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HENRIK PETERSSON

Department of Mathematics
CHALMERS UNIVERSITY OF TECHNOLOGY
GÖTEBORG UNIVERSITY
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Henrik Petersson

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Mathematics
Department of Mathematics
Chalmers University of Technology and Göteborg University
SE-412 96 Göteborg, Sweden
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Henrik Petersson*

Abstract

A continuous linear operator T is hypercyclic/supercyclic if there is a vector f such that the orbit $\operatorname{Orb}(T,f) = \{T^n f\}$ respective the set of scalar-multiples of the orbit elements, forms a dense set. A famous theorem, due to G. Godefroy & J. Shapiro, states that every non-scalar convolution operator, on the space $\mathscr{H}$ of entire functions in d variables, is hypercyclic (and thus supercyclic). This motivates us to study cyclicity of operators on $\mathscr{H}$ outside the set of convolution operators. We establish large classes of supercyclic and hypercyclic non-convolution operators.

Key words and Phrases: Cyclic, Hypercyclic, Invariant, Backward shift, Convolution operator, Exponential type, PDE-preserving, Fischer pair.

1. Introduction and Preliminaries

Let $\mathbb{T} = (T_n)$ be a sequence of continuous linear operators on a TVS X. Let $\operatorname{Orb}(\mathbb{T}, f) \equiv \{T_n f : n \geq 0\}$ denote the orbit of $f \in X$ under $\mathbb{T}$ and by $\operatorname{Orb}_l(\mathbb{T}, f)$ and $\operatorname{Orb}_s(\mathbb{T}, f)$ we denote the linear hull respective the set of scalar multiples of the elements in $\operatorname{Orb}(\mathbb{T}, f)$. Recall that $\mathbb{T}$ is cyclic/supercyclic/hypercyclic if, respectively, $\operatorname{Orb}_l(\mathbb{T}, f)/\operatorname{Orb}_s(\mathbb{T}, f)/\operatorname{Orb}(\mathbb{T}, f)$ is dense in X for some $f \in X$. (Thus hypercyclic implies supercyclic which, in turn, implies cyclic.) The vector f is said to be of corresponding cyclic type (for $\mathbb{T}$). An operator T is cyclic (with cyclic vector f) when $\mathbb{T} \equiv (T^n)$ is cyclic (with cyclic vector f), and analogous for super- and hypercyclicity. In this case of a single operator we write, simply, $\operatorname{Orb}(T, f)$ etc. (A fuller account of the significance of all these notions is given in [4] and we refer to [5] for a nice overview of the invariant subspace theory.)

We let d be a fixed arbitrary natural number and denote by $\mathscr{H}$ the Fréchet space of entire functions in d variables, equipped with the compact-open topology. Thus a generating family of semi-norms is obtained by $\|f\|_n \equiv \sup_{|z| \leq n} |f|$, $n \in \mathbb{N}$. In 1929 Birkhoff proved that, in the case of one variable, every translation operator τ_a, $a \neq 0$, is hypercyclic on $\mathscr{H}$. ($\tau_a f(z) \equiv f(z+a)$.) MacLane obtained in 1952 the analogue result for the differentiation operator D (see [4] for further references to these two classical results). Both τ_a and D are convolution operators, i.e., continuous linear operators that commute with translations. In 1991, Godefroy & Shapiro

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generalized Birkhoff's and MacLane's results considerably by proving: Every non-scalar convolution operator on $\mathscr{H}$ (d is arbitrary) is hypercyclic (cf. [9]). Their proof [4] is based on the well-known Hypercyclicity Criterion, which we formulate in Proposition 3. Godefroy-Shapiro's Theorem motivates us, and others [1], to study cyclic properties of operators outside the set $\mathscr{C}$ of convolution operators on $\mathscr{H}$. The objective in this note is to establish supercyclic and hypercyclic non-convolution operators on $\mathscr{H}$, by applying results from our study on PDE-preserving operators [10]-[13]:

Definition 1. A continuous linear operator $T : \mathscr{H} \to \mathscr{H}$ is PDE-preserving for a set $\mathbb{P} \subseteq \mathbb{C}[\xi_1, ..., \xi_d]$ if it maps ker $P(D)$ invariantly for all $P \in \mathbb{P}$. The set, and algebra!, of PDE-preserving operators for $\mathbb{P}$ is denoted by $\mathscr{O}(\mathbb{P})$. (Note, $\mathscr{O}(\mathbb{P}) = \cap_{\mathbb{P}} \mathscr{O}(P)$.)

(Later we extend this definition.) Since any differential operator $P(D) \in \mathscr{C}$ and operators in $\mathscr{C}$ commute, $\mathscr{C}$ forms a commutative subalgebra of $\mathscr{O}(\mathbb{P})$ for *any* set $\mathbb{P}$. To explain the connection between the notion PDE-preserving operators and the study under consideration, we first recall the following result (see Theorem 2): An operator T is PDE-preserving for a given $P \neq 0$ iff T "almost commutes" with $P(D)$ in the sense that $P(D)T = T^{(P)}P(D)$ for some continuous linear operator $T^{(P)}$. In fact, by Malgrange's Theorem [7], $P(D)$ is surjective so $T^{(P)}$ is unique and is called the derivative of $T \in \mathscr{O}(P)$ w.r.t. P. The following is now elementary:

Theorem 1. *Let $P \neq 0$ and assume $T \in \mathscr{O}(P)$ is cyclic and f a corresponding cyclic vector. Then $T^{(P)}$ is also cyclic and $P(D)f$ forms a cyclic vector. The analogue holds true for both super- and hypercyclicity.* (*See also Remark 2.*)

Proof. Put $S \equiv T^{(P)}$. We note that $P(D)T^n = S^n P(D)$ for all $n \geq 0$. Hence $P(D)\operatorname{Orb}(T, f) = \{P(D)T^n f\} = \operatorname{Orb}(S, P(D)f)$, and from this we also deduce, $P(D)\operatorname{Orb}_\nu(T, f) \subseteq \operatorname{Orb}_\nu(S, P(D)f)$, $\nu = s, l$. Since $P(D)$ is surjective (Malgrange), and thus maps dense sets onto dense sets, our claim follows. ■

Thus by studying PDE-preserving properties, and corresponding derivatives, of operators of given cyclic type, it is possible to get new such operators. Unfortunately, by commutativity, any derivative of any convolution operator T is a new convolution operator (in fact, equal to T), so Theorem 1 does not provides us with any non-convolution operators by starting out of operators in $\mathscr{C}$. Thus, to apply Theorem 1 in this way, we must first find a, say, hypercyclic operator outside $\mathscr{C}$, and there are very few such examples in the literature. However, we shall establish a set $\mathscr{O}_{\mathrm{S}}$ of supercyclic operators on $\mathscr{H}$ and a multiplicative closed subset $\mathscr{O}_{\mathrm{H}}$ formed by hypercyclic operators, where $\mathscr{O}_{\mathrm{S}} \setminus \mathscr{C}$ and $\mathscr{O}_{\mathrm{H}} \setminus \mathscr{C}$ are large and we can apply Theorem 1 in the way that we have indicated. Moreover, Theorem 1 gives information about how cyclic vectors are transferred, and from this we derive some "internal" structures of the set of supercyclic/hypercyclic vectors for the operators in $\mathscr{O}_{\mathrm{S}}/\mathscr{O}_{\mathrm{H}}$.

An important concept, and tool, in the invariant subspace theory is the notion of backward shifts [4, 6]. A general theory for cyclic properties of operators that commute with a so called generalized backward shift B, and thus with any of its power B^n, is developed in [4] (in particular, see Theorem 3.6). Now, $B = D$ is a generalized backward shift on $\mathscr{H}$ in the case of one variable but, unfortunately, in view of our purposes, an operator T commutes with D iff $T \in \mathscr{C}$ [4, Prop. 5.2]. Thus their theory is not applicable to obtain, say, hypercyclic operators outside $\mathscr{C}$. Now, our result(s), when $d = 1$, is based on the fact that, roughly, it is possible to extend their ideas on backward shifts for operators that almost commute with any power of B, i.e., PDE-preserving operators for the homogeneous polynomials $\{1, \xi, \xi^2, ...\}$. (This particular algebra of PDE-preserving operators (for homogeneous polynomials) has been studied more comprehensively (see Proposition 2), which implies that we can provide the operators in $\mathscr{O}_{\mathrm{S}}/\mathscr{O}_{\mathrm{H}}$ with explicit representations - we think this is a strength of our results.) Moreover, if $d > 1$, there is no analogue of the backward shift $B = D$. However, we can extend our one variable result(s) by showing that, for any non-constant homogeneous polynomial P, $P(D)$ may serve as some sort of a backward shift. The key is to apply results from the theory of so called Fischer decompositions (Fischer pairs) developed by H. Shapiro and others [3, 8, 14]. In fact, we show that even other Fischer pairs, i.e., not necessarily formed by homogeneous polynomials (see page 6 for further explanation), provide us with alternative "backward shifts" in the same way, see Remarks (i) and (iii) at the end.

The paper is organized as follows: First we recall some fundamental results from our study on PDE-preserving operators. Our main results are Theorems A, B and C, which are exposed in Section 3. In Theorem A, we establish the class $\mathscr{O}_{\mathrm{S}}$ of supercyclic operators. Next, by applying Theorem 1, we prove in Theorem B that $\mathscr{O}_{\mathrm{S}}$ is stable under certain operations and that the set of supercyclic vectors for any $T \in \mathscr{O}_{\mathrm{S}}$ admits internal vector space structures and invariant properties. Then we consider the more delicate problem - the existence of hypercyclic non-convolution operators. Our results obtained so far motivate us to study if, in particular, there are any such operators in $\mathscr{O}_{\mathrm{S}}$. In Theorem C we establish the multiplicative closed subset $\mathscr{O}_{\mathrm{H}}$ ($\not\subseteq \mathscr{C}$) of $\mathscr{O}_{\mathrm{S}}$ formed by hypercyclic operators, and prove internal properties, of the type above, of the corresponding sets of hypercyclic vectors.

2. Fundamentals

Given $n \in \mathbb{N}$, Exp_n denotes the Banach space of functions $\varphi \in \mathscr{H}$ such that $\|\varphi\|_n \equiv \sup_\xi |\varphi(\xi)|e^{-n|\xi|} < \infty$, equipped with the norm $\|\cdot\|_n$ thus defined. The space of exponential type functions, Exp, is the union $\cup_{n\geq 0}\mathrm{Exp}_n$ provided with the corresponding inductive-limit topology. We put $e_\xi \equiv e^{\langle \cdot,\xi\rangle} \in \mathscr{H}$, where $\langle z, \xi\rangle \equiv \sum z_i\xi_i$, and recall that the Fourier-Borel transform $\mathcal{F}$, $\mathscr{H}' \ni \lambda \mapsto \mathcal{F}\lambda(\xi) \equiv \lambda(e_\xi)$, is a topological isomorphism between $\mathscr{H}'$ (strong topology) and Exp. Thus $\mathscr{H}$ and Exp form a dual pair by $\langle f, \varphi\rangle \equiv \mathcal{F}^{-1}\varphi(f)$ (the *Martineau-duality*).

We denote by $\mathfrak{S}$ the set of entire mappings $P = P(z, \xi)$, in $2d$ variables $(z, \xi) \in \mathbb{C}^d \times \mathbb{C}^d$, with the following property: For every $n \geq 0$ there are $m = m_n$, $M = M_n \geq 0$ such that $\|P(\cdot, \xi)\|_n \leq Me^{m|\xi|}$ (thus $P(z, \cdot) \in \mathrm{Exp}$). The algebra of continuous linear operators on $\mathscr{H}$ is denoted by $\mathscr{L} = \mathscr{L}(\mathscr{H})$ and we have the following Kernel-Theorem:

Proposition 1. *$T \mapsto P(z, \xi) \equiv e^{-\langle z,\xi\rangle}Te_\xi(z)$ defines a bijection between $\mathscr{L}$ and $\mathfrak{S}$. P is called the* symbol *for T, we write $T = P(\cdot, D)$ and have $Tf(z) = \langle f, P(z, \cdot)e_z\rangle$ and $P(\cdot, D)Q(\cdot, D) = R(\cdot, D)$, where $R(z, \xi) = e^{-\langle z,\xi\rangle}\langle Q(\cdot, \xi)e_\xi, P(z, \cdot)e_z\rangle$. The set of convolution operators, $\mathscr{C}$, is a commutative subalgebra of $\mathscr{L}$ and is formed by the operators with symbols in* Exp $\subseteq \mathfrak{S}$*, and we write $\varphi(D) \equiv \varphi(\cdot, D)$, $\varphi \in$* Exp*. (Thus, $\mathscr{H}' \simeq \mathrm{Exp} \simeq \mathscr{C}$.)*

Proof. Let $T \in \mathscr{L}$. We must prove that $P(z, \xi) \equiv e^{-\langle z,\xi\rangle}Te_\xi(z) \in \mathfrak{S}$. Clearly, $P(\cdot, \xi) \in \mathscr{H}$ and from $Te_\xi(z) = {}^tTe_z(\xi)$, $P(z, \cdot) \in \mathscr{H}$. By Hartog's Theorem, P

is entire on $\mathbb{C}^d \times \mathbb{C}^d$ and it remains to prove that P is bounded as required, i.e. $z \mapsto P(z, \cdot)$ maps any bounded set in $\mathbb{C}^d$ into a bounded set in Exp_m for some m. But one can prove that a set in Exp is bounded iff it is contained and bounded in some Exp_m and hence, $P \in \mathfrak{S}$ since $\mathbb{C}^d \ni z \mapsto e_z \in \mathrm{Exp}$ is continuous and thus so is $\mathbb{C}^d \ni z \mapsto P(z, \cdot) \in \mathrm{Exp}$. Conversely, let $P \in \mathfrak{S}$ and define $Tf(z) \equiv \langle f, P(z, \cdot)e_z \rangle$. It is easily checked that $T \in \mathscr{L}$ and $e^{-\langle z,\xi \rangle} Te_\xi(z) = P(z, \xi)$. Thus, the map $\mathscr{L} \ni T \mapsto e^{-\langle z,\xi \rangle} Te_\xi(z) \in \mathfrak{S}$ is onto and since $\{e_\xi : \xi \in \mathbb{C}^d\}$ forms a total set in $\mathscr{H}$, it is one-to-one. The composition formula is elementary.

The last part is well-known and a proof is exposed in [4]. ■

Remark 1. If $P(z, \xi) = \sum_{\alpha\beta} P_{\alpha\beta} z^\alpha \xi^\beta \in \mathfrak{S}$, then $P(\cdot, D)f = \sum_{\alpha\beta} P_{\alpha\beta} z^\alpha D^\beta f$ in $\mathscr{H}$ (Cauchy's Estimates!). Thus every operator in $\mathscr{L}$ can be written as an infinite type of differential operator with variable coefficients, and the elements of $\mathscr{C}$ are those with constant coefficients. (Here, and in the sequel, we use standard multi-index notation and thus, in particular, $\alpha! \equiv \prod \alpha_i!$, $|\alpha| \equiv \sum \alpha_i$, $D^\alpha \equiv \prod D^{\alpha_i}$ where $D_i \equiv \partial/\partial z_i$ and $\alpha \in \mathbb{N}^d$.)

We now extend Definition 1 by allowing $\mathbb{P}$ to be any set in Exp. Thus, in particular, if $\varphi \in \mathrm{Exp}$, $\mathscr{O}(\varphi)$ is formed by all continuous operators that map $\ker \varphi(D)$ invariantly. We remark that the transpose of $\varphi(D) \in \mathscr{C}$ is the multiplication operator $\varphi : \psi \mapsto \psi\varphi$ on Exp. Moreover, the general version of Malgrange's Theorem [7] states that any $\varphi(D) \neq 0$ is surjective and it is convenient to note the following consequence $\ker \varphi(D)^\perp = \mathrm{Im}\, \varphi = \mathrm{Exp} \cdot \varphi$.

A main result in our study of PDE-preserving operators is the theorem that follows. The technical part in our proof is the following lemma and division property for $\mathfrak{S}$ ($\mathscr{L}$): (d) Let $0 \neq \varphi \in \mathrm{Exp}$, $P \in \mathfrak{S}$ and assume $P(z, \xi) = \varphi(\xi) Q(z, \xi)$ where $Q(z, \cdot) \in \mathrm{Exp}$ for all $z \in \mathbb{C}^d$, then $Q \in \mathfrak{S}$. (A proof of (d) is obtained from [13] where we prove an analogous statement.)

Theorem 2 (Characterization-Theorem). *Let $\varphi \in \mathrm{Exp}$ and $T = P(\cdot, D) \in \mathscr{L}$. Then the following are equivalent:*

1. *T is* PDE*-preserving for φ,*
2. *$\varphi(D)T = S\varphi(D)$ for some $S \in \mathscr{L}$,*
3. *$\varphi | \varphi(\xi + D)P(\cdot, \xi)(z)$ in $\mathfrak{S}$, i.e., $\varphi(\xi + D)P(\cdot, \xi)(z) = \varphi(\xi)Q(z, \xi)$ for some $Q \in \mathfrak{S}$.*

($\varphi(\xi + D) \equiv (\tau_\xi \varphi)(D) \in \mathscr{C}$.) *If $\varphi \neq 0$ the operator S is unique and is called the derivative of $T \in \mathscr{O}(\varphi)$ w.r.t. φ and is denoted by $T^{(\varphi)}$.*

Proof. We may assume $\varphi \neq 0$ and note that the uniqueness of S follows by the surjectivity of $\varphi(D)$. The equivalence between 2 and 3 follows by the observation $\varphi(D)Te_\xi(z) = \varphi(D)P(\cdot, \xi)e_\xi(z) = e^{\langle z,\xi \rangle} \varphi(\xi + D)P(\cdot, \xi)(z)$. Since 2 obviously implies 1, it remains to prove that if 1 holds true, then $\varphi | R$ in $\mathfrak{S}$ where $R(z, \xi) \equiv \varphi(D)Te_\xi(z)$ ($\in \mathfrak{S}$). For fixed $z \in \mathbb{C}^d$ let $\lambda_z(f) \equiv \varphi(D)Tf(z)$. Then $\lambda_z \in \mathscr{H}'$ and $\mathcal{F}\lambda_z(\xi) = R(z, \xi)$. We prove that $\mathcal{F}\lambda_z \in \mathrm{Im}\, \varphi = \ker \varphi(D)^\perp$. But if $f \in \ker \varphi(D)$,

$$\langle f, \mathcal{F}\lambda_z \rangle = \lambda_z(f) = \varphi(D)Tf(z) = 0$$

since $T \in \mathscr{O}(\varphi)$. Thus, for every $z \in \mathbb{C}^d$ there is a unique $Q(z, \cdot) \in \mathrm{Exp}$ such that $R(z, \xi) = \varphi(\xi)Q(z, \xi)$, $\xi \in \mathbb{C}^d$, and (d) completes the proof. ■

Remark 2. Recall, Theorem 1 was derived from Theorem 2 for a polynomial $\varphi = P$, and by this more general version we deduce that Theorem 1 also holds true for any $\varphi \in \mathrm{Exp}$ and $T \in \mathscr{O}(\varphi)$.

Let $\mathscr{P}$ denote the algebra of (complex) polynomials in d variables and by $\mathscr{P}_n$ we denote the vector space of polynomials in $\mathscr{P}$ of degree at most n. $\mathscr{H}_n$ denotes the set, and vector space, of n-homogeneous polynomials in $\mathscr{P}$ and by $\mathbb{H}$ we denote the set of all homogeneous polynomials $\cup_{n\geq 0}\mathscr{H}_n$. ($\mathscr{P}_0 = \mathscr{H}_0 \equiv \mathbb{C}$.)

Lemma 1. *If $T \in \mathscr{O}(\mathbb{H})$, then T maps every $\mathscr{P}_n$ invariantly. If $d = 1$ the converse holds true, i.e., $T \in \mathscr{O}(\mathbb{H})$ iff $T \in \mathscr{L}$ and maps every $\mathscr{P}_n$ invariantly.*

Proof. Let $n \geq 0$ and $P \in \mathscr{P}_n$. We must prove that $TP \in \mathscr{P}_n$. By virtue of Taylor's Formula, $f \in \mathscr{P}_n$ iff $Q(D)f = 0$ for all $Q \in \mathscr{H}_{n+1}$. So, for any such Q, $Q(D)P = 0$ and hence, $Q(D)TP = 0$ since $T \in \mathscr{O}(Q)$. The converse part, when $d = 1$, follows by the observation $\ker D^{n+1} = \mathscr{P}_n$. ■

We denote by $\mathcal{S}$ the set of sequences $\Phi = (\varphi_n) = (\varphi_0, ...)$ in Exp such that $|\!|\!|\varphi_n|\!|\!|_m \leq RM^n$ for some $R, M, m \geq 0$. H_m denotes the projector in $\mathscr{H}$ onto $\mathscr{H}_m$ defined by $f = \sum f_n \mapsto f_m$, where $\sum f_n$ is the power series expansion of $f \in \mathscr{H}$. We have the following one-to-one correspondence between $\mathscr{O}(\mathbb{H})$ and $\mathcal{S}$:

Proposition 2. *The algebra $\mathscr{O}(\mathbb{H})$ is formed by the operators of the form $\Phi(D)f \equiv \sum_{n\geq 0} H_n\varphi_n(D)f$, where $\Phi = (\varphi_n) \in \mathcal{S}$ and is unique. (Note, $\varphi(D) = \Phi(D) \in \mathscr{C}$, $\Phi = (\varphi, \varphi, ...)$.) If $P \in \mathscr{H}_m$, $\Phi(D)^{(P)} = \Phi^{(m)}(D) \in \mathscr{O}(\mathbb{H})$ where $\Phi^{(m)} \equiv (\varphi_{n+m}) \in \mathcal{S}$. (Thus the derivative only depends on m, not on P, and $\mathscr{O}(\mathbb{H})$ is closed under derivations.)*

Proof. A proof of the first part can be found in [11, 12] (in fact, the result is there extended to infinite-dimensional holomorphy). We prove the claim about the derivative. We note that, for any m-homogeneous polynomial P, $P(D)H_n = H_{n-m}P(D)$ if $n \geq m$ and $P(D)H_n = 0$ otherwise. Thus,

$$P(D)\Phi(D) = \sum_{n\geq 0} P(D)H_n\varphi_n(D) = \sum_{n\geq m} H_{n-m}P(D)\varphi_n(D) = \Phi^{(m)}(D)P(D)$$

since $P(D)$ and $\varphi_n(D)$ commute. ■

Example 1. With $\varphi_n = 1$ if $n \leq m$ and $\varphi_n = 0$ otherwise, $\Phi(D)$ is the *Taylor projector*, i.e. the operator obtained by mapping a function into its Taylor polynomial of order m at the origin. The *Euler operator* $\langle \cdot, D\rangle \equiv z_1 D_1 + ... + z_d D_d$, i.e. the operator with symbol $\langle z, \xi\rangle \in \mathfrak{S}$, belongs to $\mathscr{O}(\mathbb{H})$. Indeed, for any power $m \geq 1$, $\langle \cdot, D\rangle^m = \Phi(D)$ where $\Phi = (\varphi_n = n^m)$.

We equip $\mathcal{S}$ with the algebra structure induced by $\mathscr{O}(\mathbb{H})$ so that $(\Phi\Psi)(D) = \Phi(D)\Psi(D)$. In fact, one can prove [11, Theorem 6] that if $(\xi_n) = \Phi\Psi$ in $\mathcal{S}$, then

$$\xi_n = \sum_{i=0}^{\infty} H_i(\varphi_n)\psi_{n+i}, \qquad \Phi = (\varphi_n),\ \Psi = (\psi_n). \tag{1}$$

An element $\varphi \in \mathrm{Exp}$ is non-degenerate if $\varphi(0) \neq 0$ and a sequence $\Phi = (\varphi_n)$ in Exp is non-degenerate if all the elements φ_n are. From (1) we deduce that the product $\Phi\Psi$ of any non-degenerate sequences Φ and Ψ in $\mathcal{S}$ is again non-degenerate ($\xi_n(0) = \varphi_n(0)\psi_n(0)$).

Lemma 2. *Let $\Phi = (\varphi_n) \in \mathcal{S}$ be non-degenerate. Then $\Phi(D)$ maps every $\mathscr{P}_n$ isomorphically (cf. Lemma 1). Thus, the restriction of $\Phi(D)$ to $\mathscr{P}$ is an isomorphism.*

Proof. $\Phi(D)$ is surjective on $\mathscr{P}_0 = \mathbb{C}$ for $\Phi(D)1 = \varphi_0(0) \neq 0$. Next we note that if $|\alpha| = m \geq 1$: $(*)$ $\Phi(D)z^\alpha = \varphi_m(0)z^\alpha +$ (lower degree terms). Assume $\Phi(D)$ is surjective on every $\mathscr{P}_m$, $m \leq n-1$ and let $P \in \mathscr{P}_n$. By $(*)$, we may find a $Q_n \in \mathscr{H}_n$ such that $\Phi(D)Q_n - P \in \mathscr{P}_{n-1}$ and hence, by the inductive hypothesis, $\Phi(D)Q_{n-1} = \Phi(D)Q_n - P$ for some $Q \in \mathscr{P}_{n-1}$. Thus $\Phi(D)$ maps $\mathscr{P}_n$ onto $\mathscr{P}_n$ for all n. To prove that $\Phi(D)$ is one-to-one on $\mathscr{P}_n$, it is clearly enough to prove that $\Phi(D)$ is injective on $\mathscr{P}$, which is obvious in view of $(*)$. ∎

The following result is due to H. Shapiro [14]: For any homogeneous polynomial $P \neq 0$, $P(D)P^*$ is a bijection on $\mathscr{H}$, where P^* is the homogeneous polynomial obtained by conjugating the coefficients in P and $P^* : f \mapsto P^*f$. (We say that $(P(D), P^*)$ forms a Fischer pair for $\mathscr{H}$, and it is an easy exercise to prove that a pair $(P(D), Q)$, where $P, Q \in \mathscr{P}$, forms a Fischer pair iff $\mathscr{H} = \ker P(D) \oplus \operatorname{Im} Q$ (Fischer decomposition).) This is an extension of Fischer's classical Theorem [3]: $P(D)P^*$ maps every $\mathscr{H}_n$ bijectively. (Note, $P^*\mathscr{H}_n \subseteq \mathscr{H}_{n+m}$ and $P(D)\mathscr{H}_{n+m} \subseteq \mathscr{H}_n$ if $P \in \mathscr{H}_m$.) In view of our purposes, we require estimates:

Lemma 3. *For given m and dimension d, there is a constant $k = k(m,d)$ such that for any $0 \neq P \in \mathscr{H}_m$ and $Q \in \mathscr{H}_n$, $\|P^*(P(D)P^*)^{-1}Q\|_1 \leq k^n\|Q\|_1/m!\|P\|_1$. (We assume the Euclidian norm on $\mathbb{C}^d$ in the sup-norm $\|\cdot\|_1 = \sup_{|z|\leq 1}|\cdot|$.)*

Proof. Consider the inner-product $(P,Q) \equiv \sum_\alpha P_\alpha \bar{Q}_\alpha \alpha!$ on $\mathscr{P}$ where $P = \sum_\alpha P_\alpha z^\alpha$ and the coefficients Q_α are defined analogously. By $\|\cdot\|$ we denote the corresponding (Fischer) norm. The key is to note that P^* is the Hilbert-adjoint of $P(D) : \mathscr{H}_{n+m} \to \mathscr{H}_n$, $P \in \mathscr{H}_m$, w.r.t. the inner-products induced by $(\cdot,\cdot)$. Indeed, let $f \in \mathscr{H}_n$ and put $g \equiv (P(D)P^*)^{-1}f \in \mathscr{H}_n$. Then, with $A \equiv P^*(P(D)P^*)^{-1}$, $P^*g = Af$ and Cauchy-Schwartz' Inequality gives

$$\|f\|\|g\| = \|P(D)P^*g\|\|g\| \geq (P(D)P^*g, g) = \|P^*g\|^2 \geq \|P\|\|Af\|\|g\|,$$

since $\|P^*\| = \|P\|$ and, by the formula in the proof of [14, Lemma 4], $\|P^*f\| \geq \|P^*\|\|f\|$. Thus the norm of $A : \mathscr{H}_n \to \mathscr{H}_{n+m}$ is not greater than $1/\|P\|$ for the Fischer norm and we only have to translate all this to the (equivalent) sup-norm. To do so we refer to [14, p. 519], where the arguments show that

$$\|Q\|_1 \leq \|Q\|/\sqrt{n!} \leq (n+1)^{d/2} d^{n/2} \|Q\|_1$$

for any $Q \in \mathscr{H}_n$. (However, they are there dealing with the supremun norm over polydiscs and to the readers convenience we remark that $\sup_{|z_i|\leq 1}|Q| \leq d^{n/2}\|Q\|_1$ if $Q \in \mathscr{H}_n$.) Now, there is a constant $k = k(m,d)$ such that $k^n \geq (n+1)^{d/2}d^{n/2}$ for all $n \geq 0$. From $\|AQ\| \leq \|Q\|/\|P\|$ a straight forward computation gives the lemma. ∎

Finally we formulate the Fréchet space version of [4, Corollary 1.4]:

Proposition 3 (Godefroy-Shapiro). *Let X be a separable Fréchet space and $\mathbb{T} = (T_n)$ a sequence of continuous linear operators on X. Assume there are dense subsets $Z, Y \subseteq X$ (not necessarily linear) and a sequence of maps $\mathbb{S} = (S_n : Y \to Y)$ (not necessarily continuous) such that:*

1. *$T_n z \to 0$ for all $z \in Z$,*
2. *$S_n y \to 0$ for all $y \in Y$,*
3. *$T_n S_n y = y$ for all $y \in Y$.*

Then $\mathbb{T}$ is hypercyclic.

3. The Main Results

We are now ready to prove our first main result. Recall, we shall first prove it in the case of one variable, where the proof is based on the theory of backward shifts.

Theorem A (one variable). *Let $\Phi = (\varphi_n) \in \mathcal{S}$ be a sequence such that $\varphi_n = \xi^m \psi_n$, i.e. $\Phi(D) = \Psi(D)D^m$, for some $m \geq 1$ and non-degenerate $\Psi = (\psi_n) \in \mathcal{S}$. Then $\Phi(D)$ is supercyclic.*

Proof. Let $e_n \equiv z^n/n!$ denote the monomial basis vectors in $\mathscr{P}$ and define the "forward shift" $A : \mathscr{P} \to \mathscr{P}$ by: $Ae_n \equiv e_{n+1}$ and then extended linearly. Then BA is the identity on $\mathscr{P}$ where $B \equiv D$ (backward shift). We can find a non-degenerate $\Phi_0 = (\phi_n) \in \mathcal{S}$ such that $\Phi(D) = B^m\Phi_0(D) = \Phi_0^{(m)}(D)B^m$ (i.e. $\Phi_0^{(m)} = \Psi$). Indeed, let ϕ_n, $n = 0, ..., m-1$, be arbitrary non-degenerate elements in Exp and put $\phi_n \equiv \psi_{n-m}$, $n \geq m$. Then $\Phi_0 \equiv (\phi_n)$ is non-degenerate and $\Phi_0^{(m)} = \Psi$. From this point we apply the technique of Godefroy & Shapiro in [4, Theorem 3.6.b] (however, they are dealing with Banach spaces and we must complement with some arguments). Let $\Phi_0^{-1}(D)$ denote the inverse of $\Phi_0(D)$ as a mapping $\mathscr{P} \to \mathscr{P}$ (Lemma 2) and put $C \equiv \Phi_0^{-1}(D)A^m$ so that $(*)$ $\Phi(D)C = B^mA^m = \mathrm{Id}_{\mathscr{P}}$. C maps $\mathscr{P}_n$ into $\mathscr{P}_{n+m}$ and, with notation as in [4], we let $\sigma(n)$ denote the operator norm of this restriction. Here we assume every finite-dimensional space $\mathscr{P}_n$ is equipped with its unique Banach space topology, and thus with the topology defined by the norm $|P|_n \equiv \sum_0^n \|H_i(P)\|_1$, (recall $\|\cdot\|_1 \equiv \sup_{|z|\leq 1} |\cdot|$). Now, C^n maps $\mathscr{P}_k$ into $\mathscr{P}_{k+nm}$ with norm $\leq \sigma(k+(n-1)m)^n \equiv \sigma_{kn}$ [4, p. 246]. Let $r_n \equiv n!\sigma_{nn}$ and put $T_n \equiv r_n\Phi(D)^n$. Then $T_n \in \mathscr{L}$ and it suffices to prove that $\mathbb{T} \equiv (T_n)$ is hypercyclic. We shall apply Proposition 3. Define $S_n \equiv r_n^{-1}C^n : \mathscr{P} \to \mathscr{P}$. Then with $Z = Y = \mathscr{P}$, $T_n \to 0$ pointwise on Z, since $m \geq 1$, and $T_nS_n = \mathrm{Id}_Y$ in view of $(*)$. (In fact, $T_nP = 0$ for all n sufficiently large if $P \in \mathscr{P}$.) Thus, by virtue of Proposition 3, we only have to prove that $S_n \to 0$ pointwise on $\mathscr{P}$. But if $0 \neq P \in \mathscr{P}_k$ and $n \geq k$,

$$|S^nP|_{k+nm}/|P|_k \leq r_n^{-1}\sigma_{kn} \leq r_n^{-1}\sigma_{nn} = 1/n!,$$

since σ is increasing. Hence, for any given semi-norm $\|\cdot\|_\nu$ $(\nu \geq 1)$,

$$\|S_nP\|_\nu \leq \sum_{i=0}^{k+nm} \nu^i\|H_i(S_nP)\|_1 \leq \nu^{k+nm}|S_nP|_{k+nm} \leq \nu^{k+nm}|P|_k/n! \to 0.$$

■

Remark 3. The attentive reader note that we have proved a little bit more than what is claimed. We have proved that $T = \Phi(D)$ is *strongly supercyclic* in the sense that there is a fixed sequence $r = (r_n)$ and $f \in \mathscr{H}$ such that $r \cdot \mathrm{Orb}(T, f) \equiv \{r_n T^n f\}$ ($\subseteq \mathrm{Orb}_s(T, f)$) is dense.

The following example shows that some of the operators in Theorem A are in fact hypercyclic, and we shall extend this fact later (Theorem C).

Example 2. Aron and Markrose proved recently [1] that in the case of one variable T_λ, $T_\lambda f(z) \equiv f'(\lambda z)$, is a hypercyclic operator for any $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1$. They also discuss hypercyclicity of $T_{\lambda:a} \equiv T_\lambda \tau_a : f \mapsto f'(\lambda z + a)$ (see below). Note that $T_{1:a} = D\tau_a \in \mathscr{C}$ and, in fact, $T_{\lambda:a} \in \mathscr{C}$ iff $\lambda = 1$. Now, we note that for arbitray λ and a, $T_{\lambda:a} = \Phi(D) \in \mathscr{O}(\mathbb{H})$ where $\Phi = (\varphi_n(\xi) = \xi e^{\langle a,\xi\rangle} \lambda^n)$. Thus, for any $\lambda \neq 0$, $T_{\lambda:a}$ belongs to the class of operators in Theorem A and is thus supercyclic. However, assume $|\lambda| \geq 1$ so that $T_\lambda = T_{\lambda:0}$ is hypercyclic. By Theorem 1, $T_\lambda^{(P)}$ also forms a hypercyclic operator for any $P = \xi^m \in \mathbb{H}$. We deduce that $T_\lambda^{(P)} = \lambda^m T_\lambda$ so $\lambda^m T_\lambda$ is a hypercyclic operator, and moreover, $P(D)f = f^{(m)}$ is a hypercyclic vector for any such vector f for T_λ. A simple argument [1] shows that $T_{\lambda:a}$ is hypercyclic for any root of unity λ, $\lambda^m = 1$. We note that, for such λ, $T_{\lambda:a} \in \mathscr{O}(P)$ for any $P = \sum_i a_i z^i \in \mathscr{P}$ such that $m|i$ whenever $i, a_i \neq 0$. However, $T_{\lambda:a}^{(P)} = T_{\lambda:a}$ so this does not provide us with any new hypercyclic operator. On the other hand, Theorem 1 gives that $P(D)$ maps the set of hypercyclic vectors for $T_{\lambda:a}$ invariantly.

Finally, an interesting phenomenon is that T_λ is *not* (and presumably not any $T_{\lambda:a}$) hypercyclic if $|\lambda| < 1$ [1, Prop. 14], but we know that T_λ is supercyclic for any $\lambda \neq 0$.

Let us note that Theorem A covers some facts we already know. We know that $\varphi(D)$ is hypercyclic, and thus supercyclic, for any non-scalar $\varphi \in \mathrm{Exp}$. In particular, if $\varphi(0) = 0$, $\varphi(\xi) = \xi^m \psi(\xi)$ for some unique $m > 0$ and non-degenerate $\psi \in \mathrm{Exp}$ and now, $\varphi(D) = \psi(D)D^m = \Psi(D)D^m$, $\Psi \equiv (\psi, \psi, ...)$. Thus the class of operators in Theorem A above contains all $T = \varphi(D) \in \mathscr{C}$ such that $\varphi(0) = 0$, i.e., $T1 = 0$.

Next we shall, as promised, extend Theorem A to an arbitrary number of variables d.

Let $\mathcal{S}_\mathrm{S}$ denote the set of sequences $\Phi \in \mathcal{S}$ of the form $\Phi = (\psi_n P_n)$ where $\Psi = (\psi_n)$ is non-degenerate and $0 \neq P_n \in \mathscr{H}_m$, $n = 0, ...$ for some $m \geq 1$. By $\mathscr{O}_\mathrm{S}$ we denote the corresponding class of operators $\Phi(D)$. It is convenient to clarify the following. Let $\Psi = (\psi_n)$ be a non-degenerate sequence in Exp and $0 \neq P_n, P \in \mathscr{H}_m$ where $m \geq 1$, then:

1. If $\Psi \in \mathcal{S}$ and $\|P_n\|_1 \leq MR^n$, then $\Phi \equiv (\psi_n P_n) \in \mathcal{S}_\mathrm{S}$.

2. $\Phi \equiv (\psi_n P) \in \mathcal{S}_\mathrm{S}$ iff $\Psi \in \mathcal{S}$.

(1 is elementary and, by Cauchy's Estimates, a sequence $(\varphi_n) \in \mathcal{S}$ iff $\|H_i(\varphi_n)\|_1 \leq MR^n r^i / i!$ for some $r, R, M \geq 0$, hence 2 is an easy consequence of the following: If $P \in \mathscr{H}_n$ and $Q \in \mathscr{H}_m$, then $\|P\|_1 \|Q\|_1 \leq (2e)^{n+m} \|PQ\|_1$ [2, p. 72].) In particular, 2 implies that when $d = 1$ then $\mathscr{O}_\mathrm{S}$ is precisely the class of operators in Theorem A above, that we thus extend by:

Theorem A. *Every operator $\Phi(D) \in \mathscr{O}_\mathrm{S}$ is (strongly) supercyclic. Thus, in particular, any operator $\Phi(D) = \Psi(D)P(D)$, where $\Psi \in \mathcal{S}$ is non-degenerate and $0 \neq P \in \mathscr{H}_m$, $m \geq 1$, is (strongly) supercyclic.*

Proof. Let us first prove the special case, i.e., assume all the homogeneous P_n in Φ are equal to some $P \in \mathscr{H}_m$ so that $\Phi(D) = \Psi(D)P(D)$, $\Psi \in \mathcal{S}$. First of all we note that $\Phi(D)^n P = 0$ for all n sufficiently large if $P \in \mathscr{P}$ since $m \geq 1$. Next, as in the one variable proof, we can find a non-degenerate $\Phi_0 \in \mathcal{S}$ such that $\Phi_0^{(m)} = \Psi$ and thus $\Phi(D) = P(D)\Phi_0(D) = \Phi_0^{(m)}(D)P(D)$. Next, $P(D)P^*$ is a bijection on $\mathscr{H}$ by H. Shapiro's result and, by Fischer's Theorem, $P(D)P^*$ maps $\mathscr{P}_n$ into $\mathscr{P}_n$ isomorphically (see page 6). Thus, if $(P(D)P^*)^{-1}$ denotes the inverse of the restriction of $P(D)P^*$ to $\mathscr{P}$, $A \equiv P^*(P(D)P^*)^{-1} : \mathscr{P} \to \mathscr{P}$ maps $\mathscr{P}_n$ into $\mathscr{P}_{n+m}$. Now, $\Phi_0^{-1}(D) : \mathscr{P} \to \mathscr{P}$ exists by Lemma 2 and with $C \equiv \Phi_0^{-1}(D)A : \mathscr{P} \to \mathscr{P}$, $\Phi(D)C = \mathrm{Id}_{\mathscr{P}}$ and $C\mathscr{P}_n \subseteq \mathscr{P}_{n+m}$. From this point the arguments in the proof above for $d = 1$ proves the theorem for this particular case.

Next consider the general case, i.e. $\Phi = (\psi_n P_n)$ where $P_n \in \mathscr{H}_m$. Again, as a starting point we conclude that $\Phi(D)^n P = 0$ for large n if $P \in \mathscr{P}$. We define $\mathcal{B} : \mathscr{P} \to \mathscr{P}$ by $\mathcal{B}Q = \sum_{n \geq m} P_{n-m}(D)Q_n$ where $Q = \sum Q_n$, $Q_n \in \mathscr{H}_n$. Let $\Phi_0 = (\phi_n)$ be a non-degenerate sequence in Exp with $\Phi_0^{(m)} = \Psi$. Since Ψ may not be in $\mathcal{S}$, it is possible that $\Phi_0 \notin \mathcal{S}$, however $\Phi_0(D) = \sum_{n \geq 0} H_n \phi_n(D)$ is a well-defined map on $\mathscr{P}$ and we claim that $\Phi(D) = \mathcal{B}\Phi_0(D)$ on $\mathscr{P}$. Indeed,

$$\mathcal{B}\Phi_0(D) = \sum_{n \geq m} P_{n-m}(D)H_n\phi_n(D) = \sum_{n \geq m} H_{n-m}P_{n-m}(D)\phi_n(D) = \Phi(D)$$

since $\phi_n = \psi_{n-m}$ for $n \geq m$. Moreover, from the proof of Lemma 2, it is clear that $\Phi_0(D)^{-1} : \mathscr{P} \to \mathscr{P}$ exists. By Fischer's Theorem, we can define a map $\mathcal{A} : \mathscr{P} \to \mathscr{P}$ by $\mathcal{A}Q = \sum P_n^*(P_n(D)P_n^*)^{-1}Q_n$ with Q as above. We deduce that $\mathcal{B}\mathcal{A} = \mathrm{Id}_{\mathscr{P}}$ so with $\mathcal{C} \equiv \Phi_0^{-1}(D)\mathcal{A} : \mathscr{P} \to \mathscr{P}$, $\Phi(D)\mathcal{C} = \mathrm{Id}_{\mathscr{P}}$ and, again, from this point the arguments in the one variable proof applies. ■

Let $\mathscr{O}_{\mathrm{S}}^*$ denote the set of operators $\Phi(D) = \Psi(D)P(D)$ in Theorem A, i.e. where $\Psi \in \mathcal{S}$ is non-degenerate and $0 \neq P \in \mathscr{H}_m$, $m > 0$. By $\mathcal{S}_{\mathrm{S}}^*$ we denote the corresponding set of sequences $\Phi = (P\psi_n) = P\Psi$ in $\mathcal{S}$. Note that in the case of one variable, $\mathscr{O}_{\mathrm{S}}^* = \mathscr{O}_{\mathrm{S}}$.

Theorem B. *Assume $\Phi(D) \in \mathscr{O}_{\mathrm{S}}$ ($\Phi = (\varphi_n) \in \mathcal{S}_{\mathrm{S}}$). Then:*

1. *$\Phi^{(m)}(D) \in \mathscr{O}_{\mathrm{S}}$ for any $m \geq 0$ and conversely,*

2. *For any $m \geq 0$ there is a $\Psi(D) \in \mathscr{O}_{\mathrm{S}}$ such that $\Psi^{(m)}(D) = \Phi(D)$.*

3. *For any $\Psi \in \mathcal{S}_{\mathrm{S}}^*$ or non-degenerate $\Psi \in \mathcal{S}$, $\Phi(D)\Psi(D) \in \mathscr{O}_{\mathrm{S}}$.*

$\mathscr{O}_{\mathrm{S}}^*$ forms a multiplicative closed subset of $\mathscr{O}_{\mathrm{S}}$ and is stable in the sense of 1-3. For any set $A \subseteq \mathcal{S} \times \mathbb{H}$ such that $P \neq 0$, $\Psi^{(m)} = \Phi$ if $P \in \mathscr{H}_m$, for all $(\Psi, P) \in A$:

$$\mathscr{I}_A \equiv \cup_A \{P(D)f : f \text{ supercyclic for } \Psi(D)\} \tag{2}$$

forms an invariant set (under $\Phi(D)$) of supercyclic vectors for $\Phi(D)$. In particular, for any $m \geq 1$ there is a vector f such that

$$\mathscr{M}_m = \{P(D)f : P \in \mathscr{H}_m\}$$

forms an $\binom{m+d-1}{d-1}$ dimensional vectors space (i.e. $\simeq \mathscr{H}_m$) whose non-zero elements ($P(D)f$, $P \neq 0$) are supercyclic vectors for $\Phi(D)$.

Proof. 1 is elementary and so is 2. Indeed, let $m \geq 0$, then $\Psi^{(m)} = \Phi$ and $\Psi = (\psi_n) \in \mathcal{S}_{\mathrm{S}}$ where $\psi_{n+m} \equiv \varphi_n P_n$ if $n \geq 0$ and $\psi_n \equiv P_n \phi_n$, $n < m$, where $\phi_n \in \mathrm{Exp}$ are arbitrary non-degenerate elements. 3 follows by formula (1). For assume $\Psi = (\psi_n Q) \in \mathcal{S}_{\mathrm{S}}^*$ and let $(\xi_n) \equiv \Phi\Psi$. Then (1) gives $\xi_n = P_n Q \sum_{i \geq m} H_{i-m}(\varphi_n)\psi_{i+n} = P_n Q \phi_n$ if $P_n \in \mathscr{H}_m$. $R_n \equiv P_n Q$ are all homogeneous of the same degree > 0 and every ϕ_n is non-degenerate. Thus $\mathscr{O}_{\mathrm{S}}\mathscr{O}_{\mathrm{S}}^* \subseteq \mathscr{O}_{\mathrm{S}}$ and the other claim in 3 follows in the same way.

That (2) is formed by supercyclic vectors follows by Theorem 1. We must prove that $\mathscr{I}_A$ is invariant. So let $P(D)f \in \mathscr{I}_A$, $(\Psi, P) \in A$. Then $\Phi(D)P(D)f = P(D)\Psi(D)f$. Since f is supercyclic for $\Psi(D)$, it is elementary that $\Psi(D)f$ also forms a supercyclic vector for $\Psi(D)$, hence $\Phi(D)P(D)f \in \mathscr{I}_A$.

In particular, given m, then, in view of 2, there is a $\Psi \in \mathcal{S}_{\mathrm{S}}$ with $\Psi^{(m)} = \Phi$ and by Theorem A we can find a supercyclic vector f for $\Psi(D)$. So from (2), $\{P(D)f : 0 \neq P \in \mathscr{H}_m\}$ is formed by supercyclic vectors for $\Phi(D)$ and we deduce that $\mathscr{H}_m \ni P \mapsto P(D)f \in \mathscr{M}_m$ defines a linear isomorphism ℓ. Indeed, $P(D)f \neq 0$ for all $P \neq 0$, for otherwise 0 would be a supercyclic vector, so ℓ is one-to-one and hence a bijection. ■

Example 3. Fix m and $\Psi \in \mathcal{S}_{\mathrm{S}}$ such that $\Psi^{(m)} = \Phi \in \mathcal{S}_{\mathrm{S}}$. Then, with $A \equiv \{(\Psi, P) : 0 \neq P \in \mathscr{H}_m\}$, we obtain the invariant set $\mathscr{I}_A = \cup_{P \in \mathscr{H}_m \setminus \{0\}} P(D)\mathrm{SC}(\Psi)$ of supercyclic vectors for $\Phi(D)$. Here $\mathrm{SC}(\Psi)$ denotes the set of supercyclic vectors for $\Psi(D)$.

Remark 4. The attentive reader note that the arguments in the proof concerning the invariant set $\mathscr{I}_A$ hold more generally. Let $S \in \mathscr{L}$ and let $A = \{(T, \varphi)\}$ be any family of pairs $(T, \varphi) \in \mathscr{L} \times \mathrm{Exp}$ such that $\varphi \neq 0$, $T \in \mathscr{O}(\varphi)$ and $T^{(\varphi)} = S$. Then $\cup_A \varphi(D)\{f : f$ supercyclic for $T\}$ (possibly empty) forms an invariant set of supercyclic operators for S. The analogue holds for hypercyclicity (but *not* cyclicity, cf. [4, p. 235]).

A cyclic vector manifold for an operator T is, in the sense of Godefroy & Shapiro [4], a vector space whose non-zero elements are cyclic for T. Supercyclic and hypercyclic vector manifolds are defined similarly and thus, with this terminology, $\mathscr{M}_m$ in Theorem B is a supercyclic vector manifold for $\Phi(D)$.

Example 4. The example of Aron and Markrose (Example 2) is easily extended to d variables. Indeed, let $\lambda \in \mathbb{C}^d$ and consider the affine map $\Lambda : z \mapsto \lambda \cdot z$, $\lambda \cdot z \equiv (\lambda_i z_i)$. (We assume $a = 0$.) Define $M_\Lambda f \equiv f(\lambda \cdot z)$. Then we claim that if $|\lambda_i| \geq 1$ then $T \equiv M_\Lambda D^\alpha$ (i.e. $Tf(z) = f^{(\alpha)}(\lambda \cdot z)$) is hypercyclic for any $\alpha \neq 0$. (This follows with smaller, and obvious, modifications of the proof of [1, Theorem 13].) Now, if all λ_i are equal, $\lambda_i = \lambda$, but where λ is arbitrary we have that $T = \Phi(D) \in \mathscr{O}(\mathbb{H})$, $\Phi = (\varphi_n = \xi^\alpha \lambda^n)$. Thus, if $|\lambda| \geq 1$, T is a hypercyclic operator in $\mathscr{O}_{\mathrm{S}}^* \subseteq \mathscr{O}_{\mathrm{S}}$ and outside $\mathscr{C}$ if $\lambda \neq 1$ (since Φ is not a constant sequence). If λ_i not all are equal, it follows that $T \notin \mathscr{O}(\mathbb{H})$ and consequently $T \notin \mathscr{O}_{\mathrm{S}}$, hence if also $|\lambda_i| > 1$ then T is a hypercyclic non-convolution operator outside $\mathscr{O}_{\mathrm{S}}$.

Thus there are examples of hypercyclic non-convolution operators in $\mathscr{O}_{\mathrm{S}}$ and $\mathscr{O}_{\mathrm{S}}^*$ also when $d > 1$, we now extend this fact:

Theorem C. *Let* $\Phi = (\psi_n P_n) \in \mathcal{S}_{\mathrm{S}}$, $P_n \in \mathscr{H}_m$, *where* ψ_n *are scalars and, for some* $c, C, R > 0$: (b) $c \leq |\psi_n| \|P_n\|_1 \leq CR^n$ *(note,* $\Phi \in \mathcal{S}$ *and* $\Psi \equiv (\psi_n)$ *is non-degenerate). Then* $T \equiv \Phi(D)$ *is hypercyclic.*

If $\mathscr{O}_{\mathrm{H}}$ *denotes the set of operators of this form (and* $\mathcal{S}_{\mathrm{H}}$ *the corresponding class of sequences),* $\mathscr{O}_{\mathrm{H}}$ *is multiplicative closed and stable in the sense of 1-2 (Theorem B).*

Finally, invariant sets of hypercyclic vectors for T *are obtained analogous to (2) and, in particular, for every* $m \geq 1$ *there is an* $f \in \mathscr{H}$ *such that* $\mathscr{M}_m = \{P(D)f : P \in \mathscr{H}_m\}$ *forms a hypercyclic vector manifold for* T.

Proof. We prove that $T = \Phi(D)$ is hypercyclic and intend to apply Proposition 3 with $Z = Y = \mathscr{P}$. We choose a non-degenerate scalar sequence $\Phi_0 = (\phi_n)$ such that $\phi_{n+m} = \psi_n$, i.e. $\Phi_0^{(m)} = \Psi = (\psi_n)$, and define operators $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ as in the proof of Theorem A. Clearly, $\Phi_0^{-1}(D) = \sum \phi_n^{-1} H_n$ and from this we deduce $\mathcal{C}Q = \Phi_0^{-1}(D)\mathcal{A}Q = \sum_{n\geq 0} \psi_n^{-1} A_n Q_n$, where $A_n \equiv P_n^*(P_n(D)P_n^*)^{-1}$ and $Q_n \equiv H_n Q \in \mathscr{H}_n$. Hence,

$$\mathcal{C}^n f = \sum_{i\geq 0} (\psi_i \psi_{i+m} ... \psi_{i+m(n-1)})^{-1} A_{i+m(n-1)} ... A_{i+m} A_i Q_i, \quad Q = \sum Q_n \in \mathscr{P},$$

and thus, Lemma 3 gives,

$$\|C^n Q\|_r \leq \sum_{i\geq 0} r^{i+nm} |\psi_i ... \psi_{i+m(n-1)}|^{-1} \|A_{i+m(n-1)} ... A_i Q_i\|_1 \leq$$

$$\sum_{i\geq 0} r^{i+nm} \frac{|\psi_i ... \psi_{i+m(n-1)}|^{-1}}{\|P_i\|_1 ... \|P_{i+m(n-1)}\|_1} k(m,d)^n \frac{\|Q_i\|_1}{m!^n} \leq \frac{r^{nm} c^{-n} k(m,d)^n}{m!^n} \sum_{i\geq 0} r^i \|Q_i\|_1 \to 0$$

as $n \to \infty$. So $S_n \equiv \mathcal{C}^n \to 0$ pointwise on Y, $T^n S_n = \mathrm{Id}_Y$ and, since $m > 0$, $T^n \to 0$ pointwise on $Z = \mathscr{P}$. Thus T is hypercyclic by Proposition 3.

That the analogues of 1 and 2 in Theorem B hold true for $\mathscr{O}_{\mathrm{H}}$ is elementary and we prove that $\mathscr{O}_{\mathrm{H}}$ is multiplicative closed. But by formula (1), $(\varphi_n P_n)(\psi_n Q_n) = (\varphi_n \psi_{n+m} P_n Q_{n+m})$ if $P_n \in \mathscr{H}_m$ for all n, hence $\mathscr{O}_{\mathrm{H}}$ is stable under multiplication. ∎

Remark 5. Of course, given $\Phi = (\psi_n P_n) \in \mathcal{S}_{\mathrm{H}}$ we may always assume $\psi_n = 1$ for all n and thus that we deal with a "pure" sequence in $\mathscr{H}_m$. However, in view of the more general construction $\mathcal{S}_{\mathrm{S}}$, we chose to define $\mathcal{S}_{\mathrm{H}}$ in the way we did to underline the inclusion $\mathcal{S}_{\mathrm{H}} \subseteq \mathcal{S}_{\mathrm{S}}$. Another motivation is that we think that proof becomes more informative in this way in view of the more general problem: For what sequences $\Phi = (\psi_n P_n) \in \mathcal{S}_{\mathrm{S}}$ is $\Phi(D)$ hypercyclic?

In particular, if $\Phi = (\varphi_n)$ is a sequence of scalars such that: (b') $0 < c \leq |\varphi_n| \leq CR^n$, then $\Phi \in \mathcal{S}$ and $\Phi(D)P(D)$ $(\in \mathscr{O}_{\mathrm{H}})$ is hypercyclic for any non-constant $P \in \mathbb{H}$. ($P_n = P$ for all n in Theorem C.) In the case of one variable, every $\mathscr{H}_n$ is one dimensional and every element of $\mathscr{O}_{\mathrm{H}}$ can be factorized in this form (cf. $\mathscr{O}_{\mathrm{S}}^* = \mathscr{O}_{\mathrm{S}}$).

Example 5. Consider the Euler operator $\langle \cdot, D \rangle \in \mathscr{O}(\mathbb{H})$. We recall from Example 1 that, for any power m, $\langle \cdot, D \rangle^m = \Phi(D)$ where $\Phi = (\varphi_n = n^m)$. Thus Φ satisfies the bounds (b') above except for that $\varphi_0 = 0$. But if we add a sequence $(c, 0, ...)$, $c \neq 0$, to Φ we obtain a sequence satisfying (b') and conclude: For any $m \geq 1$, $c \neq 0$ and non-constant $P \in \mathbb{H}$, $(\langle \cdot, D \rangle^m + c\delta_0)P(D)$ is hypercyclic. $(\delta_0(f) \equiv f(0) = H_0 f.)$

Further, any derivative $\Phi^{(n)}(D)$ of $\langle \cdot, D \rangle^m$ corresponds in $\mathcal{S}$ to a sequence $\Phi^{(n)}$ of constants of the form (b') and hence: $P(D)\langle \cdot, D \rangle^m$ is hypercyclic for any $m \geq 1$ and non-constant $P \in \mathbb{H}$. Thus, for example, if $d = 1$ then $f \mapsto D(zDf) = zf''(z) + f'(z)$ forms a hypercyclic operator.

Note that with $|\lambda| \geq 1$ and $\Phi = (\varphi_n \equiv \lambda^n)$, $T \equiv \Phi(D)D^\alpha \in \mathscr{O}_{\mathrm{H}}$ if $\alpha \neq 0$. In fact, T is precisely the hypercyclic operator in Example 4 with $\lambda_i = \lambda$ for all i. In particular, if $d = 1$ and $\alpha = 1$ our result that T is hypercyclic is that of Aron and Markrose that T_λ is hypercyclic provided $|\lambda| \geq 1$ (Example 2).

REMARKS: (i) We note that the example due to Aron and Markrose, and all our examples far, of cyclic type operators T outside $\mathscr{C}$ degenerates in the sense that $T1 = 0$. Thus, the question is if there is any, say, hypercyclic $T \in \mathscr{L} \setminus \mathscr{C}$ with $T1 \neq 0$. We shall show that the answer is affirmative by once again illustrate how Fischer pairs provide us with alternative "backward shifts". (For simplicity we let $d = 1$ and in order to be brief, we tacitly assume the reader keep an eye on the (one-variable) proof of Theorem A.)

H. Shapiro's result on page 6 admits the following generalization: $(P(D) - c, P^*)$ forms a Fischer pair for $\mathscr{H}$, for any constant c and homogeneous polynomial $P \neq 0, c$ [14, Theorem 3].

Let $P = \xi$ and thus $P(D) - c = D - c$, $P^* = P$. Put $\mathscr{E}_n \equiv \ker(D - c)^{n+1}$, i.e., $\mathscr{E}_n = \mathscr{P}_n e_c = \mathscr{P}_n e^{cz}$ (finite-dimensional). Then $\mathscr{E} \equiv \cup_{n \geq 0} \mathscr{E}_n$ is dense in $\mathscr{H}$ and an operator T is PDE-preserving for $\mathbb{E} \equiv \{1, \xi - c, (\xi - c)^2, ...\}$ iff it maps every $\mathscr{E}_n$ invariantly (cf. Lemma 1). It is now easy to prove that $P(\cdot, D) \in \mathscr{O}(\mathbb{E})$ iff $P_c(\cdot, D) \in \mathscr{O}(\mathbb{H})$ where $P_c \equiv P(z, \xi + c) \in \mathfrak{S}$. From this and Proposition 2 we deduce the following. Let E_n denote the map $E_n \equiv e_c H_n e_{-c}$, i.e. $E_n f \equiv (D - c)^n f(0) z^n e_c / n! \in \mathscr{E}_n$. Then $\Phi \mapsto \Phi[D] \equiv \sum_{n \geq 0} E_n \varphi_n(D)$ defines a one-to-one correspondence between $\mathcal{S}$ and $\mathscr{O}(\mathbb{E})$ and $\Phi[D]^{(P)} = \Phi^{(n)}[D]$ if $P = (\xi - c)^n$. If Φ is non-degenerate at c, i.e., $\varphi_n(c) \neq 0$ for all n, then $\Phi[D]$ maps every $\mathscr{E}_n$ isomorphically. We obtain: *Every operator T of the form $\Phi[D](D - c)^m$ is supercyclic if Φ is non-degenerate at c and $m \geq 1$.* Indeed, $(D - c)\mathscr{E}_{n+1} \subseteq \mathscr{E}_n$ so $T^n E = 0$ for large n if $E \in \mathscr{E}$. Further, there is a factorization $T = (D - c)^m \Phi_0^{(m)}[D]$ where $\Phi_0 \in \mathcal{S}$ is non-degenerate at c. $(P(D) - c)P^*$ maps every $\mathscr{E}_n$ isomorphically and we put $A \equiv P^*[(P(D) - c)P^*]^{-1} : \mathscr{E} \to \mathscr{E}$. Thus with $C \equiv \Phi_0^{-1}[D]A^m : \mathscr{E} \to \mathscr{E}$, $TC = \mathrm{Id}_{\mathscr{E}}$ and we deduce that there is a sequence $r = (r_n)$ such that $\mathbb{T} = (T_n \equiv r_n T^n)$ is hypercyclic, hence T is (strongly) supercyclic. In particular we note that $T1 \neq 0$ for a suitable Φ (see below for a specific example), on the other hand, $Te_c = 0$. In the same way, with smaller modifications of the proof of Theorem C we obtain: $\Phi[D](D - c)^m$ *is hypercyclic for any scalar sequence* $\Phi = (\varphi_n)$ *with bounds* (b') (thus Φ is non-degenerate at c). With $\Phi = (\varphi_n = n + 1)$ and $m = 1$ we obtain the hypercyclic operator: $T = zD^2 - 2czD + c^2 z + D - c$, which with $c = 0$ reduces to the operator in the latter part of Example 5 and $T1 \neq 0$ if $c \neq 0$.

(ii) We suggest a study on in what degree the converse of Theorem 1 holds: Is every, say, hypercyclic $S \in \mathscr{L}$ the derivative, $T^{(\varphi)}$, of some hypercyclic $T \in \mathscr{O}(\varphi)$. (Note, this is true for any S in $\mathscr{C} \setminus \mathbb{C}$ and in $\mathscr{O}_{\mathrm{H}}$.) Or even stronger, is every hypercyclic vector g for S of the form $\varphi(D)f$ for some hypercyclic $T \in \mathscr{O}(\varphi)$ with $T^{(\varphi)} = S$. (This is, as far as we know, an open problem even for $S \in \mathscr{C} \setminus \mathbb{C}$.)

(iii) Our technique of working with Fischer pairs should work for other spaces, in particular, other power series spaces. Indeed, Fischer decompositions have also been studied for: Exp, germs of analytic functions, the entire ring of formal power series etc. This is interesting in view of the fact that these spaces do not in general admit backward shifts. In particular, Fischer splittings have been studied for entire function spaces in an infinite number of variables and we believe that some of the results in this note are extendible to infinite-dimensional holomorphy in this way. (cf. [9] where an infinite-dimensional analogue of Godefroy-Shapiro's Theorem is obtained.)

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Henrik Petersson
Chalmers/Göteborg University
School of Mathematical Sciences
SE-412 96 Göteborg, Sweden
henripet@math.chalmers.se