}_\nu$, then differentiation of the mapping

$$t \mapsto J_{\exp tX}(z)^{\nu/n} f((\exp tX)z)$$

at $t = 0$ shows that $\pi_\nu(X)f$ is also a polynomial, and

$$\begin{aligned}
\widetilde{\pi_\nu(X)f}(\lambda, b) &= \sum_k \langle \pi_\nu(X)f, e_{\lambda,b,k}\rangle_\nu \\
&= \sum_k \langle f, -\pi_\nu(X)e_{\lambda,b,k}\rangle_\nu \\
&= \sum_k \langle f, \pi_\lambda(X)e_{\lambda,b,k}\rangle_\nu \\
&= (f, \pi_\lambda(X)R^{-1}e_{\lambda,b})_\nu \\
&= \pi_\lambda(X)(f, R^{-1}e_{\lambda,b})_\nu
\end{aligned}$$

□

Theorem 24 (The Inversion Formula). *If f is a polynomial in $\mathscr{H}_\nu$, then*

$$f(z) = \int_\Lambda \int_B \widetilde{f}(\lambda, b)R^{-1}e_{\lambda,b}(z)dbd\mu(\lambda) \tag{91}$$

Proof. Take $h \in H$. Define

$$f_1(z) = \int_L \pi_\nu(l)\pi_\nu(h)f(z)dl$$

This is a radial function, and we have that

$$f_1(0) = J_{h^{-1}}(0)^{\frac{\nu}{n}} f(h^{-1}0). \tag{92}$$

Prop 22 gives

$$f_1(0) = \int_\Lambda \tilde{f}_1(\lambda)R^{-1}\varphi_\lambda(z)d\mu(\lambda) \tag{93}$$

$$\begin{aligned}
\widetilde{f}_1(\lambda) = (f_1, R^{-1}\varphi_\lambda)_\nu &= (\int_L \pi_\nu(l)\pi_\nu(h)fdl, R^{-1}\varphi_\lambda)_\nu \\
&= (\pi_\nu(h)f, R^{-1}\varphi_\lambda)_\nu
\end{aligned} \tag{94}$$

By Prop. (23) we have

$$\begin{aligned}
(\pi_\nu(h)f, R^{-1}\varphi_\lambda)_\nu &= (f, \pi_\nu(h^{-1})R^{-1}\varphi_\lambda)_\nu \\
&= (f, \int_B \pi_\nu(h^{-1})R^{-1}e_{\lambda,b}\, db)_\nu \\
&= (f, \int_B \pi_\nu(h^{-1})R^{-1}e_{\lambda,b}\, db)_\nu \\
&= (f, \int_B \pi_\lambda(h)R^{-1}e_{\lambda,b}\, db)_\nu \\
&= (f, \int_B J_{h^{-1}}(b)^{\frac{i\lambda+\rho}{n}} R^{-1}e_{\lambda,h^{-1}b}\, db)_\nu
\end{aligned}$$

The integrand above has a power series expansion where the coefficients are functions of b. If we integrate, we obtain a holomorphic functions for which the coefficients in the power series expansion are obtained by integrating the aforementioned coefficients over B. Hence we can proceed as follows.

$$\begin{aligned}
(f, \int_B J_{h^{-1}}(b)^{\frac{i\lambda+\rho}{n}} R^{-1}e_{\lambda,h^{-1}b}\, db)_\nu &= \int_B J_{h^{-1}}(b)^{\frac{-i\lambda+\rho}{n}} (f, R^{-1}e_{\lambda,h^{-1}b})_\nu db \\
&= \int_B J_{h^{-1}}(b)^{\frac{-i\lambda+\rho}{n}} \widetilde{f}(\lambda, h^{-1}b) db \\
&= \int_B J_{h^{-1}}(hb)^{\frac{-i\lambda+\rho}{n}} \widetilde{f}(\lambda, b) J_h(b)^{\frac{n-1}{n}} db \\
&= \int_B \widetilde{f}(\lambda, b) J_h(b)^{\frac{i\lambda+\rho}{n}} db \qquad (95)
\end{aligned}$$

It is easy to see that

$$J_h(b)^{\frac{i\lambda+\rho}{n}} = J_{h^{-1}}(0)^{\frac{\nu}{n}} R^{-1}e_{\lambda,b}(h^{-1}0), \qquad (96)$$

and so combining (92), (93) and (95) finally yields

$$f(h^{-1}0) = \int_\Lambda \int_B \widetilde{f}(\lambda, b) R^{-1}e_{\lambda,b}(h^{-1}0) db\, d\mu \qquad (97)$$

Thus the inversion formula holds for real points, hence for all points by the same argument as in the proof of Prop. 22. □

Theorem 25 (The Plancherel Theorem). *If f is a polynomial in $\mathscr{H}_\nu$, then*

$$\|f\|_\nu^2 = \int_\Lambda \int_B |\widetilde{f}(\lambda, b)|^2 db d\mu. \qquad (98)$$

Moroever, the Fourier-Helgason transform extends to an isometry from $\mathscr{H}_\nu$ onto $\int_\lambda (L^2(B))_\lambda d\mu$

Proof. If we write

$$f(z) = \sum_\alpha f_\alpha z^\alpha,$$

then, according to the Inversion Formula, we have

$$f_\alpha = \int_\Lambda \int_B \widetilde{f}(\lambda, b) c_\alpha(\lambda, b)(\lambda) db\, d\mu(\lambda)$$

Therefore

$$\begin{aligned} \|f\|_\nu^2 &= (f, f)_\nu = \sum_\alpha \overline{f_\alpha} \langle f, z^\alpha \rangle_\nu \\ &= \sum_\alpha \int_\Lambda \int_B \overline{\widetilde{f}(\lambda, b)}\, \overline{c_\alpha(\lambda, b)} \langle f, z^\alpha \rangle_\nu db\, d\mu(\lambda) \\ &= \int_\Lambda \int_B \overline{\widetilde{f}(\lambda, b)} \widetilde{f}(\lambda, b) db\, d\mu. \end{aligned}$$

This shows that the Fourier-Helgason transform extends to an isometry from $\mathscr{H}_\nu$ into $\int_\Lambda (L^2(B))_\lambda d\mu$. To see that it is surjective, note that by Propositions 22 and 23

$$\langle \pi_\nu(f) 1, 1 \rangle_\nu = \int_\Lambda \langle \pi_\lambda \tilde{1}(\lambda, \cdot), \tilde{1}(\lambda, \cdot) \rangle_\lambda d\mu,$$

for $f \in L^1(H)^\#$, i.e., we can write the positive functional

$$f \mapsto \langle \pi_\nu(f) 1, 1 \rangle_\nu$$

as an integral of pure states with respect to some measure. By uniqueness, it is the measure in theorem 15. Since the Fourier-Helgason transform intertwines the group action, it is the intertwining operator constructed in theorem 15. Thus it is surjective. $\square$

Remark. One can now use the Plancherel theorem to prove the inversion formula for arbitrary functions by the same argument as for L-invariant functions.

7 Realisation of the discrete part of the decomposition

Definition 26. For $\frac{n-2}{2n} \leq \alpha < \frac{n-1}{2n}$, $\mathscr{C}_\alpha$ is the Hilbert space completion of the C^∞-functions with respect to the norm

$$\|f\|_{\mathscr{C}_\alpha} = \int_{S^{n-1}} \int_{S^{n-1}} f(\zeta) \overline{f(\eta)} K(\zeta, \eta)^\alpha d\sigma(\zeta) d\sigma(\eta)$$

Using the action of H on S^{n-1}, we can define a unitary representation of H on $\mathscr{C}_\alpha$ of the form

$$\sigma_\alpha : f \mapsto J_{h^{-1}}(\cdot)^\beta f(h^{-1} \cdot), h \in H,$$

where $\beta = -\alpha + (n-1)/2$. The unitarity follows from

$$\begin{aligned}
&\int_{S^{n-1}}\int_{S^{n-1}} J_{h^{-1}}(\zeta)^{\beta} f(h^{-1}\zeta)\overline{J_{h^{-1}}(\eta)^{\beta} f(h^{-1}\eta)} K(\zeta,\eta)^{\alpha} d\sigma(\zeta) d\sigma(\eta) \\
&= \int_{S^{n-1}}\int_{S^{n-1}} J_h(\zeta)^{-\beta} f(\zeta)\overline{J_h(\eta)^{-\beta} f(\eta)} K(h\zeta,h\eta)^{\alpha} J_h(\zeta)^{\frac{n-1}{n}} J_h(\eta)^{\frac{n-1}{n}} d\sigma(\zeta) d\sigma(\eta) \\
&= \int_{S^{n-1}}\int_{S^{n-1}} J_h(\zeta)^{-\beta-\alpha+\frac{n-1}{n}} J_h(\eta)^{-\beta-\alpha+\frac{n-1}{n}} f(\zeta)\overline{f(\eta)} K(\zeta,\eta)^{\alpha} d\sigma(\zeta) d\sigma(\eta)
\end{aligned}$$

In fact, this representation is irreducible (cf [1]). We denote this representation by σ_α. The following theorem states that $\sigma_{\nu/n}$ is the representation corresponding to the singular point in the decomposition theorem.

Theorem 27. *The operator $T_{\nu/n}$ defined by the formula*

$$(T_{\nu/n} f)(z) = \int_{S^{n-1}} f(\zeta) K_\nu(z,\zeta) d\sigma(\zeta)$$

is a unitary H-intertwining operator from $\mathscr{C}_{\nu/n}$ onto an irreducible H-submodule of $\mathscr{H}_\nu$.

Proof. First of all we note that $T_{\nu/n}$ maps functions in $\mathscr{C}_{\nu/n}$ to holomorphic functions on $\mathscr{D}$ and thus π_ν has a meaning on the range of $T_{\nu/n}$. We start by showing that $T_{\nu/n}$ is formally intertwining. We have

$$\begin{aligned}
T_{\nu/n}(\sigma_{\nu/n}) f(z) &= \int_{S^{n-1}} J_{h^{-1}}(\zeta)^{-\nu/n+\frac{n-1}{n}} f(h^{-1}\zeta) K_\nu(z,\zeta) d\sigma(\zeta) \\
&= \int_{S^{n-1}} J_h(\zeta)^{\nu/n-\frac{n-1}{n}} f(\zeta) K_\nu(z,h\zeta) J_h(\zeta)^{\frac{n-1}{n}} d\sigma(\zeta) \\
&= \int_{S^{n-1}} J_h(\zeta)^{\nu/n} f(\zeta) K_\nu(h^{-1}z,\zeta) J_h(h^{-1}z)^{-\frac{\nu}{n}} J_h(\zeta)^{-\frac{\nu}{n}} d\sigma(\zeta) \\
&= J_{h^{-1}}(z)^{\frac{\nu}{n}} \int_{S^{n-1}} f(\zeta) K_\nu(h^{-1}z,\zeta) d\sigma(\zeta),
\end{aligned}$$

i.e.,

$$T_{\nu/n}\sigma_{\nu/n} = \pi_\nu T_{\nu/n}.$$

The next step is to prove that the constant function 1 is mapped into $\mathscr{H}_\nu$ and that its norm is preserved. Note that for $\alpha = \nu/n, K(z,\zeta)^\alpha = K_\nu(z,\zeta)$, and by lemma 12 we have an expansion

$$K_\nu(\zeta, e_1) = \sum_{m-2k\geq 0} c_{m,k}(\nu) K_{(m-k,k)}(\zeta, e_1),$$

where the coefficients $c_{m,k}(\nu)$ are given explicitly in Prop. (12). Now, since $K_\nu(\zeta, e_1)$ is $SO(n-1)$-invariant and the action of $SO(n-1)$ is linear, each

$K_{(m,k)}(\zeta, e_1)$ must also be $SO(n-1)$-invariant. Hence, $K_{(m,k)}(\zeta, e_1)$ can be assumed to be $\phi_{m-2k}(\zeta)(\zeta\zeta^t)^k$, where ϕ_{m-2k} is the unique element in E_{m-2k} that assumes the value 1 in e_1. Therefore

$$\begin{aligned}
\int_{S^{n-1}} K(\zeta,\eta)^\alpha d\sigma(\zeta) &= \int_L K_\nu(\zeta, le_1)dl \int_L K_\nu(l^{-1}\zeta, e_1)dl \quad (99)\\
&= \sum_{m-2k\geq 0} c_{m,k}(\nu) \int_L (l^{-1}\zeta(l^{-1}\zeta)^t)^k \phi_{m-2k}(l^{-1}\zeta)dl \quad (100)\\
&= \sum_{m-2k\geq 0} c_{m,k}(\nu)(\zeta\zeta^t)^k \int_L \phi_{m-2k}(l^{-1}\zeta)dl \quad (101)
\end{aligned}$$

Since $SO(n)$ acts irreducibly on E_{m-2k} and the function $\int_L \phi_{m-2k}(l^{-1}z)dl$ is an $SO(n)$-invariant element in E_{m-2k} it must be identically zero unless $m-2k=0$. Since

$$\|1\|^2_{\mathscr{C}_{\frac{\nu}{n}}} = \int_{S^{n-1}} \int_{S^{n-1}} K_\nu(\zeta,\eta)d\sigma(\zeta)d\sigma(\eta),$$

the computation above implies that

$$\begin{aligned}
\|1\|^2_{\mathscr{C}_{\frac{\nu}{n}}} &= \int_{S^{n-1}} \sum_{k=0}^\infty c_{2k,k}(\nu)(\zeta\zeta^t)^k d\sigma(\zeta)\\
&= \sum_{k=0}^\infty c_{2k,k}(\nu) = \sum_{k=0}^\infty \frac{(\nu)_k(\nu-\frac{n-2}{2})_k}{\|(zz^t)^k\|^2_{\mathscr{F}}}\\
&= \sum_{k=0}^\infty \frac{(\nu)_k(\nu-\frac{n-2}{2})_k}{k!(\frac{n}{2})_k} \quad (102)
\end{aligned}$$

On the other hand, the equalities (99)-(101) also show that

$$\begin{aligned}
T_{\nu/n}1(z) &= \sum_{k=0}^\infty \frac{(\nu)_k(\nu-\frac{n-2}{2})_k}{k!(\frac{n}{2})_k}(zz^t)^k\\
&= \sum_{k=0}^\infty \frac{((\nu)_k(\nu-\frac{n-2}{2})_k)^{1/2}}{(k!(\frac{n}{2})_k)^{1/2}} \frac{(zz^t)^k}{\|(zz^t)^k\|_\nu}. \quad (103)
\end{aligned}$$

If we compare (102) and (103), we see that $T_{\nu/n}1 \in \mathscr{H}_\nu$ and that $\|1\|_{\mathscr{C}_{\nu/n}} = \|1\|_\nu$.

Recall that

$$\mathscr{C}_{\nu/n} = \overline{\bigoplus_m E_m(S^{n-1})}$$

and that the representation of $\mathfrak{h}$ on the algebraic sum $\bigoplus_m E_m(S^{n-1})$ is irreducible. Hence

$$\bigoplus_m E_m(S^{n-1}) = \mathrm{Span}_{\mathbb{C}}\{\sigma_{\nu/n}(X_1)\cdots\sigma_{\nu/n}(X_k)1 | X_i \in \mathfrak{h}, 1 \leq i \leq k\}$$

Since $T_{\nu/n}$ interwines the representations of $\mathfrak{h}$, we have that π_ν is an irreducible representation of $\mathfrak{h}$ on the space $T_{\nu/n}(\bigoplus_m E_m(S^{n-1})) \subseteq \mathscr{H}_\nu$. By Schur's lemma ([10], ch.4)

$$\langle T_{\nu/n} f, T_{\nu/n} g\rangle_\nu = c\langle f, g\rangle_{\mathscr{C}_{\nu/n}},$$

for some real constant c. Putting, f and g equal to the constant function 1 and applying, we see that $c = 1$. Therefore, $T_{\nu/n}$ extends to a unitary operator

$$T_{\nu/n} : \mathscr{C}_{\nu/n} \to \overline{T_{\nu/n}(\bigoplus_m E_m(S^{n-1}))}$$

and we have proved the theorem. □

8 Realisation of the minimal representation $\pi_{(n-2)/2}$

In this section we show that the representation $\pi_{(n-2)/2}$ of H is irreducible by realising it as a complementary series representation.

We recall the space $\mathscr{C}_{\nu/n}$ from the previous section and the corresponding operator $T_{\alpha/n}$.

Theorem 28. *$T_{(n-2)/2n}$ is a unitary H-intertwining operator from $\mathscr{C}_{(n-2)/n}$ onto $\mathscr{H}_{(n-2)/n}$*

Proof. Recall that

$$\mathscr{C}_{(n-2)/n} = \overline{\bigoplus_m E_m(S^{n-1})} \tag{104}$$

and that the sum is a decomposition into $SO(n)$-irreducible subspaces. If we let $\mathcal{P}_{(n-2)/n}$ denote the set of all finite sums in (104), $\sigma_{(n-2)/n}$ defines a representation of $\mathfrak{l}$ on $\mathcal{P}_{(n-2)/n}$. The polynomial $(\zeta_1 + i\zeta_2)^m$ is a highest weight vector in E_m for this representation. Moreover, the power series expansion of $K_{(n-2)/n}$ shows that $T_{(n-2)/2n}$ is a polynomial in E_m. Since $T_{(n-2)/2n}$ intertwines the $\mathfrak{l}$-actions, $T_{(n-2)/2n}((\zeta_1 + i\zeta_2)^m)$ is a highest weight vector space for $\pi_{(n-2)/2}(\mathfrak{l})$, i.e.,

$$(T_{(n-2)/2n}(\zeta_1 + i\zeta_2)^m)(z) = C_m(z + iz)^m, \tag{105}$$

for some constant C_m. We now determine C_m. Choose $z = w\frac{1}{2}(1, -i, 0, \cdots, 0)$, where w is a complex number with $|w| < 1$. In this case $zz^t = 0, (z + iz)^m = w^m$. We now compute $(T_{(n-2)/2}((\zeta_1 + i\zeta_2)^m)(z)$.

$$\begin{aligned}
&\int_{S^{n-1}} K_{(n-2)/n}(z,\zeta)(\zeta_1 + i\zeta_2)^m \\
&= \int_{S^{n-1}} (1 - w(\zeta_1 - i\zeta_2))^{-(n-2)/n}(\zeta_1 + i\zeta_2)^m d\sigma(\zeta)
\end{aligned}$$

This integral only depends on the first two coordinates and can hence be converted to an integral over the unit disk, U (cf [15] Prop 1.4.4).

$$\begin{aligned}
&\int_{S^{n-1}} (1-w(\zeta_1 - i\zeta_2))^{-(n-2)/n}(\zeta_1+i\zeta_2)^m d\sigma(\zeta) \\
&= \frac{\Gamma\left(\frac{n-2}{2}\right)}{\pi\Gamma\left(\frac{n}{2}\right)} \int_U (1-w\overline{\zeta})^{-(n-2)/n}\zeta^m(1-|\zeta|^2)^{(n-4)/2} dm(\zeta).
\end{aligned}$$

We have the power series expansion

$$(1-w\overline{\zeta})^{-(n-2)/2} = \sum_{k=0}^{\infty} \left(\frac{n-2}{2}\right)_k (z\overline{\zeta})^k$$

Recall that $(1-w\overline{\zeta})^{-n/2}$ is the reproducing kernel for the weighted Bergman space $\mathscr{H}_{n/2}(U)$, defined as

$$\mathscr{H}_{n/2}(U) = \{f \in \mathcal{O}(U) \mid \frac{\Gamma\left(\frac{n}{2}\right)}{\pi\Gamma\left(\frac{n-2}{2}\right)} \int_U |f(\zeta)|^2(1-|\zeta|^2)^{(n-4)/2} dm(\zeta) < \infty\},$$

Polynomials of different degree are orthogonal in $\mathscr{H}_{n/2}(U)$ and hence we have

$$\begin{aligned}
&\int_U (1-w\overline{\zeta})^{-(n-2)/n}\zeta^m(1-|\zeta|^2)^{(n-4)/2} dm(\zeta) \\
&= \int_U \sum_{k=0}^{\infty} \left(\frac{n-2}{2}\right)_k (z\overline{\zeta})^k \zeta^m (1-|\zeta|^2)^{(n-4)/2} dm(\zeta) \\
&= \int_U \sum_{k=0}^{\infty} \left(\frac{n-2}{2}\right)\left(\frac{n}{2}\right)_m (z\overline{\zeta})^m \zeta^m (1-|\zeta|^2)^{(n-4)/2} dm(\zeta) \\
&= \pi w^m,
\end{aligned}$$

where the last equality follows from the reproducing property in $\mathscr{H}_{n/2}(U)$. Summing up, we have

$$(T_{(n-2)/2}(\zeta_1+i\zeta_2)^m)(z) = \frac{n-2}{2\pi^2}(z_1+iz_2)^m \tag{106}$$

From this and the intertwining of the $\mathfrak{l}$-action, it follows that

$$T_{(n-2)/2n}\left(\bigoplus_m E_m(S^{n-1})\right) \subseteq \bigoplus_m E_m \tag{107}$$

To compute the norm of $T_{(n-2)/2n}(p)$ where $p \in E_k(S^{n-1})$, we first fix $r<1$ and consider the polynomial $T_{(n-2)/2n}(p(rz))$. By definition

$$\begin{aligned}
T_{(n-2)/2n}(p)(rz) &= \int_{S^{n-1}} K_\nu(rz,\zeta)p(\zeta)d\sigma(\zeta) \\
&= \int_{S^{n-1}} K_\nu(z,r\zeta)p(\zeta)d\sigma(\zeta) \qquad (108)
\end{aligned}$$

The integral (108) can be approximated by a sequence of Riemann sums

$$\int_{S^{n-1}} K_\nu(z, r\zeta)p(\zeta)d\sigma(\zeta) = \lim_{l\to\infty} R_l, \tag{109}$$

$$R_l = \sum_{i=1}^{k_l} K_\nu(z, r\zeta_i)p(\zeta_i)\sigma(A_i) \tag{110}$$

It is not difficult to see that the sequence R_l is bounded in $\mathscr{H}_\nu$ and hence has a convergent subsequence. We can therefore assume that R_l converges to $T_{(n-2)/2n}(p)(rz)$. Hence

$$\|T_{(n-2)/2n}(p)(rz)\|_\nu^2 = \lim_{l\to\infty} \|R_l\|_\nu^2$$

The functions R_l are in the dense subspace on which we have an explicit formula for the norm

$$\|R_l\|_\nu^2 = \sum_{i,j} p(\zeta_i)\overline{p(\zeta_j)}\sigma(A_i)\sigma(A_j)K_\nu(r\zeta_i, r\zeta_j) \tag{111}$$

The sequence $(\|R_l\|_\nu^2)_l$ is also a Riemann sum; letting $l \to \infty$ we get

$$\|T_{(n-2)/2n}(p)(rz)\|_\nu^2 = \int_{S^{n-1}} \int_{S^{n-1}} p(\zeta)\overline{p(\eta)}K_\nu(r\zeta, r\eta)d\sigma(\zeta)d\sigma(\eta) \tag{112}$$

Finally, we let $r \to 1$ and obtain

$$\|T_{(n-2)/2n}(p)(z)\|_\nu^2 = \int_{S^{n-1}} \int_{S^{n-1}} p(\zeta)\overline{p(\eta)}K_\nu(\zeta, \eta)d\sigma(\zeta)d\sigma(\eta). \tag{113}$$

From this and the orthogonality of the spaces E_k, it follows that $T_{(n-2)/2n}$ maps $(\bigoplus_m E_m(S^{n-1})$ isometrically onto $(\bigoplus_m E_m)$. Hence it extends to a unitary operator from $\mathscr{C}_{(n-2)/2n}$ onto $\mathscr{H}_{(n-2)/2}$. □

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