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MATS ANDERSSON

Department of Mathematics
CHALMERS UNIVERSITY OF TECHNOLOGY
GÖTEBORGS UNIVERSITY
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Mats Andersson

CHALMERS | GÖTEBORGS UNIVERSITET



Mathematics
Department of Mathematics
Chalmers University of Technology and Göteborg University
SE-412 96 Göteborg, Sweden
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ABSTRACT. Given a generically surjective holomorphic vector bundle morphism $f: E \rightarrow Q$, E and Q Hermitian bundles, we construct a current R^f with values in $\text{Hom}(Q, H)$, where H is a certain derived bundle, and with support on the set Z where f is not surjective. The main property is that if ϕ is a holomorphic section of Q , and $R^f \phi = 0$, then locally $f\psi = \phi$ has a holomorphic solution ψ . In the generic case also the converse holds. This gives a generalization of the corresponding theorem for a complete intersection, due to Dickenstein-Sessa and Passare. We also present results for polynomial mappings, related to M Noether's theorem and the effective Nullstellensatz. The construction of the current is based on a generalization of the Koszul complex. By means of this complex one can also obtain new global estimates of solutions to $f\psi = \phi$, and as an example we give new results related to the H^p -corona problem.

1. INTRODUCTION

Let E and Q be holomorphic Hermitian vector bundles of ranks m and r , respectively, over the n -dimensional complex manifold X , and let $f: E \rightarrow Q$ be a generically surjective holomorphic morphism. Given a holomorphic section ϕ of Q we are interested in holomorphic solutions ψ to $f\psi = \phi$. The basic results in this area are the existence theorems due to Skoda in [21] and [22], which are based on L^2 -methods and complex geometry. They provide existence of global holomorphic solutions to the equation $f\psi = \phi$ with L^2 -estimates under appropriate geometric conditions provided f is pointwise surjective. However, applying these results to E restricted to $X \setminus Z$, where

$$Z = \{z; f(z) \text{ is not surjective}\},$$

also highly non-trivial local results at Z are obtained by these methods.

In this paper we introduce a complex of bundles

$$\cdots \rightarrow E_3 \rightarrow E_2 \rightarrow E \rightarrow Q \rightarrow 0,$$

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and define a global residue current

$$R^f = R_p^f + \cdots + R_\mu^f$$

with support on Z , $p = \text{codim } Z$ and $\mu = \min(n, m - r + 1)$, where R_k^f is a $(0, k)$ -current with values in $\text{Hom}(Q, E_k)$. It is not hard to see (e.g., by using Gauss elimination) that $p \leq m - r + 1$ with equality in the generic case; in this case, thus $R = R_{m-r+1}$. Our first result concerns existence of local holomorphic solutions of $f\psi = \phi$.

Theorem 1.1. *Let E and Q be holomorphic Hermitian vector bundles over a complex manifold X , let $f: E \rightarrow Q$ be a holomorphic generically surjective morphism, and let R^f be the corresponding residue current. If ϕ is a holomorphic section of Q such that $R^f\phi = 0$, then locally $f\psi = \phi$ has a holomorphic solution ψ .*

We have the following partial converse.

Theorem 1.2. *If $p = m - r + 1$ and $f\psi = \phi$ has a holomorphic solution, then $R^f\phi = 0$.*

If $p = m - r + 1$ thus $f\psi = \phi$ has holomorphic solutions if and only if $R^f\phi = 0$. If $r = 1$ and $p = m$, it turns out that if $f = \sum f_j e_j$ in a local holomorphic frame e_j , then $R = R_m$ is equal to the classical Coleff-Herrera current

$$T = \left[\bar{\partial} \frac{1}{f_1} \wedge \cdots \wedge \bar{\partial} \frac{1}{f_m} \right]$$

times a non-vanishing section of $(\det E) \otimes Q^*$, see [4]. In this case therefore we get back the Dickenstein-Sessa-Passare theorem, [12] and [19], stating that ϕ belongs to the ideal (f_1, \dots, f_m) if and only if $\phi T = 0$.

Instead of the usual norm $|\phi|$ of a section ϕ of Q it is natural, e.g., in view of the results in [22], to introduce the stronger pointwise norm

$$\|\phi\|^2 = \det(ff^*) |f^*(ff^*)^{-1}\phi|^2 = \langle \widetilde{ff^*}\phi, \phi \rangle,$$

where $\widetilde{ff^*} = \det(ff^*)(ff^*)^{-1}$ is the smooth endomorphism on Q whose matrix is the transpose of the comatrix of ff^* . By analyzing the singularity of R^f we obtain the following sufficient size condition on ϕ for annihilating the residue.

Proposition 1.3. *Let $f: E \rightarrow Q$ be a holomorphic generically surjective morphism. If ϕ is a holomorphic section of Q such that*

$$(1.1) \quad \|\phi\|^2 \leq C \det(ff^*)^{\min(n, m-r+1)},$$

then $R^f\phi = 0$.

As an immediate consequence we get the following generalization of the Briançon-Skoda theorem.

Theorem 1.4. *If ϕ is a holomorphic section of Q such that (1.1) holds, then locally $f\psi = \phi$ has a holomorphic solution ψ .*

In the case when $r = 1$, (1.1) means precisely that $|\phi| \leq C|f|^{\min(n,m)}$, and the conclusion is then that ϕ is locally in the ideal (f) generated by f (i.e., the ideal generated by f_j if $f = \sum f_j e_j$ in some local holomorphic frame e_j). This immediately implies the classical Briancon-Skoda theorem, [9], which states that $\phi^{\min(m,n)}$ belongs to (f) if $|\phi| \leq C|f|$. For $m - r + 1 \leq n$, Theorem 1.3 also follows directly from Skoda's L^2 -estimate, [22], but when $m - r + 1 > n$, the L^2 -estimate only gives the conclusion if the power of the right hand side of (1.1) is $n + 1$. By an additional argument in the case when $m > n$, the classical Briancon-Skoda theorem follows from the L^2 -estimate; the case when $m > n$ and $r > 1$ can be reduced to the classical result by means of the Fuhrmann trick, see Section 7.

Demaily has extended Skoda's L^2 -theorems to $\bar{\partial}$ -closed sections, see [10] and [11]. Our method also admits such an extension of the local result.

Theorem 1.5. *Assume that ϕ is a smooth $\bar{\partial}$ -closed $(0, q)$ -section of Q . If $R^f \phi = 0$, then locally $f\psi = \phi$ has $\bar{\partial}$ -closed (current) solutions ψ .*

For degree reasons we see that $R^f \phi = 0$ if $q > n - p$. Thus we get

Corollary 1.6. *If ϕ is any smooth $\bar{\partial}$ -closed $(0, q)$ -form with values in Q , and $q > n - p = \dim Z$, then locally $f\psi = \phi$ has $\bar{\partial}$ -closed current solutions.*

In analogy with Theorems 1.3 and (1.4) we also have

Theorem 1.7. *If ϕ is a smooth $\bar{\partial}$ -closed $(0, q)$ -form with values in Q such that*

$$(1.2) \quad \|\phi\|^2 \leq C \det(ff^*)^{\min(n-q, m-1+r)},$$

then $R^f \phi = 0$, and locally there are integrable $\bar{\partial}$ -closed solutions ψ to $f\psi = \phi$.

We can also obtain global results and first we turn our attention to polynomial ideals and generalize the approach in [5]. Let $[z] = [z_0, \dots, z_n]$ be homogeneous coordinates on \mathbb{P}^n , and let $z' = (z_1, z_2, \dots, z_n)$ be the standard coordinates in the standard affinization $\mathbb{C}^n \simeq \{[z], z_0 \neq 0\}$. Let P be a polynomial mapping $\mathbb{C}^n \rightarrow \text{Hom}(\mathbb{C}^m, \mathbb{C}^r)$ with columns P^j such that $\deg P^j \leq d_j$, $j = 1, \dots, m$. If f is the matrix whose columns are the d_j -homogenized forms $f^j(z) = z_0^{d_j} P^j(z'/z_0)$ in \mathbb{C}^{n+1} , then f defines a morphism

$$f: \bigoplus_1^m \mathcal{O}(-d_j) \rightarrow \mathbb{C}^r.$$

Let Z be the algebraic variety in \mathbb{P}^n where f is not surjective, and let R^f be the associated residue current with respect to the natural metric.

Theorem 1.8. *Assume that P is a polynomial mapping as above, and let Φ be a r -column of polynomials of degrees $\leq \rho$. Moreover, assume that*

$$(1.3) \quad m \leq n + r - 1 \quad \text{or} \quad \rho \geq \sum_{j=1}^{n+r} d_j - n,$$

where $d_1 \geq d_2 \geq \dots \geq d_m$. If $R^f \phi = 0$, then there are polynomials Q_j such that $\sum_1^m P^j Q_j = \Phi$, and $\deg P^j Q_j \leq \rho$.

Corollary 1.9. *Assume that Z is empty. Then we can find a matrix Q of polynomials with rows Q_k such that $PQ = \sum P^k Q_k = I_r$, and $\deg P^k Q_k \leq \sum_{j=1}^{n+r} d_j - n$.*

This is a generalization of a classical theorem of Macaulay, [17].

Corollary 1.10. *Let Φ be a column of polynomials, $\deg \Phi \leq \rho$, and let ϕ be its ρ -homogenization. If*

$$\|\phi\|^2 \leq C \det(f f^*)^{\min(n, m-r+1)}$$

in \mathbb{P}^n , and (1.3) is fulfilled, then $PQ = \Phi$ has a solution with $\deg P^k Q_k \leq \rho$.

Assume that P is pointwise surjective in \mathbb{C}^n , and let $P^j = (P_1^j, \dots, P_r^j)^t$. By the local Lojasiewicz inequality there is a constant M such that

$$(1.4) \quad \sum_{|I|=r} \frac{|\det(P_k^{I_j}(z'))|^2}{(1 + |z'|^2)^{\sum_1^r d_{I_j}}} \geq C \frac{1}{(1 + |z'|^2)^M},$$

where the sum is over increasing multiindices.

Corollary 1.11. *Assume that $P: \mathbb{C}^n \rightarrow \text{Hom}(\mathbb{C}^m, \mathbb{C}^r)$ is surjective in \mathbb{C}^n , $\deg P \leq d$ and that (1.4) holds. Then there is a matrix Q of polynomials such that $PQ = I_r$ and $\deg P^j Q_j \leq M \min(n, m - r + 1)$.*

From Kollar's famous theorem, [16], we can get an estimate of M . For simplicity we assume $d_j = d$ for all j .

Proposition 1.12. *If $d_j = d$ for all j , then the inequality (1.4) holds with*

$$M = (rd)^{\min(n, m!/(m-r)!r!)},$$

provided that $rd \geq 3$.

It should be pointed out that the bound

$$\deg Q + d \leq \min(n, m - r + 1)M$$

we obtain in this way for a solution to $PQ = I_r$ (even) is not optimal when $r = 1$. It is proved in [16] that one actually have $\deg Q + d \leq M$

when $r = 1$. We do not know if it is possible to modify Kollar's proof as to include the case $r > 1$ directly and get a sharper bound.

Now assume that $p = m - r + 1 = n$ and that Z is contained in \mathbb{C}^n ; thus a discrete set. Moreover, assume that $\Phi = PQ$ is solvable in \mathbb{C}^n . Then it follows from Theorem 1.2 that $R^f\phi = 0$ in \mathbb{C}^n , and hence $R^f\phi = 0$ in \mathbb{P}^n , and since (1.3) is fulfilled we can take $\rho = \deg \Phi$. Therefore there is a solution to $PQ = \Phi$ such that $\deg P^j Q_j \leq \deg \Phi$. When $r = 1$ this is a classical theorem due to Max Noether, [18]. We have the following generalization that appeared in [5] in the case $r = 1$; however we suspect that this case could be proved algebraically, e.g., by the methods in [15].

Theorem 1.13. *Assume that $P: \mathbb{C}^n \rightarrow \text{Hom}(\mathbb{C}^m, \mathbb{C}^r)$ and that $p = m - r + 1$ and that Z has no irreducible component contained in the hyperplane at infinity. Moreover, assume that $\Phi = PQ$ is solvable in \mathbb{C}^n . Then there is a solution Q such that $\deg P^j Q_j \leq \deg \Phi$.*

We can also obtain new global results in open bounded domains even when f is pointwise surjective, and as an example we present in Section 7 new sharpened estimates of solutions to the H^p -corona problem in a strictly pseudoconvex domain.

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2. A GENERALIZED KOSZUL COMPLEX

Let $f: E \rightarrow Q$ be a holomorphic morphism as above. Assume that we have a complex

$$(2.1) \quad \cdots \rightarrow E_3 \rightarrow E_2 \rightarrow E \rightarrow Q \rightarrow 0.$$

of vector bundles where all the morphisms, which we denote by δ , are holomorphic. Let $E_0 = Q$, $E_1 = E$, and let

$$H = \bigoplus_{k=0}^{\infty} E_k.$$

We will also consider $(0, *)$ -form-valued sections of H , i.e., sections of $T_{0,1}^*(X) \otimes H$. We denote this space of sections by $\mathcal{E}_{0,*}(X, H)$. Notice that it is a module over the ring (algebra) $\mathcal{E}_{0,*}(X)$. We extend the action of δ to sections of $T_{0,1}^*(X) \otimes H$ by requiring that

$$(2.2) \quad \delta\xi \otimes w = (-1)^{\deg \xi} \xi \otimes \delta w$$

if ξ is a differential form. Then $\delta\bar{\partial} = -\bar{\partial}\delta$.

Now suppose that we have $(0, k-1)$ -forms, or currents, v_k with values in E_k , $k \geq 1$, such that

$$(2.3) \quad (\delta - \bar{\partial})(v_1 + v_2 + \dots) = \phi,$$

i.e.,

$$(2.4) \quad \delta v_{k+1} = \bar{\partial}v_k, \quad k \geq 1, \quad \delta v_1 (= f v_1) = \phi.$$

For degree reasons, $\bar{\partial}v_k = 0$ if k is large enough, and if there are no obstructions for solving $\bar{\partial}$, we can successively find $(0, k-2)$ -forms (currents) w_k with values in E_k such that

$$(2.5) \quad \bar{\partial}w_k = v_k + \delta w_{k+1}, \quad k \geq 2.$$

Then finally

$$(2.6) \quad \psi = v_1 + \delta w_2$$

is a holomorphic solution to $f\psi = \phi$. Since the $\bar{\partial}$ -equations always are solvable locally we have

Lemma 2.1. *Suppose that we have a current solution $v = v_1 + v_2 + \dots$ to (2.3). Then locally there are holomorphic solutions to $f\psi = \phi$.*

If f is surjective, then obviously there are local holomorphic solutions to $f\psi = \phi$ so the interesting case is when f is just generically surjective. In view of the argument in the proof, one also gets a global holomorphic solution provided all the $\bar{\partial}$ -equations have global solutions. Before we proceed with our construction let us consider some examples.

Example 1. If the complex (2.1) is exact (in particular f is surjective), then we can always find such a solution to (2.3). In fact, given a holomorphic section ϕ of Q , let v_1 be any pointwise solution to $\delta v = \phi$. Then $\bar{\partial}v_1$ is δ -exact and hence there is a v_2 such that $\delta v_2 = \bar{\partial}v_1$ etc. \square

Example 2. If $r = 1$, i.e., Q is a line bundle, then one can take $E_k = \Lambda^k E \otimes Q^*$ and δ as interior multiplication with f . One then gets the usual Koszul complex

$$(2.7) \quad \dots \rightarrow \Lambda^3 E \otimes (Q^*)^2 \xrightarrow{\delta} \Lambda^2 E \otimes Q^* \xrightarrow{\delta} E \xrightarrow{f} Q \rightarrow 0,$$

which is exact if (and only if) f is non-vanishing. In fact, if we chose any section of $\text{Hom}(Q, E) \simeq E \otimes Q^*$ $f\sigma = I_Q$ (if E has a Hermitian metric we can, e.g., chose the section with pointwise minimal norm),

then there is an induced mapping $\sigma: \Lambda^k E \otimes (Q^*)^{k-1} \rightarrow \Lambda^{k+1} E \otimes (Q^*)^k$ such that $\delta \circ \sigma + \sigma \circ \delta = I$, and thus (2.7) is exact. \square

Example 3. Provided that f is surjective, a simple way to find an exact complex (2.1) is by taking $E_2 = \text{Ker } f$ and $E_k = 0$ for $k > 2$. However, E_2 is usually not trivial in a neighborhood of a singular point, i.e., E_2 usually cannot be extended as a vector bundle across the set Z where f is not surjective. To carry out the scheme in the proof of Lemma 2.1, one therefore has to solve a $\bar{\partial}$ -equation in the bundle $E_2 = \text{Ker } f$ over $X \setminus Z$. This is only possible under certain geometric conditions; this is precisely what is investigated and explained in [22]. \square

Let us now describe our generalized Koszul complex. Notice that our holomorphic morphism $f: E \rightarrow Q$ is a holomorphic section of the bundle $\text{Hom}(E, Q)$, which we identify with $E^* \otimes Q$. If ϵ_j is a local holomorphic frame for Q , then

$$f = \sum f_j \otimes \epsilon_j,$$

where f_j are sections of E^* . If η is a section of E , then $f\eta = \sum_1^r \delta_{f_j} \epsilon_j$, where δ_{f_j} denotes interior multiplication with f_j . We can associate to f the section

$$F = f_1 \wedge f_2 \dots \wedge f_r \otimes \epsilon_1 \wedge \dots \wedge \epsilon_r$$

of $\Lambda^r E^* \otimes \det Q^*$. It is independent of the particular choice of frame, and will be called the determinant section of f . Notice that f is surjective at a point if and only if F is non-vanishing at that point. There is an induced mapping

$$\delta_F: \Lambda^{r+1} E \otimes \det Q^* \rightarrow E$$

defined by

$$\delta_F(\xi \otimes \epsilon_1^* \wedge \dots \wedge \epsilon_r^*) = \delta_{f_r} \dots \delta_{f_1} \xi,$$

which is also easily seen to be independent of the particular local frame ϵ_j for Q ; here ϵ_j^* denotes the dual frame for Q^* . Moreover, it is also clear that

$$(2.8) \quad f \circ \delta_F = 0.$$

In order to proceed with the construction of our complex we have to recall some facts about symmetric tensors. Let $S^\ell Q^*$ be the subbundle of $\otimes Q^*$ consisting of symmetric ℓ -tensors of Q^* . If $u, v \in Q^*$ then $u \dot{\otimes} v = u \otimes v + v \otimes u$, etc. This extends to a commutative mapping $SQ^* \times SQ^* \rightarrow SQ^*$. If q is a section of Q , then it induces the usual interior multiplication on $\otimes Q^*$ (say from the left), and, in particular, if $u^p = u \dot{\otimes} \dots \dot{\otimes} u$, then $\delta_q u^p = p u^{p-1} (q \cdot u)$.

We now define

$$E_k = \Lambda^{r+k-1} E \otimes S^{k-2} Q^* \otimes \det Q^*, \quad k \geq 2.$$

Given the local frame ϵ_j , a section ξ of E_k can be written

$$\xi = \sum_{|\alpha|=k-2} \xi_\alpha \otimes \epsilon_\alpha^* \otimes \epsilon^*,$$

where

$$\epsilon_\alpha^* = \frac{(\epsilon_1^*)^{\alpha_1} \dot{\otimes} \dots \dot{\otimes} (\epsilon_r^*)^{\alpha_r}}{\alpha_1! \dots \alpha_r!}$$

and $\epsilon^* = \epsilon_1^* \wedge \dots \wedge \epsilon_r^*$. For $k \geq 2$ we have mappings $\delta: E_{k+1} \rightarrow E_k$ defined by

$$\xi \otimes q^* \otimes \epsilon^* \mapsto \sum_{j=1}^r \delta_{f_j} \xi \otimes \delta_{\epsilon_j} q^* \otimes \epsilon^*,$$

which are also independent of the specific choice of local frame ϵ_j . Since δ_{f_j} anti-commute and δ_{ϵ_j} commute, it follows that $\delta^2 = 0$. Moreover, if the section ξ of E_2 is in the image of δ , i.e., $\xi = \delta\eta = \sum \delta_{f_j} \eta_j \otimes \epsilon^*$, then clearly $\delta_F \xi = 0$. In view of (2.8) we thus have a complex

$$(2.9) \quad \dots \xrightarrow{\delta} E_3 \xrightarrow{\delta} E_2 \xrightarrow{\delta_F} E \xrightarrow{f} Q \rightarrow 0.$$

In the sequel, we will often denote all the mappings in (2.9) by δ . Observe that if $r = 1$, then (2.9) is just the Koszul complex (2.7).

If we let ξ above take values in $\Lambda(T^*(X)_{0,1} \oplus E)$ rather than just ΛE , then we get an extension of all the mappings δ and δ_F to forms and currents with values in E_k . The mappings $\delta: E_{k+1} \rightarrow E_k$, $k \geq 2$, will automatically satisfy (2.2) so that $\delta \bar{\partial} = -\bar{\partial} \delta$, but we should have to insert the factor $(-1)^{(r+1)q}$ in the definition of δ_F , when it acts on $\xi \otimes w$, and ξ is a $(0, q)$ -form. However, we will not do that, and therefore we have instead that $\bar{\partial} \delta_F = (-1)^r \delta_F \bar{\partial}$. This means that the final solution ψ in (2.6) is $\psi = v_1 + (-1)^{r+1} \delta w_2$.

3. SURJECTIVE MORPHISMS

Now we assume that f is surjective and that E and Q are equipped with Hermitian metrics. Let σ be the section of $\text{Hom}(Q, E) = E \otimes Q^*$ with pointwise minimal norm (i.e., such that $\text{Im } \sigma$ is orthogonal to $\text{Ker } f$) such that $f \circ \sigma = I_Q$. If ϵ_j^* denotes the dual frame for Q^* , then

$$\sigma = \sum \sigma_j \otimes \epsilon_j^*,$$

where σ_j are the sections of E with minimal norms such that $f_j \cdot \sigma_k = \delta_{jk}$. Moreover,

$$\mathcal{O} = \sigma_1 \wedge \dots \wedge \sigma_r \otimes \epsilon_1^* \wedge \dots \wedge \epsilon_r^*$$

is a well-defined section of $\Lambda E^r \otimes \det Q^*$, and it induces a mapping $E \rightarrow E_2 = \Lambda^{r+1} E \otimes \det Q^*$ defined by

$$\xi \mapsto \mathcal{O} \wedge \xi = \sigma_1 \wedge \dots \wedge \sigma_r \wedge \xi \otimes \epsilon_1^* \wedge \dots \wedge \epsilon_r^*.$$

Now, $\delta_F \mathcal{O} \xi = \xi$ provided that $\delta_{f_j} \xi = 0$ for all j , i.e., ξ is in $\text{Ker } f$. Thus (2.9) is exact at E if f is surjective. We also have

Lemma 3.1. *If f is surjective, then (2.9) is exact up to E_2 .*

Proof. It remains to check the exactness at E_2 . Suppose that $\xi \otimes \epsilon^*$ is a section of E_2 such that $0 = \delta_F \xi \otimes \epsilon^*$. Thus ξ is a section of $\Lambda^{r+1}E$, such that $\delta_{f_r} \cdots \delta_{f_1} \xi = 0$. By the surjectivity of f , σ_j are linearly independent and therefore they form a part of a basis $\sigma_1, \dots, \sigma_m$ for E . With respect to this basis we can write

$$\xi = \sigma_1 \wedge \dots \wedge \sigma_r \wedge \xi' + \xi'',$$

where ξ'' does not contain all the σ_j , $j = 1, \dots, r$. It follows that $\xi' = 0$ and hence $\xi = \sum_1^r \xi_j$, where ξ_j does not contain σ_j . Therefore

$$\xi \otimes \epsilon^* = \sum_1^r \delta_{f_j}(\sigma_j \wedge \xi_j) \otimes \epsilon^* = \delta\left(\sum_1^r \sigma_j \wedge \xi_j \otimes \epsilon_j^* \otimes \epsilon^*\right),$$

and thus (2.9) is exact at E_2 . \square

In order to find a solution to (2.3) it is natural to start with $v_1 = \sigma\phi = \sum \phi_j \sigma_j$, and $v_2 = \sigma \wedge \bar{\partial}\sigma\phi$. We can just as well suppress ϕ and define u_k with values in $\text{Hom}(Q, E_k)$ such that $u_k\phi$ satisfies (2.3). Notice that $\text{Hom}(Q, E_1) \simeq E \otimes Q^*$ and

$$\text{Hom}(Q, E_k) \simeq \Lambda^{r+k-1}E \otimes S^{k-2}Q^* \otimes \det Q^* \otimes Q^*, \quad k \geq 2.$$

Definition 1. We define the $(0, k-1)$ -forms u_k with values in $\text{Hom}(Q, E_k)$ as

$$(3.1) \quad u_1 = \sigma, \quad u_k = (\bar{\partial}\sigma)^{\otimes(k-2)} \otimes \sigma \otimes \bar{\partial}\sigma, \quad k \geq 2.$$

Notice that $(\bar{\partial}\sigma)^{\otimes(k-2)}$ is indeed a symmetric tensor, and that the definition of u_k is invariant. Moreover,

$$\left(\sum_1^r \bar{\partial}\sigma_\ell \otimes \epsilon_\ell^*\right)^{\otimes(k-2)} = \frac{(\sum_1^r \bar{\partial}\sigma_\ell \otimes \epsilon_\ell^*)^{\otimes(k-2)}}{(k-2)!} = \sum_{|\alpha|=k-2} (\bar{\partial}\sigma)^\alpha \otimes \epsilon_\alpha^*,$$

where

$$(\bar{\partial}\sigma)^\alpha = (\bar{\partial}\sigma_1)^{\alpha_1} \wedge \dots \wedge (\bar{\partial}\sigma_r)^{\alpha_r}.$$

Since $\bar{\partial}\sigma_j$ have degree 2 in $\Lambda(T_{0,1}^*(X) \oplus E)$ and therefore commute, we thus have that

$$(3.2) \quad u_1 = \sigma = \sum_j \sigma_j \otimes \epsilon_j^*,$$

$$u_k = \sigma_1 \wedge \dots \wedge \sigma_r \wedge \sum_{|\alpha|=k-2} \sum_j (\bar{\partial}\sigma)^\alpha \wedge \bar{\partial}\sigma_j \otimes \epsilon_\alpha^* \otimes \epsilon^* \otimes \epsilon_j^*, \quad k \geq 2.$$

Proposition 3.2. *Assume that f is surjective. If u is defined by (3.1), then*

$$(\delta - \bar{\partial})(u_1 + u_2 + \dots) = I_Q.$$

Proof. We have already seen that $\delta u_1 = f\sigma = I_Q$ and $\delta_F \sigma \otimes \bar{\partial}\sigma = \bar{\partial}\sigma = \bar{\partial}u_1$, so we have to verify that $\delta u_{k+1} = \bar{\partial}u_k$ for $k \geq 2$. Now,

$$u_{k+1} = \sigma_1 \wedge \dots \wedge \sigma_r \wedge \left(\sum_1^r \bar{\partial}\sigma_\ell \otimes \epsilon_\ell^* \right)^{\otimes(k-1)} \otimes \epsilon^* \otimes \bar{\partial}\sigma.$$

Recalling that $\delta = \sum \delta_{f_j} \otimes \delta_{\epsilon_j}$, and that δ_{ϵ_j} acts from the left, we get

$$\delta u_{k+1} = \sum_{j=1}^r \delta_{f_j} (\sigma_1 \wedge \dots \wedge \sigma_r) \wedge \bar{\partial}\sigma_j \wedge \left(\sum_1^r \bar{\partial}\sigma_\ell \otimes \epsilon_\ell^* \right)^{\otimes(k-2)} \otimes \epsilon^* \otimes \bar{\partial}\sigma.$$

On the other hand,

$$\bar{\partial}u_k = \bar{\partial}(\sigma_1 \wedge \dots \wedge \sigma_r) \wedge \left(\sum_1^r \bar{\partial}\sigma_\ell \otimes \epsilon_\ell^* \right)^{\otimes(k-2)} \otimes \epsilon^* \otimes \bar{\partial}\sigma.$$

It is now clear that $\delta u_{k+1} = \bar{\partial}u_k$. \square

It follows that if $\phi = \sum \phi_j \epsilon_j$ is a holomorphic section of Q and $v_k = u_k \phi$, then $v = u\phi$ satisfies (2.3). For later purpose we rewrite the expression for $u\phi$ so that it only “depends” on $\sigma\phi = \sum_j \phi_j \sigma_j$. In fact, applying $\bar{\partial}$ to the equality $0 = \sigma \otimes \sigma$ we get $0 = \sigma \otimes \bar{\partial}\sigma + (-1)^r \bar{\partial}\sigma \otimes \sigma$. Therefore,

$$(3.3) \quad u_k \phi = (-1)^{r+1} (\bar{\partial}\sigma)^{\otimes(k-2)} \otimes \bar{\partial}\sigma \otimes \sigma\phi, \quad k \geq 2,$$

or more explicitly (the possible minus sign cancels out)

$$\begin{aligned} u_k \phi &= \\ & \left(\sum_j \phi_j \sigma_j \right) \wedge \bar{\partial}(\sigma_1 \wedge \dots \wedge \sigma_r) \wedge \sum_{|\alpha|=k-2} (\bar{\partial}\sigma)^\alpha \otimes \epsilon_\alpha^* \otimes \epsilon^* = \\ & \left(\sum_j \phi_j \sigma_j \right) \wedge \bar{\partial}(\sigma_1 \wedge \dots \wedge \sigma_r) \wedge \left(\sum_1^r \bar{\partial}\sigma_\ell \otimes \epsilon_\ell^* \right)^{\otimes(k-2)} \otimes \epsilon^*, \quad k \geq 2. \end{aligned}$$

4. ANALYSIS OF SINGULARITIES

We now consider an f that is not necessarily surjective everywhere. Then we can define u as above in $X \setminus Z$, where Z is the set where f is not surjective, which is equal to the zero set of the holomorphic section F and hence an analytic subvariety of X . To analyze the singularities of u at Z we will use the following lemma.

Lemma 4.1. (i) *Let s be the section of $\text{Hom}(Q, E) \simeq E \otimes Q^*$ with pointwise minimal norm such that*

$$fs = |F|^2 I_Q,$$

and let S be the section of $\Lambda^r E \otimes \det Q^$ with pointwise minimal norm such that*

$$FS = 1.$$

Then s and S are smooth across Z , and

$$s = |F|^2 \sigma, \quad S = |F|^2 \mathcal{O} \quad \text{in } X \setminus Z.$$

(ii) If in addition $F = F_0 F'$ for some holomorphic function F_0 and non-vanishing holomorphic section F' , then

$$s' = F_0 \sigma, \quad S' = F_0 \mathcal{O}$$

are smooth across Z .

Given a section η of E^* , let η^* denote the dual section with respect to the Hermitian metric, i.e., $\langle \xi, \eta^* \rangle = \eta \cdot \xi$ for sections ξ of E . The mapping $\eta \mapsto \eta^*$ is conjugate-linear and extends to a conjugate-linear mapping $\Lambda E^* \rightarrow \Lambda E$ by

$$\eta_1 \wedge \dots \wedge \eta_r \mapsto \eta_1^* \wedge \dots \wedge \eta_r^*.$$

Proof. Assume that ϵ_j is a local frame for Q as before, let ϵ_j^* be its dual frame, and assume $f = \sum f_j \otimes \epsilon_j$. Now,

$$S = |\epsilon_1 \wedge \dots \wedge \epsilon_r|^2 f_1^* \wedge \dots \wedge f_r^* \otimes \epsilon_1^* \wedge \dots \wedge \epsilon_r^*$$

is a section of $\Lambda^r E \otimes \det Q^*$ that is independent of the particular choice of frame. This is checked by considering a change of frame $\epsilon' = \epsilon g$, where g is an invertible $r \times r$ -matrix. Notice that

$$(4.1) \quad FS = |\epsilon_1 \wedge \dots \wedge \epsilon_r|^2 |f_1 \wedge \dots \wedge f_r|^2 = |F|^2.$$

We can choose the frame ϵ_j such that f_j are orthogonal at any given point outside Z . Thus $f_j = \alpha_j e_j^*$, $j = 1, \dots, r$ for some ON -frame e_j^* of E^* and $f_j^* = \bar{\alpha}_j e_j$, and it is then easy to see that S is in fact the section with minimal norm such that $FS = 1$. In particular this means that S is the dual section of F . Moreover, at this point, $\sigma_j = (1/\alpha_j)e_j$, $j = 1, \dots, r$, and thus

$$\mathcal{O} = \frac{e_1 \wedge \dots \wedge e_r}{\alpha_1 \cdots \alpha_r} \otimes \epsilon_1^* \wedge \dots \wedge \epsilon_r^*.$$

Therefore,

$$|F|^2 \mathcal{O} = |\epsilon_1 \wedge \dots \wedge \epsilon_r|^2 \bar{\alpha}_1 \cdots \bar{\alpha}_r e_1 \wedge \dots \wedge e_r \otimes \epsilon_1^* \wedge \dots \wedge \epsilon_r^* = S.$$

Now assume that $F = F_0 F'$. Since S is the dual section of F it follows that $S = \bar{F}_0 S''$, where S'' is the dual section of F' , and thus S'' is smooth even across Z . Therefore $F_0 \mathcal{O} = F_0 S / |F|^2 = S'' / |F'|^2$ is smooth across Z as well.

Now, define

$$s = \left(\sum_1^r \delta_{f_j} \otimes \delta_{\epsilon_j} \right)^{r-1} S / r! =$$

$$|\epsilon_1 \wedge \dots \wedge \epsilon_r|^2 \sum_1^r (-1)^{\ell+1} \delta_{f_r} \cdots \delta_{f_{\ell+1}} \delta_{f_{\ell-1}} \cdots \delta_{f_1} (f_1^* \wedge \dots \wedge f_r^*) \otimes \epsilon_j^*.$$

Clearly,

$$fs = \sum_1^r \delta_{f_j} \otimes \epsilon_j s = |\epsilon_1 \wedge \dots \wedge \epsilon_r|^2 |f_1 \wedge \dots \wedge f_r|^2 \sum_j \epsilon_j \otimes \epsilon_j^* = |F|^2 I_Q.$$

Moreover, it is readily checked that s is orthogonal to $\text{Ker } f$ so that s is the minimal section such that $fs = |F|^2 I_Q$. One can also check this by choosing a frame as above such that that f_j are orthogonal. Thus, $s/|f|^2 = \sigma$. Finally, if $F_0 \sigma$ is smooth across Z it follows that

$$F_0 \sigma = \left(\sum_1^r \delta_{f_j} \otimes \delta_{\epsilon_j} \right)^{r-1} F_0 \sigma$$

is smooth as well. \square

Remark 1. Assume that Q as well as E are trivial and equipped with the trivial metrics, and e_j and ϵ_j are ON-frames. If

$$F = \sum_{|I|=r}^l F_I \otimes e_I^*,$$

(suppressing the factor $\epsilon_1 \wedge \dots \wedge \epsilon_r$ and its dual), we have that

$$S = \sum_{|I|=r}^l \bar{F}_I \otimes e_I.$$

\square

Since $\sigma \otimes \sigma = 0$ we have, for any scalar function ξ , that

$$(4.2) \quad (\bar{\partial}(\xi\sigma))^{\otimes(k-2)} \otimes \xi\sigma \otimes \bar{\partial}(\xi\sigma) = (-1)^{r+1} (\bar{\partial}(\xi\sigma))^{\otimes(k-2)} \otimes \bar{\partial}(\xi\sigma) \otimes \xi\sigma = \xi^k u_k.$$

In view of Lemma 4.1 we have in particular that

$$(4.3) \quad u_1 = \frac{s}{|F|^2},$$

$$u_k = \frac{(\bar{\partial}s)^{\otimes(k-2)} \otimes S \otimes \bar{\partial}s}{|F|^{2k}} = (-1)^{r+1} \frac{(\bar{\partial}s)^{\otimes(k-2)} \otimes \bar{\partial}S \otimes s}{|F|^{2k}}, \quad k \geq 2$$

Lemma 4.2. *If $f^*: Q \rightarrow E$ is the adjoint morphism with respect to the Hermitian structures on Q and E , then*

$$(4.4) \quad |F|^2 = \det(ff^*).$$

Proof. Since both sides of (4.4) are invariant pointwise statements, we can assume that ϵ_j is a ON-frame. Let ξ be any section of E . Then

$$f_\ell \cdot \xi = \left\langle \sum_j (f_j \cdot \xi) \epsilon_j, \epsilon_\ell \right\rangle = \langle f\xi, \epsilon_\ell \rangle = \langle \xi, f^* \epsilon_\ell \rangle$$

which means that $f^* = \sum_j f_j^* \otimes \epsilon_j^*$. It follows that

$$ff^* = \sum (f_j \cdot f_k^*) \epsilon_j \otimes \epsilon_k^* = \langle f_j, f_k \rangle \epsilon_j \otimes \epsilon_k^*.$$

Thus

$$\det(ff^*) = \det \langle f_j, f_k \rangle = |f_1 \wedge \dots \wedge f_r|^2,$$

which implies the statement since $|\epsilon_1 \wedge \dots \wedge \epsilon_r| = 1$, cf., (4.1). \square

Since $(ff^*)^{-1} = \widetilde{ff^*} / \det(ff^*)$ and $s/|F|^2 = \sigma = f^*(ff^*)^{-1}$ we can conclude that

$$s = f^* \widetilde{ff^*}.$$

Moreover, if $F = F_0 F'$ as in Lemma 4.1 above, then by (4.4),

$$(4.5) \quad |s' \phi| = |\sigma \phi| |F_0| = |f^*(ff^*)^{-1} \phi| |F| / |F'| = \|\phi\| / |F'|.$$

5. THE RESIDUE CURRENT OF A GENERICALLY SURJECTIVE HOLOMORPHIC MORPHISM

We say that f is generically surjective if Z has positive codimension. Let u be the section of $\text{Hom}(Q, H)$ over $X \setminus Z$ defined by (3.2). Following [4] we shall now extend it to a current U across Z and define the corresponding residue current.

Theorem 5.1. *Assume that $f: E \rightarrow Q$ is a generically surjective holomorphic morphism and let u be the associated section defined in $X \setminus Z$. The function $\lambda \mapsto |F|^{2\lambda} u$ is holomorphic for $\text{Re } \lambda > -\epsilon$ and*

$$U = |F|^{2\lambda} u|_{\lambda=0}$$

is a current extension of u across Z . Moreover,

$$(\delta - \bar{\partial})U = I_Q - R^f,$$

where

$$R^f = \bar{\partial}|F|^{2\lambda} \wedge u|_{\lambda=0}.$$

The current R^f has support on Z and

$$R^f = R_p^f + \dots + R_\mu^f,$$

where $p = \text{codim } Z$ and $\mu = \min(n, m - r + 1)$, and R_k^f is a current of bidegree $(0, k)$ with values in E_k .

Proof. The proof is more or less identical to the proof of Theorem 1.1 in [4], so we only sketch it. After an appropriate resolution of singularities we may assume that $F = F_0 F'$, where F_0 is a holomorphic function and F' is a non-vanishing section of (the pullback of) $\Lambda^r E^* \otimes \det Q$. According to Lemma 4.1 (ii), then $s' = F_0 \sigma$ and $S' = F_0 \mathcal{O}$ are smooth across the singularity, and hence by (4.2),

$$u_1 = \frac{s'}{F_0}, \quad u_k = (-1)^{r+1} \frac{(\bar{\partial}s')^{\otimes(k-2)} \otimes \bar{\partial}S' \otimes s'}{F_0^k}, \quad k \geq 2.$$

It is now easy to see that the analytic continuations of $|F|^{2\lambda}u_k$ exist, and in this resolution the values at $\lambda = 0$ are just the principal value currents

$$u_1 = \left[\frac{1}{F_0} \right] s', \quad \left[\frac{1}{F_0^k} \right] (\bar{\partial} s')^{\otimes(k-2)} \otimes \bar{\partial} S' \otimes s'.$$

Precisely as in [4], by the way following [7] and [20], one can show that

$$R_k^f = \bar{\partial} |F|^{2\lambda} \wedge u_k \Big|_{\lambda=0} = 0$$

if $k < p = \text{codim } Z$. Thus Theorem 5.1 is proved. \square

Proof of Theorem 1.2. If

$$\phi = f\psi = \sum_1^r (\delta_{f_j} \psi) \epsilon_j,$$

then

$$\begin{aligned} u_{m-r+1} \phi &= \left(\sum_1^r \phi_j \bar{\partial} \sigma_j \right) \wedge \sigma_1 \wedge \dots \wedge \sigma_r \wedge (\bar{\partial} \sigma)^{\otimes(m-r-1)} \otimes \epsilon^* = \\ &= \left(\sum_{j=1}^r \delta_{f_j} \psi \wedge \bar{\partial} \sigma_j \right) \wedge \sigma_1 \wedge \dots \wedge \sigma_r \wedge (\bar{\partial} \sigma)^{\otimes(m-r-1)} \otimes \epsilon^* = \\ &= \psi \wedge \sum_{j=1}^r \delta_{f_j} (\sigma_1 \wedge \dots \wedge \sigma_r) \wedge \bar{\partial} \sigma_j \wedge (\bar{\partial} \sigma)^{\otimes(m-r-1)} \otimes \epsilon^* = \\ &= \bar{\partial} \left(\psi \wedge \sigma_1 \wedge \dots \wedge \sigma_r \wedge (\bar{\partial} \sigma)^{\otimes(m-r-1)} \otimes \epsilon^* \right) = \bar{\partial} (u' \psi), \end{aligned}$$

where we have used that the form has maximal degree m in ΛE . If we define

$$R' \psi = \bar{\partial} |f|^{2\lambda} \wedge u' \wedge \psi \Big|_{\lambda=0},$$

it follows that

$$R_{m-r+1}(f\phi) = \bar{\partial}(R' \psi).$$

However, since $\text{codim } Z = m - r + 1$ it follows as in the proof of Theorem 5.1 above that $R' \psi$, being a $(0, m - r)$ -current, vanishes for degree reasons. Thus the theorem is proved. \square

Proof of Proposition 1.3. Let ξ be a test form with support contained in neighborhood where we have the resolution of singularities. In view of (4.2) we see that $R_k^f \cdot \xi$ is a sum of terms like

$$\bar{\partial} |F_0|^{2\lambda} v^\lambda \wedge \frac{(\bar{\partial} s')^{\otimes(k-2)} \otimes \bar{\partial} S' \otimes s' \phi}{F_0^k} \rho \Big|_{\lambda=0},$$

where ρ is a cut-off function, and v is a smooth strictly positive function. By the hypothesis, (4.5), and (4.2),

$$|s' \phi| \sim \|\phi\| \lesssim |F|^{\min(n, m-r+1)} \lesssim |F_0|^k,$$

since $k \leq \min(n, m - r + 1)$. Thus we must check that

$$\int \bar{\partial} |F_0|^{2\lambda} v^\lambda \wedge \frac{\eta}{F_0^\ell}$$

vanishes at $\lambda = 0$ if v is a test form that is $\mathcal{O}(|F_0|^\ell)$. However, we may as well assume that F_0 is a monomial in the local coordinates, and therefore this statement follows from the corresponding one-variable statement; that

$$\int_\tau \bar{\partial} |\tau|^{2\lambda} v^\lambda \wedge \frac{\eta(\tau) d\tau}{\tau^\ell}$$

vanishes at $\lambda = 0$ if $\eta = \mathcal{O}(|\tau|^\ell)$. If we write

$$\bar{\partial} |\tau|^{2\lambda} = \lambda |\tau|^{2\lambda} \frac{d\bar{\tau}}{\tau}$$

this follows by dominated convergence. \square

Proofs of Theorems 1.5 and 1.7. These theorems are proved in much the same way as the corresponding results for holomorphic functions. In fact, if ϕ is a $\bar{\partial}$ -closed smooth form with values in Q , then $(\delta - \bar{\partial})U\phi = \phi - R^f\phi$ so $(\delta - \bar{\partial})U\phi = \phi$ if $R^f\phi = 0$. Following an apparent modification of the procedure in (2.5) we get a $\bar{\partial}$ -closed current ψ with values in E such that $f\psi = \phi$. However, if (1.2) holds, then it is not hard to verify that $U\phi$ is locally integrable; in fact in a resolution as above $U\phi$ is then bounded, in particular it is locally integrable, and therefore $U\phi$ is locally integrable in X . Therefore, we can get a locally integrable solution ψ . \square

For further reference we state the following proposition that shows that the principal term R_p^f of the current R^f (where $p = \text{codim } Z$) is robust in certain sense. It is proved precisely as Theorem 2.1 in [5].

Proposition 5.2. *Let f be a generically surjective morphism, let R^f be the associated residue current, and let $p = \text{codim } Z$. Assume that h is a holomorphic section of some line bundle such that $\{h = 0\} \cap Z$ has codimension $p + 1$. If ϕ is a holomorphic section such that $R_p^f\phi = 0$ in $X \setminus \{h = 0\}$, then $R_p^f\phi = 0$ in X .*

6. POLYNOMIAL MAPPINGS

Let $L^s = \mathcal{O}(s)$ be the line bundle over \mathbb{P}^n whose sections are represented by s -homogeneous functions in $z = (z_0, \dots, z_n)$. If ϕ is a section of L^s its natural norm is

$$\|\phi(z)\| = \frac{|\phi(z)|}{|z|^s}.$$

Let E_j , $j = 1, \dots, m$, be trivial line bundles over \mathbb{P}^n with basis elements e_j . If f^j are the d_j -homogenizations of the columns of polynomials P^j as in Theorem 1.8, then $f_k = \sum_1^m f_k^j e_j^*$ are sections of

$$E^* = E_1^* \otimes L^{d_1} \oplus \dots \oplus E_m^* \otimes L^{d_m}.$$

Observe that the section $F = f_1 \dots \wedge f_r$ of $\Lambda^r E^*$ can be written

$$F = \sum_{|I|=r}^l F_I e_{I_1}^* \wedge \dots \wedge e_{I_r}^*,$$

where $F_I = \det(f_k^{I_j})$. Thus

$$\|F(z)\|^2 = \frac{\sum_{|I|=r}^l |F_I(z)|^2}{|z|^{2 \sum_1^r d_{I_j}}}.$$

If we write this expression in the affine coordinates z' we get precisely the left hand side of (1.4). Let Q be the trivial bundle $\mathbb{C}^r \rightarrow \mathbb{P}^n$. Then f defines a morphism $f: E \rightarrow Q$ such that

$$\psi = (\psi_1, \dots, \psi_m) \mapsto \sum_1^m f^j \psi_j$$

where f^j are the homogenizations of the given (columns) of polynomials P_j of degrees d_j . We can now prove Theorem 1.8.

Proof of Theorem 1.8. Let

$$\dots \rightarrow E_2 \rightarrow E \rightarrow \mathbb{C}^r \rightarrow 0.$$

be the induced complex defined in Section 2. We can take tensor products with L^ρ and get the complex

$$\dots \rightarrow E_2 \otimes L^\rho \rightarrow E \otimes L^\rho \rightarrow \mathbb{C}^r \otimes L^\rho \rightarrow 0.$$

Let U and R^f be the corresponding currents from Section 5 with respect to the natural Hermitian metrics of E and Q . If ϕ is a section of $Q \otimes L^\rho$, and $R^f \phi = 0$, then $v = U\phi$ solves the equations

$$(\delta - \bar{\partial})(v_1 + v_2 + \dots + v_{\min(n+1, m-r+1)}) = \phi.$$

In order to get a holomorphic solution ψ to $f\psi = \phi$ we have to solve all the equations $\bar{\partial}w_k = v_k - \delta v_{k+1}$. Notice that $v_k - \delta v_{k+1}$ is a $(0, k-1)$ -current with values in $E_k \otimes L^\rho$. It is well-known that $H^{0,k}(\mathbb{P}^n, L^\nu) = 0$ for all ν if $1 \leq k \leq n-1$, whereas $H^{0,n}(\mathbb{P}^n, L^\nu) = 0$ (if and only) if $\nu \geq -n$. If $m-r+1 \leq n$ there is therefore no problems at all, and the only possible obstruction may appear when $k = n+1$. Notice that v_{n+1} takes values in

$$E_{n+1} \otimes L^\rho = \Lambda^{r+n} E \otimes S^{n-1} Q^* \otimes Q^* \otimes L^\rho.$$

Since Q is trivial, E_{n+1} is a direct sum of line bundles

$$L^{\rho - (d_{I_1} + \dots + d_{I_{r+n}})},$$

where I is an increasing multiindex. Therefore the crucial $\bar{\partial}$ -equation is solvable if $\rho - (d_1 + \dots + d_{n+r}) \geq -n$. Finally we express the relation $\sum f^j \psi_j = f\psi = \phi$ in affine coordinates and get the desired polynomials Q_j as $Q_j(z') = \psi_j(1, z')$. \square

Proof of Proposition 1.12. If we have m' polynomials P'_ν of degrees at most d' with no common zeros in \mathbb{C}^n , then it is proved in [16] that

$$\frac{|P'_\nu(z')|^2}{(1 + |z'|^2)^{d'}} \geq C \frac{1}{(1 + |z'|^2)^M},$$

where $M = (d')^{\min(n, m')}$, provided that $d' \geq 3$. We have $m'/(m-r)!r!$ polynomials $P_I = \det(P_{I_j}^k)$ of degrees $d' = rd$ and hence the proposition follows. \square

Proof of Theorem 1.13. Since $\phi = f\psi$ is solvable in \mathbb{C}^n and $p = m - r + 1$, $R^f \phi = 0$ in \mathbb{C}^n by Theorem 1.2. If we take the section $h = z_0$ of $\mathcal{O}(1)$, then since assumption Z has no irreducible component in the plane at infinity, $\text{codim } Z \cap \{z_0 = 0\} = m - r + 2$. By Proposition 5.2 we therefore have that $R^f \phi = 0$ in \mathbb{P}^n . Since (1.3) is fulfilled, the desired solution is given by Theorem 1.8. \square

7. ESTIMATES FOR A POINTWISE SURJECTIVE MORPHISM

In this section we indicate that our method can be used to get new quite sharp estimates even when f is pointwise surjective. Let us assume that $E \simeq \mathbb{C}^m$ and $Q \simeq \mathbb{C}^r$ are trivial bundles over a smoothly bounded domain $D = \{\rho < 0\}$ in \mathbb{C}^n , and equipped with the trivial metrics. Then a morphism $f: E \rightarrow Q$ is just a matrix of holomorphic functions in D . We assume that $f \in H^\infty(D, \text{Hom}(E, Q))$, and that moreover

$$|f_1 \wedge \dots \wedge f_r| \geq \delta > 0;$$

this means that f is uniformly surjective. Notice that since Q is trivial, $\det Q^*$ is just the trivial line bundle, so $F = f_1 \wedge \dots \wedge f_r$. Let

$$\|\phi\|_{H^p}^p = \sup_{\epsilon > 0} \int_{\rho = -\epsilon} |\phi(z)|^p dS, \quad 0 < p < \infty.$$

It was proved in [2] and [3] that if D admits a plurisubharmonic defining function ρ , and $p \leq 2$, then for any $\phi \in H^p(D, Q)$ there is a $\psi \in H^p(E)$ such that $f\psi = \phi$. Moreover, the norm of ϕ is bounded by a constant times $\log(1/\delta)/\delta^{1+\mu}$, where $\mu = \min(n, m - r)$. To be precise, this sharp estimate is only explicitly given in the case $r = 1$, but it follows (hopefully) in the general case as well with a similar argument. This result is proved by a combination of the L^2 -methods in [22], the refined L^2 estimate for $\bar{\partial}_b$ introduced in [8], and Wolff type estimates.

The case $p > 2$ and $r = 1$ has been studied by several authors in strictly pseudoconvex domains, e.g., [1] and [6], and a generalization to $r > 1$ is made in [14]. These works are based on integral representation

and harmonic analysis. There are also similar results for other spaces of functions, see, e.g., the references in [6]. To show how the ideas in this paper can be applied in a situation like this we present the new result Theorem 7.1 below. It is clear that other known results that are proved by means of the Koszul complex in the case $r = 1$ can be generalized to the case $r > 1$ in an analogous way.

Theorem 7.1. *Let D be a strictly pseudoconvex domain with reasonably smooth (C^3 is enough) boundary and $p < \infty$. For any $\phi \in H^p(D, Q)$ there is a $\psi \in H^p(D, E)$ such that $f\psi = \phi$ and*

$$(7.1) \quad \|\psi\|_{H^p} \leq C_\delta \|\phi\|_{H^p},$$

where

$$C_\delta \leq C(\log(1/\delta))^{\mu/2}/\delta^{1+\mu}, \quad \text{if } \mu = \min(n, m-r) > 1$$

and

$$C_\delta \leq C \log(1/\delta)/\delta^2 \quad \text{if } \min(n, m-r) = 1.$$

and $C_\delta = 1/\delta$ if $\min(n, m-r) = 0$, i.e., $m = r$, and C is a constant that is independent of m .

In the case $r = 1$ this coincides with the result in [6]. Since the proof of this generalization follows the proof in [6] quite closely we just give a sketch and indicate the necessary modifications.

Sketch of proof. With the notation as before we have that $v = u\phi$, where $u = u_1 + \cdots + u_{1+\mu}$ and

$$u_{k+1} = (\bar{\partial}\sigma)^{\otimes(k-1)} \otimes \sigma \otimes \bar{\partial}\sigma.$$

Since Q is trivial and ϵ_j is a ON-frame we have, cf., Remark 1, that $S = \sum' \bar{F}_I e_I \otimes \epsilon^*$ of $F = \sum' F_I e_I^* \otimes \epsilon$, so the coefficients in S are anti-holomorphic. Moreover, $\sigma = S/|F|^2$ and

$$\sigma = \left(\sum \delta_{f_j} \otimes \delta_{\epsilon_j} \right) \sigma / k!$$

so that

$$\bar{\partial}\sigma = \pm \left(\sum \delta_{f_j} \otimes \delta_{\epsilon_j} \right) \bar{\partial}\sigma / k!.$$

The norm of forms will be taken with respect to the non-isotropic metric $\Omega = -\rho i \partial \bar{\partial} \log(1/-\rho)$; we assume that ρ is a strictly plurisubharmonic defining function. For $k \geq 2$ we can estimate the Carleson norm of $(-\rho)^{\frac{k-2}{2}} |u_{k+1}|$ precisely as in the proof of Proposition 5.2 in [6], with F instead of g and σ instead of γ , and obtain the same estimate $C\delta^{-k-1}(\log(1/\delta))^{k/2}$.

The necessary estimate of u_2 is of Wolff type and involves $\mathcal{L}u_2$ where \mathcal{L} is a smooth $(1, 0)$ -vector field. More precisely we need to know that the Carleson norm of $|u_2|^2$ is bounded by C^2 and the Carleson norm of

$\sqrt{-\rho}|\mathcal{L}u_2|$ is bounded by C , where $C = \delta^{-2} \log(1/\delta)$. It is easily seen that

$$|u_2| \lesssim \frac{|\partial F|}{|F|^3} \leq \frac{1}{\delta} \frac{|\partial F|}{|F|^2},$$

and the desired Carleson estimate of $|u_2|^2$ now follows from Proposition 5.2 in [6]. For any holomorphic function ψ we have that $\sqrt{-\rho}|\mathcal{L}\psi| \lesssim ||\text{partial}\psi|$. When we compute $\mathcal{L}u_2$ we get either derivatives on the factor $1/|F|^2$ or on the functions f_j in $(\sum \delta_{f_j} \otimes \delta_{\epsilon_j})^{r-1}$. Therefore we have that

$$\sqrt{-\rho}|\mathcal{L}u_2| \lesssim \frac{\partial h |\partial F|}{|F|^3} + \frac{|\partial F|^2}{|F|^4} \lesssim \frac{1}{\delta^2} + \frac{|\partial F|^2}{|F|^4},$$

where h is a holomorphic and bounded. It now follows from Proposition 5.2 in [6] that the Carleson norm is $\lesssim \delta^{-2} \log(1/\delta)$ as wanted. \square

Remark 2. If we just assume that ϕ is in H^p with respect to the stronger pointwise norm $\|\phi\|$ instead of $|\phi|$, the same proof would give a solution in H^p provided that we could say that the maximal function of $\|\phi\|$ is in $L^p(\partial D)$. However, we do not know if this is true. \square

As mentioned above, a similar result has previously been obtained by Hergoualch, [14], using an idea of Fuhrmann, [13], to reduce to the case $r = 1$. However, by this method one has some loss of precision in the dependence of δ . To see this let us describe this method. Assume that $\phi = \sum \phi_j \epsilon_j$ is in $H^p(D, Q)$. Now $F = f_1 \wedge \dots \wedge f_r \in H^\infty(D, \Lambda^r E^*)$ and $|F| \geq \delta$, so by the corresponding result for $r = 1$, for each ϕ_j we can find a section H^j of $H^p(D, \Lambda^r E)$ such that $F \cdot H^j = \phi_j$, $j = 1, \dots, r$, with $\|H^j\|_{H^p} \leq C'_\delta \|\phi\|_{H^p}$. Since the rank of $\Lambda^r E^*$ is $m!/(m-r)!r!$, we get

$$C'_\delta \leq C \frac{(\log(1/\delta))^{\mu'/2}}{\delta^{1+\mu'}}, \quad \mu' = \min(n, m!/(m-r)!r!),$$

if $\mu' > 1$. Since $f_1 \wedge \dots \wedge f_r \cdot H^j = \phi_j$, we also have that $f_j \cdot \psi_j = \phi_j$ if

$$\psi_j = (-1)^{j+1} \delta_{f_r} \cdots \delta_{f_{j+1}} \delta_{f_{j-1}} \cdots \delta_{f_1} H^j.$$

Now $\psi = \psi_1 + \dots + \psi_r$ solves $f_j \psi = \phi_j$ for each j , i.e., $f\psi = \phi$ as wanted, and since f_j are bounded, we get the same estimate

$$\|\psi\|_{H^p} \leq C'_\delta \|\phi\|_{H^p}.$$

However, $\mu' > \mu$ as soon as $m - r < n$ as in this case it is strictly weaker than (7.1).

Remark 3. It is actually possible to solve the equation $F\Psi = \phi$ above with a sharper estimate, by means of the complex

$$\begin{aligned} \rightarrow \Lambda^{r+2} E \otimes S^2 Q^* \otimes \det Q^* &\rightarrow \\ \Lambda^{r+1} E \otimes Q^* \otimes \det Q^* &\rightarrow \Lambda^r E \otimes \det Q^* \rightarrow \mathbb{C} \rightarrow 0 \end{aligned}$$

with the same mappings as before. Combined with the Fuhrmann trick one can then obtain Theorem 7.1. This complex, and the corresponding residues that appear when f is only generically surjective, will be studied in a forthcoming paper. \square

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DEPARTMENT OF MATHEMATICS, CHALMERS UNIVERSITY OF TECHNOLOGY
AND THE UNIVERSITY OF GÖTEBORG, S-412 96 GÖTEBORG, SWEDEN
E-mail address: matsa@math.chalmers.se