Thesis for the Degree of Licentiate of Philosophy

On the Asymptotic Behaviour of Levy Processes

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 [3]

[1]En gubbe! Sättarens anmärkning.
[2]En gubbe säger jag ju!
[3]Där ser ni!

Abstract: This thesis deals with the asymptotic behavior of Lévy processes. More precisely, the probability tails of suprema over compact intervals are studied. Lévy processes are divided into a handfull of classes, depending on the weight of the tails of their univariate marginal distributions. Different methods are used for the different classes. The methods we use include generalizations of known techniques, as well as completely new techniques, developed by us. The result is a quite complete treatment of the mentioned asymptotic problem. Several of the processes, the asymptotics of which are studied here for the first time, have recently become important in the field of mathematical finance. This means that our results could have impact on, for example, the assesments of financial risk.

Keywords: Abelian theorem; Brownian motion; CGMY process; Domain of attraction; Esscher transform; Exponential distribution; Extreme value theory; Generalized hyperbolic process; Generalized z-process; Gumbel distribution; de Haan's class Γ; Hyperbolic process; Infinitely divisibile distribution; Infinitely divisible process; Lévy process; Local Extrema; Long-tailed distribution; Meixner process; Normal inverse Gaussian process; O-regular variation; Regular variation; Semi-heavy tailed distribution; Stable process; Subexponential distribution; Superexponential distribution; Tauberian theorem; Variance gamma process; z-process.

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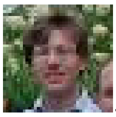
My secondary supervisor, Professor Bernt Wennberg , for helping me in return for nothing. I will start working on the problem you suggested, I promise.

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See the Acknowledgement in Part II for further acknowledgements.

Mattias Bengtsson, November 2004

Contents

Part I

Introduction

Chapter 1

Introduction to Lévy Processes

As this thesis deals with the asymptotic behavior of Lévy processes, we will here review some of the theory for such processes.

1.1 Lévy Processes

1.1.1 Basic Definitions and Fundamental Theorems

Definition 1.1 (E.G. SATO[26], DEFINITION 1.6). *An adapted stochastic process $\xi = \{\xi(t)\}_{t\geq 0}$, defined on a filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ *is called a* Lévy process *if the following conditions hold:*

1. *ξ has independent increments, that is, for any $0 \leq s < t$, $\xi(t) - \xi(s)$ is independent of $\mathcal{F}_s$;*

2. *$\xi(0) = 0$ a.s.;*

3. *$\mathcal{L}(\xi(s+t) - \xi(s))$, the distribution of $\xi(s+t) - \xi(s)$, does not depend on s;*

4. *ξ is stochastically continuous, that is, $\lim_{t\to 0} \mathbf{P}\{|\xi(s+t) - \xi(s)| > \varepsilon\} = 0$ for $\varepsilon > 0$;*

5. *ξ is* càdlàg*, that is, right-continuous with left limits.*

Definition 1.2. *A* stopping time*, is a random variable,* $T \in [0, \infty]$*, such that* $\{T < t\}$ *is* $\mathcal{F}_t$*-measurable for* $t \geq 0$*.*

Theorem 1.3 (The strong Markov property. E.G. APPLEBAUM [5] THEOREM 2.2.11)**.** *If* $\{\xi(t)\}_{t\geq 0}$ *is a Lévy process and* T *a stopping time, then the process* $\xi_T = \{\xi_T(t)\}_{t\geq 0}$*, defined by* $\xi_T(t) = \xi(T+t) - \xi(T)$*, is an* $\{\mathcal{F}_{T+t}\}_{t\geq 0}$*-adapted Lévy process, independent of* $\mathcal{F}_T$*, and with the same law as* ξ*.*

As will be seen in subsequent chapters, the strong Markov property is essential for our analysis of the asymptotic behaviour of Lévy processes.

1.1.2 Infinite Divisibility

Definition 1.4 (E.G. SATO[26], DEFINITION 7.1)**.** *A probability measure* μ *on* $\mathbb{R}$ *is* infinitely divisible *if, for all positive integers* n*, there exist a probability measure* μ_n*, a so called* convolution root *of* μ*, such that* μ *is the* n*-fold convolution of* μ_n*.*

A random variable ξ is infinitely divisible if, for all positive integers n, there exists an i.i.d. sequence $\xi_1, \ldots, \xi_n$ of random variables, such that $\xi \overset{\mathrm{d}}{=} \xi_1 + \cdots + \xi_n$. A characteristic function is infinite divisible if it can be expressed as an n-fold product of characteristic functions, for any n.

Theorem 1.5 (E.G. SATO[26], THEOREM 7.10)**.** *For a Lévy process* $\{\xi(t)\}_{t\geq 0}$*,* $\xi(1)$ *is infinitely divisible, and* $\mathcal{L}(\xi(t)) = \mu_t$*, where* μ_t *is the probability measure with characteristic function* $\psi_{\mu_t}(\theta) = \psi_\mu(\theta)^t$*. Conversely, if* μ *is an infinitely divisible probability measure, then there exists a unique in law Lévy process* $\{\xi_t\}_{t\geq 0}$ *such that* $\mathcal{L}(\xi(1)) = \mu$*.*

In order to identify Lévy processes and infinitely divisible distributions, the following representation is fundametal:.

Theorem 1.6 (Lévy-Khintchine representation. E.G. SATO[26], THEOREM 8.1)**.** *If the probability measure μ is infinitely divisible, then it has characteristic function*

$$\psi_\mu(\theta) = \exp\left\{ i\theta m + \int_{\mathbb{R}} \left(e^{i\theta x} - 1 - i\theta\kappa(x)\right) d\nu(x) - \frac{\theta^2 s^2}{2} \right\} \quad \text{for } \theta \in \mathbb{R},$$

where $\kappa(x) = x/(1 \vee |x|)$ for $x \in \mathbb{R}$, while $m \in \mathbb{R}$ and $s^2 \geq 0$ are constants, and ν is a measure such that $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} 1 \wedge x^2 d\nu(x) < \infty$. This so called characteristic triple (ν, m, s^2) *determines the law of a Lévy process $\{\xi(t)\}_{t\geq 0}$, through the relation $\psi_{\xi(t)}(\theta) = \psi_\mu(\theta)^t$.*

1.1.3 Examples of Lévy Processes

Example 1.7 (Brownian motion). *Browninan motion* is a Lévy process, $\{B(t)\}_{t\geq 0}$, with charachteristic triple $(0, 0, s^2)$. The increments of Browninan motion $B(t) - B(s)$ are normal $N(0, t-s)$ distributed.

Example 1.8 (Poisson process). A *Poisson process* with intensity $\lambda > 0$ is a Lévy process, $\{N(t)\}_{t\geq 0}$, with charachteristic triple $(\lambda\delta_1, 0, 0)$, where δ_1 is an atom at location 1 with mass 1. The increments of a Poisson process $N(t) - N(s)$ are Poisson $Po(\lambda(t-s))$ distributed.

Example 1.9 (Compound Poisson process). A *compound Poisson* process, is a Lévy process, $\{X(t)\}_{t\geq 0}$, with charachteristic triple $(\lambda\sigma, 0, 0)$, where $\lambda > 0$ and σ is a probability measure on $\mathbb{R}$ such that $\sigma(\{0\}) = 0$. The characteristic function is given by

$$\psi_{X(t)}(\theta) = \exp\left\{ t\lambda \int_{\mathbb{R}} \left(e^{i\theta x} - 1\right) d\sigma(x) \right\}.$$

Taking $\sigma = \delta_1$, this gives a Poisson process.

Example 1.10 (α-stable Lévy motion. E.G. SAMORODNITSKY AND TAQQU [25])**.** An *α-stable Lévy motion* is a Lévy process, $\{A(t)\}_{t\geq 0}$, with characteristic triple $(\nu, m, 0)$, where

$$d\nu(x) = \sigma^\alpha \left(c_1 \mathbf{1}_{(0,\infty)}(x) + c_2 \mathbf{1}_{(-\infty,0)}(x)\right) |x|^{-1-\alpha} dx.$$

Here $c_1 \geq 0$, $c_2 \geq 0$, and $\sigma, c_1 + c_2 > 0$. It follows that

$$\psi_{A(1)}(\theta) = \exp\left\{-\sigma^\alpha |\theta|^\alpha \left(1 - i\beta \operatorname{sign}(\theta) \tan\frac{\pi\alpha}{2}\right) + im\theta\right\} \quad \text{if } \alpha \neq 1,$$

and

$$\psi_{A(1)}(\theta) = \exp\left\{-\sigma|\theta| \left(1 + i\beta\frac{2}{\pi} \operatorname{sign}(\theta) \ln|\theta|\right) + im\theta\right\} \quad \text{if } \alpha = 1,$$

where $\beta = (c_1 - c_2)/(c_1 + c_2)$ is a *skewness parameter.* Moreover, $\alpha = 2$ gives a normal $N(m, 2s^2)$ distribution.

1.2 Some Lévy Processes of Recent Interest

Lately, some of the processes featured below have been introduced, for example, to model log returns of financial asset prices. This is due to the fact that these processes can provide models more true the stylized features, such as heavy or semi-heavy tails and skewness, displayed by then log returns of asset prices.

1.2.1 Generalized Hyperbolic Process

Definition 1.11 (The generalized hyperbolic process. E.G. BARNDORFF-NIELSEN AND HALGREEN [6] AND SCHOUTENS [27]**).** *The* generalized hyperbolic *GH($\alpha, \beta, \delta, \gamma, \mu$) process, $\{\xi(t)\}_{t\geq 0}$, is defined by the characteristic function*

$$\psi_{\xi(1)}(\theta) = \left(\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + i\theta)^2}\right)^{\gamma/2} \frac{K_\gamma(\delta\sqrt{\alpha^2 - (\beta + i\theta)^2})}{K_\gamma(\delta\sqrt{\alpha^2 - \beta^2})} \mathrm{e}^{i\theta\mu},$$

where K_γ is the modified Bessel function of the third kind. The parameters satisfy $\gamma, \mu \in \mathbb{R}$, together with $\delta \geq 0$ and $|\beta| < \alpha$ if $\gamma > 0$, $\delta > 0$ and $|\beta| < \alpha$ if $\gamma = 0$, while $\delta \geq 0$ and $|\beta| \leq \alpha$ if $\gamma < 0$.

The class of GH processes was introduced in the late seventies by Ole Barndorff-Nielsen in order to model wind blown sands. This class contains many interesting special cases.

Example 1.12 (The hyperbolic process. E.G. SCHOUTENS [27]**).** The *hyperbolic* $\mathrm{H}(\alpha,\beta,\delta,\mu)$ process, $\{\xi(t)\}_{t\geq 0}$, is defined by the characteristic function

$$\psi_{\xi(1)}(\theta)=\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+i\theta)^2}\right)^{1/2}\frac{K_1(\delta\sqrt{\alpha^2-(\beta+i\theta)^2})}{K_1(\delta\sqrt{\alpha^2-\beta^2})}\mathrm{e}^{i\theta\mu}.$$

We see that $\mathrm{H}(\alpha,\beta,\delta,\mu)$=$\mathrm{GH}(\alpha,\beta,\delta,1,\mu)$.

Example 1.13 (The normal inverse Gaussian process. E.G. SCHOUTENS [27]**).** The *normal inverse Gaussian* $\mathrm{NIG}(\alpha,\beta,\delta,\mu)$ process, $\{\xi(t)\}_{t\geq 0}$, is defined by its characteristic function

$$\psi_{\xi(1)}(\theta)=\exp\{-\delta(\sqrt{\alpha^2-(\beta+i\theta)^2})-\sqrt{\alpha^2-\beta^2}+i\theta\mu\}.$$

In a later section we will see that the NIG process can be viewed as time changed Brownian motion.

1.2.2 The Generalized z Processes

Definition 1.14 (The generalized z process. E.G. GRIGELIONIS[18]**).** *The* generalized z *GZ($\alpha,\beta_1,\beta_2,\delta,\mu$) process, $\{\xi(t)\}_{t\geq 0}$ is defined by the characteristic function*

$$\psi_{\xi(1)}(\theta)=\left(\frac{B(\beta_1+i\alpha\theta/2\pi,\beta_2-i\alpha\theta/2\pi)}{B(\beta_1,\beta_2)}\right)\mathrm{e}^{i\theta\mu},$$

where B is the beta function. For the parameters, we have that $\alpha,\beta_1,\beta_2,\delta>0$, and $\mu\in\mathbb{R}$.

The generalized z process was introduced by Grigelionis in 2000. The most well-known special case of the generalized z process is the Meixner procss:

Example 1.15 (The Meixner process. E.G. SCHOUTENS AND TEUGELS [28]**).** The *Meixner*$(\alpha,\beta,\delta,\mu)$ process, $\{\xi(t)\}_{t\geq 0}$, is defined by its characteristic function

$$\psi_{\xi(1)}(\theta)=\frac{\cos\left(\frac{\beta}{2}\right)}{\cosh\left(\frac{\alpha\theta-i\beta}{2}\right)}\mathrm{e}^{i\theta\mu}.$$

1.2.3 The Carr Geman Madan Yor Process

This process is named after its inventors Peter Carr, Helyette Geman, Dilip Madan and Marc Yor.

Definition 1.16 (The Carr Geman Madan Yor process. [13] AND [14]**).** *The* Carr Geman Madan Yor *CGMY(C_-, C_+, G, M, Y_-, Y_+) process, is defined by the characteristic triple* $(\nu, m, 0)$*, where*

$$d\nu(x) = \left(C_-|x|^{-1-Y_-}\mathrm{e}^{Gx}\mathbf{1}_{(-\infty,0)}(x) + C_+|x|^{-1-Y_+}\mathrm{e}^{-Mx}\mathbf{1}_{(0,\infty)}(x)\right)dx,$$

and $m = \int_{\mathbb{R}} \kappa(x)d\nu(x)$*. The parameters satisfy* $C_-, C_+, G, M > 0$ *and* $Y_-, Y_+ < 2$*.*

The above definition is a generalization of the CGMY(C, G, M, Y) process introduced in by Carr, Geman, Madan and Yor [13], for which $C_- = C_+ = C$ and $Y_- + Y_+ = Y$. Due to the appearance of the Lévy measure, the CGMY process is also called an *exponentially damped* α*-stable* process. Still another name is *KoBoL* process, after Koponen [19] and Boyarchenko and Levendorskiĭ [12].

An important special case of the CGMY class is the *variance gamma* process. This process if the difference between two gamma processes:

Example 1.17 (Variance Gamma process. E.G. SCHOUTENS [27]**).** A *variance gamma* VG(C,G,M) process is the special case of the CGMY(C, G, M, Y) process where $Y = 0$.

Chapter 2

Introduction to Extremes

2.1 Elements of Classical Extreme Value Theory

Let $\xi_1, \ldots, \xi_n$ be a sequence of i.i.d. random variables. In classical extreme value theory, the issue of concern is to study the asymptotic behaviour of maxima $M_n = \max(\xi_1, \ldots, \xi_n)$, as $n \to \infty$. Pioneers in this area were Fréchet [16], Fisher and Tippet [15], and Gnedenko [17]. These people worked on finding normalizing constants $a_n > 0$ and $b_n \in \mathbb{R}$, such that

$$\mathbf{P}\big\{a_n(M_n - b_n) \le x\big\} \xrightarrow{\mathrm{d}} G(x), \tag{2.1}$$

where G is a non-degenerate limit distribution.

It turns out that, see for example Leadbetter, Lindgren and Rootzén [20], that there are three types of possible limit distribution functions $G(x)$, namely (normalizations of)

$$\text{Type I} : G(x) = \exp\{-\mathrm{e}^{-x}\}, \text{ for } -\infty < x < \infty$$

$$\text{Type II} : G(x) = \begin{cases} \exp\{-x^{-\alpha}\}, & \text{for some } \alpha > 0 \text{ and if } x > 0 \\ 0, & \text{if } x \le 0 \end{cases}$$

$$\text{Type III} : G(x) = \begin{cases} \exp\{-(-x)^{\alpha}\}, & \text{for some } \alpha > 0 \text{ and if } x \le 0 \\ 1, & \text{if } x > 0. \end{cases}$$

This is known as the *Extremal Types* Theorem, and the possible limit distributions are also called the *Gumbel, Fréchet* and *Weibull* distributions, respectively. One says that a random variable belongs to the Type I (Type II/Type III) domain of attraction of extremes, if (2.1) holds with the Type I (Type II/Type III) limit distribution.

The account on criteria and theorems for verifying that a certain distribution belongs to one of the three domains of attraction is quite extensive, see Leadbetter, Lindgren and Rootzén [20].

2.2 Extremes for dependent sequences

As sequences of random variables in many situations are dependent, efforts have been made, see for example Leadbetter, Lindgren and Rootzén [20], to determine normalizing sequences and possible limit distribution functions in such a setting. This is of course more intricate and requires extra assumptions.

Condition $D(u_n)$. Let $\{\eta_n\}_{n\geq 1}$ be a sequence of (dependent) random variables, and $\{u_n\}_{n\geq 1}$ a sequence of real numbers. For any integers $1 \leq i_1 < \cdots < i_p < j_1 < \cdots < j_q \leq n$ such that $j_1 - i_p \geq m$, it holds that

$$\left|\mathbf{P}\left\{\bigcap_{k=1}^{p}\{\eta_{i_k} \leq u_n\}, \bigcap_{l=1}^{q}\{\eta_{i_l} \leq u_n\}\right\} - \mathbf{P}\left\{\bigcap_{k=1}^{p}\{\eta_{i_k} \leq u_n\}\right\}\mathbf{P}\left\{\bigcap_{l=1}^{q}\{\eta_{i_l} \leq u_n\}\right\}\right| \leq \alpha(n,m),$$

where

$$\lim_{n\to\infty} \alpha(n, m_n) = 0 \quad \text{for some sequence } \; m_n = o(n).$$

If Condition $D(u_n)$ holds, or some suitable version there off, for $u_n = a_n x + b_n$, where $x \in \mathbb{R}$ and

$$\frac{\max_{1\leq k\leq n} \eta_k - b_k}{a_k} \stackrel{\mathrm{d}}{\to} G \quad \text{as } n \to \infty,$$

then the Extremal Types Theorem holds, with the same possible limit distributions G as in the classical setting.

2.3 Extremes for Stochastic Processes in continuous time

Pioneers on the topic extreme value theory for stochastic processes were, for example, Kac, Rice, Slepian, Volkonskii and Rozanov. Later people like Belyaev, Berman, Cramér, Pickands and Watanabe made major contributions to the area. In the last two decades work on extremes for more general types of processes has been done by Leadbetter, Lindgren and Rootzén [20], Berman [7], [8], Piterbarg [23] and Albin [1]-[4]. Many of the papers of the above authors deal with Gaussian Processes. For such processes a lot nice tools are available, like, for example, the double sum method (see Piterbarg [23]).

Given a stochastic process $\{\xi(t)\}_{t\geq 0}$, the principal aim is to determine the asymptotic behavior of

$$\mathbf{P}\Big\{\sup_{0\leq t\leq T}\xi(t)>u\Big\}.$$

Here asymptotic behaviour could mean either that $T\to\infty$ or $u\to\infty$.

In the case when $T\to\infty$, for a stationary Gaussian process, $\{\xi(t)\}_{t\geq 0}$, with zero mean, unit variance and covariance function r such that

$$r(\tau)=1+\frac{r''(0)\tau^2}{2}+o(\tau^2)\quad\text{as }\ \tau\to 0,$$

and

$$r(t)\ln(t)\to 0\quad\text{as }\ t\to\infty,$$

it holds that, see for example Leadbetter, Lindgren and Rootzén [20]

$$\mathbf{P}\Big\{\sup_{0\leq t\leq T}\xi(t)\leq\frac{u}{a_T}+b_T\Big\}\overset{\mathrm{d}}{\to}\exp\{-\mathrm{e}^{-x}\}\quad\text{as }\ T\to\infty,$$

where

$$a_T=sqrt2\ln(T)\quad\text{and}\quad b_T=\sqrt{2\ln(T)}+\frac{\ln\Big(\sqrt{-r''(0)}/(2\pi)\Big)}{\sqrt{2\ln(T)}}.$$

In the case when $u\to\infty$, J. Pickands III [22], showed that if, for some $\alpha>0$

$$r(\tau)=1-|\tau|^{\alpha}+o(|\tau|^{\alpha})\quad\text{as }\ \tau\to 0,$$

and $r(t)<1$ for all $t>0$, then it holds that

$$\mathbf{P}\Big\{\sup_{0\leq t\leq T}\xi(t)>u\Big\}=H_{\alpha}Tu^{-2/\alpha}\mathbf{P}\{\mathrm{N}(0,1)>u\}(1+o(1))\quad\text{as }\ u\to\infty,\qquad(2.2)$$

where

$$H_\alpha = \lim_{T\to\infty} \mathbf{E}\exp\Big\{\sup_{0\le t\le T} \eta(t)\Big\},$$

where $\{\eta(t)\}_{t\ge 0}$ is a fractional Brownian motion with Hurst parameter $\alpha/2$ and drift $-|t|^\alpha$.

2.4 On the Extreme value Theory dealt with in this Thesis

As indicated by (2.2), the problem of concern in extreme value theory for stochastic processes, $\{\xi(t)\}_{t\ge 0}$, is to establish relations, like for example

$$\mathbf{P}\Big\{\sup_{0\le t\le h} \xi(t) > u\Big\} = C\mathbf{P}\{\xi(h) > u\} \quad \text{as } u\to\infty, \tag{2.3}$$

for a suitable some constant $C \ge 1$.

The most well-known result along the lines of (2.3) is probably the following:

Example 2.1 (E.G. PROTTER [24] THEOREM 33). For Brownian motion $\{B(t)\}_{t\ge 0}$, it holds that $C = 2$. In this case no limiting procedure is necessary: There is equality between the left-hand side and right-hand side in (2.3) for all $u > 0$.

In order to make statements like (2.3) valid in more general setting, such as for exampele Lévy processes, it seems that one has to take different routes, depending on the tail behaviour of the processes: For processes with heavy tails, such as regularly varying ones and sub-exponential ones, "everything" has been covered by people like Berman [7], Braverman [9], [10], Braverman and Samorodnitsky [11], Marcus [46] and Willekens [60]. What we are dealing with in this thesis are processes with lighter tails, and asymptotics for suprema of such Lévy processes have not yet been documented in the literature to any greater extent. In particular, this includes many of the Lévy processes of recent interest, that has been described above.

Since for many of the examples of Lévy processes presented earlier, the distribution function is not known, and it is rather the Lévy measure that is available, one typically to use employ Tauberian techniques, that relate the probability tails to the behaviour of the Laplace transform at the left end-point of its doamin.

As existing Tauberian techniques were quite insufficient for our needs, we had to develop new ones.

When deriving statements like (2.3) for a class of processes with tails in a certain class, we will use discrete approximations in the style of the great John Martin Patrik Albin. It should be observed that these techniques have been used very little on Lévy processes before, and we had to make substantial modifications of them, to make them work as desired.

As special cases, we do recover all results of pervious authors in the area. In some cases, although much more general, our proofs are only a fractions of those in the literature in terms of length.

We do also provide some new converses to results in the literature, as well as to our own new results. That is, we derive necessary consequences of (2.3). These converse can be surprisingly useful, for example to prove that $C > 1$ in (2.3).

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Part II

On the Asymptotic Behaviour of Levy Processes

Chapter 1

Concepts of Exponentiality

For the convenience of the reader, in this chapter we review the many classes of exponential and related distributions, that will feature in the following chapters.

1.1 Subexponential Distributions

Definition 1.1. *A probability distribution function F on the real line, with right end-point* $\sup\{x : F(x) < 1\} = \infty$, *is said to belong to the class of* long-tailed *distributions $\mathcal{L}$, if*

$$\lim_{u\to\infty} \frac{1-F(u+x)}{1-F(u)} = 1 \quad \text{for } x \in \mathbb{R}. \tag{1.1}$$

For the class $\mathcal{L}$, we will need the following lemma, the proof of which is elementary:

Lemma 1.2. *We have $F \in \mathcal{L}$ if*

$$\liminf_{u\to\infty} \frac{1-F(u+x)}{1-F(u)} \geq 1 \quad \text{for some } x > 0.$$

Embrechts and Goldie [31], p. 245, made the following very natural conjecture:

Conjecture 1.3. *$\mathcal{L}$ is closed under convolution roots.*

The truth of this conjecture is still unknown.

Definition 1.4. *A probability distribution function F on the real line, with right end-point* $\sup\{x : F(x) < 1\} = \infty$, *is said to belong to the class of* subexponential *distributions $\mathcal{S}$, if*

$$\lim_{u \to \infty} \frac{1 - F \star F(u)}{1 - F(u)} = 2.$$

It is known that $\mathcal{S} \subseteq \mathcal{L}$ (see Athreya and Ney [5], p. 148. This inclusion is strict, as Embrechts and Goldie [31], Section 3, give an example of a distribution that is in $\mathcal{L}$, but not in $\mathcal{S}$.

Remark 1.5. It was argued convincingly by Pitman [47], p. 338, that, rather than the class $\mathcal{S}$, it is the class $\mathcal{L}$ that should be called subexponential. As Pitman's argument is the same as that which makes us call certain processes exponential, and others superexponential, we found it natural to use subexponential in the label of this section, rather than long-tailed. However, it is really $\mathcal{L}$ we are concerned with here, rather than the more narrow class $\mathcal{S}$.

Most natural examples of distributions in $\mathcal{L}$ and $\mathcal{S}$ come from one of the following two classes of distributions (cf. Bingham, Goldie and Teugels [19], p. 18 and p. 65):

Definition 1.6. *A probability distribution function F on the real line, with right end-point* $\sup\{x : F(x) < 1\} = \infty$, *is said to belong to the class of* regularly varying *distributions with* index $\alpha \leq 0$, $\mathcal{R}(\alpha)$, *if*

$$\lim_{u \to \infty} \frac{1 - F(ux)}{1 - F(u)} = x^{\alpha} \quad \text{for } x > 0.$$

Definition 1.7. *A probability distribution function F on the real line, with right end-point* $\sup\{x : F(x) < 1\} = \infty$, *is said to belong to the class of* extended regularly varying *distributions $\mathcal{ER}$, if*

$$\lim_{x \uparrow 1} \liminf_{u \to \infty} \frac{1 - F(ux)}{1 - F(u)} = \lim_{x \downarrow 1} \limsup_{u \to \infty} \frac{1 - F(ux)}{1 - F(u)} = 1.$$

It is trivial that $\mathcal{R}(\alpha) \subseteq \mathcal{ER}$ for $\alpha \leq 0$, and elementary that $\mathcal{ER} \subseteq \mathcal{S}$.

Within the class of infinitely divisible distributions, the canonical examples of regularly varying distributions, and thus of distributions in $\mathcal{S}$ and $\mathcal{L}$, are *alpha-stable* distributions:

Example 1.8. An α-stable $S_\alpha(\sigma, \beta, \mu)$ random variable ξ has characteristic function (cf. e.g., Samorodnitsky and Taqqu [53], Eq. 1.1.6)

$$\mathbf{E}\{e^{i\theta\xi}\} = \begin{cases} \exp\left\{-|\theta|^\alpha \sigma^\alpha \left(1 - i\beta \operatorname{sign}(\theta) \tan(\frac{\pi\alpha}{2})\right) + i\mu\theta\right\} & \text{if} \quad \alpha \neq 1 \\ \exp\left\{-|\theta|\sigma \left(1 - i\frac{2}{\pi}\beta \operatorname{sign}(\theta) \ln(|\theta|)\right) + i\mu\theta\right\} & \text{if} \quad \alpha = 1 \end{cases} \tag{1.2}$$

for $\theta \in \mathbb{R}$. Here $\alpha \in (0, 2]$, $\beta \in [-1, 1]$, $\sigma > 0$ and $\mu \in \mathbb{R}$ are parameters.

Notice that $S_2(\sigma, \beta, \mu)$-distributions are, in fact, normal $N(\mu, 2\sigma^2)$-distributions.

Now, for $\alpha < 2$ and $\beta > -1$, an $S_\alpha(\sigma, \beta, \mu)$-distributed random variable ξ has the right probability tail (see e.g., Samorodnitsky and Taqqu [53], Eq. 1.2.8)

$$\mathbf{P}\{\xi > u\} \sim \begin{cases} \dfrac{(1-\alpha)(1+\beta)\sigma^\alpha}{2\Gamma(2-\alpha)\cos(\frac{\pi\alpha}{2})} u^{-\alpha} & \text{if} \quad \alpha \neq 1 \\ \dfrac{(1+\beta)\sigma}{\pi} u^{-1} & \text{if} \quad \alpha = 1 \end{cases} \quad \text{as } u \to \infty.$$

Hence $\xi \in \mathcal{R}(-\alpha) \subseteq \mathcal{ER} \subseteq \mathcal{S} \subseteq \mathcal{L}$.

1.2 Exponential Distributions

Guided by the notation $\mathcal{OR}$ and $\mathcal{O}\Pi$ of Bingham, Goldie and Teugels [19], p. 65 and p. 128, for $\mathcal{O}$-regularly varying functions and $\mathcal{O}$-de Haan classes, respectively, we make the following definition:

Definition 1.9. *A probability distribution function F on the real line, with right end-point $\sup\{x : F(x) < 1\} = \infty$, is said to belong to the class $\mathcal{OL}$, if*

$$0 < \liminf_{u\to\infty} \frac{1 - F(u + x)}{1 - F(u)} \leq \limsup_{u\to\infty} \frac{1 - F(u + x)}{1 - F(u)} < \infty \quad \textit{for } x \in \mathbb{R}.$$

For the class $\mathcal{OL}$, we have the following lemma, corresponding to Lemma 1.2 for the class $\mathcal{L}$, the proof of which is again elementary:

Lemma 1.10. *We have $F \in \mathcal{OL}$ if*

$$\liminf_{u\to\infty} \frac{1-F(u+x)}{1-F(u)} > 0 \quad \textit{for some } x > 0.$$

Consider a distribution function F such that the limit

$$\ell_1(x) = \lim_{u\to\infty} \frac{1-F(u+x)}{1-F(u)} \quad \text{exists with} \quad \ell(x) \in (0,\infty) \quad \text{for } x \in \mathbb{R}. \tag{1.3}$$

It is easy to see that (1.3) implies that

$$\lim_{u\to\infty} \frac{1-F(u+x)}{1-F(u)} = \mathrm{e}^{-\alpha x} \quad \text{for } x \in \mathbb{R}, \tag{1.4}$$

for some $\alpha \geq 0$:

Definition 1.11. *A probability distribution function F on the real line, with right end-point* $\sup\{x : F(x) < 1\} = \infty$, *is said to belong to the class $\mathcal{L}(\alpha)$ if (1.4) holds, for some $\alpha \geq 0$.*

Notice that $\mathcal{L}(0) = \mathcal{L}$, and the trivial inclusion $\mathcal{L}(\alpha) \subseteq \mathcal{OL}$ for $\alpha \geq 0$.

Definition 1.12. *A probability distribution function F on the real line, with right end-point* $\sup\{x : F(x) < 1\} = \infty$, *is said to belong to the* exponential *class $\mathcal{S}(\alpha)$ of distributions, if $F \in \mathcal{L}(\alpha)$ and*

$$\ell_2 = \lim_{u\to\infty} \frac{1-F\star F(u)}{1-F(u)} \quad \textit{exists with} \quad \ell_2 < \infty. \tag{1.5}$$

Notice that $\mathcal{S}(0) = \mathcal{S}$, and the trivial inclusion $\mathcal{S}(\alpha) \subseteq \mathcal{L}(\alpha)$ for $\alpha \geq 0$.

Within $\mathcal{S}(0)$, one must have $\ell_2 = 2$. (This highly non-trivial result is due to Chover, Ney and Wainger [25], Theorem 3. See also Embrechts and Goldie [32], Section 2.) But as it is known that (1.5) with $\ell_2 = 2$ implies $F \in \mathcal{L}$ (see Athreya and Ney [5], p. 148), it is more convenient to define $\mathcal{S}$ by this requirement alone.

The following proposition, which extends to $\mathcal{S}(\alpha)$ a similar result of Embrechts, Goldie and Veraverbeke [33], Theorem 1 for the class $\mathcal{S}$, was given by Samorodnitsky and Braverman [22], Equation 3.37. However, Samorodnitsky and Braverman noted that, although not recorded in the literature, their result was undoubtly known to Embrechts and Goldie [32], and added little in essence to Theorem 4.2

ii of theirs. As Samorodnitsky and Braverman only sketch the proof, which is in fact rather advanced, we have chosen to give it in complete detail here.

Proposition 1.13 (Embrechts and Goldie [32], Theorem 4.2 i, Samorodnitsky and Braverman [22], Equation 3.37). *Let* $\{\xi(t)\}_{t\geq 0}$ *be a Lévy process, starting at* $\xi(0)=0$*, such that, for* $h>0$ *a constant,* $\xi(h)$ *has Lévy measure* ν*. If*

$$\frac{\nu((1\vee\cdot,\infty))}{\nu((1,\infty))}\in\mathcal{S}(\alpha)\quad \textit{for some}\ \ \alpha>0, \tag{1.6}$$

then

$$\xi(t)\in\mathcal{S}(\alpha)\quad \textit{and}\quad \lim_{u\to\infty}\frac{\mathbf{P}\{\xi(t)>u\}}{\nu((u,\infty))}=\frac{t}{h}\,\mathbf{E}\{\mathrm{e}^{\alpha\xi(t)}\}\quad \textit{for}\ \ t>0.$$

Proof. Assume that (1.6) holds. Suppose for simplicity that $h=1$. Let ξ have characteristic triple (ν,m,s^2), and pick an $t>0$. As the Lévy measure $\nu_1(\cdot)=\nu((0,1]\cap\cdot)$ trivially satisfies

$$\int_{|x|>1}\mathrm{e}^{cx}d\nu_1(x)<\infty\quad \text{for}\ \ c\in\mathbb{R},$$

an infinitely divisible random variable ξ_1 with characteristic triple $(t\nu_1,tm,ts^2)$ will have

$$\mathbf{E}\{\mathrm{e}^{c\xi_1}\}<\infty\quad \text{for}\ \ c\in\mathbb{R} \tag{1.7}$$

(see e.g., Sato [54], Theorem 25.17). Further, letting ξ_2 be an infinitely divisible random variable, independent of ξ_1, and with characteristic triple $(t\nu((1,\infty)\cap\cdot),0,0)$, we have $\xi_2\in\mathcal{S}(\alpha)$, by Embrechts and Goldie [32], Theorem 4.2 ii, together with (1.6), because (with obvious notation) the compound Poisson distribution

$$\mathrm{e}^{-\nu((1,\infty))}\sum_{n=0}^{\infty}\frac{(t\nu((1,\infty)))^n}{n!}\left(\frac{\nu((1,\infty)\cap x)}{\nu((1,\infty))}\right)^{\star n}$$

has the same characteristic function

$$\exp\left\{t\nu((1,\infty))\int_{\mathbb{R}}\left(\mathrm{e}^{i\theta x}-1\right)\frac{d\nu((1,\infty)\cap\cdot)(x)}{\nu((1,\infty))}\right\}$$

as ξ_1. As this gives

$$\lim_{u\to\infty}\mathrm{e}^{(\alpha+\varepsilon)u}\,\mathbf{P}\{\xi_2>u\}=\infty\quad \text{for}\ \ \varepsilon>0,$$

by Embrechts and Goldie [32], Lemma 2.4 i, we may now use (1.7) together with Chebysjev's inequality, to conclude that

$$\limsup_{u\to\infty}\frac{\mathbf{P}\{\xi_1>u\}}{\mathbf{P}\{\xi_2>u\}}\le\limsup_{u\to\infty}\frac{\mathbf{E}\{\mathrm{e}^{(\alpha+\varepsilon)\xi_1}\}}{\mathrm{e}^{(\alpha+\varepsilon)u}\,\mathbf{P}\{\xi_2>u\}}=0. \tag{1.8}$$

However, by Cline [26], Corollary 2.7, for $\xi_2\in\mathcal{S}(\alpha)$ and ξ_1 such that

$$\lim_{u\to\infty}\frac{\mathbf{P}\{\xi_1>u\}}{\mathbf{P}\{\xi_2>u\}}\quad\text{exists and is finite,}$$

we have $\xi_1+\xi_2\in\mathcal{S}(\alpha)$. And so $\xi(t)\overset{\mathrm{d}}{=}\xi_1+\xi_2\in\mathcal{S}(\alpha)$. Moreover, again by Cline [26], Corollary 2.7, in the presence of (1.8), we have

$$\mathbf{P}\{\xi(t)>u\}=\mathbf{P}\{\xi_1+\xi_2>u\}\sim\mathbf{E}\left\{\mathrm{e}^{\alpha\xi_1}\right\}\mathbf{P}\{\xi_2>u\}\quad\text{as }\ u\to\infty,$$

where, by Embrechts and Goldie [32], Theorem 4.2 ii,

$$\mathbf{P}\{\xi_2>u\}\sim\mathbf{E}\left\{\mathrm{e}^{\alpha\xi_1}\right\}\,t\nu((u,\infty)).$$

Since $\mathbf{E}\{\mathrm{e}^{\alpha\xi_1}\}\mathbf{E}\{\mathrm{e}^{\alpha\xi_2}\}=\mathbf{E}\{\mathrm{e}^{\alpha\xi(t)}\}$, this gives the asymptotic relation for $\mathbf{P}\{\xi(t)>u\}$ desired, when $h=1$. □

Many authors in the field of mathematical finance argue that distributions to model log increments of asset prices, such as, for example, stock prices, should have *semi-heavy tails*, that is, the probability density function f of such a distribution should satisfy

$$f(u)\sim C\,u^{\rho}\mathrm{e}^{-\eta u}\quad\text{as }\ u\to\infty \tag{1.9}$$

for some constants $C,\eta>0$ and $\rho\in\mathbb{R}$. See Schoutens [56], p. 36 and Section 5.3, for an overview.

Albeit the label "semi-heavy" tails was coined as late as 1998 by Barndorff-Nielsen [13], Section 3, the interest for such tails goes back at least to 1977, with Barndorff-Nielsen [10].

Example 1.14. For a semi-heavy tailed distribution F, with density function satisfying (1.9), we have, as $u\to\infty$,

$$\frac{1-F\star F(u)}{1-F(u)}\sim\begin{cases}2\int_0^\infty \mathrm{e}^{\eta x}f(x)\,dx & \text{for}\quad \rho<-1\\ 2C\ln(u) & \text{for}\quad \rho=-1\\ C\,B(\rho+1,\rho+1)u^{\rho+1} & \text{for}\quad \rho>-1\end{cases}$$

(where B is the Beta function). So albeit all semi-heavy tailed distributions belongs to $\mathcal{L}(\eta)$, and thus to $\mathcal{OL}$, theay are in $\mathcal{S}(\eta)$ only when $\rho<-1$.

The exponential distribution itself is semi-heavy tailed with $\rho=0$. Hence the exponential distribution is not exponential! This, quite convincingly, shows that it is really the classes $\mathcal{L}(\alpha)$, rather than the classes $\mathcal{S}(\alpha)$, that should be called exponential. Also, recall Remark 1.5.

For semi-heavy tails that are in $\mathcal{L}(\alpha)\setminus\mathcal{S}(\alpha)$, rather than $\mathcal{S}(\alpha)$, there exists a result corresponding to Proposition 1.13. However, the analysis is now much more difficult, and requires a the Theorem 1.16, that is presented below.

Remark 1.15. According to the Tauberian result by Barndorff-Nielsen, Kent and Sørensen [15], Theorem 5.2, if for a probability distribution function F, the function

$$F_k(u)=\int_0^u x^k\mathrm{e}^{\alpha x}dF(u),\quad u>0,$$

has an ultimately monotone derivative for some $k\in\mathbb{N}$, such that its moment generating function

$$\phi(s)=\int_{\mathbb{R}}\mathrm{e}^{sx}dF(x)$$

has a k:th derivative that satisfies

$$\phi^{(k)}(\alpha+s)\sim C\,(-s)^{\rho-k}\quad\text{as}\ s\uparrow 0,\tag{1.10}$$

for some constants $C>0$ and $\rho<k$, then F has a probability density function f that satisfies

$$f(u)\sim\frac{Cu^{-\rho-1}\mathrm{e}^{-\alpha u}}{\Gamma(k+1+\rho)}\quad\text{as}\ u\to\infty.$$

But if only the moment generating function ϕ is known, then it seems quite impossible to check that F_k has an ultimately monotone derivative. If, on the other hand, that information on f is available, then the Tauberian result should typically not be needed anyway. Moreover, Abelian theory only give information about the asymptotics of the moment generating function, while Theorems 1.16 and 1.17 below indicate that rather it is the asymptotics of the Lévy measure that is the right tool.

For example, Grigelionis [41], p. 242, use the Tauberian result by Barndorff-Nielsen, Kent and Sørensen, to establish semi-heavyness of GZ distributions (see Example 5.7 below), with only the information (1.10) readily available, but the details about how it is shown that F_k has an ultimately monotone derivative omitted, as being standard calculations: We do not belive in this!!!

The distribution of an infinitely divisible random variable Z is characterized by its characteristic triple (ν, m, s^2), given by

$$\mathbf{E}\{\mathrm{e}^{i\theta Z}\} = \exp\left\{ i\theta m + \int_{\mathbb{R}} \big(\mathrm{e}^{i\theta x} - 1 - i\theta\kappa(x)\big)\, d\nu(x) - \frac{\theta^2 s^2}{2} \right\} \quad \text{for } \theta \in \mathbb{R}.$$

Here $\kappa(x) = x/(1 \vee |x|)$ for $x \in \mathbb{R}$, while $m \in \mathbb{R}$ and $s^2 \geq 0$ are constants, and ν is the (Borel) Lévy measure on $\mathbb{R}$, satisfying

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} 1 \wedge |x|^2 d\nu(x) < \infty.$$

The finite dimensional distributions of a Lévy process $\{\xi(t)\}_{t\geq 0}$ are determined by its characteristic triple (ν, m, s^2) of the process, given by

$$\mathbf{E}\{\mathrm{e}^{i\theta\xi(t)}\} = \exp\left\{ it\theta m + t\int_{\mathbb{R}} \big(\mathrm{e}^{i\theta x} - 1 - i\theta\kappa(x)\big)\, d\nu(x) - \frac{t\theta^2 s^2}{2} \right\}$$

for $\theta \in \mathbb{R}$ and $t \geq 0$.

Although absolute continuity is a time-dependent property for a Lévy process, the set $\{t \geq 0 : \xi(t)$ is not absolutely continuous$\}$ is a interval of the type $[0, a)$ or $[0, a]$, where possibly $a = \infty$ (see e.g., Sato [54], Remark 25.22). Of course, for most Lévy processes encountered in practice, that interval is $[0, 0] = \{0\}$, as such processes are non-degenerate selfdecomposable, which implies that marginal distributions are absolutely continuous (see e.g., Sato [54], Theorem 27.13)

Recall that the upper end-point $\sup\{x : \mathbf{P}\{\xi(t) \leq x\} < 1\}$ is infinite for some $t > 0$ if and only if it is infinite for all $t > 0$ (see e.g., Sato [54], Theorem 24.10).

The following Theorems 1.16 and 1.17 are very important for our treatment of exponential Lévy processes. See the introduction to our superexponential Tauberian result Theorem 1.21 below, for bibliographic information.

Theorem 1.16. *Let $\{\xi(t)\}_{t\geq 0}$ be a Lévy process with characteristic triple (ν, m, s^2). Suppose that ν is absolutely continuous with a density function that has semi-heavy tails*

$$\frac{d\nu(u)}{du} \sim C\, u^{\rho} e^{-\alpha u} \quad \textit{as } u \to \infty, \tag{1.11}$$

for some constants $C, \alpha > 0$ and $\rho > -1$. If $\xi(t)$ is absolutely continuous for an $t > 0$, then we have $\xi(t) \in \mathcal{L}(\alpha) \setminus \mathcal{S}(\alpha)$. Moreover, we have

$$\xi(h) \in \mathcal{L}(\alpha) \setminus \mathcal{S}(\alpha) \quad \textit{with} \quad \lim_{u\to\infty} \frac{\mathbf{P}\{\xi(t) > u\}}{\mathbf{P}\{\xi(h) > u\}} = 0 \quad \textit{for } t < h. \tag{1.12}$$

Proof. We start with some preparations: By Sato [54], Theorem 24.7, ξ has infinite upper endpoint

$$\sup\{x : \mathbf{P}\{\xi(t) \leq x\} < 1\} = \infty \quad \text{for } t > 0,$$

Consider the Laplace transform

$$\phi_t(\lambda) = \mathbf{E}\{\mathrm{e}^{-\lambda\xi(t)}\} = \left(\mathbf{E}\{\mathrm{e}^{-\lambda\xi(1)}\}\right)^t \quad \text{for } \lambda \in (-\alpha, 0] \text{ and } t > 0, \tag{1.13}$$

which is finite for $\lambda \in (-\alpha, 0]$ (see e.g., Sato [54], Theorem 25.17). Denote

$$\begin{cases} \mu(\lambda) = -\dfrac{\phi_1'(\lambda)}{\phi_1(\lambda)} = \displaystyle\int_{\mathbb{R}} \left(x\,\mathrm{e}^{-\lambda x} - \kappa(x)\right) d\nu(x) + m - \lambda s^2 \\ \sigma(\lambda)^2 = -\mu'(\lambda) = \displaystyle\int_{\mathbb{R}} x^2 \mathrm{e}^{-\lambda x} d\nu(x) + s^2 \end{cases} \tag{1.14}$$

for $\lambda \in (-\lambda, 0]$. Observe that, by (1.11),

$$\sigma(\lambda)^2 \sim \int_0^\infty \frac{x^2}{\sigma(\lambda)^2}\,\mathrm{e}^{-(\alpha+\lambda)x} dx \sim \frac{\Gamma(3+\rho)}{(\alpha+\lambda)^{3+\rho}} \to \infty \quad \text{as } \lambda \downarrow -\alpha, \tag{1.15}$$

which in turn gives

$$\begin{aligned} \limsup_{\lambda\downarrow-\alpha} \int_{|x|>\varepsilon\sigma(\lambda)} \frac{x^2}{\sigma(\lambda)^2}\,\mathrm{e}^{-\lambda x} d\nu(x) &= \limsup_{\lambda\downarrow-\alpha} \int_{\varepsilon\sigma(\lambda)}^\infty \frac{x^{2+\rho}}{\sigma(\lambda)^2}\,\mathrm{e}^{-(\alpha+\lambda)x} dx \\ &= \limsup_{\lambda\downarrow-\alpha} \int_{\varepsilon\sigma(\lambda)(\alpha+\lambda)}^\infty \frac{y^{2+\rho}}{\Gamma(3+\rho)}\,\mathrm{e}^{-y} dy = 0 \end{aligned} \tag{1.16}$$

for $\varepsilon > 0$. By arguing as for (1.15), we further get

$$\begin{aligned}\mu(\lambda)-m &= \int_{-\infty}^{0} -\big(\kappa(x)-\mathrm{e}^{-\lambda x}x\big)\,d\nu(x)+\int_{0}^{\infty}\big(\mathrm{e}^{-\lambda x}x-\kappa(x)\big)\,d\nu(x)+(-\lambda)s^2\\ &\geq \int_{-\infty}^{-1}\big(1+\mathrm{e}^{-\lambda x}x\big)\,d\nu(x)+\int_{0}^{\infty}\big(\mathrm{e}^{-\lambda x}x-1)\big)\,d\nu(x)\\ &\sim \frac{\Gamma(2+\rho)}{(\alpha+\lambda)^{2+\rho}}\to\infty \quad \text{as } \lambda\downarrow-\alpha.\end{aligned} \tag{1.17}$$

Moreover, we have

$$\lim_{\lambda\downarrow-\alpha}\int_{\mathbb{R}}\Big(\frac{\theta x}{\sigma(\lambda)}-\sin\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\,\mathrm{e}^{-\lambda x}d\nu(x)=0 \quad \text{for } \theta\in\mathbb{R}. \tag{1.18}$$

This is so because (1.16) gives

$$\begin{aligned}&\limsup_{\lambda\downarrow-\alpha}\int_{|x|>\varepsilon\sigma(\lambda)}\Big|\frac{\theta x}{\sigma(\lambda)}-\sin\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big|\,\mathrm{e}^{-\lambda x}d\nu(x)\\ &\qquad\leq \frac{2|\theta|}{\varepsilon}\limsup_{\lambda\downarrow-\alpha}\int_{|x|>\varepsilon\sigma(\lambda)}\frac{x^2}{\sigma(\lambda)^2}\,\mathrm{e}^{-\lambda x}d\nu(x)=0 \quad \text{for } \varepsilon>0,\end{aligned}$$

while, by Taylor expansion, given any $\delta>0$, for $\varepsilon=\varepsilon(\theta)>0$ sufficiently small,

$$\begin{aligned}&\limsup_{\lambda\downarrow-\alpha}\int_{|x|\leq\varepsilon\sigma(\lambda)}\Big|\frac{\theta x}{\sigma(\lambda)}-\sin\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big|\,\mathrm{e}^{-\lambda x}d\nu(x)\\ &\qquad\leq \delta\limsup_{\lambda\downarrow-\alpha}\int_{|x|\leq\varepsilon\sigma(\lambda)}\frac{\theta^2x^2}{\sigma(\lambda)^2}\,\mathrm{e}^{-\lambda x}d\nu(x)\leq\delta\theta^2.\end{aligned}$$

As a final preparation we show that

$$\lim_{K\to\infty}\limsup_{\lambda\downarrow-\alpha}\int_{|\theta|>K}\exp\Big\{-t\Big[\int_{\mathbb{R}}\Big(1-\cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\,\mathrm{e}^{-\lambda x}d\nu(x)+\frac{\theta^2s^2}{2\sigma(\lambda)^2}\Big]\Big\}d\theta=0: \tag{1.19}$$

To that end it is clearly enough to prove (1.19) for $s^2=0$. Now, since $1-\cos(x)\geq\frac{1}{4}x^2$ for $|x|\leq1$, (1.11) together with (1.15) show that, for some constant $A>0$,

$$\begin{aligned}\int_{\mathbb{R}}\Big(1-\cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\,\mathrm{e}^{-\lambda x}d\nu(x) &\geq \frac{A\theta^2}{4}\int_0^{1/(\alpha+\lambda)}\frac{x^{2+\rho}\mathrm{e}^{-(\alpha+\lambda)x}}{\sigma(\lambda)^2}\,dx\\ &\geq \frac{A\theta^2}{8\Gamma(3+\rho)}\int_0^1 x^{3+\rho}\mathrm{e}^{-x}dx\end{aligned}$$

for $|\theta|\leq(\alpha+\lambda)\sigma(\lambda)$ and $\lambda\in(-\alpha,0]$ sufficiently small. This immediately gives

$$\lim_{K\to\infty}\overline{\lim_{\lambda\downarrow-\alpha}}\int_{K<|\theta|\leq(\alpha+\lambda)\sigma(\lambda)}\exp\Big\{-t\int_{\mathbb{R}}\Big(1-\cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\,\mathrm{e}^{-\lambda x}d\nu(x)\Big\}\,d\theta=0. \tag{1.20}$$

For the case when $|\theta| \geq (\alpha+\lambda)\sigma(\lambda)$, we pick a constant $B > 1$ such that,

$$\frac{B^{1+\rho}C}{4} - 2 \geq 2, \quad \frac{B^{1+\rho}C}{4}\ln(2) \geq 2\ln(B) + \frac{1}{3} \quad \text{and} \quad \frac{d\nu(x)}{dx} \geq \frac{Cx^{\rho}\mathrm{e}^{-\alpha x}}{2} \quad \text{for } x \geq B.$$

Also, notice that, since $1-\cos(x) \leq \frac{1}{2}x^2$ for $|x| \leq 1$, we have

$$\begin{aligned}
\int_0^B \left(1-\cos\left(\frac{\theta x}{\sigma(\lambda)}\right)\right)\frac{\mathrm{e}^{-(\alpha+\lambda)x}}{x}dx &\leq \int_1^{B|\theta/\sigma(\lambda)|}\frac{1-\cos(x)}{x}dx + \int_0^1 \frac{1-\cos(x)}{x}dx \\
&\leq \int_1^{1\vee(B|\theta/\sigma(\lambda)|)}\frac{dx}{x}dx + \int_0^1 \frac{x^2}{2}dx \\
&\leq \ln\left(1 \vee \left|\frac{\theta}{\sigma(\lambda)}\right|\right) + \ln(B) + \frac{1}{6} \quad \text{for } \lambda \in (-\alpha, 0].
\end{aligned} \tag{1.21}$$

Hence Erdélyi, Magnus, Oberhettinger and Tricomi [36], Equation 4.7.59 gives

$$\begin{aligned}
&\int_{\mathbb{R}} \left(1-\cos\left(\frac{\theta x}{\sigma(\lambda)}\right)\right)\mathrm{e}^{-\lambda x}d\nu(x) - \ln\left(1 \vee \left|\frac{\theta}{\sigma(\lambda)}\right|\right) - \ln(B) - \frac{1}{6} \\
&\geq \frac{B^{1+\rho}C}{2}\int_B^{\infty}\left(1-\cos\left(\frac{\theta x}{\sigma(\lambda)}\right)\right)\frac{\mathrm{e}^{-(\alpha+\lambda)x}}{x}dx - \ln\left(1 \vee \left|\frac{\theta}{\sigma(\lambda)}\right|\right) - \ln(B) - \frac{1}{6} \\
&\geq \frac{B^{1+\rho}C}{2}\int_0^{\infty}\left(1-\cos\left(\frac{\theta x}{\sigma(\lambda)}\right)\right)\frac{\mathrm{e}^{-(\alpha+\lambda)x}}{x}dx - 2\ln\left(1 \vee \left|\frac{\theta}{\sigma(\lambda)}\right|\right) - 2\ln(B) - \frac{1}{3} \\
&= \frac{B^{1+\rho}C}{2}\ln\left(1+\frac{\theta^2}{(\alpha+\lambda)^2\sigma(\lambda)^2}\right) - 2\ln\left(1 \vee \left|\frac{\theta}{\sigma(\lambda)}\right|\right) - 2\ln(B) - \frac{1}{3} \\
&\geq 2\ln\left(1+\frac{\theta^2}{(\alpha+\lambda)^2\sigma(\lambda)^2}\right) \quad \text{for } |\theta| \geq (\alpha+\lambda)\sigma(\lambda) \text{ and } \lambda > -\alpha \text{ small enough.}
\end{aligned} \tag{1.22}$$

On the other hand, Erdélyi, Magnus, Oberhettinger and Tricomi [36], Equation 4.7.58 together with (1.21) readily show that, for some constant $\varepsilon > 0$,

$$\begin{aligned}
&\int_{\mathbb{R}} \left(1-\cos\left(\frac{\theta x}{\sigma(\lambda)}\right)\right)\mathrm{e}^{-\lambda x}d\nu(x) + \ln\left(1 \vee \left|\frac{\theta}{\sigma(\lambda)}\right|\right) + \frac{1}{6} \\
&\geq \frac{C}{2}\int_B^{\infty}\left(1-\cos\left(\frac{\theta x}{\sigma(\lambda)}\right)\right)x^{\rho}\mathrm{e}^{-(\alpha+\lambda)x}dx + \ln\left(1 \vee \left|\frac{\theta}{\sigma(\lambda)}\right|\right) + \ln(B) + \frac{1}{6} \\
&\geq \frac{C}{2}\int_0^{\infty}\left(1-\cos\left(\frac{\theta x}{\sigma(\lambda)}\right)\right)x^{\rho}\mathrm{e}^{-(\alpha+\lambda)x}dx \\
&= \frac{C\Gamma(\rho+1)}{2(\alpha+\lambda)^{\rho+1}}\left[1-\left(1+\frac{\theta^2}{(\alpha+\lambda)^2\sigma(\lambda)^2}\right)^{-(\rho+1)/2}\cos\left((\rho+1)\arctan\left(\frac{\theta}{(\alpha+\lambda)\sigma(\lambda)}\right)\right)\right] \\
&= \frac{C\varepsilon}{2(\alpha+\lambda)^{\rho+1}} \quad \text{for } |\theta| \geq (\alpha+\lambda)\sigma(\lambda).
\end{aligned} \tag{1.23}$$

Putting (1.22) and (1.23) together, and using (1.15), we get

$$\begin{aligned}&\varlimsup_{\lambda\downarrow-\alpha}\int_{|\theta|>(\alpha+\lambda)\sigma(\lambda)}\exp\Big\{-t\int_{\mathbb{R}}\Big(1-\cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\,\mathrm{e}^{-\lambda x}d\nu(x)\Big\}\,d\theta\\&\qquad\le\varlimsup_{\lambda\downarrow-\alpha}2\int_{(\alpha+\lambda)\sigma(\lambda)}^{\infty}\frac{(\alpha+\lambda)^2\sigma(\lambda)^2}{\theta^2}\exp\Big\{-\frac{C\varepsilon}{4(\alpha+\lambda)^{\rho+1}}\Big\}\,d\theta=0.\end{aligned}$$

From this in turn, together with (1.20), we get (1.19).

Let $Z_{t,\lambda}$ be a random variable with probability density function

$$f_{Z_{t,\lambda}}(x)=\frac{\mathrm{e}^{-\lambda x}f_{\xi(t)}(x)}{\phi_t(\lambda)}\quad\text{for } x\in\mathbb{R}\text{ and }\lambda\in(-\alpha,0],\tag{1.24}$$

where $f_{\xi(t)}$ is the probability density function of $\xi(t)$. Notice that, writing

$$\begin{cases}m_{t,\lambda} &= t\Big(m-\int_{\mathbb{R}}\kappa(x)(1-\mathrm{e}^{-\lambda x})d\nu(x)-\lambda s^2\Big)\\ d\nu_{t,\lambda}(x) &= t\mathrm{e}^{-\lambda x}d\nu(x)\\ s_{t,\lambda}^2 &= ts^2\end{cases},$$

the random variable $Z_{t,\lambda}$ has characteristic function

$$\begin{aligned}&\mathbf{E}\left\{\mathrm{e}^{i\theta Z_{t,\lambda}}\right\}\\&=\Big(\frac{\mathbf{E}\{\mathrm{e}^{(i\theta-\lambda)\xi(1)}\}}{\phi_1(\lambda)}\Big)^t\\&=\Big(\frac{1}{\phi_1(\lambda)}\exp\Big\{(i\theta-\lambda)m+\int_{\mathbb{R}}\big(\mathrm{e}^{(i\theta-\lambda)x}-1-(i\theta-\lambda)\kappa(x)\big)\,d\nu(x)-\frac{(i\theta-\lambda)^2s^2}{2}\Big\}\Big)^t\\&=\exp\Big\{i\theta m_{t,\lambda}+\int_{\mathbb{R}}\big(\mathrm{e}^{i\theta x}-1-i\theta\kappa(x)\big)\,d\nu_{t,\lambda}(x)-\frac{\theta^2s_{t,\lambda}^2}{2}\Big\}\end{aligned}$$

for $\theta\in\mathbb{R}$ and $\lambda\in(-\alpha,0]$. Hence $Z_{t,\lambda}$ is infinitely divisible with characteristic triple $\big(\nu_{t,\lambda},m_{t,\lambda},s_{t,\lambda}^2\big)$. Since $\mathbf{E}\{Z_{t,\lambda}\}=t\mu(\lambda)$, this gives (see e.g., Sato [54], p. 39)

$$\mathbf{E}\left\{\mathrm{e}^{i\theta Z_{t,\lambda}}\right\}=\exp\Big\{i\theta t\mu(\lambda)+\int_{\mathbb{R}}\big(\mathrm{e}^{i\theta x}-1-i\theta x\big)\,d\nu_{t,\lambda}(x)-\frac{\theta^2s_{t,\lambda}^2}{2}\Big\}.$$

Hence the characteristic function $g_{t,\lambda}$ of $(Z_{t,\lambda}-t\mu(\lambda))/\sigma(\lambda)$ is given by

$$\begin{aligned}g_{t,\lambda}(\theta)=\Big(\exp\Big\{&-\int_{\mathbb{R}}\Big(1-\cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\,\mathrm{e}^{-\lambda x}d\nu(x)\\&-i\int_{\mathbb{R}}\Big(\frac{\theta x}{\sigma(\lambda)}-\sin\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\,\mathrm{e}^{-\lambda x}d\nu(x)-\frac{\theta^2s^2}{2\sigma(\lambda)^2}\Big\}\Big)^t\end{aligned}\tag{1.25}$$

for $\theta\in\mathbb{R}$ and $\lambda\in(-\alpha,0]$. And so (1.16) and (1.18), together with a Taylor expansion, readily give

$$\lim_{\lambda\downarrow-\alpha} g_{t,\lambda}(\theta) = \mathrm{e}^{-t\theta^2/2} \quad \text{for } \theta\in\mathbb{R}.$$

Using this together with (1.19), in turn, it follows that

$$\begin{aligned}
\limsup_{\lambda\downarrow-\alpha}\sup_{x\in\mathbb{R}} &\left| f_{(Z_{t,\lambda}-\mu_t(\lambda))/\sigma(\lambda)}(x) - \frac{1}{\sqrt{2\pi t}}\,\mathrm{e}^{-x^2/(2t)}\right| \\
&\leq \limsup_{\lambda\downarrow-\alpha}\int_{\mathbb{R}}\left|g_{t,\lambda}(\theta)-\mathrm{e}^{-t\theta^2/2}\right| d\theta \\
&\leq \limsup_{K\to\infty}\limsup_{\lambda\downarrow-\alpha}\int_{|\theta|\leq K}\left|g_{t,\lambda}(\theta)-\mathrm{e}^{-t\theta^2/2}\right| d\theta \\
&\qquad + \limsup_{K\to\infty}\limsup_{\lambda\downarrow-\alpha}\int_{|\theta|>K}\left(|g_{t,\lambda}(\theta)|+\mathrm{e}^{-t\theta^2/2}\right) d\theta \\
&= 0.
\end{aligned} \tag{1.26}$$

Now observe that

$$f_{(Z_{t,\lambda}-t\mu(\lambda))/\sigma(\lambda)}(x) = \frac{\mathrm{e}^{-\lambda(\sigma(\lambda)x+t\mu(\lambda))}\, f_{\xi(t)}(t\mu(\lambda)+\sigma(\lambda)x)\,\sigma(\lambda)}{\phi_t(\lambda)} \tag{1.27}$$

for $x\in\mathbb{R}$ and $\lambda\in(-\alpha,0]$. Hence (1.15) together with (1.26) show that

$$\begin{aligned}
f_{\xi(t)}(t\mu(\lambda)+x/\lambda) &= \mathrm{e}^{x}\,\frac{f_{(Z_{t,\lambda}-t\mu(\lambda))/\sigma(\lambda)}(x/(\lambda\sigma(\lambda)))\,\mathrm{e}^{\lambda t\mu(\lambda)}\phi_t(\lambda)}{\sigma(\lambda)} \\
&\sim \mathrm{e}^{x}\,\frac{\mathrm{e}^{\lambda t\mu(\lambda)}\phi_1(\lambda)^t}{\sqrt{2\pi t}\,\sigma(\lambda)} \quad \text{as } \lambda\downarrow-\alpha.
\end{aligned} \tag{1.28}$$

From this in turn, together with another application of (1.26) and , we get

$$\begin{aligned}
\lim_{\lambda\downarrow-\alpha}\frac{\alpha\mathbf{P}\{\xi(t)>t\mu(\lambda)-y/\lambda\}}{f_{\xi(t)}(t\mu(\lambda)-x/\lambda)} &= \lim_{\lambda\downarrow-\alpha}\frac{(-\lambda)\mathbf{P}\{\xi(t)>t\mu(\lambda)-y/\lambda\}}{f_{\xi(t)}(t\mu(\lambda)-x/\lambda)} \\
&= \mathrm{e}^{x}\lim_{\lambda\downarrow-\alpha}\frac{(-\lambda)\mathbf{P}\{\xi(t)>t\mu(\lambda)-y/\lambda\}}{f_{\xi(t)}(t\mu(\lambda))} \\
&= \mathrm{e}^{x}\lim_{\lambda\downarrow-\alpha}\int_y^\infty\frac{f_{\xi(t)}(t\mu(\lambda)-z/\lambda)}{f_{\xi(t)}(t\mu(\lambda))}\,dz \\
&= \mathrm{e}^{x}\lim_{\lambda\downarrow-\alpha}\int_y^\infty \mathrm{e}^{-z}\,\frac{f_{(Z_{t,\lambda}-t\mu(\lambda))/\sigma(\lambda)}(-z/(\lambda\sigma(\lambda)))}{f_{(Z_{t,\lambda}-t\mu(\lambda))/\sigma(\lambda)}(0)}\,dz \\
&= \mathrm{e}^{x-y} \quad \text{for } x,y\in\mathbb{R}.
\end{aligned} \tag{1.29}$$

Observing that

$$\frac{f_{\xi(t)}(t\mu(\lambda)-y/\lambda)}{-\lambda\mathbf{P}\{\xi(t)>t\mu(\lambda)\}},\quad y\geq 0,$$

is a probability density function, (1.29) together with the theorem of Scheffé [55] show that

$$\begin{aligned}\lim_{\lambda\downarrow-\alpha}\frac{\mathbf{P}\{\xi(t)>t\mu(\lambda)-x/\lambda\}}{\mathbf{P}\{\xi(t)>t\mu(\lambda)\}}&=\lim_{\lambda\downarrow-\alpha}\int_x^\infty\frac{f_{\xi(t)}(t\mu(\lambda)-y/\lambda)}{-\lambda\mathbf{P}\{\xi(t)>t\mu(\lambda)\}}\,dy\\&=\int_x^\infty \mathrm{e}^{-y}dy=\mathrm{e}^{-x}\quad\text{for } x\geq 0.\end{aligned}\tag{1.30}$$

From this in turn, together with (1.28) and (1.29), we readily obtain

$$\lim_{\lambda\downarrow-\alpha}\frac{\mathbf{P}\{\xi(t)>t\mu(\lambda)-x/\lambda\}}{\mathbf{P}\{\xi(t)>t\mu(\lambda)\}}=\mathrm{e}^{-x}\quad\text{for } x\geq 0.$$

And so we get

$$\begin{aligned}\limsup_{\lambda\downarrow-\alpha}\frac{\mathbf{P}\{\xi(t)>t\mu(\lambda)+x/\alpha\}}{\mathbf{P}\{\xi(t)>t\mu(\lambda)\}}&\leq\limsup_{\varepsilon\downarrow 0}\limsup_{\lambda\downarrow-\alpha}\frac{\mathbf{P}\{\xi(t)>t\mu(\lambda)-(x-\varepsilon)/\lambda\}}{\mathbf{P}\{\xi(t)>t\mu(\lambda)\}}\\&=\mathrm{e}^{-x}\end{aligned}$$

and

$$\begin{aligned}\liminf_{\lambda\downarrow-\alpha}\frac{\mathbf{P}\{\xi(t)>t\mu(\lambda)+x/\alpha\}}{\mathbf{P}\{\xi(t)>t\mu(\lambda)\}}&\leq\liminf_{\varepsilon\downarrow 0}\liminf_{\lambda\downarrow-\alpha}\frac{\mathbf{P}\{\xi(t)>t\mu(\lambda)-(x+\varepsilon)/\lambda\}}{\mathbf{P}\{\xi(t)>t\mu(\lambda)\}}\\&=\mathrm{e}^{-x},\end{aligned}$$

so that

$$\lim_{\lambda\downarrow-\alpha}\frac{\mathbf{P}\{\xi(t)>t\mu(\lambda)+x\}}{\mathbf{P}\{\xi(t)>t\mu(\lambda)\}}=\mathrm{e}^{-\alpha x}\quad\text{for } x\geq 0.$$

From this it is a straightforward matter to see that this equation holds for all $x\in\mathbb{R}$, which in turn means that $\xi(t)\in\mathcal{L}(\alpha)$.

To prove (1.12), we notice that $\xi(h)\in\mathcal{L}(\alpha)$ follows from what has been proven already (remembering that $\xi(h)$ is absolutely continuous when $\xi(t)$ is). For the second part of (1.12), we notice that

$$\begin{aligned}&\lambda\mu(\lambda)+\ln(\phi_1(\lambda))\\&=\int_{-\infty}^0\big(1-\mathrm{e}^{-\lambda x}(1+\lambda x)\big)\,d\nu(x)+\int_0^\infty\big(\mathrm{e}^{-\lambda x}((-\lambda)x-1)+1\big)\,d\nu(x)+\frac{\lambda^2s^2}{2}\\&\to-\infty\quad\text{as }\lambda\downarrow-\alpha.\end{aligned}\tag{1.31}$$

By application of (1.28) and (1.28), together with (1.15), (1.17) and the continuity of μ, we therefore get the second part of (1.12) in the following way:

$$\begin{aligned}\limsup_{u\to\infty}\frac{\mathbf{P}\{\xi(t)>u\}}{\mathbf{P}\{\xi(h)>u\}} &= \limsup_{\lambda\downarrow-\alpha}\frac{\mathbf{P}\{\xi(t)>t\mu(\lambda)\}}{\mathbf{P}\{\xi(h)>t\mu(\lambda)\}}\\ &= \limsup_{\lambda\downarrow-\alpha}\frac{f_{\xi(t)}(t\mu(\lambda))}{f_{\xi(h)}(h\mu(\lambda))}\\ &= \sqrt{\frac{h}{t}}\limsup_{\lambda\downarrow-\alpha}\exp\left\{(h-t)\left(\lambda\mu(\lambda)+\ln(\phi_1(\lambda))\right)\right\}=0.\end{aligned}$$

From the second part of (1.12), in turn, we get that $\xi(t), \xi(h) \in \mathcal{L}(\alpha)\setminus\mathcal{S}(\alpha)$, as apparently (1.5) is not satisfied. □

The case when $\rho = -1$ in (1.11) requires special care, as can be easily seen that (1.16) and (1.18)-(1.19) all fail in that case. In fact, $\rho = -1$ implies a qualitatively different type of behaviour of the Laplace transform (Esscher transform), that is employed in the proof, which results in a local limit theorem with a different limit distribution than the Gaussian one in Theorem 1.17. It turns out that important classes of Lèvy processes are of this type, for example, the class of generalized z-processes.

Notice that, by Proposition 1.13 and Theorem 1.16, one is on the rim between $\mathcal{S}(\alpha)$ and $\mathcal{L}(\alpha)\setminus\mathcal{S}(\alpha)$ when $\rho = -1$ in (1.11). And we now from Example 1.14 that this rim is special.

Theorem 1.17. *Let $\{\xi(t)\}_{t\geq 0}$ be a Lévy process with characteristic triple (ν, m, s^2). Suppose that ν is absolutely continuous with a density function that has semi-heavy tails*

$$\frac{d\nu(u)}{du}\sim\frac{C\,e^{-\alpha u}}{u}\quad \text{as } u\to\infty, \tag{1.32}$$

for some constants $C, \alpha > 0$. If $\xi(t)$ is absolutely continuous for an $t > 0$, then we have $\xi(t) \in \mathcal{L}(\alpha)\setminus\mathcal{S}(\alpha)$. Moreover, we have

$$\xi(h)\in\mathcal{L}(\alpha)\setminus\mathcal{S}(\alpha)\quad \text{with}\quad \lim_{u\to\infty}\frac{\mathbf{P}\{\xi(t)>u\}}{\mathbf{P}\{\xi(h)>u\}}=0\quad \text{for } t<h. \tag{1.33}$$

Proof. As in the proof of Theorem 1.16, ξ has infinite upper endpoint, with a finite Laplace transform $\phi_t(\lambda)$ for $\lambda \in (-\alpha, 0]$. Keeping the notation (1.13), we have, by (1.15) and (1.17),

$$\mu(\lambda) \sim \frac{1}{\alpha+\lambda} \quad \text{and} \quad \sigma(\lambda)^2 \sim \frac{1}{(\alpha+\lambda)^2} \quad \text{as } \lambda \downarrow -\alpha. \tag{1.34}$$

However, instead of (1.18), we have, by (1.32) and (1.34), together with Erdélyi, Magnus, Oberhettinger and Tricomi [36], Equations 4.2.1 and 4.7.82,

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\frac{\theta x}{\sigma(\lambda)} - \sin\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\right) \mathrm{e}^{-\lambda x} d\nu(x) \\
&\sim C \int_0^\infty \left(\frac{\theta x}{\sigma(\lambda)} - \sin\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\right) \frac{\mathrm{e}^{-(\alpha+\lambda)x}}{x} dx \\
&= \frac{C\theta}{(\alpha+\lambda)\sigma(\lambda)} - \frac{C}{2}\arctan\left[\frac{2\theta}{(\alpha+\lambda)\sigma(\lambda)} \Big/ \left(1 - \frac{\theta^2}{(\alpha+\lambda)^2\sigma(\lambda)^2}\right)\right] \\
&\to C\theta - \frac{C}{2}\arctan\left(\frac{2\theta}{1-\theta^2}\right) = C\theta - C\arctan(\theta) \quad \text{as } \lambda \downarrow -\alpha \text{ for } \theta \in \mathbb{R}.
\end{aligned} \tag{1.35}$$

Moreover, instead of (1.16), we have, by (1.32) and (1.15) together with Erdélyi, Magnus, Oberhettinger and Tricomi [36], Equation 4.7.59,

$$\begin{aligned}
\int_{\mathbb{R}} \left(1 - \cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\right) \mathrm{e}^{-\lambda x} d\nu(x) &\sim C \int_0^\infty \left(1 - \cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\right) \frac{\mathrm{e}^{-(\alpha+\lambda)x}}{x} dx \\
&= \frac{C}{2}\ln\left(1 + \frac{\theta^2}{(\alpha+\lambda)^2\sigma(\lambda)^2}\right) \\
&\to \frac{C}{2}\ln(1+\theta^2) \quad \text{as } \lambda \downarrow -\alpha \text{ for } \theta \in \mathbb{R}.
\end{aligned} \tag{1.36}$$

Keeping also the notation (1.24) and $g_{t,\lambda}$ from the proof of Theorem 1.16, (1.25) together with (1.35) and (1.36) give

$$\lim_{\lambda\downarrow-\alpha} g_{t,\lambda}(\theta) = \exp\left\{-Ct\left[\frac{\ln(1+\theta^2)}{2} + i\,(\theta - \arctan(\theta))\right]\right\} \equiv g_t(\theta) \quad \text{for } \theta \in \mathbb{R}. \tag{1.37}$$

Let Y_t be a random variable with the above charactersitic function g. As (1.19) is not available, to be able to get a version of the crucial uniform convergence (1.26), we employ an auxiliary random variable X, that is independent of $Z_{t,\lambda}$ and Y_t, with probability density function

$$f_X(x) = \frac{3(x+1)(1-x)}{4} \quad \text{for } x \in (-1,1),$$

and characteristic function

$$\psi_X(\theta) = \frac{3\sin(\theta) - 3\theta\cos(\theta)}{\theta^3} \quad \text{for } \theta \in \mathbb{R}. \tag{1.38}$$

Taking a $\delta > 0$, the charactersitic function of $\delta X + (Z_{t,\lambda} - t\mu(\lambda))/\sigma(\lambda)$ satisfies

$$\psi_{\delta X+(Z_{t,\lambda}-t\mu(\lambda))/\sigma(\lambda)}(\theta) = \psi_X(\theta) g_{t,\lambda}(\theta) \to \psi_X(\theta) g_t(\theta) \quad \text{for } \theta \in \mathbb{R}.$$

And so it is obvious from an inspection of (1.26), together with elementary arguments, that we have the uniform convergence desired

$$\limsup_{\lambda\downarrow-\alpha} \sup_{x\in\mathbb{R}} \left| f_{\delta X+(Z_{t,\lambda}-t\mu(\lambda))/\sigma(\lambda)}(x) - f_{\delta X+Y_t}(x) \right| = 0. \tag{1.39}$$

Now observe that, because of (1.27), we have

$$f_{\delta X+(Z_{t,\lambda}-t\mu(\lambda))/\sigma(\lambda)}(x) = \frac{\mathrm{e}^{-\lambda(\sigma(\lambda)x+t\mu(\lambda))} f_{\delta X+\xi(t)}(t\mu(\lambda)+\sigma(\lambda)x)\,\sigma(\lambda)}{\phi_t(\lambda)}$$

for $x \in \mathbb{R}$ and $\lambda \in (-\alpha, 0]$. Hence (1.34) together with (1.39) show that

$$f_{\delta X+\xi(t)}(t\mu(\lambda) + x/\lambda) \sim \mathrm{e}^x \frac{\mathrm{e}^{\lambda t\mu(\lambda)} \phi_1(\lambda)^t f_{\delta X+Y_t}(0)}{\sigma(\lambda)} \quad \text{as } \lambda \downarrow -\alpha \text{ for } x \in \mathbb{R}. \tag{1.40}$$

And so the arguments from the proof of Theorem 1.16 carry over with only trivial modifications, to show that $\delta X + \xi(t) \in \mathcal{L}(\alpha)$, and that

$$\delta X + \xi(h) \in \mathcal{L}(\alpha) \quad \text{with} \quad \lim_{u\to\infty} \frac{\mathbf{P}\{\delta X + \xi(t) > u\}}{\mathbf{P}\{\delta X + \xi(h) > u\}} = 0 \quad \text{for } t < h.$$

From this in turn, sending $\delta \downarrow 0$, it now follows from an elementary argument, that also $\xi(t), \xi(h) \in \mathcal{L}(\alpha)$, and that the second part of (1.33) holds. And so we get $\xi(t), \xi(h) \in \mathcal{L}(\alpha) \setminus \mathcal{S}(\alpha)$ as in the proof of Theorem 1.16. □

1.3 Superexponential Infinitely Divisible Distributions

We have previously encountered subexponential and exponential distributions, and now it is time to consider *superexponential* distributions:

Definition 1.18. *A Lévy process* $\{\xi(t)\}_{t\geq 0}$ *is* superexponential *if*

$$\mathbf{E}\{\mathrm{e}^{\alpha\xi(1)}\} < \infty \quad \textit{for } \alpha \geq 0.$$

It is quite basic (see e.g., Sato [54], Theorem 25.17), that ξ is superexponential if and only if

$$\mathbf{E}\{\mathrm{e}^{\alpha\xi(t)}\} = \left(\mathbf{E}\{\mathrm{e}^{\alpha\xi(1)}\}\right)^t < \infty \quad \text{for } \alpha \geq 0 \text{ and } t > 0.$$

So a Lévy process ξ is superexponential if it has a well-defined Laplace transform

$$\phi_t(\lambda) = \mathbf{E}\{\mathrm{e}^{-\lambda\xi(t)}\} = \big(\mathbf{E}\{\mathrm{e}^{-\lambda\xi(1)}\}\big)^t < \infty \quad \text{for } \lambda \leq 0 \text{ and } t > 0. \tag{1.41}$$

Example 1.19. The normal $\mathrm{N}(\mu, \sigma^2)$ distribution is superexponential.

Definition 1.20. *A random variable X with infinite upper end-point*

$$\sup\{x : \mathbf{P}\{X \leq x\} < 1\} = \infty$$

belongs to the Type I domain of attraction *of extremes (also called the* Gumbel domain *or the* de Haan class Γ*), with* auxiliary function $w(u) > 0$*, if*

$$\lim_{u\to\infty} \frac{\mathbf{P}\{X > u + xw(u)\}}{\mathbf{P}\{X > u\}} = \mathrm{e}^{-x} \quad \textit{for } x \in \mathbb{R}.$$

The auxiliary function w of a Type I attracted can be chosen continuous and must satsify $w(u) = o(u)$ as $u \to \infty$ (see e.g., Bingham, Goldie and Teugels [19], Lemma 3.10.1 and Corollary 3.10.9). Further, it is quite elementary that $\tilde{w}(u) > 0$ is another auxiliary function if and only if $\tilde{w}(u) \sim w(u)$ as $u \to \infty$.

Feigin and Yashchin, [37], Theorem 2 and 3, gave a scheme to recover the asymptotic behaviour of the right tail of a probability distribution function or a probability density funtion, from that of the left tail of its Laplace transform. The usefulness of this to establish Type I attraction was noted in a particular case by Davis and Resnick, [27], Section 3. See also Rootzén [49] and [50]. It is this idea of approach that we used in Theorem 1.16.

A somewhat different line of research, pursued by A.A. Balkema, C. Klüppelberg, S.I. Resnick and U. Stadtmüller, in a series of articles, culminating in Balkema, Klüppelberg and Resnick [8], is to characterize when the convergence of Esscher transforms (exponential families), which are the key ingredient of proofs in this area, take place, and what limits then are possible. See also the earlier contributions in Balkema, Klüppelberg and Resnick [6], Balkema, Klüppelberg and Stadtmüller [9], and Balkema, Klüppelberg and Resnick [7]. It should be noted that they impose conditions on densities that we do not feel comfortable with, in our infinitely divisible setting, and that it is not the mentioned convergence that is our goal to analyze, but to find the actual tail behaviour, and establish Type I attraction. In fact, we have to deal with random variables, the distribution of which depend on how far out we are in the tail, that is, an "external parameter", which make results in the literature completely non-applicable anyway.

The main result of this section, Theorem 1.21 below, is a considerable development, as well as adaption to our particular needs, in the present context of superexponential Lévy processes [namely (1.49) and (1.80)-(1.82) below], of the scheme of Albin [4], Section 3, to establish Type I attraction for selfdecomposable distributions. That scheme of Albin, in turn, were an adpation to the selfdecomposable setting, of the scheme of Feigin and Yashchin, with additional input from Albin [3], Sections 2 and 3.

Here is the main result of this section, which will be very important for our study of superexponential processes:

Theorem 1.21. *Let $\{\xi(t)\}_{t\geq 0}$ be a superexponential Lévy process with $\xi(0) = 0$ and characteristic triple (ν, m, s^2). Let ξ have infinite upper end-point*

$$\sup\{x : \mathbf{P}\{\xi(t) \leq x\} < 1\} = \infty \quad \text{for } t > 0, \tag{1.42}$$

and Laplace transform given by (1.41). Denote

$$\begin{cases} \mu(\lambda) &= -\dfrac{\phi_1'(\lambda)}{\phi_1(\lambda)} = \displaystyle\int_{\mathbb{R}} \big(x\mathrm{e}^{-\lambda x} - \kappa(x)\big)\, d\nu(x) + m - \lambda s^2 \\ \sigma(\lambda)^2 &= -\mu'(\lambda) \;=\; \displaystyle\int_{\mathbb{R}} x^2 \mathrm{e}^{-\lambda x} d\nu(x) + s^2 \end{cases} \tag{1.43}$$

for $\lambda \leq 0$. Given an $h > 0$, assume that the following conditions hold:

$$\lim_{\lambda\to-\infty} \lambda^2\sigma(\lambda)^2 = \infty; \tag{1.44}$$

$$\lim_{\lambda\to-\infty} \int_{|x|>\varepsilon\sigma(\lambda)} \frac{x^2}{\sigma(\lambda)^2}\, \mathrm{e}^{-\lambda x} d\nu(x) = 0 \quad \text{for } \varepsilon > 0; \tag{1.45}$$

$$\lim_{K\to\infty} \limsup_{\lambda\to-\infty} \int_{|\theta|>K} \exp\bigg\{-t\bigg[\int_{\mathbb{R}} \Big(1 - \cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\, \mathrm{e}^{-\lambda x} d\nu(x) + \frac{\theta^2 s^2}{2\sigma(\lambda)^2}\bigg]\bigg\} d\theta$$
$$= 0 \quad \text{for } t \text{ in a neighborhood of } h. \tag{1.46}$$

Assume that the following limit exists

$$\lim_{\lambda\to-\infty}\frac{\lambda\mu(\lambda)}{\lambda\mu(\lambda)+\ln(\phi_1(\lambda))}=L. \tag{1.47}$$

Let $\mu^{\leftarrow}$ *denote the inverse of* μ*, and define*

$$w(u)=-\frac{1}{\mu^{\leftarrow}(u/h)} \quad \textit{and} \quad q(u)=\frac{1}{\ln(\phi_1(\mu^{\leftarrow}(u/h)))}, \tag{1.48}$$

If $\xi(t)$ *is absolutely continuous for* t *in a neighborhood of* h*, then we have*

$$\lim_{u\to\infty}\frac{\mathbf{P}\{\xi(h-q(u)t)>u+xw(u)\}}{\mathbf{P}\{\xi(h)>u\}}\to \mathrm{e}^{-t-x} \quad \textit{for } x\in\mathbb{R} \textit{ and } t>0. \tag{1.49}$$

Proof. By (1.42), we have (see e.g., Sato [54], Theorem 24.7)

$$\int_{-\infty}^{0}(-x)d\nu(x)=\infty \quad \text{or} \quad \nu((0,\infty))>0 \quad \text{or} \quad s>0. \tag{1.50}$$

For the function μ we therefore have (see e.g., Sato [54], p. 39)

$$\mu(\lambda)-m=\int_{-\infty}^{0}-\left(\kappa(x)-\mathrm{e}^{-\lambda x}x\right)d\nu(x)+\int_{0}^{\infty}\left(\mathrm{e}^{-\lambda x}x-\kappa(x)\right)d\nu(x)+(-\lambda)s^2.$$

Here all terms on the right-hand side are non-negative, for λ sufficiently small. Moreover, the first, second or third term go to ∞ as $\lambda\to-\infty$, if the first, second or third option in (1.50) holds, respectively, so that

$$\lim_{\lambda\to-\infty}\mu(\lambda)=\infty. \tag{1.51}$$

Also notice that the function

$$Q(\lambda)\equiv\frac{1}{\ln(\phi_1(\lambda))}=\left(m\lambda+\int_{\mathbb{R}}\left(\mathrm{e}^{-\lambda x}-1+\lambda\kappa(x)\right)d\nu(x)+\frac{\lambda^2s^2}{2}\right)^{-1}$$

satisfies

$$Q(\lambda)>0 \quad \text{eventually, with} \quad \lim_{\lambda\to-\infty}Q(\lambda)=0: \tag{1.52}$$

This follows readily when $\nu((0,\infty))>0$ or $s^2=0$, by observing that

$$\int_{-1}^{0}\left(\mathrm{e}^{-\lambda x}-1+\lambda x\right)d\nu(x)=o(\lambda^2) \quad \text{as} \quad \lambda\to-\infty, \tag{1.53}$$

because

$$\begin{aligned}\int_{-1}^{0}\left(\mathrm{e}^{-\lambda x}-1+\lambda x\right)d\nu(x) &= \left[\frac{\mathrm{e}^{-\lambda x}-1+\lambda x}{x^2}\int_{-1}^{x}y^2d\nu(y)\right]_{-1}^{0}\\ &\quad-\lambda^2\int_0^{-\lambda}\frac{x\,\mathrm{e}^{-x}+x+2\mathrm{e}^{-x}-2}{x^3}\bigg(\int_{-1}^{x/\lambda}y^2d\nu(y)\bigg)dx\\ &\sim\lambda^2\int_{-1}^{0}y^2d\nu(y)\bigg(\frac{1}{2}-\int_0^{\infty}\frac{x\,\mathrm{e}^{-x}+x+2\mathrm{e}^{-x}-2}{x^3}dx\bigg)\end{aligned} \tag{1.54}$$

as $\lambda\to-\infty$, since the second integral on the righ-hand side is $\frac{1}{2}$. Further, when $\nu((0,\infty))=0$ and $s^2=0$, then (1.52) holds since (1.31) ensures that,

$$\lim_{\lambda\to-\infty}\frac{1}{|\lambda|}\int_{-\infty}^{0}\left(\mathrm{e}^{-\lambda x}-1+\lambda\kappa(x)\right)d\nu(x)=\infty. \tag{1.55}$$

As a final preparation, we make the observation that, by the argument (1.18),

$$\lim_{\lambda\to-\infty}\int_{\mathbb{R}}\bigg(\frac{\theta x}{\sigma(\lambda)}-\sin\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\bigg)\mathrm{e}^{-\lambda x}d\nu(x)=0\quad\text{for }\theta\in\mathbb{R}. \tag{1.56}$$

Let $Z_{t,\lambda}$ be a random variable with probability density function

$$f_{Z_{t,\lambda}}(x)=\frac{\mathrm{e}^{-\lambda x}f_{\xi(h-Q(\lambda)t)}(x)}{\phi_{h-Q(\lambda)t}(\lambda)}\quad\text{for }x\in\mathbb{R}\text{ and }t>0,$$

for $\lambda\leq 0$ sufficiently small [recall (1.52)]. Notice that, writing

$$\left\{\begin{array}{rcc} m_{t,\lambda} &=& (h-Q(\lambda)t)\Big(m-\int_{\mathbb{R}}\kappa(x)(1-\mathrm{e}^{-\lambda x})d\nu(x)-\lambda s^2\Big)\\ d\nu_{t,\lambda}(x) &=& (h-Q(\lambda)t)\mathrm{e}^{-\lambda x}d\nu(x)\\ s_{t,\lambda}^2 &=& (h-Q(\lambda)t)s^2\end{array}\right.,$$

the random variable $Z_{t,\lambda}$ has characteristic function

$$\begin{aligned}\mathbf{E}\left\{\mathrm{e}^{i\theta Z_{t,\lambda}}\right\} &= \bigg(\frac{\mathbf{E}\{\mathrm{e}^{(i\theta-\lambda)\xi(1)}\}}{\phi_1(\lambda)}\bigg)^{h-Q(\lambda)t}\\ &=\bigg(\frac{1}{\phi_1(\lambda)}\exp\bigg\{(i\theta-\lambda)m+\int_{\mathbb{R}}\left(\mathrm{e}^{(i\theta-\lambda)x}-1-(i\theta-\lambda)\kappa(x)\right)d\nu(x)\\ &\qquad\qquad-\frac{(i\theta-\lambda)^2s^2}{2}\bigg\}\bigg)^{h-Q(\lambda)t}\\ &=\exp\bigg\{i\theta m_{t,\lambda}+\int_{\mathbb{R}}\left(\mathrm{e}^{i\theta x}-1-i\theta\kappa(x)\right)d\nu_{t,\lambda}(x)-\frac{\theta^2s_{t,\lambda}^2}{2}\bigg\}\end{aligned}$$

for $\theta\in\mathbb{R}$ and $t>0$, for $\lambda\leq 0$ sufficiently small. Hence the random variable $Z_{t,\lambda}$ is infinitely divisible with characteristic triple $\left(\nu_{t,\lambda},m_{t,\lambda},s_{t,\lambda}^2\right)$. Observing that

$$\mathbf{E}\{Z_{t,\lambda}\}=(h-Q(\lambda)t)\mu(\lambda)\equiv\mu_{t,\lambda}$$

it follows that (see e.g., Sato [54], p. 39)

$$\mathbf{E}\left\{\mathrm{e}^{i\theta Z_{t,\lambda}}\right\} = \exp\left\{i\theta\mu_{t,\lambda} + \int_{\mathbb{R}} \left(\mathrm{e}^{i\theta x} - 1 - i\theta x\right) d\nu_{t,\lambda}(x) - \frac{\theta^2 s_{t,\lambda}^2}{2}\right\}.$$

And so the characteristic function $g_{t,\lambda}$ of $(Z_{t,\lambda} - \mu_{t,\lambda})/\sigma(\lambda)$ is given by

$$\begin{aligned} g_{t,\lambda}(\theta) = \Bigg(\exp\Bigg\{ &- \int_{\mathbb{R}} \left(1 - \cos\left(\frac{\theta x}{\sigma(\lambda)}\right)\right) \mathrm{e}^{-\lambda x} d\nu(x) \\ &- i \int_{\mathbb{R}} \left(\frac{\theta x}{\sigma(\lambda)} - \sin\left(\frac{\theta x}{\sigma(\lambda)}\right)\right) \mathrm{e}^{-\lambda x} d\nu(x) - \frac{\theta^2 s^2}{2\sigma(\lambda)^2}\Bigg\}\Bigg)^{h-Q(\lambda)t} \end{aligned}$$

for $\theta \in \mathbb{R}$ and $t > 0$, for $\lambda \leq 0$ sufficiently small. And so (1.45) and (1.56), together with (1.52) and a Taylor expansion, readily give

$$\lim_{\lambda\to-\infty} g_{t,\lambda}(\theta) = \mathrm{e}^{-h\theta^2/2} \quad \text{for } \theta \in \mathbb{R} \text{ and } t > 0.$$

Using this together with (1.46), we get as in (1.26), that

$$\limsup_{\lambda\to-\infty} \sup_{x\in\mathbb{R}} \left| f_{(Z_{t,\lambda}-\mu_{t,\lambda})/\sigma(\lambda)}(x) - \frac{1}{\sqrt{2\pi h}} \mathrm{e}^{-x^2/(2h)} \right| = 0. \tag{1.57}$$

Now observe that

$$f_{(Z_{t,\lambda}-\mu_{t,\lambda})/\sigma(\lambda)}(x) = \frac{\mathrm{e}^{-\lambda(\sigma(\lambda)x+\mu_{t,\lambda})} f_{\xi(h-Q(\lambda)t)}(\mu_{t,\lambda} + \sigma(\lambda)x)\sigma(\lambda)}{\phi_{h-Q(\lambda)t}(\lambda)}$$

for $x \in \mathbb{R}$ and $t \geq 0$. Hence (1.51) together with (1.44) and (1.57) show that

$$\begin{aligned} f_{\xi(h-Q(\lambda)t)}(\mu_{t,\lambda} + x/\lambda) &= \mathrm{e}^x \frac{f_{(Z_{t,\lambda}-\mu_{t,\lambda})/\sigma(\lambda)}(x/(\lambda\sigma(\lambda)))\,\mathrm{e}^{\lambda\mu_{t,\lambda}}\phi_{h-Q(\lambda)t}(\lambda)}{\sigma(\lambda)} \\ &\sim \mathrm{e}^x \frac{\mathrm{e}^{\lambda\mu_{t,\lambda}}\phi_1(\lambda)^{h-Q(\lambda)t}}{\sqrt{2\pi h}\,\sigma(\lambda)} \\ &\sim \mathrm{e}^{x-t} \frac{\mathrm{e}^{h\lambda\mu(\lambda)}\phi_1(\lambda)^h}{\sqrt{2\pi h}\,\sigma(\lambda)} \quad \text{as } \lambda \to -\infty. \end{aligned} \tag{1.58}$$

From this in turn, together with another application of (1.44) and (1.57), we get

$$\begin{aligned} &\lim_{\lambda\to-\infty} \frac{-\lambda\,\mathbf{P}\{\xi(h-Q(\lambda)t) > \mu_{t,\lambda} - y/\lambda\}}{f_{\xi(h-Q(\lambda)t)}(\mu_{t,\lambda} - x/\lambda)} \\ &= \mathrm{e}^x \lim_{\lambda\to-\infty} \frac{-\lambda\,\mathbf{P}\{\xi(h-Q(\lambda)t) > \mu_{t,\lambda} - y/\lambda\}}{f_{\xi(h-Q(\lambda)t)}(\mu_{t,\lambda})} \\ &= \mathrm{e}^x \lim_{\lambda\to-\infty} \int_y^\infty \frac{f_{\xi(h-Q(\lambda)t)}(\mu_{t,\lambda} - z/\lambda)}{f_{\xi(h-Q(\lambda)t)}(\mu(\lambda))} dz \\ &= \mathrm{e}^x \lim_{\lambda\to-\infty} \int_y^\infty \mathrm{e}^{-z} \frac{f_{(Z_{t,\lambda}-\mu_{t,\lambda})/\sigma(\lambda)}(-z/(\lambda\sigma(\lambda)))}{f_{(Z_{t,\lambda}-\mu_{t,\lambda})/\sigma(\lambda)}(0)} dz = \mathrm{e}^{x-y} \quad \text{for } x, y \in \mathbb{R}. \end{aligned} \tag{1.59}$$

Observing that

$$\frac{-\lambda f_{\xi(h-Q(\lambda)t)}(\mu_{t,\lambda}+(Lt-y)/\lambda)}{\mathbf{P}\{\xi(h-Q(\lambda)t)>\mu_{t,\lambda}+Lt/\lambda\}} \quad y\geq 0, \tag{1.60}$$

is a probability density function, (1.59) and the theorem of Scheffé [55] show that

$$\begin{aligned}
&\lim_{\lambda\to-\infty}\frac{\mathbf{P}\{\xi(h-Q(\lambda)t)>\mu_{t,\lambda}+(Lt-x)/\lambda\}}{\mathbf{P}\{\xi(h-Q(\lambda)t)>\mu_{t,\lambda}+Lt/\lambda\}}\\
&\quad=\lim_{\lambda\to-\infty}\int_x^\infty \frac{f_{\xi(h-Q(\lambda)t)}(\mu_{t,\lambda}+(Lt-y)/\lambda)}{-Q(\lambda)\mu(\lambda)\mathbf{P}\{\xi(h-Q(\lambda)t)>\mu_{t,\lambda}+Lt/\lambda\}}\,dy\\
&\quad=\int_x^\infty \mathrm{e}^{-y}dy=\mathrm{e}^{-x} \quad \text{for } x\geq 0.
\end{aligned} \tag{1.61}$$

From this in turn, together with (1.58) and (1.59), we readily obtain

$$\lim_{\lambda\to-\infty}\frac{\mathbf{P}\{\xi(h-Q(\lambda)t)>\mu_{t,\lambda}+(Lt-x)/\lambda\}}{\mathbf{P}\{\xi(h)>\mu_{t,\lambda}+Lt/\lambda\}}=\mathrm{e}^{-x-t} \quad \text{for } x,t\geq 0.$$

As () shows that, given any $\varepsilon>0$,

$$\mu_{t,\lambda}+\frac{Lt+\varepsilon}{\lambda}\leq h\mu(\lambda)\leq \mu_{t,\lambda}+\frac{Lt-\varepsilon}{\lambda} \quad \text{for } \lambda \text{ small enough,}$$

we may conclude from this that

$$\begin{aligned}
&\limsup_{\lambda\to-\infty}\frac{\mathbf{P}\{\xi(h-Q(\lambda)t)>h\mu(\lambda)-x\lambda\}}{\mathbf{P}\{\xi(h)>h\mu(\lambda)\}}\\
&\leq\limsup_{\lambda\to-\infty}\frac{\mathbf{P}\{\xi(h-Q(\lambda)t)>\mu_{t,\lambda}+(Lt+\varepsilon-x)/\lambda\}}{\mathbf{P}\{\xi(h)>\mu_{t,\lambda}+(Lt-\varepsilon)/\lambda\}}\\
&=\limsup_{\lambda\to-\infty}\frac{\mathbf{P}\{\xi(h-Q(\lambda)t)>\mu_{t,\lambda}+(Lt+\varepsilon-x)/\lambda\}}{\mathbf{P}\{\xi(h)>\mu_{t,\lambda}+Lt/\lambda\}}\,\frac{\mathbf{P}\{\xi(h)>\mu_{t,\lambda}+Lt/\lambda\}}{\mathbf{P}\{\xi(h)>\mu_{t,\lambda}+(Lt-\varepsilon)/\lambda\}}\\
&=\mathrm{e}^{2\varepsilon-x-t}\to\mathrm{e}^{-x-t} \quad \text{as } \varepsilon\downarrow 0.
\end{aligned}$$

In a similar fashion

$$\liminf_{\lambda\to-\infty}\frac{\mathbf{P}\{\xi(h-Q(\lambda)t)>h\mu(\lambda)-x\lambda\}}{\mathbf{P}\{\xi(h)>h\mu(\lambda)\}}=\mathrm{e}^{-2\varepsilon-x-t}\to\mathrm{e}^{-x-t} \quad \text{as } \varepsilon\downarrow 0,$$

so that, in fact,

$$\lim_{\lambda\to-\infty}\frac{\mathbf{P}\{\xi(h-Q(\lambda)t)>h\mu(\lambda)-x\lambda\}}{\mathbf{P}\{\xi(h)>h\mu(\lambda)\}}\mathrm{e}^{-x-t} \quad \text{for } x,t\geq 0. \tag{1.62}$$

As μ is continuous and eventually strictly decreasing [by (1.44)], with $\mu(\lambda)\to\infty$ if and only if $\lambda\to-\infty$, we may substitute $\lambda=\mu^{\leftarrow}(u)$ in (1.62), to obtain

$$\lim_{u\to\infty}\frac{\mathbf{P}\{\xi(h-q(hu)t)>hu+xw(hu)\}}{\mathbf{P}\{\xi(h)>hu\}}=\mathrm{e}^{-x-t} \quad \text{for } x,t\geq 0. \tag{1.63}$$

From this in turn, it is a simple matter to establish (1.49). □

Remark 1.22. Given $0 < \hat{h} < h$, it is possible, with just a little extra work, to prove a version of Theorem 1.21, where (1.49) holds uniformly (in an obvious sense) for $t \in [0, (h - \hat{h})/q(u)]$, provided that $\xi(t)$ is absolutely continuous for $t \in [\hat{h}, h]$. As we have no need for this extension, we have selected not to elaborate on it.

Corollary 1.23. *Under the hypothesis of Theorem 1.21, we have*

$$\mathbf{P}\{\xi(h) > u\} \sim \frac{\mathrm{e}^{u\mu^{\leftarrow}(u/h)}\phi_1(\mu^{\leftarrow}(u/h))^h}{\sqrt{2\pi h}\,\sigma(\mu^{\leftarrow}(u/h))(-\mu^{\leftarrow}(u/h))} \quad \textit{as } u \to \infty.$$

Proof. This follows by inspection of (1.58) and (1.59). □

Corollary 1.24. *Under the hypothesis of Theorem 1.21, we have*

$$\lim_{u\to\infty} \frac{q(u)}{w(u)} = 0.$$

Proof. By inspection of (1.48), it is sufficient to prove that

$$\lim_{\lambda\to-\infty} \frac{-\lambda}{\ln(\phi_1(\lambda))} = 0.$$

However, this follows from the arguments we used to establish (1.52). □

To derive sufficient criteria for the conditions in Theorem 1.21, we require the concepts of *regular variation* and *O-regular variation* at 0:

Definition 1.25. *A monotone function $f : (-\infty, 0) \to (0, \infty)$ is* regularly varying *as $x \uparrow 0$ with index $\alpha \in \mathbb{R}$, $f \in \mathcal{R}_{0^-}(\alpha)$, if*

$$\lim_{x\uparrow 0} \frac{f(yx)}{f(x)} = y^\alpha \quad \textit{for } y > 0.$$

Definition 1.26. *A monotone function $f : (-\infty, 0) \to (0, \infty)$ is O-*regularly varying *as $x \uparrow 0$, with Matuszewska indices $-\infty < \alpha \leq \beta < \infty$, $f \in \mathcal{OR}_{0^-}(\alpha, \beta)$, for some constant $x_0 < 0$, to each constant $\varepsilon > 0$, there exists a constant $C \geq 1$, such that*

$$\frac{y^{\beta+\varepsilon}}{C} \leq \frac{f(yx)}{f(x)} \leq C y^{\alpha-\varepsilon} \quad \textit{for } x \in [x_0, 0) \textit{ and } y \in (0,1],$$

and if α *and* β *are the largest and smallest numbers with that property, respectively.*

By Potter's theorem (see e.g., Bingham, Goldie and Teugels [19], Theorem 1.5.6), we have $\mathcal{R}_{0^-}(\alpha) \subseteq \mathcal{OR}_{0^-}(\alpha, \alpha)$ for $\alpha \in \mathbb{R}$.

As the literature is a bit incomplete regarding results for O-regular variation at 0, and it is not trivial how results for O-regular variation at ∞ transfer should be transfered to that at 0, we will prove two lemmas, that are important for us. However, it should be noticed that these lemma are certainly well-known by any expert on regular variation. The first lemma is a version at 0 of the Stieltjes version of Karamata's theorem for one-sided indices at ∞ (see e.g., Bingham, Goldie and Teugels [19], Section 2.6.2).

Lemma 1.27. *For a nondecreasing function* $U \in \mathcal{OR}_{0^-}(\alpha, \beta)$ *with* $-2 < \alpha \leq \beta < 0$, *we have*

$$0 < \liminf_{x \uparrow 0} \frac{1}{x^2 U(x)} \int_x^0 y^2 dU(y) \leq \limsup_{x \uparrow 0} \frac{1}{x^2 U(x)} \int_x^0 y^2 dU(y) < \infty. \quad (1.64)$$

Proof. As

$$\limsup_{x \uparrow 0} x^2 \frac{U(x)}{U(x_0)} \leq \limsup_{x \uparrow 0} \frac{C x^{2+\alpha-\varepsilon}}{x_0^{\beta+\varepsilon}} = 0$$

and

$$\liminf_{x \uparrow 0} x^2 \frac{U(x)}{U(x_0)} \geq \liminf_{x \uparrow 0} \frac{x^{2+\beta+\varepsilon}}{C x_0^{\beta+\varepsilon}} = 0$$

for $\varepsilon > 0$ small enough, we have

$$\lim_{x \uparrow 0} x^2 U(x) = 0.$$

From this in turn, we get the upper bound, noticing that

$$\begin{aligned} \frac{1}{x^2 U(x)} \int_x^0 y^2 dU(y) &= -1 + 2 \int_x^0 \frac{(-y)U(y)}{x^2 U(x)} dy \\ &= -1 + 2 \int_0^1 \frac{z U(zx)}{U(x)} dz \\ &\leq -1 + 2 \int_0^1 C z^{\alpha+1-\varepsilon} dz < \infty \quad \text{for } \varepsilon > 0 \text{ small enough.} \end{aligned}$$

Further, as for $x \in [x_0, 0)$,

$$\limsup_{z\downarrow 0} z \int_z^1 \frac{U(yx)}{U(x)}\,dy \le \limsup_{z\downarrow 0} z \int_z^1 Cy^{\alpha-\varepsilon} dy = \limsup_{z\downarrow 0} \frac{C(z - z^{\alpha+2-\varepsilon})}{\alpha+1-\varepsilon} = 0$$

for $\varepsilon > 0$ small enough, Fatou's Lemma gives

$$\begin{aligned} \liminf_{x\uparrow 0} \frac{1}{x^2U(x)} \int_x^0 y^2 dU(y) &= -1 + 2 \liminf_{x\uparrow 0} \int_x^0 \frac{(-y)U(y)}{x^2U(x)} dy \\ &= -1 + 2 \liminf_{x\uparrow 0} \int_0^1 \frac{zU(zx)}{U(x)} dz \qquad (1.65) \\ &\ge -1 + 2 \int_0^1 \left(\int_z^1 \liminf_{x\uparrow 0} \frac{U(yx)}{U(x)} dy \right) dz. \end{aligned}$$

As $U(yx)/U(x) \ge 1$ is a nondecreasing function of $y \in (0,1)$, the lim inf on the left in (1.64) can be 0 only if

$$\liminf_{x\uparrow 0} \frac{U(yx)}{U(x)} = 1 \quad \text{for } y \in (0,1),$$

as otherwise the right-hand side of (1.65) is strictly greater than

$$-1 + 2 \int_0^1 \left(\int_z^1 dy \right) dz = 0.$$

However, as

$$\liminf_{x\uparrow 0} \frac{U(yx)}{U(x)} \ge \frac{y^{\beta+\varepsilon}}{C} > 1 \quad \text{for } \varepsilon, y > 0 \text{ small enough,}$$

the lim inf on the left in (1.64) must be strictly greater than 0. □

The second lemma is an O-version of the Feller's Tauberian theorem. For O-regular variation at ∞, this result is the so called de Haan-Stadtmüller theorem (see e.g., Bingham, Goldie and Teugels [19], Theorem 2.10.2):

Lemma 1.28. *For a nonincreasing function U such that*

$$\int_{-\infty}^1 e^{-\lambda x} d(-U)(x) < \infty \quad \textit{for } \lambda \textit{ small enough}$$

and $U \in \mathcal{OR}_{0^-}(\alpha, \beta)$ with $0 < \alpha \le \beta < \infty$, we have

$$\begin{aligned} 0 &< \liminf_{\lambda\to-\infty} \frac{1}{U(1/\lambda)} \int_{-\infty}^0 e^{-\lambda x} d(-U)(x) \\ &\le \limsup_{\lambda\to-\infty} \frac{1}{U(1/\lambda)} \int_{-\infty}^0 e^{-\lambda x} d(-U)(x) < \infty. \end{aligned}$$

Proof. As $U(-x/\lambda)/U(1/\lambda) \leq 1$ with

$$\frac{U(-x/\lambda)}{U(1/\lambda)} \leq 1 \quad \text{and} \quad \frac{U(-x/\lambda)}{U(1/\lambda)} \leq C(-x)^{\alpha-\varepsilon} < 1 \quad \text{for } x \in [-1,0) \tag{1.66}$$

and λ small enough, we have

$$\begin{aligned}\liminf_{\lambda\to-\infty} \frac{1}{U(1/\lambda)} \int_{-\infty}^{0} \mathrm{e}^{-\lambda x} d(-U)(x) &\geq \liminf_{\lambda\to-\infty} \frac{1}{U(1/\lambda)} \int_{1/\lambda}^{0} \mathrm{e}^{-\lambda x} d(-U)(x) \\ &= \mathrm{e}^{-1} - \limsup_{\lambda\to-\infty} \int_{-1}^{0} \mathrm{e}^{x} \frac{U(-x/\lambda)}{U(1/\lambda)} dx > 0\end{aligned}$$

[cf. the concluding argument for (1.64) in the proof of Lemma 1.64].

In the other direction, using the easily established fact that

$$\lim_{\lambda\to-\infty} \frac{\mathrm{e}^{\lambda/2}}{U(1/\lambda)} = 0,$$

we obtain

$$\begin{aligned}&\frac{1}{U(1/\lambda)} \int_{-\infty}^{0} \mathrm{e}^{-\lambda x} d(-U)(x) \\ &\quad = \frac{\mathrm{e}^{\lambda} U(-1)}{U(1/\lambda)} - \int_{-1}^{0} \mathrm{e}^{-\lambda x} \frac{U(x)}{U(1/\lambda)} dx + \frac{\mathrm{e}^{\lambda/2}}{U(1/\lambda)} \int_{-\infty}^{1} \mathrm{e}^{-\lambda x/2} d(-U)(x) \\ &\quad \leq \frac{\mathrm{e}^{\lambda} U(-1)}{U(1/\lambda)} + \frac{\mathrm{e}^{\lambda/2}}{U(1/\lambda)} \int_{-\infty}^{1} \mathrm{e}^{-\lambda x/2} d(-U)(x) \to 0 \quad \text{as } \lambda \to -\infty. \quad \square\end{aligned}$$

We are now prepared to prove the key Proposition 1.29, for verification of the conditions of Theorem 1.21. This proposition is of crucial importance for our applications to superexponential processes.

Proposition 1.29. *Let $\{\xi(t)\}_{t\geq 0}$ be a superexponential Lévy process, with characteristic triple (ν, m, s^2), and infinite upper end-point (1.42). We have the following implications:*

1. *If $s^2 > 0$, then (1.44) and (1.46) hold;*
2. *if $s^2 > 0$ and $\nu((0,\infty)) = 0$, then (1.44)-(1.47) hold;*
3. *if $\nu((0,\infty)) > 0$, then (1.44) and (1.47) hold;*

4. *if* $\nu((0,\infty)) > 0$ *and there exists a non-decreasing function* G *such that*

$$\lim_{x\to\infty}\frac{G(x)}{\ln(x)} = \infty \quad \textit{and} \quad \int_1^\infty \exp\{G(x^2)x\}d\nu(x) < \infty, \tag{1.67}$$

then (1.44), (1.45) and (1.47) hold;

5. *if*

$$\nu((-\infty,\cdot)) \in \mathcal{OR}_{0^-}(\alpha,\beta) \quad \textit{for some} \ -2 < \alpha \le \beta < 0, \tag{1.68}$$

then (1.44) holds;

6. *if* $\nu((0,\infty)) = 0$ *and (1.44) holds, then (1.45) holds;*

7. *if* $\nu((0,\infty)) = 0$ *and (1.68) holds, then (1.44)-(1.46) hold;*

8.

9. *if* ξ *is selfdecomposable, then (1.44) and (1.45) hold;*

10. *if* $\nu((0,\infty)) = 0$ *and*

$$d\nu(x) = \frac{k(x)dx}{|x|^2} \quad \textit{where } k : (-\infty,0) \to [0,\infty) \textit{ is non-decreasing,} \tag{1.69}$$

then (1.44)-(1.46) hold;

11. *if* $\nu((0,\infty)) = 0$ *and*

$$\nu((-\infty,\cdot)) \in \mathcal{R}_{0^-}(\alpha) \quad \textit{for some} \ -2 < \alpha < -1, \tag{1.70}$$

then (1.44)-(1.47) hold.

Proof. Statement 1 of the proposition is quite immediate.

To prove Statement 2, we notice that

$$\limsup_{\lambda\to-\infty}\int_{|x|>\varepsilon\sigma(\lambda)}\frac{x^2}{\sigma(\lambda)^2}\,\mathrm{e}^{-\lambda x}d\nu(x) \le \limsup_{\lambda\to-\infty}\frac{1}{s^2}\int_{-\infty}^0 x^2\mathrm{e}^{-\lambda x}d\nu(x) = 0, \tag{1.71}$$

when $s^2 > 0$ and $\nu((0,\infty)) = 0$ [because $\int 1\wedge x^2 d\nu(x) < \infty$], so that (1.46) holds. In view of Statement 1, it remains to prove (1.47). To that end, in turn, it is sufficient to show that the limit

$$\lim_{\lambda\to-\infty}\frac{\ln(\phi_1(\lambda))}{\lambda\mu(\lambda)}=\lim_{\lambda\to-\infty}\frac{\displaystyle\int_{\mathbb{R}}\big(\mathrm{e}^{-\lambda x}-1+\lambda\kappa(x)\big)\,d\nu(x)+m\lambda-\frac{\lambda s^2}{2}}{\displaystyle\int_{\mathbb{R}}\big(\lambda x\,\mathrm{e}^{-\lambda x}-\lambda\kappa(x)\big)\,d\nu(x)+m\lambda-\lambda^2 s^2}=\tilde{L}\neq-1 \tag{1.72}$$

exists. As it is obvious that

$$\int_{-\infty}^{-1}(\mathrm{e}^{-\lambda x}-1+\lambda\kappa(x))d\nu(x)=O(\lambda)\quad\text{and}\quad\int_{-\infty}^{-1}(\lambda x\ \mathrm{e}^{-\lambda x}-\lambda\kappa(x))d\nu(x)=O(\lambda)$$

as $\lambda\to-\infty$, (1.72) in turn, with $\tilde{L}=\frac{1}{2}$, will follow provided that we can prove

$$\int_{-1}^{0}\big(\mathrm{e}^{-\lambda x}-1+\lambda x\big)\,d\nu(x)=o(\lambda^2)\quad\text{and}\quad\int_{-1}^{0}\big(\lambda x\,\mathrm{e}^{-\lambda x}-\lambda x\big)\,d\nu(x)=o(\lambda^2)$$

as $\lambda\to-\infty$. The first of these asymptotic relations is established in (1.54). The second asymptotic relation follows in a simlar fashion, noticing that

$$\begin{aligned}\int_{-1}^{0}\big(\lambda\mathrm{e}^{-\lambda x}-\lambda\big)\,d\nu(x)&=\left[\frac{\lambda x\,\mathrm{e}^{-\lambda x}-\lambda x}{x}\int_{-1}^{x}y^2d\nu(y)\right]_{-1}^{0}\\&\quad-\lambda^2\int_0^{-\lambda}\frac{x\,\mathrm{e}^{-x}+\mathrm{e}^{-x}-1}{x^2}\left(\int_{-1}^{x/\lambda}y^2d\nu(y)\right)dx\\&\sim\lambda^2\int_{-1}^{0}y^2d\nu(y)\left(-1-\int_0^{\infty}\frac{x\,\mathrm{e}^{-x}+\mathrm{e}^{-x}-1}{x^2}dx\right)\end{aligned}$$

as $\lambda\to-\infty$, since the second integral on the righ-hand side is -1.

To prove Statement 3, we assume that $\nu((0,\infty))>0$. Then (1.53) readily gives (1.44) holds. Further, by inspection of the proof of Statement 2, (1.47) will follow, with $L=1$, provided that we can show that

$$\lim_{\lambda\to-\infty}\frac{\displaystyle\int_0^{\infty}\big(\mathrm{e}^{-\lambda x}-1+\lambda\kappa(x)\big)\,d\nu(x)}{\displaystyle\int_0^{\infty}\big(\lambda\kappa(x)-\lambda x\,\mathrm{e}^{-\lambda x}\big)\,d\nu(x)}=0$$

and

$$\lim_{\lambda\to-\infty}\frac{1}{\lambda^2}\int_0^{\infty}(\mathrm{e}^{-\lambda x}-1+\lambda\kappa(x))d\nu(x)=\lim_{\lambda\to-\infty}\frac{1}{\lambda^2}\int_0^{\infty}(\lambda\kappa(x)-\lambda x\,\mathrm{e}^{-\lambda x})d\nu(x)=\infty.$$

However, both these requirements are quite obvious consequences of the elementary fact that

$$\int_0^1\big(\lambda\kappa(x)-\lambda x\,\mathrm{e}^{-\lambda x}\big)\,d\nu(x)\geq\int_0^1\big(\mathrm{e}^{-\lambda x}-1+\lambda\kappa(x)\big)\,d\nu(x)\geq 0.$$

To prove Statement 4, assume that (1.67) holds with $\nu((0,\infty)) > 0$. As we must have $\nu((\underline{x},\infty)) > 0$ for some $\underline{x} > 0$, (1.67) gives

$$\begin{aligned}\liminf_{\lambda\to-\infty} \frac{G(\varepsilon^2\sigma(\lambda)^2)}{-\lambda} &\geq \liminf_{\lambda\to-\infty} \frac{1}{-\lambda} G\left(\varepsilon^2 \int_{\underline{x}}^{\infty} x^2 \mathrm{e}^{-\lambda x} d\nu(x)\right) \\ &\geq \liminf_{\lambda\to-\infty} \frac{1}{-\lambda} G\left(\varepsilon^2 \underline{x}^2 \nu((\underline{x},\infty)) \mathrm{e}^{-\lambda \underline{x}}\right) = \infty \quad \text{for } \varepsilon > 0.\end{aligned}$$

From this in turn, we readily obtain, making use of (1.67) again [see also (1.71)],

$$\begin{aligned}&\limsup_{\lambda\to-\infty} \int_{|x|>\varepsilon\sigma(\lambda)} \frac{x^2}{\sigma(\lambda)^2} \mathrm{e}^{-\lambda x} d\nu(x) \\ &\leq \limsup_{\lambda\to-\infty} \frac{1}{\sigma(\lambda)^2} \int_{-\infty}^{0} x^2 \mathrm{e}^{-\lambda x} d\nu(x) + \sup_{x<0} x^2 \mathrm{e}^{x} \limsup_{\lambda\to-\infty} \frac{1}{\lambda^2\sigma(\lambda)^2} \int_{|x|>\varepsilon\sigma(\lambda)} \mathrm{e}^{-2\lambda x} d\nu(x) \\ &\leq 0 + \sup_{x<0} x^2 \mathrm{e}^{x} \int_{1}^{\infty} \exp\{G(x^2)x\} d\nu(x) \limsup_{\lambda\to-\infty} \sup_{x>\varepsilon\sigma(\lambda)} \exp\left\{-2\lambda x - G(x^2)x\right\} = 0\end{aligned}$$

for $\varepsilon > 0$. Hence (1.45) holds. The statement now follows from Statement 3.

To prove Statement 5, assume that (1.68) holds. By Lemma 1.27, we have

$$\begin{aligned}0 < \frac{1}{C_1} &\leq \liminf_{x\uparrow 0} \frac{1}{x^2\nu((-\infty,x))} \int_x^0 y^2 d\nu(y) \\ &\leq \limsup_{x\uparrow 0} \frac{1}{x^2\nu((-\infty,x))} \int_x^0 y^2 d\nu(y) \leq C_1 < \infty\end{aligned} \tag{1.73}$$

for some constant $C_1 \geq 1$. As this shows that also

$$\int_{\cdot}^{0} y^2 d\nu(y) \in \mathcal{OR}_{0^-}(\alpha+2,\beta+2),$$

Lemma 1.28 now in turn gives

$$\begin{aligned}0 < \frac{1}{C_2} &\leq \liminf_{\lambda\to-\infty} \left(\int_{1/\lambda}^{0} y^2 d\nu(y)\right)^{-1} \int_{-\infty}^{0} \mathrm{e}^{-\lambda x} d\left(-\int_x^0 y^2 d\nu(y)\right) \\ &\leq \limsup_{\lambda\to-\infty} \left(\int_{1/\lambda}^{0} y^2 d\nu(y)\right)^{-1} \int_{-\infty}^{0} \mathrm{e}^{-\lambda x} d\left(-\int_x^0 y^2 d\nu(y)\right) \leq C_2 < \infty\end{aligned}$$

for some constant $C_2 \geq 1$. And so we may deduce (1.44) in the following manner:

$$\begin{aligned}\liminf_{\lambda\to-\infty} \lambda^2\sigma(\lambda)^2 &\geq \liminf_{\lambda\to-\infty} \lambda^2 \int_{-\infty}^{0} x^2 \mathrm{e}^{-\lambda x} d\nu(x) \\ &= \liminf_{\lambda\to-\infty} \lambda^2 \int_{-\infty}^{0} \mathrm{e}^{-\lambda x} d\left(-\int_x^0 y^2 d\nu(y)\right) \\ &\geq \frac{1}{C_2} \liminf_{\lambda\to-\infty} \lambda^2 \int_{1/\lambda}^{0} y^2 d\nu(y) \\ &\geq \frac{1}{C_1C_2} \liminf_{\lambda\to-\infty} \nu((-\infty,1/\lambda)) = \infty\end{aligned} \tag{1.74}$$

[using (1.50) for the last step].

To prove Statement 6, we assume that (1.44) holds. As $-\varepsilon\sigma(\lambda) < 1/\lambda$ for λ small enough, we get (1.45) in the following manner:

$$\limsup_{\lambda\to-\infty} \int_{-\infty}^{-\varepsilon\sigma(\lambda)} \frac{x^2}{\sigma(\lambda)^2}\,\mathrm{e}^{-\lambda x} d\nu(x)$$
$$\leq \Big(\sup_{x<0} x^2 \mathrm{e}^{x/2}\Big) \limsup_{\lambda\to-\infty} \mathrm{e}^{\varepsilon\lambda\sigma(\lambda)/2} \nu((-\infty,-\varepsilon\sigma(\lambda)))\Big/\Big(\mathrm{e}^{-1}\int_{1/\lambda}^{0} x^2 d\nu(x)\Big) = 0$$

for $\varepsilon > 0$.

To prove Statement 7, assume that (1.68) holds with $\nu((0,\infty)) = 0$. In view of Statements 5 and 6, it is enough to prove that (1.46) holds. Notice that, since $\nu((0,\infty)) = 0$, the arguments that were use to establish (1.74) carry over to show that

$$\frac{1}{C_1C_2} \leq \liminf_{\lambda\to-\infty} \frac{\nu((-\infty,1/\lambda))}{\lambda^2\sigma(\lambda)^2} \leq \limsup_{\lambda\to-\infty} \frac{\nu((-\infty,1/\lambda))}{\lambda^2\sigma(\lambda)^2} \leq C_1C_2. \tag{1.75}$$

Now, for the proof of (1.46), since $1-\cos(x) \geq \frac{1}{4}x^2$ for $|x| \leq 1$, we have, by (1.73) together with (1.75),

$$\int_{\mathbb{R}} \Big(1-\cos\Big(\frac{\theta x}{\sigma(\lambda)}\Big)\Big)\,\mathrm{e}^{-\lambda x} d\nu(x)$$
$$\geq \frac{\theta^2}{4\,\mathrm{e}\,\sigma(\lambda)^2} \int_{\max\{-\sigma(\lambda)/|\theta|,1/\lambda\}}^{0} x^2 d\nu(x)$$
$$\geq \frac{1}{8C_1\mathrm{e}} \min\Big\{\nu((-\infty,-\sigma(\lambda)/|\theta|)), \frac{\nu((-\infty,1/\lambda))\,\theta^2}{\lambda^2\sigma(\lambda)^2}\Big\}$$
$$\geq \frac{1}{8C_1\mathrm{e}} \min\Big\{\frac{|\theta|^{-\beta-\varepsilon}\nu((-\infty,-\sigma(\lambda)))}{C}, \frac{\theta^2}{2C_1C_2}\Big\}$$

for $|\theta| > 1$ and λ small enough. As $\lim_{\lambda\to-\infty} \nu((-\infty,-\sigma(\lambda))) = \infty$ [recall (1.50)], since $\lim_{\lambda\to-\infty} \sigma(\lambda) = 0$, this establishes (1.46).

To prove Statement 8 we assume that ξ is selfdecomposable. By Statements 1 and 3, we may assume that $\nu((0,\infty)) = 0$ and $s^2 = 0$. Notice that it is enough to prove (1.44), as Statement 6 then gives (1.45). By selfdecomposability, we have

$$d\nu(x) = \frac{k(x)}{|x|} \quad \text{where } k : (-\infty,0) \to [0,\infty) \text{ is non-decreasing}, \tag{1.76}$$

(see e.g., Sato, [54], Corollary 15.11). From (1.50), we get in addition that

$\lim_{x\uparrow 0} k(x) = \infty$. And so, with obvious notation, we get (1.44) as follows:

$$\begin{aligned}
&\liminf_{\lambda\to-\infty} \lambda^2\sigma(\lambda)^2 \\
&\qquad \geq \liminf_{\lambda\to-\infty} \int_{1/\lambda}^{0} \lambda^2 x^2 \mathrm{e}^{-\lambda x} d\nu(x) \\
&\qquad \geq \frac{1}{\mathrm{e}} \liminf_{\lambda\to-\infty} \int_{1/\lambda}^{0} \lambda^2 (-x) k(x) dx \\
&\qquad \geq \frac{1}{\mathrm{e}} \liminf_{\lambda\to-\infty} \lambda^2(-(0-\varepsilon)) \int_{1/\lambda}^{0-\varepsilon} k(y)dy + \mathrm{e}^{-1} \liminf_{\lambda\to-\infty} \lambda^2 \int_{1/\lambda}^{0} \Big(\int_{1/\lambda}^{x} k(y)dy \Big) dx \\
&\qquad \geq \frac{1}{2\,\mathrm{e}} \liminf_{\lambda\to-\infty} k(1/\lambda) = \infty.
\end{aligned}$$

To prove Statement 9 we take $\nu((0,\infty)) = 0$, and assume that (1.69) holds. By Statement 1, we may assume that $s^2 = 0$. Further, as (1.69) implies (1.76), ξ is selfdecomposable. Therefore Statement 8 gives (1.45). Notice that

$$\begin{aligned}
\frac{d}{dx} \frac{1}{-x} \int_x^0 y^2 \mathrm{e}^{-\lambda y} d\nu(y) &= \frac{1}{x^2} \int_x^0 \mathrm{e}^{-\lambda y} k(y) dy + \frac{1}{x} \mathrm{e}^{-\lambda x} k(x) \\
&= \frac{1}{x^2} \int_x^0 (-y) \frac{d}{dy} \left(\mathrm{e}^{-\lambda y} k(y) \right) dy \geq 0.
\end{aligned} \tag{1.77}$$

It is now an easy matter to finish off the proof: Since $1 - \cos(x) \geq \frac{1}{4}x^2$ for $|x| \leq 1$, we get (1.46), noticing that (1.77) together with (1.45) give

$$\begin{aligned}
\int_{\mathbb{R}} \Big(1 - \cos\Big(\frac{\theta x}{\sigma(\lambda)} \Big) \Big) \mathrm{e}^{-\lambda x} d\nu(x) &\geq \int_{-\sigma(\lambda)/|\theta|}^{0} \frac{\theta^2 x^2}{4\sigma(\lambda)^2} \mathrm{e}^{-\lambda x} d\nu(x) \\
&\geq |\theta| \int_{-\sigma(\lambda)}^{0} \frac{x^2}{4\sigma(\lambda)^2} \mathrm{e}^{-\lambda x} d\nu(x) \\
&\geq \frac{|\theta|}{8} \quad \text{for } \lambda \text{ small enough, for } |\theta| \geq 1.
\end{aligned}$$

To prove Statement 10, assume that $\nu((0,\infty)) = 0$ and that (1.70) holds. In view of Statement 2, we may further assume that $s^2 = 0$. By (1.86) below, we have

$$\int_{-\infty}^{0} \left(\mathrm{e}^{-\lambda x} - 1 + \lambda\kappa(x) \right) d\nu(x) \sim -\Gamma(1+\alpha)\nu((-\infty, 1/\lambda)) \quad \text{as } \lambda \to -\infty. \tag{1.78}$$

Moreover, by (1.85) below, together with Feller's Tauberian theorem (see e.g., Bingham, Goldie and Teugels [19], Theorem 1.7.1′), we have

$$\begin{aligned}
&\int_{-\infty}^{0} \big(\lambda x\, \mathrm{e}^{-\lambda x} - \lambda\kappa(x)\big)\, d\nu(x) \\
&= \int_{-1}^{0} \lambda x\, (\mathrm{e}^{-\lambda x} - 1) d\nu(x) + \int_{-\infty}^{-1} \big(\lambda x\, \mathrm{e}^{-\lambda x} + \lambda\big)\, d\nu(x) \\
&= \lambda(\mathrm{e}^{\lambda} - 1)\nu((-\infty,-1)) + \int_{-1}^{0} \big((\lambda^2 x - \lambda)\,\mathrm{e}^{-\lambda x} + \lambda\big)\, \nu((-\infty,x))dx + O(\lambda) \\
&= \lambda^2 \int_{-1}^{0} (\lambda x - 2)\,\mathrm{e}^{-\lambda x} d\Big(\int_{x}^{0}\Big(\int_{-1}^{y} \nu((-\infty,z))dz\Big)dy\Big) + O(\lambda) \\
&\sim \lambda^3 \frac{(1/\lambda)^3 \Gamma(4+\alpha)\nu((-\infty,-1/\lambda))}{-(\alpha+1)(2+\alpha)} - 2\lambda^2 \frac{(1/\lambda)^2 \Gamma(3+\alpha)\nu((-\infty,-1/\lambda))}{-(\alpha+1)(2+\alpha)} \\
&= -\Gamma(2+\alpha)\nu((-\infty,-1/\lambda)) \quad \text{as } \lambda \to -\infty,
\end{aligned} \tag{1.79}$$

because $\lim_{\lambda\to-\infty} \nu((-\infty,1/\lambda))/(-\lambda) = \infty$, since $\alpha < -1$. Now, putting (1.78) and (1.79) together, we see that (1.72) holds, with $\tilde{L} = 1 + \alpha$. □

As ξ is selfdecomposable if and only if (1.76) holds, and taking (1.50) in to account, it seems fair to expect that most selfdecomposable processes with infinite upper end-point (1.42), that have $\nu((0,\infty)) = 0$ and $s^2 = 0$, should satisfy (1.69).

Proposition 1.30. *Let $\{\xi(t)\}_{t\geq 0}$ be a superexponential Lévy process with characteristic triple (ν, m, s^2), and infinite upper end-point (1.42). With the notation (1.48), we have the following implications:*

1. *If $\nu((0,\infty)) > 0$, then (with obvious notation)*
$$\frac{\xi(aq(u))}{w(u)} \xrightarrow{d} 0 \quad \text{as } u \to \infty; \tag{1.80}$$
2. *if $\nu((0,\infty)) = 0$ and $s^2 > 0$, then*
$$\frac{\xi(aq(u))}{w(u)} \xrightarrow{d} N(0,2a) \quad \text{as } u \to \infty \quad \text{for } a > 0; \tag{1.81}$$
3. *if $\nu((0,\infty)) = 0$ and $s^2 = 0$, and (1.70) holds, then*
$$\frac{\xi(aq(u))}{w(u)} \xrightarrow{d} S_{-\alpha}\Big((a\cos(-\frac{\pi\alpha}{2}))^{-1/\alpha}, -1, 0\Big) \quad \text{as } u \to \infty \quad \text{for } a > 0. \tag{1.82}$$

Proof. We have weak convergence of $\xi(aq(u))/w(u)$ to a random variable X as

$u \to \infty$, if and only if we have convergence of the Laplace transform

$$
\begin{aligned}
&\lim_{u\to\infty} \mathbf{E}\Big\{\exp\Big[-t\,\frac{\xi(aq(u))}{w(u)}\Big]\Big\}\\
&= \lim_{u\to\infty} \phi_1(t/w(u))^{aq(u)}\\
&= \lim_{u\to\infty} \exp\{aq(u)\ln(\phi_1(t/w(u)))\}\\
&= \lim_{u\to\infty} \exp\left\{\frac{a\ln(\phi_1(-t\mu^{\leftarrow}(u/h)))}{-(u/h)\mu^{\leftarrow}(u/h)-\ln(\phi_1(\mu^{\leftarrow}(u/h)))}\right\}\\
&= \lim_{\lambda\to-\infty} \exp\left\{\frac{a\ln(\phi_1(-t\lambda))}{-\lambda\mu(\lambda)-\ln(\phi_1(\lambda))}\right\}\\
&= \lim_{\lambda\to-\infty} \exp\left\{a\,\frac{\int_{-\infty}^0 (\mathrm{e}^{t\lambda x}-1-t\lambda\kappa(x))\,d\nu(x)+\int_0^\infty (\mathrm{e}^{t\lambda x}-1-t\lambda\kappa(x))\,d\nu(x)+mt\lambda+\frac{t^2\lambda^2s^2}{2}}{\int_{-\infty}^0 \big(1-\mathrm{e}^{-\lambda x}(1+\lambda x)\big)\,d\nu(x)+\int_0^\infty \big(\mathrm{e}^{-\lambda x}((-\lambda)x-1)+1\big)\,d\nu(x)+\frac{\lambda^2s^2}{2}}\right\}\\
&= \mathbf{E}\{\mathrm{e}^{-tX}\}\quad \text{for } t\in(-1,0)
\end{aligned}
\tag{1.83}
$$

(see e.g., Hoffmann-Jørgensen [42], pp. 377-378).

To prove Statement 1, notice that, by arguing as for 1.47) in Statement 3 of Proposition 1.29, the limit in (1.83) is 1 when $\nu((0,\infty)) > 0$, implying weak convergence to a degenerate random variable $X = 0$.

To prove Statement 2, notice that, by arguing as for (1.47) in Statement 2 of Proposition 1.29, the limit in (1.83) is e^{at^2} when $\nu((0,\infty)) = 0$ and $s^2 > 0$, implying weak convergence to a normal $\mathrm{N}(0, 2a)$ distributed random variable X.

To prove Statement 3, assume that $\nu((0,\infty)) = 0$ and $s^2 = 0$. Notice that, by Karamata's theorem (see e.g., Bingham, Goldie and Teugels [19], Section 1.5.6),

$$
-\int_x^0 y\nu((-\infty,y))dy \sim \frac{x^2\nu((-\infty,x))}{2+\alpha} \in \mathcal{R}_{0^-}(2+\alpha) \quad \text{as } x\uparrow 0.
$$

Hence Feller's Tauberian theorem (see e.g., Bingham, Goldie and Teugels [19], Theorem 1.7.1′), gives

$$
\begin{aligned}
\int_{-\infty}^0 \big(1-\mathrm{e}^{-\lambda x}(1+\lambda x)\big)\,d\nu(x) &= \int_{-\infty}^0 \lambda^2\mathrm{e}^{-\lambda x}d\Big(-\int_x^0 y\nu((-\infty,y))dy\Big)\\
&\sim \Gamma(2+\alpha)\nu((-\infty,1/\lambda)) \quad \text{as } \lambda\to-\infty.
\end{aligned}
\tag{1.84}
$$

Moreover, using Karamata's theorem again, we get

$$
\int_x^0 \Big(\int_{-1}^y \nu((-\infty,z))dz\Big)dy \sim \frac{x^2\nu((-\infty,x))}{-(\alpha+1)(2+\alpha)} \in \mathcal{R}_{0^-}(2+\alpha) \quad \text{as } x\uparrow 0, \tag{1.85}
$$

so that, by Feller's Tauberian theorem,

$$\begin{aligned}
&\int_{-\infty}^{0} \big(\mathrm{e}^{t\lambda x} - 1 - t\lambda\kappa(x)\big)\, d\nu(x) \\
&\quad = \int_{-\infty}^{-1} \mathrm{e}^{t\lambda x} d\nu(x) - \mathrm{e}^{-t\lambda}\nu((-\infty,-1)) + \int_{-1}^{0} \big(t\lambda - \mathrm{e}^{t\lambda x} t\lambda\big)\, \nu((-\infty,x))dx \\
&\quad = o(1) + (t\lambda)^2 \int_{-1}^{0} \mathrm{e}^{t\lambda x} d\bigg(\int_{x}^{0}\bigg(\int_{-1}^{y} \nu((-\infty,z))dz\bigg) dy\bigg) \\
&\quad \sim \frac{\Gamma(2+\alpha)\nu((-\infty,-1/(t\lambda)))}{-(\alpha+1)} \\
&\quad \sim -(-t)^{-\alpha}\Gamma(1+\alpha)\nu((-\infty,1/\lambda)) \quad \text{as } \lambda\to-\infty \quad \text{for } t\in[-1,0).
\end{aligned} \tag{1.86}$$

As $\lim_{\lambda\to-\infty}\nu((-\infty,1/\lambda))/(-\lambda) = \infty$, since $\alpha < -1$, it follows readily that the limit in (1.83) is $\mathrm{e}^{-a(-t)^{-\alpha}}$. This is the Laplace transform of the $-\alpha$-stable distribution at the right-hand side of (1.82) (see e.g., Samorodnitsky and Taqqu [53], proposition 1.2.12). □

As have been noted in the proof of Proposition 1.29, we must have $\alpha \le -1$ in (1.68), when $\nu((0,\infty)) = 0$ and $s^2 = 0$, to get an infinite upper end-point (1.42). The case when $\alpha = -1$ was not covered in Proposition 1.30, and does in fact turn out to behave in a qualitatively different way from that with $\alpha < -1$:

Proposition 1.31. *Let $\{\xi(t)\}_{t\ge0}$ be a superexponential Lévy process with characteristic triple $(\nu, m, 0)$, and infinite upper end-point (1.42). Assume that $\nu((0,\infty)) = 0$, and that*

$$\nu((-\infty,\cdot)) \in \mathcal{R}_{0^-}(-1). \tag{1.87}$$

Denoting

$$w(u) = -\frac{1}{\mu^{\leftarrow}(u/h)} \quad \text{and} \quad q(u) = \frac{1}{\ln(\phi_1(\mu^{\leftarrow}(u/h)))},$$

we have

$$\lim_{u\to\infty} \frac{\mathbf{P}\{\xi(h-q(u)t) > u + xw(u)\}}{\mathbf{P}\{\xi(h) > u\}} = \begin{cases} \mathrm{e}^{-x} & \text{for } x\in\mathbb{R} \text{ and } t=0, \\ 0 & \text{for } x\in\mathbb{R} \text{ and } t>0. \end{cases} \tag{1.88}$$

and

$$\frac{\xi(aq(u))}{w(u)} \xrightarrow{d} a \quad \textit{as } u\to\infty \quad \textit{for } a>0. \tag{1.89}$$

Proof. Under the hypothesis of the proposition, we still have (1.84), with $\alpha=-1$. However, by so called de Haan theory (see e.g., Bingham, Goldie and Teugels [19], Proposition 1.5.9a), (1.85) changes to

$$\int_{\cdot}^{0}\biggl(\int_{-1}^{y}\nu((-\infty,z))dz\biggr)dy \in \mathcal{R}_{0^-}(1) \tag{1.90}$$

with

$$\lim_{x\uparrow 0}\frac{1}{x^2\nu((-\infty,x))}\int_x^0\biggl(\int_{-1}^{y}\nu((-\infty,z))dz\biggr)dy=\infty. \tag{1.91}$$

And so, by Feller's Tauberian theorem, the corresponding modification of (1.86) becomes

$$\begin{aligned}\int_{-\infty}^{0}\bigl(\mathrm{e}^{-\lambda x}-1+\lambda\kappa(x)\bigr)\,d\nu(x) &= o(1)+\lambda^2\int_{-1}^{0}\mathrm{e}^{-\lambda x}d\biggl(\int_x^0\biggl(\int_{-1}^{y}\nu((-\infty,z))dz\biggr)dy\biggr)\\ &\sim \Gamma(2)\lambda^2\int_{1/\lambda}^{0}\biggl(\int_{-1}^{y}\nu((-\infty,z))dz\biggr) \quad \text{as } \lambda\to-\infty,\end{aligned} \tag{1.92}$$

which is regularly varying, by (1.90). Since (1.50) shows that

$$\lim_{\lambda\to-\infty}\frac{1}{(-\lambda)}\int_{-\infty}^{0}\bigl(\mathrm{e}^{-\lambda x}-1+\lambda\kappa(x)\bigr)\,d\nu(x)=\infty,$$

we now readily get (1.89) in the following manner:

$$\begin{aligned}\lim_{u\to\infty}\mathbf{E}\biggl\{\exp\biggl[-t\,\frac{\xi(aq(u))}{w(u)}\biggr]\biggr\} &= \lim_{u\to\infty}\exp\biggl\{\frac{a\ln(\phi_1(-t\mu^{\leftarrow}(u/h)))}{\ln(\phi_1(\mu^{\leftarrow}(u/h)))}\biggr\}\\ &= \lim_{\lambda\to-\infty}\exp\biggl\{\frac{a\ln(\phi_1(-t\lambda))}{\ln(\phi_1(\lambda))}\biggr\}\\ &= \lim_{\lambda\to-\infty}\exp\left\{a\,\frac{\int_{-\infty}^{0}\bigl(\mathrm{e}^{t\lambda x}-1-t\lambda\kappa(x)\bigr)\,d\nu(x)+mt\lambda}{\int_{-\infty}^{0}\bigl(\mathrm{e}^{-\lambda x}-1+\lambda\kappa(x)\bigr)\,d\nu(x)-m\lambda}\right\}\\ &= \lim_{\lambda\to-\infty}\exp\left\{a\,\frac{(-t)\int_{-\infty}^{0}\bigl(\mathrm{e}^{-\lambda x}-1+\lambda\kappa(x)\bigr)\,d\nu(x)}{\int_{-\infty}^{0}\bigl(\mathrm{e}^{-\lambda x}-1+\lambda\kappa(x)\bigr)\,d\nu(x)}\right\}\\ &= \mathrm{e}^{-at} \quad \text{for } t\in(-1,0).\end{aligned}$$

Changing the definition of Q to $Q(\lambda)=1/\ln(\phi_1(\lambda))$ in the proof of Theorem (1.21), that proof still goes through, in essence. The only important change is that, since

$$\lim_{\lambda\to-\infty}\frac{-\lambda\mu(\lambda)-\ln(\phi_1(\lambda))}{\ln(\phi_1(\lambda))}=0,$$

by (1.84) together with (1.91) and (1.92) [recall that $\lim_{\lambda\to-\infty}\nu((-\infty,1/\lambda))/(-\lambda) = \infty$], (1.58) changes to

$$f_{\xi(h-Q(\lambda)t)}(\mu_t(\lambda)+x/\lambda)\sim \mathrm{e}^{x-t\ln(\phi_1(\lambda))/(-\lambda\mu(\lambda)-\ln(\phi_1(\lambda)))}\,\frac{\mathrm{e}^{h\lambda\mu(\lambda)}\phi_1(\lambda)^h}{\sqrt{2\pi h}\,\sigma(\lambda)}\quad\text{as }\lambda\to-\infty.$$

However, this does not affect the validity of the statements (1.59)-(1.61), while (1.62) and (1.63) change to

$$\lim_{u\to\infty}\frac{\mathbf{P}\{\xi(h-q(hu)t)>hu-xw(hu)\}}{\mathbf{P}\{\xi(h)>hu\}}=\begin{cases}\mathrm{e}^{-x} & \text{for } x\in\mathbb{R} \text{ and } t=0,\\ 0 & \text{for } x\in\mathbb{R} \text{ and } t>0.\end{cases}$$

From this in turn, it follows readily that (1.88) holds, as claimed. □

Chapter 2

A General Upper Bound

Throughout this treatment, we will be concerned with statements about the probability

$$\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u\Big\},$$

for a separable Lévy process $\{\xi(t)\}_{t\geq 0}$, and $h>0$ a constant. Since the distribution of $\sup_{t\in[0,h]}\xi(t)$ will be the same for all separable Lévy process, with the same finite dimensional distributions as $\{\xi(t)\}_{t\geq 0}$, it is enough to consider one specific such separable version. Therefore we can without loss of generality assume, in proofs, that ξ is cádlág.

As for the constant $h>0$, it is enough to consider the case when $h=1$, since results for $\sup_{t\in[0,h]}\xi(t)$ then will follow from considering $\sup_{t\in[0,1]}\xi(ht)$, using that $\{\xi(ht)\}_{t\geq 0}$ is a separable Lévy process when ξ is. However, we have chosen to keep a general h in our results, as this does not cause any extra labour.

We have the following useful upper bound, for the asymptotic behaviour of suprema of Lévy processes, the argument for which builds on an argument developed by Doob [28], p. 106, for symmetric processes.

Theorem 2.1. *For a separable Lévy process $\{\xi(t)\}_{t\geq 0}$, starting at $\xi(0)=0$, that is not supported on $(-\infty,0]$, we have*

$$\sup_{u\in\mathbb{R}}\frac{1}{\mathbf{P}\{\xi(h)>u-\varepsilon\}}\,\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u\Big\}\leq\frac{1}{\inf_{t\in(0,h)}\mathbf{P}\{\xi(t)\geq-\varepsilon\}}\qquad(2.1)$$

for $h>0$ and $\varepsilon\geq 0$.

Proof. As Lévy processes are continuous in probability, any dense subset of $[0,h]$ is a separator for $\{\xi(t)\}_{t\in[0,h]}$ (see e.g., Samorodnitsky and Taqqu [53], Exercise 9.3). From this we readily get

$$
\begin{aligned}
&\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u\Big\}-\mathbf{P}\{\xi(h)>u-\varepsilon\}\\
&\quad\le \mathbf{P}\Big\{\Big\{\sup_{t\in[0,h)}\xi(t)>u\Big\}\cup\{\xi(h)>u-\varepsilon\}\Big\}-\mathbf{P}\{\xi(h)>u-\varepsilon\}\\
&\quad= \mathbf{P}\Big\{\Big\{\sup_{t\in[0,h)}\xi(t)>u\Big\}\cap\{\xi(h)\le u-\varepsilon\}\Big\}\\
&\quad= \lim_{n\to\infty}\mathbf{P}\Big\{\bigcup_{k=1}^{n-1}\{\xi(\tfrac{k}{n}h)>u\},\xi(h)\le u-\varepsilon\Big\}\\
&\quad= \lim_{n\to\infty}\sum_{k=1}^{n-1}\mathbf{P}\Big\{\bigcap_{\ell=1}^{k-1}\{\xi(\tfrac{\ell}{n}h)\le u\},\xi(\tfrac{k}{n}h)>u,\xi(h)\le u-\varepsilon\Big\}\\
&\quad\le \limsup_{n\to\infty}\sum_{k=1}^{n-1}\mathbf{P}\Big\{\bigcap_{\ell=1}^{k-1}\{\xi(\tfrac{\ell}{n}h)\le u\},\xi(\tfrac{k}{n}h)>u,\xi(h)-\xi(\tfrac{k}{n}h)<-\varepsilon\Big\}\\
&\quad= \limsup_{n\to\infty}\sum_{k=1}^{n-1}\mathbf{P}\Big\{\bigcap_{\ell=1}^{k-1}\{\xi(\tfrac{\ell}{n}h)\le u\},\xi(\tfrac{k}{n}h)>u,\xi(h)-\xi(\tfrac{k}{n}h)\ge-\varepsilon\Big\}\\
&\qquad\qquad\times\frac{\mathbf{P}\{\xi(h)-\xi(\tfrac{k}{n}h)<-\varepsilon\}}{\mathbf{P}\{\xi(h)-\xi(\tfrac{k}{n}h)\ge-\varepsilon\}}\\
&\quad\le \limsup_{n\to\infty}\sum_{k=1}^{n-1}\mathbf{P}\Big\{\bigcap_{\ell=1}^{k-1}\{\xi(\tfrac{\ell}{n}h)\le u\},\xi(\tfrac{k}{n}h)>u,\xi(h)>u-\varepsilon\Big\}\\
&\qquad\qquad\times\bigg(\frac{1}{\mathbf{P}\{\xi(h)-\xi(\tfrac{k}{n}h)\ge-\varepsilon\}}-1\bigg)\\
&\quad\le \mathbf{P}\{\xi(h)>u-\varepsilon\}\bigg(\frac{1}{\inf_{t\in(0,h)}\mathbf{P}\{\xi(t)\ge-\varepsilon\}}-1\bigg),
\end{aligned}
\tag{2.2}
$$

which gives (2.1) after rearrangement. □

Remark 2.2. For Lévy process ξ that Theorem 2.1 does not apply to, we have ξ supported on $(-\infty,0]$, so that

$$\mathbf{P}\Big\{\sup_{t\ge0}\xi(t)>0\Big\}=0.$$

And so (2.1) does not make sense. However, in this case ξ nonincreasing with

$$\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u\Big\}=0\quad\text{for }\ h,u>0,$$

so that there is no need for an upper bound like (2.1).

A weaker version of Theorem 2.1 features in the proof of Theorem 25.18 in Sato [54]: For any $x,y>0$:

$$\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u+\varepsilon\Big\}\leq\bigg(\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)\leq\frac{\varepsilon}{2}\Big\}\bigg)^{-1}\mathbf{P}\{|\xi(h)|>u\}.$$

However, in order for this result to be useful for studies of asymptotics, one must have comparability of the tails of $\xi(h)$ and $|\xi(h)|$

$$\limsup_{u\to\infty}\frac{\mathbf{P}\{|\xi(h)|>u\}}{\mathbf{P}\{\xi(h)>u\}}<\infty.$$

This in turn never happens for the superexponential Lévy processes that we will apply inequalities of this type to, except if they are Wiener processes. This is so because

$$\mathbf{E}\{\mathrm{e}^{\alpha|\xi|}\}<\infty\quad\text{for }\ \alpha\geq 0$$

if and only if ξ has zero Lévy measure (see e.g., Sato [54], Theorem 26.1).

Corollary 2.3. *Let $\{\xi(t)\}_{t\geq 0}$ be a separable Lévy process, starting at $\xi(0)=0$, such that*

$$\liminf_{t\downarrow 0}\mathbf{P}\{\xi(t)>0\}>0\quad\textit{or more generally}\quad\inf_{t\in(0,h)}\mathbf{P}\{\xi(t)\geq 0\}>0.\quad(2.3)$$

For $h>0$ a constant, we have

$$\sup_{u\in\mathbb{R}}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u\Big\}<\infty.$$

Proof. By Theorem 2.1, it is enough to show that the condition to the left in (2.3) implies the condition to the right. To that, suppose that the left condition holds, and that the right does not hold. Then there exists a sequence $\{t_n\}_{n=1}^{\infty}\subseteq[0,h]$ such that

$$\mathbf{P}\{\xi(t_n)\geq 0\}\to\inf_{t\in(0,h)}\mathbf{P}\{\xi(t)\geq 0\}=0\quad\text{as }\ n\to\infty.$$

Picking a convegent subsequence $\{t'_n\}_{n=1}^{\infty}\subseteq\{t_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty}t'_n=t_0$, we get

$$\mathbf{P}\{\xi(t_0) > 0\} \leq \liminf_{n\to\infty} \mathbf{P}\{\xi(t'_n) > 0\} \leq \liminf_{n\to\infty} \mathbf{P}\{\xi(t'_n) \leq 0\} = 0, \tag{2.4}$$

by continuity in probability of ξ. Hence the left condition in (2.3) ensures that $t_0 > 0$. And so, using (2.4) again, ξ must be supported on $(-\infty, 0]$. This of course contradicts the left condition in (2.3). □

Albeit quite fine, in a way, the difference between Theorem 2.1, with $\varepsilon > 0$, and Corollary 2.3 is quite important, as illustrated by the following example:

Example 2.4. Let $\{N(t)\}_{t\geq 0}$ be a Poisson process with rate 1, and $\{\eta_k\}_{k=1}^{\infty}$ independent Bernoulli distributed random variables

$$\mathbf{P}\{\eta_k = 1\} = \mathbf{P}\{\eta_k = -1\} = \frac{1}{2} \quad \text{for } k \geq 1.$$

Quite ingeniously, Braverman [21], Section 4, shows that for the Lévy process

$$\xi(t) = \sum_{k=1}^{N(t)} \eta_k - t \quad \text{for } t \geq 0,$$

it holds that

$$\begin{aligned} 1 &= \liminf_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\Big\{\sup_{t\in[0,h]} \xi(t) > u\Big\} \\ &< \limsup_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\Big\{\sup_{t\in[0,h]} \xi(t) > u\Big\} = \infty. \end{aligned}$$

Hence neither Corollary 2.3 or (2.3) hold for this process.

For a Lévy process that does not satisfy (2.3), it is enough to consider $\xi(t) = N(t) - t$.

Chapter 3

Subexponential Lévy Processes

For a Lévy process $\{\xi(t)\}_{t\geq 0}$, starting at $\xi(0)=0$, and a constant $h>0$, such that $\xi(h)$ has a long-tailed distribution, we have

$$\lim_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\,\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u\Big\}=1. \tag{3.1}$$

The result (3.1) is due to Willekens [60] (see Theorem 3.2 below). But it was proven earlier, under the stronger assumptions of symmetry and regularly varying tails, by Berman [16], Theorem 1. See also Marcus [46], Lemma 7.6.

Remark 3.1. Rosiński and Samorodnitsky [51], Theorem 2.1, gave a version of (3.1) valid for much more general infinitely divisible process than Lévy process, and for more general functionals of the sample paths of ξ than the supremum $\sup_{t\in[0,h]}\xi(t)$, but under the more restrictive assumption subexponentiality on the tails.

For completeness, we now give the statement and proof of Willekens' result, before turning to establish a converse to that result, which is new. In later chapters, we will extend Willekens' result to processes with increasingly light tails, which require increasingly complicated proofs. However, in a way, the very simple arguments in this chapter remains the point of take off.

A slight difference compared with Willekens theorem, is that we, for consistency with our other results, as well as with extreme value theory in general, formulate the result for probabilities $>u$, while Willekens considered probabilities $\geq u$.

Theorem 3.2 (Berman [16], Theorem 1, Willekens [60]). *For a separable Lévy process $\{\xi(t)\}_{t\geq 0}$, starting at $\xi(0)=0$, and a constant $h>0$, we have*

$$\xi(h)\in\mathcal{L}\Leftrightarrow \sup_{t\in[0,h]}\xi(t)\in\mathcal{L}.$$

Moreover, if any of these memberships hold (so that both of them hold), then (3.1) holds.

Proof. Assume that $\xi(h)\in\mathcal{L}$. By the strong Markov property, we have

$$\begin{aligned}\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)\geq u\Big\}\\ &\leq \mathbf{P}\{\xi(h)\geq u-K\}+\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)\geq u,\xi(h)<u-K\Big\}\\ &\leq \mathbf{P}\{\xi(h)>u-2K\}+\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)\geq u\Big\}\mathbf{P}\Big\{\inf_{t\in[0,h]}\xi(t)\leq -K\Big\}.\end{aligned}$$

This gives

$$\begin{aligned}\limsup_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u\Big\}\\ &\leq \limsup_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)\geq u\Big\}\\ &\leq \Big(\mathbf{P}\Big\{\inf_{t\in[0,h]}\xi(t)>-K\Big\}\Big)^{-1}\limsup_{u\to\infty}\frac{\mathbf{P}\{\xi(h)>u-2K\}}{\mathbf{P}\{\xi(h)>u\}}\\ \to 1 \quad \text{as } K\to\infty.\end{aligned}\tag{3.2}$$

It follows that (3.1) holds [since the corresponding lower bound for the lim inf of the probability ratio in (3.1) is trivial]. And so we must have $\sup_{t\in[0,h]}\xi(t)\in\mathcal{L}$.

Assume that $\sup_{t\in[0,h]}\xi(t)\in\mathcal{L}$. By inspection of (3.2), we have

$$\begin{aligned}&\liminf_{u\to\infty}\frac{\mathbf{P}\{\xi(h)>u+x\}}{\mathbf{P}\{\xi(h)>u\}}\\ &\geq \liminf_{u\to\infty}\Big(\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u\Big\}\Big)^{-1}\mathbf{P}\{\xi(h)\geq u+2x\}\\ &\geq \mathbf{P}\Big\{\inf_{t\in[0,h]}\xi(t)>-K\Big\}\liminf_{u\to\infty}\Big(\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u\Big\}\Big)^{-1}\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u+2x+K\Big\}\\ &\to 1 \quad \text{as } K\to\infty \text{ for } x>0.\end{aligned}\tag{3.3}$$

Thus Lemma 1.2 gives $\xi(h) \in \mathcal{L}$, so that the first part of the proof gives (3.1). $\square$

Also for completeness, we provide the canonical application of Willekens result:

Example 3.3. For an α-stable Lévy motion $\{\xi(t)\}_{t\geq 0}$, $(\alpha,\beta) \in (0,2]\times[-1,1]$ (giving Brownian motion for $\alpha = 2$), $\xi(t)$ is $S_\alpha((t-s)^{1/\alpha},\beta)$ distributed (cf. Example 1.8 together with Samorodnitsky and Taqqu [53], Example 3.1.3).

By Example1.8, for $\alpha < 2$ and $\beta > -1$, we have $\xi(h) \in \mathcal{R}(-\alpha)$, so that $\xi(h) \in \mathcal{L}$, and (3.1) holds by Theorem 3.2.

The discovery that (3.1) holds for α-stable Lévy motion, with $\alpha < 2$ and $\beta > -1$, is due to Berman [16], Theorem 1, Marcus [46], Lemma 7.6, and Willekens [60], p. 173.

The following converse to Theorem 3.2 is new:

Theorem 3.4. *For a Lévy process $\{\xi(t)\}_{t\geq 0}$, starting at $\xi(0) = 0$, such that (3.1) holds, one of the following conditions must hold:*

1. *ξ is a subordinator;*
2. *the following limit cannot exist and be strictly positive for any $t \in (0,h)$:*
$$\ell(t) = \lim_{u\to\infty} \frac{\mathbf{P}\{\xi(t) > u\}}{\mathbf{P}\{\xi(h) > u\}};$$
3. *$\xi(h) \in \mathcal{L}$.*

Proof. Let (3.1) hold, and assume that the limit $\ell(t) > 0$ exists for some $t \in (0,h)$ in Condition 2. To show that one of Conditions 1 and 3 must hold, notice that

$$\begin{aligned}
0 &= \lim_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\Big\{\sup_{s\in[0,h]} \xi(s) > u\Big\} - 1 \\
&\geq \limsup_{u\to\infty} \frac{\mathbf{P}\{\{\xi(h) > u\} \cup \{\xi(t) > u\}\}}{\mathbf{P}\{\xi(h) > u\}} - 1 \\
&\geq \limsup_{u\to\infty} \frac{\mathbf{P}\{\xi(h) \leq u, u < \xi(t) \leq u+\varepsilon\}}{\mathbf{P}\{\xi(h) > u\}} \\
&\geq \mathbf{P}\{\xi(h-t) \leq -\varepsilon\} \limsup_{u\to\infty} \frac{\mathbf{P}\{u < \xi(t) \leq u+\varepsilon\}}{\mathbf{P}\{\xi(h) > u\}} \\
&\geq \ell(t)\,\mathbf{P}\{\xi(h-t) \leq -\varepsilon\} \limsup_{u\to\infty} \frac{\mathbf{P}\{u < \xi(t) \leq u+\varepsilon\}}{\mathbf{P}\{\xi(t) > u\}} \quad \text{for } \varepsilon > 0.
\end{aligned} \tag{3.4}$$

Now, a subordinated ξ has the representation $\xi(t) = X(t) + \mu t$ for $t \geq 0$, where $\mu \geq 0$ is a constant and X a subordinator with support

$$\inf\Big\{x \in \mathbb{R} : \mathbf{P}\{X(t) > x\} = 1\Big\} = 0 \quad \text{for } t > 0.$$

Hence, if Condition 1 does not hold, then either $\xi(t) = X(t) + \mu t$, where X is a subordinator as above, but $\mu < 0$. This gives

$$\mathbf{P}\{\xi(h-t) \leq -\varepsilon\} = \mathbf{P}\{X(h-t) \leq (-\mu)(h-t) - \varepsilon\} > 0 \quad \text{for } \varepsilon < (-\mu)(h-t).$$

Or else we do not have $\xi(t) = X(t) + \mu t$ at all, for any subordinator X, in which case well-known properties of infinite divisibile distributions gives $\mathbf{P}\{\xi(h-t) \leq -\varepsilon\} > 0$ again (see e.g., Sato [54], Theorem 24.7). In either case, (3.4) now yields

$$\limsup_{u\to\infty} \frac{\mathbf{P}\{u < \xi(t) \leq u+\varepsilon\}}{\mathbf{P}\{\xi(t) > u\}} = 0,$$

so that

$$\liminf_{u\to\infty} \frac{\mathbf{P}\{\xi(t) > u+\varepsilon\}}{\mathbf{P}\{\xi(t) > u\}} = 1.$$

Hence $\xi(t) \in \mathcal{L}$, by Lemma 1.2. But from this in turn, we get $\xi(h) \in \mathcal{L}$ (using Lemma 1.2 again), as

$$\liminf_{u\to\infty} \frac{\mathbf{P}\{\xi(h) > u+x\}}{\mathbf{P}\{\xi(h) > u\}} = \liminf_{u\to\infty} \frac{\mathbf{P}\{\xi(t) > u+x\}/\ell(t)}{\mathbf{P}\{\xi(t) > u\}/\ell(t)} = 1. \qquad \square$$

Assuming closedness of convolution roots (recall Conjecture 1.3), we can prove a more appealing version of Theorem 3.4:

Corollary 3.5. *Consider a Lévy process $\{\xi(t)\}_{t\geq 0}$, starting at $\xi(0) = 0$, such that (3.1) holds. If all convolution roots of $\xi(t)$ are in $\mathcal{L}$ if $\xi(t)$ is, for any t, then Theorem 3.4 holds with the following stronger version of Condition 2:*

2. $\displaystyle\liminf_{u\to\infty} \frac{\mathbf{P}\{\xi(t) > u\}}{\mathbf{P}\{\xi(h) > u\}} = 0$ *for* $t \in (0,h)$.

Proof. Assume that (3.1) holds, that Condition 1 of Theorem 3.4 does not hold, and that the lim inf in Condition 2 has value $\ell(t) > 0$ for some $t \in (0,h)$. By (3.4), we then have

$$\ell(t)\,\mathbf{P}\{\xi(h-t)\leq -\varepsilon\}\limsup_{u\to\infty}\frac{\mathbf{P}\{u<\xi(t)\leq u+\varepsilon\}}{\mathbf{P}\{\xi(t)>u\}}=0.$$

Consequently, using the fact that Condition 1 does not hold, as in the proof of Theorem 3.4,

$$\liminf_{u\to\infty}\frac{\mathbf{P}\{\xi(t)>u+\varepsilon\}}{\mathbf{P}\{\xi(t)>u\}}=1.$$

This gives $\xi(t)\in\mathcal{L}$, by Lemma 1.2. Using that convolution roots of $\xi(t)$ are in $\mathcal{L}$, together with the fact that $\mathcal{L}$ is closed under convolutions, by Embrechts and Goldie [31], Theorem 3, it follows that $\xi(k)\in\mathcal{L}$ for $k>h$ such that $k=qt$ for some positive $q\in\mathbb{Q}$. This in turn gives $\sup_{s\in[0,k]}\xi(s)\in\mathcal{L}$, by Theorem 3.2. It follows that, for any $\varepsilon>0$,

$$\begin{aligned}
&\liminf_{u\to\infty}\Big(\mathbf{P}\Big\{\sup_{s\in[0,h]}\xi(s)>u\Big\}\Big)^{-1}\mathbf{P}\Big\{\sup_{s\in[0,h]}\xi(s)>u+\varepsilon\Big\}\\
&\geq \mathbf{P}\Big\{\xi(k-h)>-\frac{\varepsilon}{2}\Big\}\liminf_{u\to\infty}\Big(\mathbf{P}\Big\{\sup_{s\in[0,k]}\xi(s)>u\Big\}\Big)^{-1}\mathbf{P}\Big\{\sup_{s\in[0,k]}\xi(s)>u+\frac{3\varepsilon}{2}\Big\}\\
&=\mathbf{P}\Big\{\xi(k-h)>-\frac{\varepsilon}{2}\Big\}\to 1\quad\text{as }\ k\downarrow h.
\end{aligned}$$

Hence we have $\sup_{s\in[0,h]}\xi(s)\in\mathcal{L}$, by Lemma 1.2, so that also $\xi(h)\in\mathcal{L}$, by Theorem 3.2, so that Condition 3 of Theorem 3.4 holds. □

As of lately, the Normal Inverse Gaussian (NIG) process, introduced by O.E. Barndorff-Nielsen [11], Section 3, and [12], Section 2, has become very popular in mathematical finance, mainly as a substitute for Brownian motion in Black-Scholes models for asset prices. Example 3.6 below shows that, for a NIG process,

$$\limsup_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\,\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u\Big\}>1. \tag{3.5}$$

See Schoutens [56], Section 5.3.8, for more information on NIG processes.

Example 3.6. The Normal Inverse Gaussian (NIG) process is a Lévy process $\{\xi(t)\}_{t\geq 0}$ with $\xi(t)$ NIG$(\alpha,\beta,\delta t,\mu t)$ distributed, with probability density function given by (see Barndorff-Nielsen [12], p. 59)

$$f_{\mathrm{NIG}}(x;\alpha,\beta,\delta,\mu)=\frac{\alpha\delta}{\pi}\,\mathrm{e}^{\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)}\,\frac{K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})}{\sqrt{\delta^2+(x-\mu)^2}}\quad\text{for }\ x\in\mathbb{R}.$$

Here the parameters $\alpha, \beta, \delta, \mu \in \mathbb{R}$ satisfy $\alpha > |\beta|$, while K_λ is the modified Bessel function of the third kind.

Using Erdélyi et al. [35], p. 23, we get

$$f_{\xi(t)}(x) = f_{\mathrm{NIG}}(x; \alpha, \beta, \delta t, \mu t) \sim \sqrt{\frac{\alpha}{2\pi}}\, \mathrm{e}^{\delta t\sqrt{\alpha^2-\beta^2}}\, \frac{\delta t}{x^{3/2}}\, \mathrm{e}^{-(\alpha-\beta)(x-\mu t)} \tag{3.6}$$

as $x \to \infty$. Hence it follows readily that

$$\mathbf{P}\{\xi(t) > u\} \sim \sqrt{\frac{\alpha}{2\pi}}\, \mathrm{e}^{\delta t\sqrt{\alpha^2-\beta^2}}\, \frac{\delta t}{(\alpha-\beta)\, u^{3/2}}\, \mathrm{e}^{-(\alpha-\beta)(u-\mu t)} \quad \text{as } u \to \infty. \tag{3.7}$$

Consequently, Condition 2 of Theorem 3.4 does not hold, as we have

$$\ell(t) = \frac{t}{h} \exp\left\{\left((\alpha-\beta)\mu + \delta\sqrt{\alpha^2-\beta^2}\right)(t-h)\right\}. \tag{3.8}$$

As Conditions 1 and 3 do not hold either, (3.1) does not hold for the NIG process, by Theorem 3.4.

Remark 3.7. For μ negative enough, we have $\ell(t) > \ell(h)$ for some $t \in (0, h)$, for the NIG process in Example 3.6, so that (3.5) holds trivially. But for moderately negative values of μ, as well as positive ones, there is no such trivial argument for (3.5).

As we will see later, it follows from Samorodnitsky and Braverman [22], Theorem 3.1, and Braverman [20], Theorem 2.1, that, for the NIG process, the limit

$$\lim_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}}\, \mathbf{P}\Big\{\sup_{t\in[0,h]} \xi(t) > u\Big\} \tag{3.9}$$

exists. However, we do not belive that it has been recorded before in the literature, or can be seen from the references mentioned, that the limit is strictly greater than 1.

Albeit Condition 1 of Theorem 3.4 implies (3.1), trivially, and Condition 3 implies (3.1), by Theorem 3.2, it is not the case that Condition 2 implies (3.1): The following example gives an important counter example to this implication.

Example 3.8. Consider totally skewed to the left strictly α-stable Lévy motion $\{\xi(t)\}_{t\geq 0}$, $\alpha \in (1,2]$, where $\xi(t)$ has an $S_\alpha(t^{1/\alpha},-1,0)$-distribution (cf.

Example 1.8). Now $\xi(t)$ has the right probability tail (see e.g., Samorodnitsky and Taqqu [53], Eq. 1.2.11)

$$\mathbf{P}\{\xi(t)>u\} \sim \frac{\big(\alpha^\alpha t/\cos(\frac{\pi\alpha}{2})\big)^{1/(2(\alpha-1))}}{\sqrt{2\pi\alpha(\alpha-1)}\,u^{\alpha/(2(\alpha-1))}} \exp\left\{-(\alpha-1)\Big(\frac{\cos(\frac{\pi\alpha}{2})\,u^\alpha}{\alpha^\alpha t}\Big)^{1/(\alpha-1)}\right\}$$

as $u\to\infty$. Hence we have $\ell(t)=0$, so that Condition 2 of Theorem 3.4 holds. However, by Albin [2], Theorem 1, the limit (3.9) exists, and is strictly greater than 1. Of course, for Brownian motion ($\alpha=2$), the probability ratio in (3.9) is well-known to be 2, exactly.

In Theorem 5.2 below, we show that for exponential processes, Condition 2 is sufficient for (3.9).

As we have given an example of a process for which Condition 2 holds but (3.9) does not, it is suitable to round of this chapter with a "natural" example of a process that satisfies both Condition 2 and (3.9).

Example 3.9. Consider totally skewed to the left 1-stable Lévy motion $\{\xi(t)\}_{t\geq 0}$, where $\xi(1)$ has an $S_1(\gamma,-1,0)$-distribution (cf. Example 1.8), $\gamma>0$. Now $\xi(t)$ has the right probability tail (see e.g., Samorodnitsky and Taqqu [53], Eq. 1.2.11)

$$\mathbf{P}\{\xi(t)>u\} \sim \sqrt{\frac{\mathrm{e}}{2\pi\gamma t}}\,\exp\left\{-\frac{\pi u}{4\gamma t}-\gamma t\exp\left[\frac{\pi u}{2\gamma t}-1\right]\right\} \quad \text{as } u\to\infty. \quad (3.10)$$

Hence we have $\ell(t)=0$, so that Condition 2 of Theorem 3.4 holds. Further, (3.9) holds, by Albin [2], Theorem 2.

Chapter 4

O-exponential Lévy Processes

In this chapter, we will extend Willekens' result Theorem 3.2 to the class $\mathcal{OL}$. Instead of (3.1), for processes of this type, we have that the tails of $\xi(h)$ and $\sup_{t\in[0,h]}\xi(t)$ are comparable, in the sense that

$$\begin{aligned}1 &\le \liminf_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u\Big\} \\ &\le \limsup_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u\Big\}<\infty.\end{aligned} \tag{4.1}$$

[Of course, the lim inf of the probability ratio in (4.1) is at least 1, trivially.]

Theorem 4.1. *For a separable Lévy process $\{\xi(t)\}_{t\ge0}$, starting at $\xi(0)=0$, and a constant $h>0$, we have*

$$\xi(h)\in\mathcal{OL}\Leftrightarrow\sup_{t\in[0,h]}\xi(t)\in\mathcal{OL}.$$

Moreover, if any of these memberships hold (so that both of them hold), then (4.1) holds.

Proof. For $\xi(h)\in\mathcal{OL}$, (3.2) gives that the lim sup in (4.1) is finite. Further, (3.2) shows that

$$\begin{aligned}&\liminf_{u\to\infty}\bigg(\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u\Big\}\bigg)^{-1}\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u+x\Big\} \\ &\qquad\ge\liminf_{u\to\infty}\frac{\mathbf{P}\{\xi(h)>u+x\}}{\mathbf{P}\{\xi(h)>u-2K\}}\mathbf{P}\Big\{\inf_{t\in[0,h]}\xi(t)>-K\Big\}>0\end{aligned}$$

for K large enough, so that $\sup_{t\in[0,h]}\xi(t)\in\mathcal{L}$.

For $\sup_{t\in[0,h]}\xi(t)\in\mathcal{L}$, taking K large enough in (3.3), we get $\xi(h)\in\mathcal{OL}$ from Lemma 1.10. And so we get (4.1) from what has been proven already. □

Remark 4.2. From the literature, it is known that $\xi(h)\in\mathcal{S}(\alpha)$ implies (4.1), and in fact, that the limit

$$\lim_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u\Big\}=H \quad \text{exists vid value } H\in[1,\infty). \tag{4.2}$$

This was proven by Braverman and Samorodnitsky [22], Theorem 3.1, with some futher input from Braverman [20], Theorem 2.1. See the next chapter for more information, where we will extend this result to $\mathcal{L}(\alpha)$.

Here is a converse to Theorem 4.1, similar to Theorem 3.4 for the class $\mathcal{L}$:

Theorem 4.3. *For a Lévy process $\{\xi(t)\}_{t\geq 0}$, starting at $\xi(0)=0$, such that (4.1) holds, one of the following conditions must hold:*

2. $\liminf_{u\to\infty}\dfrac{\mathbf{P}\{\xi(t)>u\}}{\mathbf{P}\{\xi(h)>u\}}=0$ *for* $t\in(0,h)$*;*

3. $\xi(h)\in\mathcal{OL}$.

Proof. Assume that (4.1) holds, and that the lim inf in Condition 2 has value $\ell(t)>0$, for some $t\in(0,h)$. Notice that

$$\begin{aligned}\infty &> \limsup_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u\Big\}\\ &\geq \mathbf{P}\{\xi(h-t)\geq\varepsilon\}\limsup_{u\to\infty}\frac{\mathbf{P}\{\xi(t)>u-\varepsilon\}}{\mathbf{P}\{\xi(h)>u\}}\\ &\geq \ell(t)\mathbf{P}\{\xi(h-t)\geq\varepsilon\}\limsup_{u\to\infty}\frac{\mathbf{P}\{\xi(h)>u-\varepsilon\}}{\mathbf{P}\{\xi(h)>u\}}\quad \text{for } \varepsilon>0.\end{aligned}$$

Now, $-\xi$ cannot be a subordinator, as this would make the limits in (4.1) make no sense. Therefore, we get as in the proof of Theorem 3.4, that $\mathbf{P}\{\xi(h-t)\geq\varepsilon\}>0$ for $\varepsilon>0$ small enough. And so $\xi(h)\in\mathcal{OL}$ follows from Lemma 1.10. □

Example 4.4. As semi-heavy tailed distributions are in $\mathcal{L}(\alpha)$ (see Example 1.14), they are in $\mathcal{OL}$. We will see many examples of Lévy processes with semi-heavy tails in the next chapter.

Chapter 5

Exponential Lévy Processes

In this chapter we consider Lévy processes such that $\xi(h) \in \mathcal{L}(\alpha)$ for some $\alpha > 0$.

Samorodnitsky and Braverman [22], Theorem 3.1, proved that (4.2) holds for Lévy processes $\{\xi(t)\}_{t\geq 0}$ with exponential tails $\xi(t) \in \mathcal{S}(\alpha)$ for $t \geq 0$, for some $\alpha > 0$.

Samorodnitsky and Braverman showed the existence of the limit H in (4.2), but did not really express H in terms of characterstics of the process ξ. However, this was done by Braverman [20], Theorem 2.1, making use of the sojourn approach to extremes developed by Berman (see Berman, [17]). Still, Braverman's formula for H is typically not really useful for explicit computation of H in closed form: At least Braverman claims so himself in Section 6 of his paper.

Remark 5.1. Braverman [20] Braverman's completion of the result of Samorodnitsky and Braverman, with a formula for the constant, mirrors what happened earlier for extremes of an α-stable stochastic process $\{X(t)\}_{t\in T}$, where de Acosta [1], Theorem 4.1, proved that the limit

$$\lim_{u\to\infty} u^{\alpha}\mathbf{P}\Big\{\sup_{t\in T} X(t) > u\Big\}$$

exists (possibly infinite), and where Samorodnitsky, [52], Theorem 4.1, provided information about the limit expressed in terms of characterstics of the process X. The calculation of the limit could then completed, with the appearance of Rosiński and Samorodnitsky [51], Theorem 2.1.

As illustrated by Example 1.14, there are important examples of processes with tails in $\mathcal{L}(\alpha)$, that do not have exponential tails [in the sense of belonging to some $\mathcal{S}(\alpha)$]. In particular, this happens for many of the semi-heavy tailed Lévy processes are important in applications to mathematical finance, see below.

When we generalize the result (4.2) of Samorodnitsky and Braverman [22] and Braverman [20] from $\mathcal{S}(\alpha)$ to $\mathcal{L}(\alpha)$, we cover some important ground, as was the case with our earlier generalization to $\mathcal{OL}$. In addition, we provide more complete results, in the fashion of the result of Willekens' [60], cited in Theorem 3.2, with a partial converse, in the fashion of Theorem 3.4.

Our proofs are very short and transparent, while proofs in the literature are more like 10 pages of technically complicated arguments, whcih in turn build on additional complicated asymptotic theory.

Theorem 5.2. *Consider a Lévy process $\{\xi(t)\}_{t\geq 0}$, starting at $\xi(0)=0$, such that the limit*

$$L(t) = \lim_{u\to\infty} \frac{\mathbf{P}\{\xi(t)>u\}}{\mathbf{P}\{\xi(h)>u\}} \quad \textit{exists for } t\in(0,h). \tag{5.1}$$

We have

$$\xi(h)\in\mathcal{L}(\alpha) \Leftrightarrow \sup_{t\in[0,h]} \xi(t)\in\mathcal{L}(\alpha) \quad \textit{for } \alpha\geq 0.$$

Moreover, if any of these memberships hold (so that both of them holds), then the following limit exists, with value $H\in[1,\infty)$:

$$H = \lim_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h)>u\}}\mathbf{P}\Big\{\sup_{t\in[0,h]} \xi(t)>u\Big\}. \tag{5.2}$$

If, in addition to the above requirements, $L(t)=0$ for $t\in(0,h)$, then $H=1$, so that (3.1) holds.

Proof. Assume that $\xi(h)\in\mathcal{L}(\alpha)$, and notice that, by (5.2), we have

$$\begin{aligned}\lim_{u\to\infty}\mathbf{P}\{\xi(t)-u>x|\xi(t)>u\} &= \lim_{u\to\infty}\frac{\mathbf{P}\{\xi(t)>u+x\}}{\mathbf{P}\{\xi(t)>u\}}\\ &= \lim_{u\to\infty}\frac{\mathbf{P}\{\xi(h)>u+x\}}{\mathbf{P}\{\xi(h)>u\}}\\ &= \mathrm{e}^{-\alpha x} \quad \text{for } x\geq 0,\end{aligned} \tag{5.3}$$

unless $L(t)=0$. Letting η denote a $\exp(\alpha)$ distributed random variable, independent of ξ, we thus have the following lower bound

$$
\begin{aligned}
&\liminf_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h)>u\}}\,\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u\Big\}\\
&\geq \limsup_{a\downarrow 0}\liminf_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\,\mathbf{P}\Big\{\max_{k=0,\ldots,\lfloor h/a\rfloor}\xi(h-ka)>u\Big\}\\
&= \limsup_{a\downarrow 0}\sum_{k=0}^{\lfloor h/a\rfloor}\liminf_{u\to\infty}\frac{L(h-ka)}{\mathbf{P}\{\xi(h-ka)>u\}}\,\mathbf{P}\Big\{\xi(h-ka)>u,\bigcap_{\ell=k-1}^{0}\{\xi(h-\ell a)\leq u\}\Big\}\\
&= \limsup_{a\downarrow 0}\sum_{k=0}^{\lfloor h/a\rfloor}L(h-ka)\\
&\qquad\times\liminf_{u\to\infty}\mathbf{P}\Big\{\bigcap_{\ell=k-1}^{0}\{\xi(h-\ell a)-\xi(h-ka)+\xi(h-ka)-u\leq 0\}\,\Big|\,\xi(h-ka)>u\Big\}\\
&\geq \limsup_{a\downarrow 0}\sum_{k=0}^{\lfloor h/a\rfloor}L(h-ka)\mathbf{P}\Big\{\bigcap_{\ell=k-1}^{0}\{\xi((k-\ell)a)+\eta\leq 0\}\Big\}.
\end{aligned}
\tag{5.4}
$$

For a matching upper bound, notice that, by the strong Markov property,

$$
\begin{aligned}
&\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u+x\Big\}\\
&\qquad\leq \mathbf{P}\Big\{\max_{k=0,\ldots,\lfloor h/a\rfloor}\xi(h-ka)>u\Big\}+\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u\Big\}\,\mathbf{P}\Big\{\inf_{t\in[0,a]}\xi(t)<-x\Big\}
\end{aligned}
$$

for $x>0$. From this together with (5.2) and (5.3), remembering that $\xi(h)\in\mathcal{L}(h)$, we get in the fashion of (5.4), that

$$
\begin{aligned}
&\limsup_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\,\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u\Big\}\\
&\qquad=\limsup_{x\to\infty}\limsup_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u+x\}}\,\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u+x\Big\}\\
&\qquad=\limsup_{x\to\infty}\limsup_{u\to\infty}\frac{\mathrm{e}^{\alpha x}}{\mathbf{P}\{\xi(h)>u\}}\,\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u+x\Big\}\\
&\qquad\leq\limsup_{x\to\infty}\liminf_{a\downarrow 0}\bigg(\mathbf{P}\Big\{\inf_{t\in[0,a]}\xi(t)>-x\Big\}\bigg)^{-1}\\
&\qquad\qquad\times\limsup_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\,\mathbf{P}\Big\{\max_{k=0,\ldots,\lfloor h/a\rfloor}\xi(h-ka)>u\Big\}\\
&\qquad\leq\liminf_{a\downarrow 0}\sum_{k=0}^{\lfloor h/a\rfloor}L(h-ka)\,\mathbf{P}\Big\{\bigcap_{\ell=k-1}^{0}\{\xi((k-\ell)a)+\eta\leq 0\}\Big\}.
\end{aligned}
\tag{5.5}
$$

From (5.4) together with (5.5), we have that the following limits exist and coincide

$$H = \lim_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h)>u\}}\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u\Big\}$$
$$= \lim_{a\downarrow 0}\sum_{k=0}^{\lfloor h/a\rfloor} L(h-ka)\,\mathbf{P}\Big\{\bigcap_{\ell=k-1}^{0}\{\xi((k-\ell)a)+\eta\le 0\}\Big\}.$$

As $H \geq 1$ trivially, and $H = 1$ if $L(t) = 0$ for $t \in (0, h)$, it only remains to show that $H < \infty$. However, this follows from the following version of (5.5):

$$\limsup_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u\Big\}$$
$$\le \limsup_{x\to\infty}\bigg(\mathbf{P}\Big\{\inf_{t\in[0,a]}\xi(t)>-x\Big\}\bigg)^{-1} \tag{5.6}$$
$$\times\limsup_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\mathbf{P}\Big\{\max_{k=0,\ldots,\lfloor h/a\rfloor}\xi(h-ka)>u\Big\}\quad\text{for } a>0.$$

Conversely, assume that $\sup_{t\in[0,h]}\xi(t)\in\mathcal{L}(\alpha)$. Assume in addition that $\alpha > 0$, as we are done otherwise, by Theorem 3.2. Consider the function

$$g(u,x)=\frac{\mathbf{P}\{\xi(h)>u+x\}}{\mathbf{P}\{\xi(h)>u\}}\quad\text{for } x\in\mathbb{R} \text{ and } u>0.$$

As $\sup_{t\in[0,h]}\xi(t)\in\mathcal{OL}$, so that $\xi(h)\in\mathcal{OL}$, by Theorem 4.1, we have

$$0<\liminf_{u\to\infty}g(u,x)<\limsup_{u\to\infty}g(u,x)<\infty\quad\text{for } x\in\mathbb{R}.$$

Letting $\{q_i\}_{i=1}^{\infty}$ be an enumeration of the members of $\mathbb{Q}$, we can therefore find a sequence $\{u_n^{(1)}\}_{n=1}^{\infty}$ such that

$$g(q_1)=\lim_{n\to\infty}g(u_n^{(1)},q_1)\quad\text{exists.}$$

And a further subsequence $\{u_n^{(2)}\}_{n=1}^{\infty}$ to $\{u_n^{(1)}\}_{n=1}^{\infty}$ such that

$$g(q_2)=\lim_{n\to\infty}g(u_n^{(2)},q_1)\quad\text{exists ... .}$$

Proceeding in this way, at stage $k+1$ we find a subsequence $\{u_n^{(k+1)}\}_{n=1}^{\infty}$ to $\{u_n^{(k)}\}_{n=1}^{\infty}$ such that

$$g(q_{k+1})=\lim_{n\to\infty}g(u_n^{(k+1)},q_{k+1})\quad\text{exists.}$$

Diagonalizing, by introducing the sequence $u_n = u^{(n)}$ for $n\in\mathbb{N}$, it follows that

$$g(q)=\lim_{n\to\infty}g(u_n,q)\quad\text{exists for } q\in\mathbb{Q}. \tag{5.7}$$

Notice that, since $\sup_{t\in[0,h]}\xi(t)\in\mathcal{OL}$, the fact that (4.1) holds, by Theorem 4.1, gives, writing ℓ for the limit on the right-hand side of (4.1),

$$\begin{aligned}\limsup_{\mathbb{Q}\ni q\to\infty} g(q) &\leq \limsup_{x\to\infty}\limsup_{u\to\infty}\frac{\mathbf{P}\{\xi(h)>u+x\}}{\mathbf{P}\{\xi(h)>u\}}\\ &\leq \ell\limsup_{x\to\infty}\limsup_{u\to\infty}\bigg(\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u\Big\}\bigg)^{-1}\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u+x\Big\}\\ &= \ell\lim_{x\to\infty}\mathrm{e}^{-\alpha x}=0.\end{aligned} \tag{5.8}$$

Turning things upside down, this also shows that

$$\liminf_{\mathbb{Q}\ni q\to-\infty} g(q)=\infty. \tag{5.9}$$

Now let $M(k)$ denote the maximum of k independent random variables, distributed as $\xi(h)$. Taking

$$n=\left\lfloor\frac{1}{\mathbf{P}\{\xi(h)>u_n\}}\right\rfloor,$$

(5.7) then gives

$$\lim_{n\to\infty}\mathbf{P}\{M(n)-u_n\leq q\}=\lim_{n\to\infty}\left(1-\mathbf{P}\{\xi(h)>u_n+q\}\right)^n=\exp\{-g(q)\} \tag{5.10}$$

for $q\in\mathbb{Q}$. Since g is nondecresing, with the properties (5.8) and (5.9), $\exp\{-g\}$ is a nondegenerate probability distribution function on $\mathbb{R}$. Hence (5.10) shows that $M(n)-u_n$ converges in distribution to $\exp\{-g\}$. However, by classical extreme value theory, this means that $\exp\{-g\}$ must be an extreme value distribution. In addition, by the characteristics of the normalizing $M(n)-u_n$ of $M(n)$, in this case, $\exp\{-g\}$ must be a so called Type I extreme value distribution, that is $g(x)=\mathrm{e}^{-\alpha x}$, for some $\alpha>0$: See e.g., Leadbetter, Lindgren and Rootzén, [43], Chapter 1, on classical extreme value theory.

Now, consider any sequence $\{u_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty}u_n=\infty$, and, with obvious notation, $(\xi(h)-u_n|\xi(h)>u_n)$ converges in distribution to a subprobability measure as $n\to\infty$. By applying the arguments developed above [thinning $\{u_n\}_{n=1}^{\infty}$ suitably to get the convergence (5.7)], we conclude that the limit distribution must be nondegenerate and exponentially distributed, say $\exp(\beta)$. Using that $\sup_{t\in[0,h]}\xi(t)\in\mathcal{L}(\alpha)$, together with (5.1), it follows that, by more or less repeating the arguments of (5.4) and (5.5),

$$
\begin{aligned}
\mathrm{e}^{-\alpha y} &= \limsup_{n\to\infty}\bigg(\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u_n\Big\}\bigg)^{-1}\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u_n+y\Big\}\\
&= \limsup_{x\downarrow 0}\limsup_{n\to\infty}\bigg(\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u_n\Big\}\bigg)^{-1}\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u_n+x+y\Big\}\\
&= \mathrm{e}^{-\beta y}\limsup_{x\downarrow 0}\limsup_{n\to\infty}\bigg(\frac{1}{\mathbf{P}\{\xi(h)>u_n\}}\,\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u_n\Big\}\bigg)^{-1}\\
&\qquad\times\frac{1}{\mathbf{P}\{\xi(h)>u_n+y+y\}}\,\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u_n+x+y\Big\}\\
&\le \mathrm{e}^{-\beta y}\limsup_{x\downarrow 0}\bigg(\mathbf{P}\Big\{\inf_{t\in[0,a]}\xi(t)>-x\Big\}\bigg)^{-1}\\
&\qquad\times\liminf_{a\downarrow 0}\limsup_{n\to\infty}\bigg(\frac{1}{\mathbf{P}\{\xi(h)>u_n\}}\mathbf{P}\Big\{\max_{k=0,\ldots,\lfloor h/a\rfloor}\xi(h-ka)>u_n\Big\}\bigg)^{-1}\\
&\qquad\times\frac{1}{\mathbf{P}\{\xi(h)>u_n+x+y\}}\,\mathbf{P}\Big\{\max_{k=0,\ldots,\lfloor h/a\rfloor}\xi(h-ka)>u_n+x+y\Big\}\\
&= \mathrm{e}^{-\beta y}\liminf_{a\downarrow 0}\bigg(\sum_{k=0}^{\lfloor h/a\rfloor}L(h-ka)\,\mathbf{P}\Big\{\bigcap_{\ell=k-1}^{0}\{\xi((k-\ell)a)+\eta\le 0\}\bigg)^{-1}\\
&\qquad\times\sum_{k=0}^{\lfloor h/a\rfloor}L(h-ka)\,\mathbf{P}\Big\{\bigcap_{\ell=k-1}^{0}\{\xi((k-\ell)a)+\eta\le 0\}=\mathrm{e}^{-\beta y},
\end{aligned}
\tag{5.11}
$$

where η denotes an $\exp(\beta)$ distributed random variable that is independent of ξ: Here we made use of the elementary fact that $(\xi(h)-u_n-x|\xi(h)>u_n+x)$ converges to a $\exp(\beta)$ distribution when $(\xi(h)-u_n|\xi(h)>u_n)$ does. (Or lack of memory for the exponential distribution, if you like!)

By (5.10), we have $\beta\le\alpha$. By a symmetric argument, we get $\alpha\le\beta$, so that $\beta=\alpha$. Thus we have proven that $(\xi(h)-u_n|\xi(h)>u_n)$ converges in distribution to a (proper) $\exp(\alpha)$ distribution, for any sequence $\{u_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty}u_n=\infty$ and $(\xi(h)-u_n|\xi(h)>u_n)$ converges to a subprobability measure as $n\to\infty$. By basic probability theory, this gives that $(\xi(h)-u|\xi(h)>u)$ converges in distribution to an $\exp(\alpha)$ distribution, as $n\to\infty$. Hence we have $\xi(h)\in\mathcal{L}(\alpha)$. This in turn gives (5.2), from what has been proven already. □

Corollary 5.3 (Samorodnitsky and Braverman [22], Theorem 3.1).

Consider a Lévy process $\{\xi(t)\}_{t\geq 0}$*, starting at* $\xi(0)=0$*, with characteristic-triple* (ν, m, s^2)*. If*

$$\frac{\nu((1\vee\cdot,\infty))}{\nu((1,\infty))}\in\mathcal{S}(\alpha)\quad \textit{for some}\ \ \alpha>0,$$

then he limit (5.2) exists, with value $H\in[1,\infty)$*, and* $\sup_{t\in[0,h]}\xi(t)\in\mathcal{S}(\alpha)$*.*

Clearly, the semi-heavy tailed Lévy process in Example 1.14 is the canonical application of Theorem 5.2. It turns out that the main difficulty in doing so, is to establish (5.1). The reason for this is that one typically only can find specific information in the literature about the tail of $\xi(1)$, for a Lévy process ξ, from which it typically cannot be deduced what is the tail of $\xi(t)$ for $t\neq 1$, except for processes where the distribution remains in the "same class", for all $t>0$.

For example, for a $\mathrm{GH}(\alpha,\beta,\delta,\gamma,\mu)$ process (see Example 5.4 below), $\xi(1)$ is $\mathrm{GH}(\alpha,\beta,\delta,\gamma,\mu)$ distributed, and the tail known to be semi-heavy from the literature [see (5.12) below]. However, unless $\gamma=-\frac{1}{2}$ (that is, the GH process is a NIG process; see Example 5.4 below), $\xi(t)$ is not $\mathrm{GH}(\alpha,\beta,\delta,\gamma,\mu)$ distributed for $t\neq 1$, and so the literature does not help to establish (5.1). It is for this reason that we rely in a crucial way on Proposition 1.13 and Theorems 1.16 and 1.17, to establish that (5.1) holds, even if it is known from the beginning that $\xi(h)$ or $\xi(1)$ is in $\mathcal{L}(\alpha)$.

We will now give three examples of general classes of semi-heavy tailed Lévy process, that are of great importance in mathmatical finance: The *generalized hyperbolic* (GH) processes, the *generalized* z-processes (GZ), and the *CGMY* processes.

The class of GH processes, in turn, contains the important classes of NIG processes (see Example 3.6) and *hyperbolic* (HYP) processes (see below) as special cases. The class of GZ processes contains the important class of *Meixner* processes (see below) as special cases. The class of CGMY processes contains the important class of *variance gamma* (VG) processes (see below) as special cases.

Example 5.4. For a $\mathrm{GH}(\alpha,\beta,\delta,\gamma,\mu)$ Lévy process $\{\xi(t)\}_{t\geq 0}$, we have $\xi(1)$ $\mathrm{GH}(\alpha,\beta,\delta,\gamma,\mu)$ distributed, with probability density function given by

$$f_{\mathrm{GH}(\alpha,\beta,\delta,\gamma,\mu)}(x) = \frac{(\alpha^2-\beta^2)^{\gamma/2}(\delta^2+(x-\mu)^2)^{\gamma/2-1/4}}{\sqrt{2\pi}\,\alpha^{\gamma-1/2}K_\gamma(\delta^2\sqrt{\alpha^2-\beta^2})}\,\mathrm{e}^{\beta(x-\mu)}K_{\gamma-1/2}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)$$

for $x \in \mathbb{R}$. Here the parameters satisfy $\alpha, \beta, \delta, \gamma, \mu \in \mathbb{R}$ with $\alpha > |\beta|$, and K_γ is the modified Bessel function of the third kind.) By Erdélyi et al. [35], p. 23, we have (cf. Example 3.6)

$$f_{\mathrm{GH}(\alpha,\beta,\delta,\gamma,\mu)}(x) \sim \frac{(\alpha^2-\beta^2)^{\gamma/2}x^{\gamma-1}}{2\,\alpha^{\gamma-1}K_\gamma(\delta^2\sqrt{\alpha^2-\beta^2})}\,\mathrm{e}^{-(\alpha-\beta)(x-\mu)} \quad \text{as } x \to \infty. \tag{5.12}$$

The Lévy measure ν of a GH distribution takes the following unusually complicated form

$$\frac{d\nu(x)}{dx} = \begin{cases} \dfrac{\mathrm{e}^{\beta x}}{|x|}\displaystyle\int_0^\infty \frac{\exp\{-|x|\sqrt{2y+\alpha^2}\}}{\pi^2 y\left(J_\gamma(\delta\sqrt{2y})^2+Y_\gamma(\delta\sqrt{2y})^2\right)}dy + \dfrac{\gamma \mathrm{e}^{\beta x-\alpha|x|}}{|x|} & \text{for } \gamma \geq 0 \\ \dfrac{\mathrm{e}^{\beta x}}{|x|}\displaystyle\int_0^\infty \frac{\exp\{-|x|\sqrt{2y+\alpha^2}\}}{\pi^2 y\left(J_{-\gamma}(\delta\sqrt{2y})^2+Y_{-\gamma}(\delta\sqrt{2y})^2\right)}dy & \text{for } \gamma < 0 \end{cases}. \tag{5.13}$$

Here J_γ is the Bessel function and Y_γ the Bessel function of the second kind (also called the Neumann function).

GH distributions were introduced by Barndorff-Nielsen [10], Appendix, and Barndorff-Nielsen [11]. Their infinite divisibility were established by Barndorff-Nielsen and Halgreen [14] and Shanbhag and Sreehari [58], Theorem 4. GH Lévy processes, and special case there off, as for example NIG and HYP (see below), have become increasingly important in mathematical finance: See Eberlein and Prause, [30], together with Schoutens Section 5.3.11, for more information on GH processes.

To show that GH processes have semi-heavy tails, we notice that (see e.g., Watson [59], Equation 3.1.8)

$$J_\gamma(x) \sim \frac{x^\gamma}{2^\gamma\Gamma(\gamma)} \quad \text{as } x \downarrow 0 \text{ for } \gamma \geq 0.$$

For the Bessel function of the second kind Y_γ matters are somewhat more complicated, mainly due to singularities in the relation between J_γ and Y_γ: For non-integer γ we have, by Watson [59], Equations 3.1.8 and 3.54.1,

$$Y_\gamma(x) \sim -\frac{2^\gamma x^{-\gamma}}{\sin(\pi\gamma)\Gamma(1-\gamma)} \quad \text{as } x \downarrow 0 \text{ for } \gamma \in [0,\infty) \setminus \mathbb{Z}.$$

For integer γ we have, by Watson [59], Equations 3.51.1, 3.52.3 and 3.54.2,

$$Y_\gamma(x) \sim -\frac{2^\gamma \Gamma(\gamma) x^{-\gamma}}{\pi} \quad \text{for } \gamma \in \mathbb{N} \setminus \{0\} \quad \text{and} \quad Y_0(x) \sim \frac{2\ln(x)}{\pi} \quad \text{as } x \downarrow 0.$$

For the integrals in (5.13) we therefore readily obtain

$$\begin{aligned}
&\int_0^\infty \frac{\exp\{-|x|\sqrt{2y+\alpha^2}\}}{\pi^2 y \left(J_{-\gamma}(\delta\sqrt{2y})^2 + Y_{-\gamma}(\delta\sqrt{2y})^2\right)} dy \\
&\sim \int_0^\infty \frac{\exp\{-x(\alpha+y/\alpha)\}}{\pi^2 y Y_{-\gamma}(\delta\sqrt{2y})^2} dy \\
&\sim \begin{cases} \displaystyle\int_0^\infty \frac{\delta^{2\gamma}[\sin(\pi\gamma)\Gamma(1-\gamma)]^2 y^{\gamma-1} \exp\{-x(\alpha+y/\alpha)\}}{\pi^2 2^\gamma} dy & \text{for } \gamma \in [0,\infty)\setminus\mathbb{Z} \\ \displaystyle\int_0^\infty \frac{\delta^{2\gamma} y^{\gamma-1} \exp\{-x(\alpha+y/\alpha)\}}{2^\gamma \Gamma(\gamma)^2} dy & \text{for } \gamma \in \mathbb{N}\setminus\{0\} \\ \displaystyle\int_0^\infty \frac{\exp\{-x(\alpha+y/\alpha)\}}{y \ln(y/\alpha)^2} dy & \text{for } \gamma = 0 \end{cases} \\
&= \begin{cases} \dfrac{\delta^{2\gamma}[\sin(\pi\gamma)\Gamma(1-\gamma)]^2 \alpha^\gamma \exp\{-\alpha x\}}{\pi^2 2^\gamma x^\gamma} & \text{for } \gamma \in [0,\infty)\setminus\mathbb{Z} \\ \dfrac{\delta^{2\gamma}\alpha^\gamma \exp\{-\alpha x\}}{2^\gamma \Gamma(\gamma)^2 x^\gamma} & \text{for } \gamma \in \mathbb{N}\setminus\{0\} \\ \dfrac{\exp\{-\alpha x\}}{\ln(2)} & \text{for } \gamma = 0 \end{cases}
\end{aligned}$$

as $x \to \infty$, so that

$$\frac{d\nu(x)}{dx} \sim \begin{cases} \dfrac{2^\gamma[\sin(\pi\gamma)\Gamma(1+\gamma)]^2 x^{\gamma-1} \exp\{-(\alpha-\beta)x\}}{\pi^2 \delta^{2\gamma}\alpha^\gamma} & \text{for } \gamma \in (-\infty,0]\setminus\mathbb{Z} \\ \dfrac{2^\gamma x^{\gamma-1} \exp\{-(\alpha-\beta)x\}}{\delta^{2\gamma}\alpha^\gamma \Gamma(-\gamma)^2} & \text{for } \gamma \in (-\mathbb{N})\setminus\{0\} \\ \dfrac{\exp\{-(\alpha-\beta)x\}}{\ln(2)x} & \text{for } \gamma = 0 \\ \dfrac{\gamma e^{-(\alpha-\beta)x}}{x} & \text{for } \gamma > 0 \end{cases} \tag{5.14}$$

as $x \to \infty$.

For $\gamma \geq 0$, (5.14) together with Theorem 1.17 shows that $\mathrm{GH}(\alpha,\beta,\delta,\gamma,\mu) \in \mathcal{L}(\alpha-\beta)\setminus\mathcal{S}(\alpha-\beta)$, and that Theorem 5.2 applies with $H = 1$. Notice that the results of Braverman and Samorodnitsky [22] and Braverman [20] (see Corollary 5.3) do not apply here.

For $\gamma < 0$, (5.14) together with Proposition 1.13 shows that GH(α, $\beta, \delta, \gamma, \mu) \in \mathcal{S}(\alpha - \beta)$, and that Corollary 5.3 applies with

$$L(t) = \lim_{u \to \infty} \frac{\mathbf{P}\{\xi(t) > u\}}{\mathbf{P}\{\xi(h) > u\}} = \frac{t}{h} \mathbf{E}\{\mathrm{e}^{(\alpha-\beta)\xi(t)}\} \quad \text{for } t > 0. \tag{5.15}$$

Further, using Theorem 3.4 in the same way as in Example 3.6, we get that $H > 1$ in (5.2). Notice that the results of Braverman and Samorodnitsky [22] and Braverman [20] apply when $\gamma < 0$, when it has been established that they are in $\mathcal{S}(\alpha - \beta)$. However, it does not follow from their results that $H > 1$.

Example 5.5. The class of NIG Lévy processes is the special case $\gamma = -\frac{1}{2}$ of the GH processes considered in Example 5.4. And so we have that, for a NIG process $\{\xi(t)\}_{t \geq 0}$, $\xi(h) \in \mathcal{S}(\alpha - \beta)$, and that (5.2) holds, with the limit $L(t)$ in (5.1) given by (5.15). Moreover, in (5.2), we have $H > 1$.

Notice that the results of Braverman and Samorodnitsky [22] and Braverman [20] apply to the NIG process, as it is in $\mathcal{S}(\alpha - \beta)$. However, it does not follow from their results that $H > 1$.

As for a NIG process, $\xi(t)$ is NIG$(\alpha, \beta, \delta t, \mu t)$ distributed (see Example 3.6), (3.6) directly shows that $\xi(t)$ is semi-heavy, and (3.7) directly shows that (5.1) holds, without using the Tauberian arguments of Example 5.4. However, NIG processes are the only GH processes that yield to such direct arguments, as for $\gamma \neq -\frac{1}{2}$ $\xi(t)$ is not GH distributed for $t \neq 1$.

Example 5.6. HYP Lévy processes are the special case $\gamma = 1$ of GH processes. HYP processes were introduced by Barndorff-Nielsen [10]. They are of importance in mathematical finance, where the were introduced by Eberlein and Keller [29]. See Schoutens [56], Section 5.3.11 for more information on HYP processes.

From properties of the GH process, established in Example 5.4, we have that (5.2) holds with $H = 1$, for a HYP process. Recall that the results of Braverman and Samorodnitsky [22] and Braverman [20] do not apply to the HYP process, by Example 5.4.

Example 5.7. For a $\mathrm{GZ}(\alpha,\beta_1,\beta_2,\delta,\mu)$ Lévy process $\{\xi(t)\}_{t\geq 0}$, $\xi(t)$ is $\mathrm{GZ}(\alpha,\beta_1,\beta_2,\delta t,\mu t)$ distributed, with characteristic triple $(\nu,m,0)$, where

$$\frac{d\nu(x)}{dx}=\begin{cases}\dfrac{2\delta}{\pi|x|}\mathrm{e}^{-2\pi\beta_2 x/\alpha}\Big/\left(1-\mathrm{e}^{-2\pi x/\alpha}\right) & \text{for}\quad x>0\\ \dfrac{2\delta}{\pi}\mathrm{e}^{2\pi\beta_1 x/\alpha}\Big/\left(1-\mathrm{e}^{2\pi x/\alpha}\right) & \text{for}\quad x<0\end{cases}\tag{5.16}$$

and

$$m=\mu+\frac{\alpha\delta}{\pi}\int_0^{2\pi/\alpha}\frac{\mathrm{e}^{-\beta_2 x}-\mathrm{e}^{-\beta_1 x}}{1-\mathrm{e}^{-x}}dx.$$

This process was introduced by Grigelionis [41], Definition 1, as a very natural generalization of the so called z processes, introduced by Prentice [48], Section 1 (see Example 5.9 below). See Schoutens [56], Section 5.3.10, for more information on GZ processes.

According to Grigelionis, Corollary 1, GZ processes have semi-heavy tails

$$f_{\xi(t)}(u)\sim\left(\frac{2\pi}{\alpha\,B(\beta_1,\beta_2)}\right)^{2\delta t}\frac{u^{2\delta t-1}}{\Gamma(2\delta t)}\exp\left\{-\frac{2\pi\beta_2(u-\mu t)}{\alpha}\right\}\tag{5.17}$$

as $u\to\infty$. This means that $\xi(h)\in\mathcal{L}(2\pi\beta_2/\alpha)\setminus\mathcal{S}(2\pi\beta_2/\alpha)$, by Example 1.14). However, as have been explained in Example 1.15, we do not trust Grigelionis' proof of (5.17), and therefore provide our own proof of the fact that $\xi(h)\in\mathcal{L}(2\pi\beta_2/\alpha)\setminus\mathcal{S}(2\pi\beta_2/\alpha)$.

By (5.16), we may apply Theorem 1.17, to obtain

$$\xi(h)\in\mathcal{L}\Big(\frac{2\pi\beta_2}{\alpha}\Big)\setminus\mathcal{S}\Big(\frac{2\pi\beta_2}{\alpha}\Big)\quad\text{and}\quad\lim_{u\to\infty}\frac{\mathbf{P}\{\xi(t)>u\}}{\mathbf{P}\{\xi(h)>u\}}=0\ \text{ for } t<h.$$

Hence Theorem 5.2 applies to the GZ process, yielding (5.2) with $H=1$. Notice that the results of Braverman and Samorodnitsky [22] and Braverman [20] (see Corollary 5.3) do not apply to this process.

Example 5.8. A $\mathrm{Meixner}(\alpha,\beta,\delta,\mu)$ Lévy process, is the same thing as a $\mathrm{GZ}(\alpha,1/2+\beta/(2\pi),1/2-\beta/(2\pi),\delta,\mu)$ Lévy process. Meixner processes are of great interest in mathematical finance (see e.g., Schoutens [56], Section (5.3.30), and were introduced by Schoutens and Teugels [57], Eq. 12. See also Grigelionis [41], Definition 2. The parameter β is now restricted to the interval $(-\pi,\pi)$, as it can be seen that other values of the parameter give nothing new. See Schoutens [56], Section 5.3.10, for more information on Meixner processes.

From Example 5.8 we have that (5.2) holds with $H = 1$, for a Meixner process.

Example 5.9. The class of $z(\alpha, \beta, \delta, \mu)$ Lévy processes is the subclass with $\delta = \frac{1}{2}$ of the class of GZ processes. The z distributions were introduced by Prentice [48], Section 1. See Schoutens [56], Section 5.3.10, for more information on GZ processes.

As z processes simply are linear (deterministic) time changed GZ processes, their extreme value theory coincide with that of GZ processes.

Example 5.10. For a CGMY$(C_-, C_+, G, M, Y_-, Y_+)$ Lévy process $\{\xi(t)\}_{t\geq 0}$, we have $\xi(1)$ CGMY$(C_-, C_+, G, M, Y_-, Y_+)$ distributed, with characteristic triple $(\nu, m, 0)$, where the Lévy measure ν and constant m are given by

$$d\nu(x) = \begin{cases} C_-(-x)^{-(1+Y_-)}\mathrm{e}^{Gx}dx & \text{for} \quad x < 0 \\ C_+ x^{-(1+Y_+)}\mathrm{e}^{-Mx}dx & \text{for} \quad x > 0 \end{cases} \qquad \text{and} \qquad m = \int_{\mathbb{R}} \kappa(x)\, d\nu(x).$$

The parameter values are restricted to $C_-, C_+, G, M > 0$ and $Y_-, Y_+ < 2$.

CGMY processes with $C_+ = C_-$ and $Y_- = Y_+$ were introduced by Carr, Geman, Madan and Yor [23], Section 2.2, and generalized to the above setting by Carr, Geman, Madan and Yor [24], Section 2.3. There the importance of CGMY Lévy processes in mathematical finance is also established. See Schoutens [56], Section 5.3.9, for more information on CGMY processes.

For $0 < Y_+ < 2$, Example 1.14 shows that

$$\frac{\nu((1 \vee \cdot, \infty))}{\nu((1, \infty))} \in \mathcal{S}(M),$$

so that $\xi(h) \in \mathcal{S}(M)$ by Proposition 1.13. Hence we may apply Corollary 5.3 to CGMY processes with $0 < Y_+ < 2$. Further, using Theorem 3.4 in the same way as in Example 3.6, we get that $H > 1$ in (5.2).

For $Y_+ = 0$, we may use Theorem 1.17 to immediately conclude (1.33). However, as we have used that approach already for GZ processes, we will show how difficult a direct calculation is, even in this simple case, when we have equality in (1.32), rather than asymptotic inequality only: Write $\xi(t) = \xi_-(t) + \xi_+(t)$, where ξ_- and ξ_+ are independent Lévy processes, with Lévy measures (with obvious notation)

$$\frac{d\nu_-(x)}{dx} = \frac{C_-}{(-x)^{1+Y_-}}\,\mathrm{e}^{Gx} I_{(-\infty,0)}(x) \quad \text{and} \quad \frac{d\nu_+(x)}{dx} = \frac{C_+}{x}\,\mathrm{e}^{Mx} I_{(0,\infty)}.$$

For $Y_- \geq 0$, Theorem 1.21 together with Proposition 1.29 7 show that the tail of $\xi_-(t)$ is superexponential. Notice that $\xi_-(t)$ is selfdecomposable, and thus absolutely continuous. For $Y_- \leq 0$, $-\xi_-(t)$ is a subordinator (see e.g., Sato [54], Theorem 24.7), with zero tail. On the other hand, ξ_+ is a $\Gamma(C_+, M)$ process, so that $\xi_+(t)$ has semi-heavy probability density function

$$f_{\xi_+(t)}(x) = \frac{M^{C_+t} x^{C_+t-1}}{\Gamma(C_+t)}\,\mathrm{e}^{-Mx} \quad \text{for } x > 0, \tag{5.18}$$

As $\xi_-(t)$ has lighter tails than $\xi_+(t) \in \mathcal{L}(\alpha)$, we may use Embrechts and Goldie [31], Theorem 3 a, to conclude that $\xi(t) \in \mathcal{L}(\alpha)$. Moreover, it follows readily by inspection of Equation 2.9 of the proof of that theorem, that

$$\mathbf{P}\{\xi(t) > u\} \sim \mathbf{E}\left\{\mathrm{e}^{G\xi_-(t)}\right\} \mathbf{P}\{\xi_+(t) > u\} \sim \mathbf{E}\left\{\mathrm{e}^{G\xi_-(t)}\right\} u^{C_+t-1}\mathrm{e}^{-Mu} \tag{5.19}$$

as $u \to \infty$ [see also Cline [26], Corollary 2.7, for a corresponding result for $\mathcal{S}(\alpha)$]. Now Example 1.14 shows that $\xi(h) \in \mathcal{L}(M) \setminus \mathcal{S}(M)$, and it follows that Theorem 5.2 applies with $L(t) = 0$ for $t < h$, so that $H = 1$ in (5.2).

For $Y_+ < 0$, the density of the Lévy measure has semi-heavy tails in $\mathcal{L}(M) \setminus \mathcal{S}(M)$

$$\frac{d\nu(x)}{dx} = C_+ x^{-(1+Y_+)}\mathrm{e}^{-Mx}.$$

Hence Theorem 1.16 shows that $\xi(h) \in \mathcal{L}(M) \setminus \mathcal{S}(M)$, and that Theorem 5.2 applies with $L(t) = 0$ for $t < h$, so that $H = 1$ in (5.2).

Notice that the results of Braverman and Samorodnitsky [22] and Braverman [20] (see Corollary 5.3) do not apply when $Y_+ \leq 0$.

Example 5.11. The class of VG(C, G, M) Lévy processes is the special case $C_- = C_+ = C$ and $Y_- = Y_+ = 0$ of the CGMY processes considered in Example 5.10. These processes can be represented as the difference between two gamma processes. VG processes were introduced by Madan and Seneta [45], Section 1. See Schoutens [56], Section 5.3.7, for more information on VG processes.

For a VG(C,G,M) process ξ, using (5.18) together with (5.19), observing that now $-\xi_-$ is a $\Gamma(C,G)$ process, we get

$$\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u\Big\}\sim\mathbf{P}\{\xi(h)>u\}\sim\Big(\frac{G}{G+M}\Big)^{Ch}u^{Ct-1}\mathrm{e}^{-Mu}\quad\text{as } u\to\infty.$$

Chapter 6

Superexponential Lévy Processes

In this chapter we study processes with superexponential tails. The well-known example of such a Lévy process is, of course, Brownian motion. But, albeit less well-known, there exist many other Lévy processes of this category. One natural example is the class of totally skewed to the left α-stable processes, which do in fact include Brownian motion (the case when $\alpha = 2$).

Theorem 6.1. *Let $\{\xi(t)\}_{t\geq 0}$ be a superexponential Lévy process with $\xi(0) = 0$ and infinite upper end-point (1.42). Assume that there exist functions $w > 0$ and $q > 0$, with w continuous, such that (with obvious notation)*

$$\frac{\xi(aq(u))}{w(u)} \stackrel{d}{\rightarrow} \zeta_a \quad \text{as } u \rightarrow \infty \text{ for } a > 0; \tag{6.1}$$

$$L(t,x) = \lim_{u\rightarrow\infty} \frac{\mathbf{P}\{\xi(h - q(u)t) > u + xw(u)\}}{\mathbf{P}\{\xi(h) > u\}} \quad \text{exists for } t > 0 \text{ and } x \in \mathbb{R}, \tag{6.2}$$

with $L(0,x) = \mathrm{e}^{-x}$ (so that $\xi(h)$ belong to the Type I domain of attraction of extremes). Further, assume that ζ_a is continuously distributed for $a > 0$, or that $L(t,x)$ is a continuous function of x for $t > 0$. If

$$\lim_{T\rightarrow\infty} \limsup_{u\rightarrow\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\Big\{\sup_{t\in[0,h-Tq(u)]} \xi(t) > u\Big\} = 0, \tag{6.3}$$

then the following limit exists, with value $H \in [1,\infty)$:

$$H = \lim_{u\rightarrow\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\Big\{\sup_{t\in[0,h]} \xi(t) > u\Big\}. \tag{6.4}$$

Proof. Let $\{\zeta_a(i)\}_{i=1}^{\infty}$ be independent random variables distriubted as ζ_a. Notice that, by repeated use of (6.2),

$$\lim_{u\to\infty} \mathbf{P}\Big\{\frac{\xi(h-q(u)t)-u}{w(u)} > x \,\Big|\, \xi(h-q(u)t) > u\Big\}$$
$$\lim_{u\to\infty} \frac{\mathbf{P}\{\xi(h-q(u)t) > u + xw(u)\}}{\mathbf{P}\{\xi(h-q(u)t) > u\}} = \frac{L(t,x)}{L(t,0)} \quad \text{for } x>0 \text{ if } L(t,0)>0.$$

Letting η_t denote a possibly infinite valued random variable, independent of $\{\zeta_a(i)\}_{i=1}^{\infty}$, with the possibly improper probability distribution function $1-L(t,x)$ $/L(t,0)$, when $L(t,0)>0$, and with $\eta_t=\infty$ when $L(t,0)=0$, (6.1) thus gives the following lower bound

$$\begin{aligned}
&\liminf_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h)>u\}} \mathbf{P}\Big\{\sup_{t\in[0,h]} \xi(t) > u\Big\} \\
&\geq \lim_{T\to\infty} \limsup_{a\downarrow 0} \liminf_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h)>u\}} \mathbf{P}\Big\{\max_{k=0,\ldots,\lfloor T/a\rfloor} \xi(h-kaq(u)) > u\Big\} \\
&= \lim_{T\to\infty} \limsup_{a\downarrow 0} \sum_{k=0}^{\lfloor T/a\rfloor} L(ka,0) \liminf_{u\to\infty} \mathbf{P}\Big\{\bigcap_{\ell=0}^{k-1}\{\xi(h-\ell aq(u)) \\
&\qquad -\xi(h-kaq(u))+\xi(h-kaq(u))-u \leq 0\} \,\Big|\, \xi(h-kaq(u))>u\Big\} \qquad (6.5) \\
&\geq \lim_{T\to\infty} \limsup_{a\downarrow 0} \sum_{k=0}^{\lfloor T/a\rfloor} L(ka,0) \mathbf{P}\Big\{\bigcap_{\ell=0}^{k-1}\Big\{\sum_{i=0}^{k-\ell} \zeta_i(a) + \eta_{ka} < 0\Big\}\Big\} \\
&= \lim_{T\to\infty} \limsup_{a\downarrow 0} \sum_{k=0}^{\lfloor T/a\rfloor} L(ka,0) \mathbf{P}\Big\{\bigcap_{\ell=0}^{k-1}\Big\{\sum_{i=0}^{k-\ell} \zeta_i(a) + \eta_{ka} \leq 0\Big\}\Big\},
\end{aligned}$$

by the assumed continuity properties of ζ_a and/or $L(t,\cdot)$.

To get a matching upper bound, we make several preparations: First, notice that, by the strong Markov property,

$$\begin{aligned}
&\mathbf{P}\Big\{\sup_{t\in[h-Tq(u),h]} \xi(t) > u+xw(u)\Big\} \\
&\quad \leq \mathbf{P}\Big\{\max_{k=0,\ldots,\lfloor T/a\rfloor} \xi(h-kaq(u)) > u\Big\} \\
&\qquad + \mathbf{P}\Big\{\sup_{t\in[h-Tq(u),h]} \xi(t) > u+xw(u)\Big\} \mathbf{P}\Big\{\inf_{t\in[0,aq(u)]} \xi(t) \leq -xw(u)\Big\} \quad \text{for } x>0.
\end{aligned} \qquad (6.6)$$

From the continuity of w, and the fact that $w(u)=o(u)$, as a consequence of the fact that $\xi(h)$ belongs to the Type I domain of attraction, as noted after after Definition 1.20), u and $u+xw(u)$ range over the same set of values as $u\to\infty$, for any constant $x>0$. Hence we have

$$\limsup_{u\to\infty} g(u) = \limsup_{u\to\infty} g(u+xw(u)) \quad \text{for } x\in\mathbb{R} \text{ for any function } g. \tag{6.7}$$

From (6.1) together with basic theory of Lévy processes (see e.g., Sato [54], Theorem 8.7, together with Fristedt [39], p. 251), we have that $\{\xi(tq(u))/w(u)\}_{t\geq 0} \stackrel{\mathrm{d}}{\to} \{\zeta(t)\}_{t\geq 0}$ in the space $D[0,1]$ of càdlàg functions equipped with the Skorohod J_1 topology, where $\{\zeta(t)\}_{t\geq 0}$ is a Lévy process such that $\zeta(t)\stackrel{\mathrm{d}}{=}\zeta_t$. From this in turn, we have

$$\begin{aligned}\liminf_{a\downarrow 0}\liminf_{u\to\infty}\mathbf{P}\Big\{\inf_{t\in[0,aq(u)]}\xi(t) > -xw(u)\Big\}\\ \geq \liminf_{a\downarrow 0}\mathbf{P}\Big\{\inf_{t\in[0,2a]}\zeta(t) > -\frac{x}{2}\Big\} = 1 \quad \text{for } x>0.\end{aligned} \tag{6.8}$$

Using (6.6)-(6.8) together with (6.2) and (6.1), get in the fashion of (6.5),

$$\begin{aligned}&\limsup_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u\Big\}\\
&= \lim_{T\to\infty}\limsup_{x\downarrow 0}\limsup_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u+xw(u)\}}\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u+xw(u)\Big\}\\
&= \lim_{T\to\infty}\limsup_{x\downarrow 0}\limsup_{u\to\infty}\frac{\mathrm{e}^x}{\mathbf{P}\{\xi(h)>u\}}\mathbf{P}\Big\{\sup_{t\in[0,h]}\xi(t)>u+xw(u)\Big\}\\
&\leq \lim_{T\to\infty}\limsup_{x\downarrow 0}\liminf_{a\downarrow 0}\limsup_{u\to\infty}\bigg(\mathbf{P}\Big\{\inf_{t\in[0,aq(u)]}\xi(t)>-xw(u)\Big\}\bigg)^{-1}\\
&\qquad\times\frac{1}{\mathbf{P}\{\xi(h)>u\}}\mathbf{P}\Big\{\max_{k=0,\ldots,\lfloor T/a\rfloor}\xi(h-kaq(u))>u\Big\}\\
&\qquad+\lim_{T\to\infty}\limsup_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\mathbf{P}\Big\{\sup_{t\in[0,h-Tq(u)]}\xi(t)>u\Big\}\\
&\leq \lim_{T\to\infty}\liminf_{a\downarrow 0}\limsup_{u\to\infty}\frac{1}{\mathbf{P}\{\xi(h)>u\}}\mathbf{P}\Big\{\max_{k=0,\ldots,\lfloor T/a\rfloor}\xi(h-kaq(u))>u\Big\}\\
&\leq \lim_{T\to\infty}\liminf_{a\downarrow 0}\sum_{k=0}^{\lfloor T/a\rfloor}L(ka,0)\,\mathbf{P}\bigg\{\bigcap_{\ell=k-1}^{0}\Big\{\sum_{i=0}^{k-l}\zeta_i(a)+\eta_{ka}\leq 0\Big\}\bigg\}.\end{aligned} \tag{6.9}$$

From (6.3) together with (6.5) and (6.9), we have that the following three limits exist and coincide

$$
\begin{aligned}
H &= \lim_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h)>u\}} \mathbf{P}\Big\{\sup_{t\in[0,h]} \xi(t)>u\Big\} \\
&= \lim_{T\to\infty} \limsup_{a\downarrow 0} \sum_{k=0}^{\lfloor T/a\rfloor} L(ka,0)\,\mathbf{P}\Big\{\bigcap_{\ell=k-1}^{0}\Big\{\sum_{i=0}^{k-\ell}\zeta_i(a)+\eta_{ka}\le 0\Big\}\Big\} \qquad (6.10)\\
&= \lim_{T\to\infty} \liminf_{a\downarrow 0} \sum_{k=0}^{\lfloor T/a\rfloor} L(ka,0)\,\mathbf{P}\Big\{\bigcap_{\ell=k-1}^{0}\Big\{\sum_{i=0}^{k-\ell}\zeta_i(a)+\eta_{ka}\le 0\Big\}\Big\}.
\end{aligned}
$$

It remains to show $H<\infty$. However, this follows from applying (6.3) and (6.8) to the following version of (6.9), with $a>0$ small enough and $T>0$ large enough,

$$
\begin{aligned}
&\limsup_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h)>u\}} \mathbf{P}\Big\{\sup_{t\in[0,h]} \xi(t)>u\Big\}\\
&\le e^{x}\Big(\mathbf{P}\Big\{\inf_{t\in[0,2a]}\zeta(t)>-\frac{x}{2}\Big\}\Big)^{-1}\\
&\qquad\times \limsup_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h)>u\}} \mathbf{P}\Big\{\max_{k=0,\ldots,\lfloor T/a\rfloor} \xi(h-kaq(u))>u\Big\}\\
&\qquad+ \limsup_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h)>u\}} \mathbf{P}\Big\{\sup_{t\in[0,h-Tq(u)]} \xi(t)>u\Big\}\\
&\le \limsup_{x\downarrow 0}\Big(\mathbf{P}\Big\{\inf_{t\in[0,2a]}\zeta(t)>-\frac{x}{2}\Big\}\Big)^{-1} \sum_{k=0}^{\lfloor T/a\rfloor} L(ka,0)\\
&\qquad+ \limsup_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h)>u\}} \mathbf{P}\Big\{\sup_{t\in[0,h-Tq(u)]} \xi(t)>u\Big\} \quad \text{for } x>0. \qquad \square
\end{aligned}
$$

Our tools for verifying (6.1) and (6.2) are Proposition 1.30 and Theorem 1.21, respectively. To check (6.3), we have two options: If Corollary (2.3) applies, then it should be used together with (1.49), to prove (6.3). This is how we work in our examples below. If Corollary (2.3) does not apply, then (6.3) can shown using Theorem 2.1 and Corollary 1.23, to show that the tail of $\sup_{t\in[0,\hat{h}]}\xi(t)$ is asymptotically neglible, for $0<\hat{h}<h$. Then the tail of $\sup_{t\in[\hat{h},h-Tq(u)]}\xi(t)$ can be dealt with using Remark 1.22 and the following version of (6.6):

$$
\begin{aligned}
&\mathbf{P}\Big\{\sup_{t\in[\hat{h},h-Tq(u)]} \xi(t)>u\Big\}\\
&\le 2\Big(\mathbf{P}\Big\{\inf_{t\in[0,2a]}\zeta(t)>-\frac{x}{2}\Big\}\Big)^{-1}\mathbf{P}\Big\{\max_{k=\lfloor T/a\rfloor,\ldots,\lfloor(h-\hat{h}/(aq(u))\rfloor} \xi(h-kaq(u))>u-xw(u)\Big\}\\
&\le 2\Big(\mathbf{P}\Big\{\inf_{t\in[0,2a]}\zeta(t)>-\frac{x}{2}\Big\}\Big)^{-1}\sum_{k=\lfloor T/a\rfloor}^{\lfloor(h-\hat{h}/(aq(u))\rfloor} \mathbf{P}\{\xi(h-kaq(u))>u-xw(u)\}.
\end{aligned}
$$

Of course, Brownian motion is the cannonical example of a superexponential Lévy process. As have been noted already (see Remark 2.2), this is the only non-trivial process which has both lower and upper tails that are superexponential.

Example 6.2. Brownian motion with drift is a superexponential Lévy process $\{\xi(t)\}_{t\geq 0}$ with characteristic triple $(0, m, s^2)$, where $m \in \mathbb{R}$ and $s^2 > 0$.

By Proposition 1.29 2, ξ satisfies (1.44)-(1.47), so that Theorem 1.21 gives (6.2). Further, Proposition 1.30 2 gives (6.1) with ζ_a $\mathrm{N}(0, 2a)$ distributed.

For $m \geq 0$ (6.3) follows readily from Corollary 2.3 together with (1.49). For $m < 0$ we cannot use Corollary 2.3 directly, as (2.3) does not hold. However, a simple trick does the job: Let $\{\xi_0(t)\}_{t\geq 0}$ be the Lévy process with characteristic triple $(0, 0, s^2)$. Then Corollary 2.3 together with (1.49) give (6.3) in the following way, using that $q(u) = o(w(u))$ by Corollary 1.24:

$$\begin{aligned}
\mathbf{P}\Big\{\sup_{t\in[0,h-Tq(u)]} \xi(t) > u\Big\} &\leq \mathbf{P}\Big\{\sup_{t\in[0,h-Tq(u)]} \xi_0(t) > u - m(h - Tq(u))\Big\} \\
&\leq \mathbf{P}\Big\{\sup_{t\in[0,h-Tq(u)]} \xi_0(t) > u - mh - w(u)\Big\} \\
&\leq 2\,\mathbf{P}\{\xi_0(h - Tq(u)) > u - mh - w(u)\} \\
&\leq 2\,\mathbf{P}\{\xi_0(h - Tq(u)) > u - m(h - Tq(u)) - w(u)\} \\
&= 2\,\mathbf{P}\{\xi(h - Tq(u)) > u - w(u)\} \\
&\sim 2\,\mathrm{e}^{1-T}\mathbf{P}\{\xi(h) > u)\} \quad \text{as } u \to \infty.
\end{aligned} \tag{6.11}$$

Notice that, by Proposition 1.29 2 together with Proposition 1.30 2, (6.1) and (6.2) hold with ζ_a $\mathrm{N}(0, 2a)$ distributed, and the functions w and q given by (1.48), for any Lévy process with characteristic triple (ν, m, s^2) such that $\nu((0, \infty)) = 0$ and $s^2 > 0$. However, we cannot hope to verify (6.3) as simply as for Brownian motion. Rather, the second of the strategies for verifying (6.3) outlined above has to be employed. But if (6.3) holds, then we have $H = 2$ in (6.4), independently of ν and m, by well-known properties of Brownian motion without drift!

Example 6.3. A totally skewed to the left α-stable, $\alpha \in (1, 2)$, Lévy motion

$\{\xi(t)\}_{t\geq 0}$, is a Lévy process with charactersitic triple $(\nu, m, 0)$, where

$$\frac{d\nu(x)}{dx} = \frac{\alpha\gamma^\alpha}{(-\Gamma(1-\alpha))(-\cos(\frac{\pi\alpha}{2}))(-x)^{\alpha+1}} \quad \text{for } x<0,$$

and $\gamma > 0$ and $m \in \mathbb{R}$ are constants. As have been mentioned in Example 3.8, Albin [2], Theorem 1, state that (6.4) holds with $H > 1$ for $m = \int_{\mathbb{R}}(\kappa(x) - x)d\nu(x)$ (the strictly stable case). We will now recover Albin's result, and extend it to a general m, as a simple application of Theorem 6.1, and without relying on difficult results from the literature about the tails of totally skewed α-stable distributions, as did Albin.

By Proposition 1.29 10, ξ satisfies (1.44)-(1.47), so that Theorem 1.21 gives (6.2). Further, Proposition 1.30 3 gives (6.1) with ζ_a $S_\alpha((-a\cos(\frac{\pi\alpha}{2}))^{1/\alpha}, -1, 0)$ distributed.

For $\mathfrak{m} \geq 0$ (6.3) follows from Corollary 2.3 together with (1.49), since

$$\begin{aligned}\mathbf{P}\{\xi(t) > 0\} \geq \mathbf{P}\{(\xi(t) - \mathfrak{m}t) > 0\} &= \mathbf{P}\left\{t^{1/\alpha}(\xi(1) - \mathfrak{m}1) > 0\right\} \\ &= \mathbf{P}\{(\xi(1) - \mathfrak{m}1) > 0\} > 0\end{aligned}$$

by self-similarity. For $\mathfrak{m} < 0$ we can use the trick (6.11) in exactly the same way as in Example 6.2, observing that again $q(u) = o(w(u))$.

Notice that, by Proposition 1.29 3 together with Proposition 1.30 3, (6.1) and (6.2) hold with ζ_a $S_\alpha((-a\cos(\frac{\pi\alpha}{2}))^{1/\alpha}, -1, 0)$ distributed, and the functions w and q given by (1.48), for any Lévy process with characteristic triple (ν, m, s^2) such that $\nu((0,\infty)) = 0$ and (1.70) holds. However, we cannot hope to verify (6.3) as simply as for α-stable processes: Rather, the second of the strategies for verifying (6.3) outlined above has to be employed. But if (6.3) turns out to hold, then we have $H > 1$ in (6.4), because H only depends on α, and was shown to satisfy $H > 1$ for totally skewed α-stable processes with $\mathfrak{m} = 0$ by Albin [2], Theorem 1.

Example 6.4. A totally skewed to the left 1-stable Lévy motion $\{\xi(t)\}_{t\geq 0}$, is a Lévy process with charactersitic triple $(\nu, m, 0)$, given by

$$\frac{d\nu(x)}{dx} = \frac{2\gamma}{\pi(-x)^2} \quad \text{for } x<0, \quad \text{where } \gamma > 0 \text{ and } m \in \mathbb{R}.$$

As have been mentioned in Example 3.9, Albin [2], Theorem 2, state that (6.4) holds with $H = 1$ for $m = 0$. We will now recover Albin's result, and extend it to a general m, as a simple application of Theorem 6.1, and without relying on difficult results from the literature about the tails of totally skewed 1-stable distributions, as did Albin.

By Proposition 1.31, (6.1) and (6.2) hold with $\zeta_a = a$. We may derive (6.3) from Corollary 2.3 together with (1.88), for all values of m, because

$$\mathbf{P}\{\xi(t) > 0\} \geq \mathbf{P}\Big\{t\xi(1) - \frac{2\gamma t \ln(t)}{\pi} + mt > 0\Big\} \to 1 \quad \text{as } t \downarrow 0.$$

Notice that, by Proposition 1.29 7 together with Proposition 1.30 4, (6.1) and (6.2) hold with $\zeta_a = a$ and the functions w and q given by (1.48), for any Lévy process with characteristic triple (ν, m, s^2) such that $\nu((0,\infty)) = 0$ and (1.87) holds. In this case, we always have $H = 1$, by inspection of (6.10).

We mention here that the methodology of Examples 6.2-6.4 readily carry over to deal with, for example, the sum of two independent totally skewed to the left stable Lévy processes, with different stability indices.

Example 6.5. An unnamed, but quite famous superexponential Lévy process $\{\xi(t)\}_{t\geq 0}$ is defined by Linnik and Ostrovskiĭ [44] pp. 52-53, see also Sato [54], Exercise 18.19. This process has characteristic triple $(\nu, m, 0)$, where

$$\frac{d\nu(x)}{dx} = \frac{\mathrm{e}^{bx}}{|x|(1-\mathrm{e}^{ax})} \quad \text{for } x < 0, \quad \text{for some constants } a, b > 0.$$

The corresponding Laplace transform is

$$\phi_1(\lambda) = \frac{\Gamma((b-\lambda)/a)c^{\lambda/a}}{\Gamma(b/a)} \quad \text{for } \lambda \leq 0,$$

where $c > 0$ is a parameter (that does not affect the Lévy measure).

By Proposition 1.31, (6.1) and (6.2) hold with $\zeta_a = a$. We may derive (6.3) from Corollary 2.3 together with (1.88), because selecting $g(t) > 0$ so that $t\ln(1/g(t))/g(t) \to 1$ as $t \downarrow 0$, we have (see e.g., Erdélyi, Magnus, Oberhettinger and Tricomi [34], Equation 1.18.2)

$$\phi_1(\lambda/g(t))^t$$

$$\sim \frac{1}{(2\pi)^{t/2}} \exp\left\{ t\left(\frac{b-\lambda/g(t)}{a} - \frac{1}{2}\right) \ln\left(\frac{b-\lambda/g(t)}{a}\right) - t\,\frac{b}{a} + t\,\frac{\lambda/g(t)}{a} \ln(c/e) \right\}$$
$$\to \mathrm{e}^{(-\lambda)/a} \quad \text{as } t \downarrow 0 \text{ for } \lambda \leq 0,$$

so that

$$\mathbf{P}\{\xi(t) > 0\} = \mathbf{P}\{\xi(t)/g(t) > 0\} \to \mathbf{P}\{1/a > 0\} = 1 \quad \text{as } t \downarrow 0. \quad \square$$

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