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JOHANNES F. BRASCHE

Department of Mathematics
CHALMERS UNIVERSITY OF TECHNOLOGY
GÖTEBORG UNIVERSITY
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Johannes F. Brasche

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Mathematics
Department of Mathematics
Chalmers University of Technology and Göteborg University
SE-412 96 Göteborg, Sweden
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INTERACTIONS ALONG BROWNIAN PATHS IN $\mathbb{R}^d$, $d \leq 5$

JOHANNES F. BRASCHE*
Department of Mathematics
CTH & GU
41296 Göteborg, Sweden

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Abstract: Via the Posilicano method Schrödinger operators with singular potentials supported by Brownian paths in the configuration space $\mathbb{R}^d$, $1 \leq d \leq 5$, are constructed. The essential, absolutely continuous and singular continuous spectra are determined almost surely (with respect to Wiener measure). It is shown, that the set of positive eigenvalues is discrete and that the wave operators exist and are asymptotically complete a.s.; if $d \geq 3$ then the set of positive eigenvalues is even empty a.s. A trace formula for the number (counting multiplicities) of negative eigenvalues is derived.

1 Introduction

In a wide variety of models in quantum field theory one studies a family (H_ω) of Schrödinger operators in $L^2(\mathbb{R}^4, \lambda^4)$ (λ^d being the Lebesgue measure) with potentials supported by a Brownian path. Here severe mathematical problems arise from the very beginning. Due to the fact that the c_1-capacity of a "typical path of a Brownian particle in $\mathbb{R}^4$" equals zero, Kato's quadratic form method cannot be used in order to define the operator H_ω (cf. the introduction in [Bra] for a detailed discussion of this point).

Instead one has worked with ultraviolett cutoff [Cher] or nonstandard analysis [AFHL]. The spectral analysis of the operators constructed via these methods is, however, extremely difficult; actually, virtually nothing is known about their spectra.

Recently A. Posilicano [Pos] presented a new method for the construction of singularly perturbed selfadjoint operators (cf. also [Bra]). It is the purpose of this note to show that this method can be applied for the construction of Schrödinger operators with a singular potential supported by a "typical Brownian path" if the dimension d of $\mathbb{R}^d$ is less

*Email: brasche@math.chalmers.se

than or equal to 5. Moreover we shall provide a detailed spectral analysis of the operators constructed via the Posilicano method. In particular, we shall determine the essential spectra, prove existence and completeness of wave operators, absence of singular continuous spectra and positive eigenvalues and derive a trace formula for the expectation value of the number (counting multiplicities) of negative eigenvalues.

2 Preliminaries and Notation

2.1 Capacity and quasi continuity

$L^2(\mathbb{R}^d) = L^2(\mathbb{R}^d, \lambda^d)$ denotes the space of (equivalence classes) of functions which are square integrable w.r.t. the Lebesgue measure λ^d and $\hat{f}$ the Fourier transform of f. Let $s > 0$. $H^s(\mathbb{R}^d)$ denotes the Sobolev space of all $f \in L^2(\mathbb{R}^d)$ such that

$$\| f \|_{H^s} := \left(\int (1 + x^2)^{s/2} |\hat{f}(x)|^2 \lambda^d(dx) \right)^{1/2} < \infty. \tag{2.1}$$

The c_s – capacity of the compact set $K \subset \mathbb{R}^d$ is defined by

$$c_s(K) := \inf \| f \|^2_{H^s},$$

where the infimum is taken over all f in the space $C_0^\infty(\mathbb{R}^d)$ of smooth functions f with compact support satisfying $f(x) \geq 1$ for all $x \in K$. The c_s – capacity of an arbitrary Borel set B is defined by

$$c_s(B) := \sup c_s(K), \tag{2.2}$$

where the supremum is taken over all compact subsets of K.

The function $g : \mathbb{R}^d \longrightarrow \mathbb{C}$ is quasi continuous w.r.t. the c_s – capacity if and only if for every $\varepsilon > 0$ there exists an open subset O_ε of $\mathbb{R}^d$ such that

$$c_s(O_\varepsilon) < \varepsilon$$

and the restriction of g to the complement $\mathbb{R}^d \setminus O_\varepsilon$ is continuous. Every $f \in H^s(\mathbb{R}^d)$ has a representative $\tilde{f}$ which is quasi continuous w.r.t. the c_s – capacity. If $\tilde{f}$ and f° are representatives of $f \in H^s(\mathbb{R}^d)$ and quasi continuous w.r.t. the c_s – capacity then the c_s – capacity of the set $\{x \in \mathbb{R}^d : \tilde{f}(x) \neq f^\circ(x)\}$ equals zero. In the present note $\tilde{f}$ denotes any representative of $f \in H^s(\mathbb{R}^d)$ which is quasi continuous w.r.t. the c_s – capacity; this notation does not indicate which s is meant, but this will always be clear from the context.
If $\mu(B) = 0$ for every Borel set B satisfying $c_s(B) = 0$ and

$$\int |\tilde{f}|^2 d\mu < \infty, \quad f \in H^s(\mathbb{R}^d), \tag{2.3}$$

then we can define the mapping $J_{s\mu} : H^s(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d, \mu)$ by

$$J_{s\mu} f := \tilde{f} \quad \mu\text{-a.e.}, \ f \in H^s(\mathbb{R}^d). \tag{2.4}$$

2.2 Wiener measure and occupation time measure

Ω denotes the space $C(\mathbb{R}_+, \mathbb{R}^d)$ of continuous functions $\omega : \mathbb{R}_+ = [0, \infty) \longrightarrow \mathbb{R}^d$ and $\mathbb{W}$ the Wiener measure on Ω. $0 < T < \infty$ is fixed, the occupation time measure μ_ω^T on the Borel algebra $\mathcal{B}(\mathbb{R}^d)$ of $\mathbb{R}^d$ is defined via

$$\mu_\omega^T(B) := \lambda^1(\{t : 0 \leq t \leq T,\ \omega(t) \in B\}), \quad B \in \mathcal{B}(\mathbb{R}^d). \tag{2.5}$$

The topological support of the occupation time measure μ_ω^T equals the set

$$\Gamma_\omega^T := \{\omega(t) : 0 \leq t \leq T\}. \tag{2.6}$$

2.3 Singular perturbations

Let $s, \alpha > 0$. We put

$$G_{s\alpha} := (-\Delta + \alpha)^{-s} \text{ and } G_\alpha := G_{1\alpha} = (-\Delta + \alpha)^{-1} \tag{2.7}$$

where $-\Delta$ is the selfadjoint operator in $L^2(\mathbb{R}^d)$ defined by

$$D(-\Delta) := H^2(\mathbb{R}^d), \quad -\Delta f := -\sum_{j=1}^{d} \frac{\partial^2 f}{\partial x_j^2}, \quad f \in H^2(\mathbb{R}^d),$$

and derivatives have to be understood in the distributional sense.

An operator H belongs to the set $\mathcal{A}_\omega^T$ if and only if

$$\begin{array}{l} H \text{ is a selfadjoint operator in } L^2(\mathbb{R}^d), \\ C_0^\infty(\mathbb{R}^d \setminus \Gamma_\omega^T) \subset D(H), \\ Hf = -\Delta f, \quad f \in C_0^\infty(\mathbb{R}^d \setminus \Gamma_\omega^T), \\ H \neq -\Delta. \end{array} \tag{2.8}$$

If there exists an $s < 2$ such that $\mu_\omega^T(B) = 0$ for every Borel set B satisfying $c_s(B) = 0$ and

$$\mu_\omega^T(B) = 0, \text{ if } c_s(B) = 0, \text{ and } \int |\tilde{f}|^2 d\mu_\omega^T < \infty, \quad f \in H^s(\mathbb{R}^d), \tag{2.9}$$

then we can define the mapping $J_\omega^T : H^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d, \mu_\omega^T)$ by

$$J_\omega^T f := \tilde{f} \quad \mu_\omega^T\text{-a.e. }, f \in H^2(\mathbb{R}^d) \qquad (\text{ i.e. } J_\omega^T = J_{2\mu_\omega^T}) \tag{2.10}$$

and there exists a unique operator $H_{\omega\,\alpha}^T \in \mathcal{A}_\omega^T$ such that $-\alpha$ belongs to the resolvent set of $H_{\omega\,\alpha}^T$ and

$$(H_{\omega\,\alpha}^T + \alpha)^{-1} = G_\alpha + (J_\omega^T G_\alpha)^*(J_\omega^T G_\alpha), \tag{2.11}$$

cf. [Bra], Theorem 9.

2.4 Wave operators and Schatten classes

The wave operators $W^\pm(H, -\Delta)$ exist provided

$$W^\pm(H,-\Delta)f := \lim_{t\mp\infty} e^{itH}e^{it\Delta}f$$

exist for every $f \in L^2(\mathbb{R}^d)$. The wave operators $W^\pm(H, -\Delta)$ are asymptotically complete if and only if

$$\operatorname{ran}(W^+(H,-\Delta)) = \operatorname{ran}(W^-(H,-\Delta)) = (\mathcal{H}^{pp}(H))^\perp$$

(i.e. every state f can be decomposed into the orthogonal sum of a bound state f_b and a state f_s such that the system behaves asymptotically as a free system provided the initial state equals f_s). The wave operators $W^\pm(H, -\Delta)$ are asymptotically complete if and only if the singular continuous spectrum $\sigma_{sc}(H)$ is empty and the operators $W^\pm(H, -\Delta)$ are complete, i.e

$$\operatorname{ran}(W^+(H,-\Delta)) = \operatorname{ran}(W^-(H,-\Delta)) = \mathcal{H}^{ac}(H).$$

Here $\mathcal{H}^{pp}(H)$ and $\mathcal{H}^{ac}(H)$ denote the pure point spectral subspace (the closure of the span of the eigenvectors) of H and the absolutely continuous spectral subspace of H, respectively.

In order to prove existence and completeness of wave operators one often uses Schatten classes. Let $C : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be a compact linear bounded mapping. There exists an orthonormal basis $\{e_i\}_{i\in I}$ of $\mathcal{H}_2$ and nonnegative numbers λ_i, $i \in I$, such that

$$\sqrt{CC^*}e_i = \lambda_i e_i, \quad i \in I.$$

The family $\{\lambda_i\}_{i\in I}$ is unique up to permutations. We put

$$\| C \|_{\mathbb{S}_p} := (\sum_{i\in I} \lambda_i^p)^{1/p} \quad (\leq \infty), \quad 0 < p < \infty.$$

C belongs to the Schatten class of order p provided $\| C \|_{\mathbb{S}_p} < \infty$. We define $\| C \|_{\mathbb{S}_p} = \infty$ if C is not compact.

We shall repeatedly use the following well known facts: Along with C also B_1CB_2 and the adjoint C^* belong to the Schatten class $\mathbb{S}_p$ for all bounded operators B_1 and B_2. Moreover $CK \in \mathbb{S}_r$ provided $C \in \mathbb{S}_p$, $K \in \mathbb{S}_q$ and $1/p + 1/q = 1/r$.

3 Compactness and Schatten norms

Let $d \leq 5$. Our first goal is to show that the condition (2.9) is satisfied for $\mathbb{W}$-a.a. $\omega \in \Omega$. As mentioned this guarantees that for $\mathbb{W}$-a.a. $\omega \in \Omega$ there exists a unique operator $H^T_{\omega\,\alpha} \in \mathcal{A}^T_\omega$ such that $-\alpha$ belongs to the resolvent set of $H^T_{\omega\,\alpha}$ and (2.11) holds. If the dimension d is larger than 5 then the c_2-capacity of the set Γ^T_ω (cf. (2.1), (2.2), (2.6)) equals zero $\mathbb{W}$-a.s. and the set $\mathcal{A}^T_\omega$ is empty $\mathbb{W}$-a.s.

We shall prove (2.3) with the aid of Lemma 3.1 below which might be useful in other contexts, too. Let $s, \alpha > 0$ and $d \in \mathbb{N}$.

There exist rotationally symmetric functions $k_{s\alpha} : \mathbb{R}^d \longrightarrow [0,\infty]$ and $g_{s\alpha} : \mathbb{R}^d \longrightarrow [0,\infty]$ satisfying

$$\hat{k}_{s\alpha}(p) = (p^2+\alpha)^{-s/2}, \quad \hat{g}_{s\alpha}(p) = (p^2+\alpha)^{-s}, \quad \lambda^d - \text{a.e.}, \tag{3.1}$$

cf. [SW]. We choose $k_{s\alpha}$ and $g_{s\alpha}$ such that they are continuous on $\mathbb{R}^d$ if possible (i.e. if $s > d$ resp. $s > d/2$); otherwise we choose them such that they are continuous on $\mathbb{R}^d \setminus \{0\}$ and equal to ∞ at 0. $g_{s\alpha}$ is the convolution kernel of the operator $(-\Delta+\alpha)^{-s}$ on $L^2(\mathbb{R}^d,\lambda^d)$.

LEMMA 3.1 *Let $G^\mu_{s\alpha}$ be the integral operator with kernel $g_{s\alpha}(x-y)$ (cf. (3.1)) in $L^2(\mathbb{R}^d,\mu)$. If $G^\mu_{s\alpha}$ is bounded then the measure μ does not charge any set with c_s – capacity zero and*

$$\int |\tilde{v}|^2 d\mu \leq \| G^\mu_{s\alpha} \| \, ((-\Delta+\alpha)^{s/2}v, (-\Delta+\alpha)^{s/2}v)_{L^2(\mathbb{R}^d,\lambda^d)}, \quad v \in H^s(\mathbb{R}^d). \tag{3.2}$$

The estimate (3.2) is sharp.

PROOF. Denote by $K^\mu_{s\alpha}$ the integral operator with kernel $k_{s\alpha}(x-y)$ (cf. (3.1)) from $L^2(\mathbb{R}^d,\mu)$ to $L^2(\mathbb{R}^d,\lambda^d)$. Then the adjoint operator $K^{\mu*}_{s\alpha}$ is the integral operator from $L^2(\mathbb{R}^d,\lambda^d)$ to $L^2(\mathbb{R}^d,\mu)$ with the same kernel $k_{s\alpha}(x-y)$.

Let $f \in L^2(\mathbb{R}^d,\mu)$, $f \geq 0$ μ – a.e. Then

$$\begin{array}{rc} & \int\int k_{s\alpha}(x-y)f(y)\mu(dy)\int k_{s\alpha}(x-z)f(z)\mu(dz)\lambda^d(dx) \\ = & \int\int\int k_{s\alpha}(x-y)k_{s\alpha}(x-z)\lambda^d(dx)f(y)\mu(dy)f(z)\mu(dz) \\ = & \int f(y)\int g_{s\alpha}(y-z)f(z)\mu(dz)\mu(dy) \\ = & (f, G^\mu_{s\alpha}f)_{L^2(\mathbb{R}^d,\mu)} \\ \leq & \| G^\mu_{s\alpha} \| \, \| f \|^2_{L^2(\mathbb{R}^d,\mu)} < \infty; \end{array}$$

in the second step we have used that

$$\int k_{s\alpha}(x-y)k_{s\alpha}(x-z)\lambda^d(dx) = g_{s\alpha}(y-z).$$

Thus we arrive at

$$\| K^{\mu*}_{s\alpha} \|^2 \leq \| G^\mu_{s\alpha} \| < \infty.$$

For every f in the Schwartz space of rapidly decreasing smooth functions the function

$$v(\cdot) := \int k_{s\alpha}(\cdot - y)f(y)\lambda^d(dy)$$

also belongs to Schwartz space $\mathcal{S}(\mathbb{R}^d)$; in particular, v is continuous. Note that v is a representative of both $(-\Delta+\alpha)^{-s/2}f$ and $K^{\mu*}_{s\alpha}f$. Moreover

$$\begin{aligned} \int |v|^2 d\mu &= \| K^{\mu*}_{s\alpha}f \|^2_{L^2(\mathbb{R}^d,\mu)} \leq \| K^{\mu*}_{s\alpha} \|^2 \, \| f \|^2_{L^2(\mathbb{R}^d,\lambda^d)} \\ &\leq \| G^\mu_{s\alpha} \| \, (v, (-\Delta+\alpha)^s v)_{L^2(\mathbb{R}^d,\lambda^d)} \leq c \, \| v \|^2_{H^s(\mathbb{R}^d)} \end{aligned} \tag{3.3}$$

for some finite constant c independent of v.

If the c_s – capacity $c_s(K)$ of the compact set K equals zero then there exist v_n in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ satisfying

$$v_n \geq 1 \text{ on } K \quad \text{ and } \| v_n \|_{H^s(\mathbb{R}^d)} \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

By (3.3), it follows that $c_s(K) = 0$. By the inner regularity of the c_s – capacity and the measure μ, this implies that $c_s(B) = 0$ for every Borel set B such that $\mu(B) = 0$.

Let $v \in H^s(\mathbb{R}^d)$. Take any v_n, $n \in \mathbb{N}$, in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ converging to v in $H^s(\mathbb{R}^d)$ as n tends to infinity. By (3.3), there exists an $h \in L^2(\mathbb{R}^d, \mu)$ such that

$$v_n \longrightarrow h \quad \text{as } n \longrightarrow \infty \quad \text{in } L^2(\mathbb{R}^d, \mu).$$

Moreover there exists a subsequence $\{v_{n_j}\}$ of $\{v_n\}$ such that

$$v_{n_j} \longrightarrow \tilde{v} \quad c_s - \text{q.e. as } n \longrightarrow \infty.$$

Since the measure μ does not charge any set with c_s – capacity zero it follows that $\tilde{v} = h$ a.e. with respect to the measure μ, that $\tilde{v} \in L^2(\mathbb{R}^d, \mu)$ and

$$\begin{aligned}
\int |\tilde{v}|^2 d\mu &= \lim_{n \longrightarrow \infty} \int |v_n|^2 d\mu \\
&\leq \lim_{n \longrightarrow \infty} \| G^\mu_{s\alpha} \| \left((v_n, (-\Delta + \alpha)^s v_n\right)_{L^2(\mathbb{R}^d, \lambda^d)} \\
&= \| G^\mu_{s\alpha} \| \left((-\Delta + \alpha)^{s/2} v, (-\Delta + \alpha)^{s/2} v\right)_{L^2(\mathbb{R}^d, \lambda^d)}.
\end{aligned}$$

Since $G^\mu_{s\alpha} = J_{s\mu}(J_{s\mu}G_{s\alpha})^* = (J_{s\mu}G^{1/2}_{s\alpha})\,(J_{s\mu}G^{1/2}_{s\alpha})^* = K^{\mu *}_{s\alpha} K^\mu_{s\alpha}$ the operator $G^\mu_{s\alpha}$ is nonnegative and selfadjoint and

$$\| G^\mu_{s\alpha} \| = \| K^\mu_{s\alpha} \|^2 . \tag{3.4}$$

We choose a sequence $\{f_n\}$ in $\mathcal{S}(\mathbb{R}^d)$ such that $\| f_n \|_{L^2(\mathbb{R}^d, \lambda^d)} = 1$ for every $n \in \mathbb{N}$ and $\| K^{\mu *}_{s\alpha} f_n \|_{L^2(\mathbb{R}^d, \lambda^d)} \longrightarrow \| K^{\mu *}_{s\alpha} \|$. We put $v_n := (-\Delta + \alpha)^{-s/2} f_n$, $n \in \mathbb{N}$. Then $(v_n, (-\Delta + \alpha)^s v_n) = 1$ for every $n \in \mathbb{N}$ and (3.3) and (3.4) yield

$$\int |v_n|^2 d\mu \longrightarrow \| G^\mu_{s\alpha} \|, \quad \text{as } n \longrightarrow \infty,$$

i.e the inequality (3.2) is sharp. □

REMARK 3.2 a) With the aid of the above lemma we can immediately rediscover a well known result on measures in Kato classes. Let $0 < s < d/2$. Let μ be a measure in the Kato class w.r.t. the operator $(-\Delta)^s$, i.e

$$\lim_{\varepsilon \longrightarrow 0} \sup_{x \in \mathbb{R}^d} \int_{|y-x| \leq \varepsilon} \frac{1}{|x-y|^{d-2s}} \mu(dy) = 0.$$

Then the Schur test in combination with the facts that $g_{s\alpha}(x)$ tends to zero uniformly on $\{x : |x| > \varepsilon\}$ as α tends to infinity and that there exists a finite constant c independent of α

such that $g_{s\alpha}(x) \leq c|x|^{2s-d}$ for all $x \in \mathbb{R}^d$ yields that the operator norm $\| G^{\mu}_{s\alpha} \|$ of $g^{\mu}_{s\alpha}$ tends to zero as α tends to infinity. Thus (3.2) implies that $\int |\tilde{v}|^2 d\mu$ is an infinitesimal small form perturbation of the operator $(-\Delta)^s$.

b) For $s \leq 1$ the operator $(-\Delta)^s$ is associated to a Dirichlet form; We refer to [Amor] for a partial generalization of the above lemma in the Dirichlet case.

By Lemma 3.1, (2.3) holds provided that the operator $G^{\mu}_{s\alpha}$ is bounded for some $s < 2$. Actually, this operator even belongs to the Schatten class of order 4 if $s > d/2 - 1$:

LEMMA 3.3 *Let $s > d/2-1$, $\alpha > 0$ and for $\omega \in \Omega$ let $G^{\mu^T_\omega}_{s\alpha}$ be the integral operator in $L^2(\mathbb{R}^d, \mu^T_\omega)$ with the kernel $g_{s\alpha}(x-y)$ defined by (3.1). Then*

$$\mathbb{E} \| G^{\mu^T_\cdot}_{s\alpha} \|^4_{\mathbb{S}_4} < \infty,$$

i.e. the expectation value (w.r.t. to the Wiener measure $\mathbb{W}$) of $\| G^{\mu^T_\omega}_{s\alpha} \|^4_{\mathbb{S}_4}$ is finite. In particular, $\mathbb{W}$-a.s. the operator $G^{\mu^T_\omega}_{s\alpha}$ belongs to the Schatten class $\mathbb{S}_4$ of order 4. Moreover

$$\mathbb{E} \| G^{\mu^T_\cdot}_{s\alpha} \|^4_{\mathbb{S}_4} \longrightarrow 0, \quad \alpha \longrightarrow \infty.$$

PROOF. First let μ be any positive Radon measure on $\mathbb{R}^d$. Then

$$\begin{aligned} & \| G^{\mu}_{s\alpha} \|^4_{\mathbb{S}_4} = \| (G^{\mu}_{s\alpha})^2 \|^2_{\mathbb{S}_2} \\ = & \int\int \left(\int g_{s\alpha}(x-z) g_{s\alpha}(z-y)\mu(dz)\right)^2 \mu(dx)\mu(dy) \\ = & \int\int\int\int g_{s\alpha}(x-z)g_{s\alpha}(z-y)g_{s\alpha}(x-a)g_{s\alpha}(a-y)\mu(dz)\mu(dx)\mu(dy)\mu(da). \end{aligned}$$

If μ equals the occupation time measure μ^T_ω then this implies, by the general transformation theorem, that

$$\begin{aligned} \| G^{\mu^T_\omega}_{s\alpha} \|^4_{\mathbb{S}_4} = & \int_0^T\int_0^T\int_0^T\int_0^T g_{s\alpha}(\omega(t_1)-\omega(t_3))g_{s\alpha}(\omega(t_3)-\omega(t_2))g_{s\alpha}(\omega(t_1)-\omega(t_4)) \\ & g_{s\alpha}(\omega(t_4)-\omega(t_2))dt_1dt_2dt_3dt_4. \end{aligned} \tag{3.5}$$

For every element π of the symmetric group S_4 let

$$M_\pi := \{(t_1,t_2,t_3,t_4) \in [0,T]^4 : t_{\pi(1)} < t_{\pi(2)} < t_{\pi(3)} < t_{\pi(4)}\}.$$

Up to a set with Lebesgue measure zero the domain of integration in (3.5), i.e. the set $[0,T]^4$, equals the disjoint union of the 24 sets M_π, $\pi \in S_4$.

By using gaussian kernels

$$p_t(x) := (2\pi|t|)^{-d/2} e^{-\frac{x^2}{2|t|}}$$

we can derive an expression for the expectation value of the integral over the set M_π for every $\pi \in S_4$. For instance, in the case $\pi(j) = j$ for $j = 1,2,3,4$ we get

$$\begin{aligned}
&\mathbb{E}\int_{0\le t_1<t_2<t_3<t_4\le T} g_{s\alpha}(\omega(t_1)-\omega(t_3))g_{s\alpha}(\omega(t_3)-\omega(t_2))g_{s\alpha}(\omega(t_1)-\omega(t_4)) \\
&\qquad g_{s\alpha}(\omega(t_4)-\omega(t_2))dt_1dt_2dt_3dt_4 \\
&= \int_{0\le t_1<t_2<t_3<t_4\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_{t_2-t_1}(x)p_{t_3-t_2}(y)p_{t_4-t_3}(z)g_{s\alpha}(x+y)g_{s\alpha}(y) \\
&\qquad g_{s\alpha}(x+y+z)g_{s\alpha}(y+z)\lambda^d(x)\lambda^d(y)\lambda^d(z)dt_1dt_2dt_3dt_4. \qquad (3.6)
\end{aligned}$$

The function $g_{s\alpha}$ tends exponentially fast to zero at infinity. Moreover it is bounded if $2s > d$, has a logarithmic singularity at 0, if $2s = d$, and satisfies

$$g_{s\alpha}(x) \le c_\alpha |x|^{2s-d}, \quad x \in \mathbb{R}^d, \tag{3.7}$$

for some finite constant c_α if $2s < d$. $\limsup_{\alpha\longrightarrow\infty} c_\alpha < \infty$. Moreover $g_{s\alpha}(x) \longrightarrow 0$, as $\alpha \longrightarrow \infty$, for every $x \in \mathbb{R}^d \setminus \{0\}$. In what follows we shall treat the last case, $2s < d$; the other two cases can be treated in an analogous way and are even more simple.

By (3.7), the integrand on the right hand side of (3.6) is, up to constant, bounded by

$$p_{t_2-t_1}(x)p_{t_3-t_2}(y)p_{t_4-t_3}(z)|x+y|^{2s-d}|y|^{2s-d}|x+y+z|^{2s-d}|y+z|^{2s-d}.$$

A straighforward but tedious computation shows that

$$\begin{aligned}
&\int_{0\le t_1<t_2<t_3<t_4\le T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_{t_2-t_1}(x)p_{t_3-t_2}(y)p_{t_4-t_3}(z)|x+y|^{2s-d}|y|^{2s-d} \qquad (3.8) \\
&\qquad |x+y+z|^{2s-d}|y+z|^{2s-d}\lambda^d(x)\lambda^d(y)\lambda^d(z)dt_1dt_2dt_3dt_4 < \infty
\end{aligned}$$

provided $s > d/2 - 1$. Thus

$$\begin{aligned}
&\mathbb{E}\int_{0\le t_1<t_2<t_3<t_4\le T} g_{s\alpha}(\omega(t_1)-\omega(t_3))g_{s\alpha}(\omega(t_3)-\omega(t_2))g_{s\alpha}(\omega(t_1)-\omega(t_4)) \\
&\qquad g_{s\alpha}(\omega(t_4)-\omega(t_2))dt_1dt_2dt_3dt_4 < \infty
\end{aligned}$$

for every $\alpha > 0$ and

$$\begin{aligned}
&\mathbb{E}\int_{0\le t_1<t_2<t_3<t_4\le T} g_{s\alpha}(\omega(t_1)-\omega(t_3))g_{s\alpha}(\omega(t_3)-\omega(t_2))g_{s\alpha}(\omega(t_1)-\omega(t_4)) \\
&\qquad g_{s\alpha}(\omega(t_4)-\omega(t_2))dt_1dt_2dt_3dt_4 \longrightarrow 0, \quad \alpha \longrightarrow \infty.
\end{aligned}$$

The remaining 23 domains of integration can be treated in a similar manner and we get that

$$\begin{aligned}
&\mathbb{E}\int_{[0,T]^4} g_{s\alpha}(\omega(t_1)-\omega(t_3))g_{s\alpha}(\omega(t_3)-\omega(t_2))g_{s\alpha}(\omega(t_1)-\omega(t_4)) \\
&\qquad g_{s\alpha}(\omega(t_4)-\omega(t_2))dt_1dt_2dt_3dt_4 < \infty
\end{aligned}$$

for every $\alpha > 0$ and

$$\mathbb{E}\int_{[0,T]^4} g_{s\alpha}(\omega(t_1)-\omega(t_3))g_{s\alpha}(\omega(t_3)-\omega(t_2))g_{s\alpha}(\omega(t_1)-\omega(t_4)) \\ g_{s\alpha}(\omega(t_4)-\omega(t_2))dt_1dt_2dt_3dt_4 \longrightarrow 0, \quad \alpha \longrightarrow \infty.$$

By (3.5), we have proved the Lemma. □

4 Wave operators, continuous spectral subspaces and positive eigenvalues

In this section we shall present results on the scattering theory for the operators $H^T_{\omega\alpha}$ and related results on their spectra. Moreover we shall prove absence of singular continuous spectra and, for $d \geq 3$, also absence of positive eigenvalues.

THEOREM 4.4 *Let the dimension d of $\mathbb{R}^d$ be less than 5 or equal to 5 and $\alpha > 0$. For $\mathbb{W}$-a.a. $\omega \in \Omega$ let $H^T_{\omega\alpha}$ be the selfadjoint operator defined by (2.7), (2.10) and (2.11). Then the following is true for $\mathbb{W}$-a.a. $\omega \in \Omega$.*

(i) The essential spectrum of $H^T_{\omega\alpha}$ equals $[0,\infty)$.

(ii) The wave operators $W^{\pm}(H^T_{\omega\alpha}, -\Delta)$ exist and are asymptotically complete.

(iii) The singular continuous spectrum of $H^T_{\omega\alpha}$ is empty, the set of the positive eigenvalues of $H^T_{\omega\alpha}$ is discrete and every positive eigenvalue of $H^T_{\omega\alpha}$ (if there is any) has finite multiplicity.

(iv) The absolutely continuous part of $H^T_{\omega\alpha}$ is unitarily equivalent to the operator $-\Delta$ and, in particular, the absolutely continuous spectrum of $H^T_{\omega\alpha}$ equals $[0,\infty)$.

(v) If $d \geq 3$ then the operator $H^T_{\omega\alpha}$ has no positive eigenvalue.

PROOF. (i) We have

$$G_{2\alpha}^{\mu_\omega^T} = J^T_\omega G_{2\alpha}^{1/2}(J^T_\omega G_{2\alpha}^{1/2})^* = J^T_\omega G_\alpha (J^T_\omega G_\alpha)^*. \tag{4.9}$$

Since $2 > d/2 - 1$, this equation and Lemma 3.3 imply that

$$J^T_\omega G_\alpha \in \mathbb{S}_8 \quad \text{for } \mathbb{W}-\text{a.a. } \omega \in \Omega. \tag{4.10}$$

By (2.11) and (4.11),

$$(H^T_{\omega\alpha}+\alpha)^{-1} - (-\Delta+\alpha)^{-1} = J^T_\omega G_\alpha (J^T_\omega G_\alpha)^* \in \mathbb{S}_4 \quad \text{for } \mathbb{W}-\text{a.a. } \omega \in \Omega. \tag{4.11}$$

Since every operator in $\mathbb{S}_p$ is, in particular, compact, and the operators $H^T_{\omega\alpha}$ and $-\Delta$ are selfadjoint, Weyl's essential spectrum theorem together with (4.11) implies the assertion (i).

(ii) The wave operators exist and are asymptotically complete provided that the singular continuous spectra are empty and the wave operators exist and are complete. We shall prove absence of singular continuous spectra below under (iii).

The wave operators exist and are complete provided that there exists an $N \in \mathbb{N}$ such that the operator

$$D^T_{\alpha\omega N} := (H_{\omega\alpha} + \alpha)^{-N} - (-\Delta + \alpha)^{-N}$$

is compact and

$$(H_{\omega\alpha} + \alpha)^{-N} D^T_{\alpha\omega N} (-\Delta + \alpha)^{-N} \in \mathbb{S}_1, \tag{4.12}$$

cf. [Dem].

It follows immediately from (4.11) and the identity

$$D^T_{\alpha\omega N} = \sum_{j=0}^{N-1} (H_{\omega\alpha} + \alpha)^{-j} \left((H_{\omega\alpha} + \alpha)^{-1} - (-\Delta + \alpha)^{-1} \right) (-\Delta + \alpha)^{-(N-1-j)}$$

that the operator $D^T_{\alpha\omega N}$ is compact for $\mathbb{W}$-a.a. $\omega \in \Omega$. Thus we need only to prove that (4.12) is true $\mathbb{W}$-a.s. for some $N \in \mathbb{N}$.

For $k > d/2$ the integral operator $J^T_\omega G^k_\alpha$ has a continuous convolution kernel vanishing exponentially fast at infinity. Thus

$$\begin{aligned} &(J^T_\omega G_\alpha)^* J^T_\omega G_\alpha G^j_\alpha \in \mathbb{S}_2, \quad j > d/2 - 1, \\ &G^j_\alpha (J^T_\omega G_\alpha)^* J^T_\omega G_\alpha \in \mathbb{S}_2, \quad j > d/2 - 1. \end{aligned} \tag{4.13}$$

Let $N \in \mathbb{N}$ and $N > d$. Since

$$(H^T_{\omega\alpha} + \alpha)^{-1} - (-\Delta + \alpha)^{-1} = (J^T_\omega G_\alpha)^* J^T_\omega G_\alpha$$

(cf. (2.11)), the operator $D^T_{\alpha\omega N}$ is the sum of $2^N - 1$ terms where every term has the form

$$A (J^T_\omega G_\alpha)^* J^T_\omega G_\alpha G^j_\alpha B \text{ or}$$

$$A G^j_\alpha (J^T_\omega G_\alpha)^* J^T_\omega G_\alpha B \text{ or}$$

$$A (J^T_\omega G_\alpha)^* J^T_\omega G_\alpha B (J^T_\omega G_\alpha)^* J^T_\omega G_\alpha C$$

for some bounded operators A, B, C and some $j > d/2 - 1$. By (4.11) and (4.13), each of these terms belongs to the Hilbert-Schmidt class $\mathbb{S}_2$. Thus

$$D^T_{\alpha\omega N} \in \mathbb{S}_2 \text{ for } \mathbb{W} - \text{a.a. } \omega \in \Omega \text{ (if } N > d). \tag{4.14}$$

We have

$$(H^T_{\omega\alpha} + \alpha)^{-N} D^T_{\alpha\omega N} (-\Delta + \alpha)^{-N} = D^T_{\alpha\omega N} D^T_{\alpha\omega N} + (-\Delta + \alpha)^{-N} D^T_{\alpha\omega N} (-\Delta + \alpha)^{-N}.$$

For $\mathbb{W}$-a.a. $\omega \in \Omega$ the first term on the right hand side belongs to the trace class $\mathbb{S}_1$ since it is the product of two Hilbert-Schmidt operators. The second term is the sum of 2^{N-1} operators where every operator has the form

$$A G^N_\alpha (J^T_\omega G_\alpha)^* B J^T_\omega G_\alpha G^N_\alpha C$$

for some bounded operators A, B, C. Applying again (4.13) we get that each of these 2^{N-1} operators is the product of two Hilbert-Schmidt operators and therefore also an operator in the trace class. Thus (4.12) holds $\mathbb{W}$-a.s. for every $N > d$.

(iii) Let $D := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0 \text{ or } \mathrm{Im}(z) > 0\}$, and $D_{ext} := D \cup \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. It is sufficient to prove that for $\mathbb{W}$-a.a. $\omega \in \Omega$ there exists a discrete set C such that for every $f \in C_0^\infty(\mathbb{R}^d)$ the mapping

$$z \mapsto (f, (H_{\omega\alpha}^T + z)^{-1} f)$$
$$D \longrightarrow \mathbb{C}$$

has an analytic continuation on D_{ext}. C may depend on ω.

In fact, suppose that such a discrete set C and such an analytic extension exist. Let $f \in C_0^\infty(\mathbb{R}^d)$. Let $-\infty < a < b < 0$ be such that $[a, b] \cap C = \emptyset$. Then there exists an $\varepsilon > 0$ such that

$$\{x + iy : a \leq x \leq b, 0 \leq y \leq \varepsilon\} \cap C = \emptyset.$$

Since continuous mappings are bounded on compact sets and the mapping $z \mapsto (f, (H_{\omega\alpha}^T + z)^{-1} f)$ has a continuous continuation on a neighbourhood of the compact set $\{x + iy : a \leq x \leq b, 0 \leq y \leq \varepsilon\}$ we get

$$\sup_{a \leq x \leq b, 0 < y \leq \varepsilon} |(f, (H_{\omega\alpha}^T + z)^{-1} f)| < \infty.$$

Since the space $C_0^\infty(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$ and by the limiting absorption principle ([RS4], Theorem XIII.19), this implies that

$$\sigma_{sc}(-H_{\omega\alpha}^T) \cap (a, b) = \emptyset = \sigma_p(-H_{\omega\alpha}^T) \cap (a, b).$$

Since C is discrete it follows that $\sigma_{sc}(-H_{\omega\alpha}^T) \cap (-\infty, 0]$ is at most countable. This is only possible if $\sigma_{sc}(-H_{\omega\alpha}^T) \cap (-\infty, 0) = \emptyset$. Moreover, by (i) and the fact that $\sigma_{sc}(-H_{\omega\alpha}^T) \subset \sigma_{ess}(-H_{\omega\alpha}^T)$, we also have $\sigma_{sc}(-H_{\omega\alpha}^T) \cap (0, \infty) = \emptyset$.

It remains to prove the existence of the mentioned continuation. By (4.10), $J_\omega^T G_\alpha$ is compact $\mathbb{W}$-a.s. and therefore J_ω^T is also compact $\mathbb{W}$-a.s. Trivially the range of J_ω^T is dense in $L^2(\mathbb{R}^d, \mu_\omega^T)$. Thus, by [Bra], Theorem 3, $(-\infty, -\alpha]$ belongs to the resolvent set of $H_{\omega\alpha}^T$ and

$$(H_{\omega\alpha}^T + \beta)^{-1} = G_\beta + (J_\omega^T G_\beta)^* (I - (\alpha - \beta) J_\omega^T G_\alpha (J_\omega^T G_\beta)^*)^{-1} J_\omega^T G_\beta, \quad \beta > \alpha, \quad \mathbb{W}\text{-a.s.} \tag{4.15}$$

In what follows let ω be any element of Ω such that J_ω^T is compact. Let

$$g_z(x) := \frac{1}{2\sqrt{z}} e^{-\sqrt{z}|x|}, \quad x \in \mathbb{R}, \quad d = 1,$$

resp.

$$\frac{1}{(2\pi)^{d/2}} \left(\frac{|x|}{-\sqrt{-z}} \right)^{1-d/2} K_{d/2-1}(-\sqrt{z}|x|), \quad x \in \mathbb{R}^d \setminus \{0\}, \quad d > 1.$$

Here we choose the root as follows: $\sqrt{r\exp(i\phi)} = \sqrt{r}\exp(i\phi/2)$ for $r > 0$ and $-\pi/2 < \phi < 3\pi/2$. Then

$$\hat{g}_z(p) = \frac{1}{p^2 + z}, \quad \mathrm{Re}(z) > 0 \text{ or } \mathrm{Im}(z) > 0,$$

(cf. [SW]) and this definition of $g_z(x)$ is in accordance with (3.1). If $\mathrm{Re}(z) < 0$ and $\mathrm{Im}(z) \leq 0$ then g_z is not square-integrable w.r.t. the Lebesgue measure. Note that the function $z \mapsto g_z(x)$ is analytic on D_{ext} for $x \neq 0$ (every x if $d = 1$).

For $z \in D$ we define the operator G_z in $L^2(\mathbb{R}^d)$ by $G_z := (-\Delta + z)^{-1}$. For $z \in D_{ext}$ let $G_z^{\mu_\omega^T}$ be the integral operator in $L^2(\mathbb{R}^d, \mu_\omega^T)$ with the kernel $g_z(x - y)$. By the preceeding considerations, we need only to prove that there exists a discrete set C such that

(α) $I - (\alpha - z)G_z^{\mu_\omega^T}$ is invertible in $L^2(\mathbb{R}^d, \mu_\omega^T)$ for every $z \in D_{ext}$ and

(β) the mapping $z \mapsto (f, G_z f + (J_\omega^T G_{\bar{z}})^*[I - (\alpha - z)G_z^{\mu_\omega^T}]^{-1} J G_z f)$ is analytic on $D_{ext} \setminus C$ for every $f \in C_0^\infty(\mathbb{R}^d)$.

A straightforward computation yields analyticity of the mapping $z \mapsto G_z^{\mu_\omega^T}$ and (α) and (β) follow from Fredholm's analytic theorem.

(iv) It is well known that the spectrum $\sigma(-\Delta)$ of $-\Delta$ equals $[0, \infty)$ and that $-\Delta$ equals its absolutely continuous part $(-\Delta)^{ac}$. Since the wave operators $W^\pm(H_{\omega\alpha}^T, -\Delta)$ exist and are complete for $\mathbb{W}$-a.a. $\omega \in \Omega$ this implies, by [RS3], XI.3, Proposition 1, that the wave operators $W^\pm(H_{\omega\alpha}^T, -\Delta)$ are unitary mappings from $L^2(\mathbb{R}^d, \lambda^d)$ onto the absolutely continuous spectral subspaces of $W^\pm(H_{\omega\alpha}^T, -\Delta)$ and

$$H_{\omega\alpha}^T = W^\pm(H_{\omega\alpha}^T, -\Delta)^{-1}(-\Delta)W^\pm(H_{\omega\alpha}^T, -\Delta), \quad \mathbb{W}\text{-a.s.}$$

In particular, the operators $H_{\omega\alpha}^T$ and $-\Delta$ have the same absolutely continuous spectrum and therefore

$$\sigma_{ac}(H_{\omega\alpha}^T) = \sigma_{ac}(-\Delta) = \sigma(-\Delta) = [0, \infty).$$

$\square$

By the last theorem, the set of positive eigenvalues of the operator $H_{\omega\alpha}^T$ is discrete. In the case when $d \geq 3$ the complement of a typical path Γ_ω^T of a Brownian particle in $\mathbb{R}^d$ is connected. Together with a unique continuation theorem this provides a much stronger statement about positive eigenvalues in the case $d \geq 3$:

THEOREM 4.5 *Let $d \geq 3$. For every $\omega \in \Omega$ let H_ω^T be any selfadjoint operator in $L^2(\mathbb{R}^d, \lambda^d)$ such that the space $C_0^\infty(\mathbb{R}^d \setminus \Gamma_\omega^T)$ is contained in the domain of H_ω^T and*

$$H_\omega^T f = -\Delta f, \quad f \in C_0^\infty(\mathbb{R}^d \setminus \Gamma_\omega^T).$$

Then $\mathbb{W}$-a.s. the operator H_ω^T has no positive eigenvalue.

PROOF. Let $\omega \in \Omega$ be such that Γ_ω^T has Lebesgue measure zero and its complement $\mathbb{R}^d \setminus \Gamma_\omega^T$ is connected. Then the set $C_0^\infty(\mathbb{R}^d \setminus \Gamma_\omega^T)$ is dense in $L^2(\mathbb{R}^d, \lambda^d)$ and the adjoint of the restriction of $-\Delta$ to this space is an extension of H_ω^T,

$$H_\omega^T \subset (-\Delta \lceil C_0^\infty(\mathbb{R}^d \setminus \Gamma_\omega^T))^* =: -\Delta_{\omega,max}^T.$$

Let $E > 0$ and $H^T_\omega f = Ef$. Then

$$\int_{\mathbb{R}^d} E\bar{f}(x)g(x)\lambda^d(dx) = (-\Delta^T_{\omega,max} f, g) = \int_{\mathbb{R}^d} \bar{f}(x)(-\Delta g)(x)\lambda^d(dx), \quad g \in C_0^\infty(\mathbb{R}^d \setminus \Gamma^T_\omega).$$

By Weyl's regularity theorem, it follows that f is infinitely differentiable on $\mathbb{R}^d \setminus \Gamma^T_\omega$ and

$$H^T_\omega f(x) = -\sum_{j=1}^d \frac{\partial^2 f}{\partial x_j^2} = Ef \quad \lambda^d\text{-}a.e. \text{ on } \mathbb{R}^d \setminus \Gamma^T_\omega. \tag{4.16}$$

Let B be any ball containing Γ^T_ω. Since $-\sum_{j=1}^d \frac{\partial^2}{\partial x_j^2} f = Ef$ λ^d-a.e. on the complement of B and $f \in L^2(\mathbb{R}^d, \lambda^d)$ we have $f = 0$ λ^d-a.e. on $\mathbb{R}^d \setminus B$ (cf., e.g., the proof of [RS4], Theorem XIII.56). By [RS4], Theorem XIII.63, and (4.16), it follows that $f = 0$ λ^d-a.e. on the connection component of $\mathbb{R}^d \setminus \Gamma^T_\omega$ containing B. Since $\mathbb{R}^d \setminus \Gamma^T_\omega$ is connected and the Lebesgue measure of the compact set Γ^T_ω equals zero it follows that $f = 0$ λ^d-a.e. Thus E is not an eigenvalue of H^T_ω.

Since $d \geq 3$ and the two-dimensional Hausdorff-measure of Γ^T_ω equals zero for $\mathbb{W}$-a.a. $\omega \in \Omega$ the Lebesgue measure of Γ^T_ω equals zero and the complement of Γ^T_ω is connected for $\mathbb{W}$-a.a. $\omega \in \Omega$. $\square$

5 A trace formula for the expectation value of the number of negative eigenvalues

In this section we shall derive a trace formula for the number of negative eigenvalues of the operators $H^T_{\omega\alpha}$ provided $3 \leq d \leq 5$. By mimicking the reasoning below and using the Klaus-Newton method (cf. [Kl], [Newt] and the extension in [BEKS]), similar results can be derived for $d = 1, 2$ as well.

Let

$$A_{\alpha 0} := \frac{(-\Delta + \alpha)(-\Delta)}{\alpha}. \tag{5.17}$$

By Lemma 3.3 and (4.9), for $\mathbb{W}$-a.a. $\omega \in \Omega$ the operator J^T_ω from $H^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d, \mu^T_\omega)$ (cf. (2.10)) is compact and for every $\varepsilon > 0$ there exists an $a(\alpha, \varepsilon, \omega) < \infty$ such that

$$\| J^T_\omega f \|^2_{L^2(\mathbb{R}^d, \mu^T_\omega)} \leq \varepsilon \| A^{1/2}_{\alpha 0} f \|^2_{L^2(\mathbb{R}^d, \lambda^d)} + a(\alpha, \varepsilon, \omega) \| f \|^2_{L^2(\mathbb{R}^d, \lambda^d)}, \quad f \in H^2(\mathbb{R}^d). \tag{5.18}$$

Thus $\mathbb{W}$-a.s. the quadratic form $\mathcal{E}^T_{\alpha 0\omega}$ in $L^2(\mathbb{R}^d, \lambda^d)$, defined by

$$D(\mathcal{E}^T_{\alpha 0\omega}) = H^2(\mathbb{R}^d), \tag{5.19}$$

$$\mathcal{E}^T_{\alpha 0\omega}(f, g) = (A^{1/2}_{\alpha 0} f, A^{1/2}_{\alpha 0} g) - \int \bar{\tilde{f}} \tilde{g} \, d\mu^T_\omega, \quad f, g \in H^2(\mathbb{R}^d), \tag{5.20}$$

is lower semibounded and closed. We denote by $A_{\alpha 0} - \mu^T_\omega$ the unique lower semibounded selfadjoint operator in $L^2(\mathbb{R}^d, \lambda^d)$ associated to $\mathcal{E}^T_{\alpha 0\omega}$.

Let $N_1(\omega, T)$ and $N_2(\omega, T)$ be the number (counting multiplicities) of negative eigenvalues of the operator $H^T_{\omega\alpha}$ and $A_{\alpha 0} - \mu^T_\omega$, respectively. By [Bra], Corollary 8,

$$N_1(\cdot, T) = N_2(\cdot, T) \quad \mathbb{W}\text{-a.s.} \tag{5.21}$$

Let

$$G_{\alpha 0\gamma} := (A_{\alpha 0} + \gamma)^{-1}. \tag{5.22}$$

By [Bra], (28),

$$(A_{\alpha 0} - \mu^T_\omega + \gamma)^{-1} = G_{\alpha 0\gamma} + (J^T_\omega G_{\alpha 0\gamma})^*(1 - J^T_\omega (J^T_\omega G_{\alpha 0\gamma})^*)^{-1} J^T_\omega G_{\alpha 0\gamma} \tag{5.23}$$

for every $\gamma > 0$ such that $-\gamma$ belongs to the resolvent set of $A_{\alpha 0} - \mu^T_\omega$. Let

$$K^T_{\omega\alpha\gamma} := 1_{(1,\infty)}(J^T_\omega (J^T_\omega G_{\alpha 0\gamma})^*). \tag{5.24}$$

Modifying the Birman-Schwinger analysis in an obvious way, we can derive from (5.23) that the number of eigenvalues below $-\gamma$ of $A_{\alpha 0} - \mu^T_\omega$ equals $\| K^T_{\cdot\alpha\gamma} \|_{\mathbb{S}_1}$ $\mathbb{W}$-a.s..

In particular, the number of negative eigenvalues of $H^T_{\omega\alpha}$ is less than or equal to $\| G_{\alpha 00} \|^4_{\mathbb{S}_4}$. By the considerations in the proof of Lemma 3.3 (cf., in particular, the formula (3.8)), the expectation value of the last expression is finite if $3 \leq d \leq 5$. Thus we have proved the following theorem.

THEOREM 5.6 *Let $3 \leq d \leq 5$ and $\alpha > 0$. For $\mathbb{W}$-a.a. $\omega \in \Omega$ let $H^T_{\omega\alpha}$ be the selfadjoint operator defined by (2.7), (2.10) and (2.11). Then for $\mathbb{W}$-a.a. $\omega \in \Omega$ the number, counting multiplicities, of negative eigenvalues of $H^T_{\omega\alpha}$ equals the trace norm of the operator $K^T_{\omega\alpha 0}$, defined by (5.24). In particular, the expectation value (w.r.t. Wiener measure) for the number, counting multiplicities, of negative eigenvalues of $H^T_{\omega\alpha}$ is finite.*

REMARK 5.7 In a forthcoming paper we shall derive further results on the negative eigenvalues. In particular, we shall show that for every $N \in \mathbb{N}$ the probability that the number of negative eigenvalues of $H^T_{\omega\alpha}$ is at least N is strictly positive. On the other hand, these probabilities tend rapidly to zero, as N tends to infinity. In fact, above theorem implies that the sum over N times the probability that the number of negative eigenvalues of $H^T_{\omega\alpha}$ equals N is finite; here the sum is taken over all positive integers N.

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