Geometric continuity and compatibility conditions

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Abstract

When considering regularity of surfaces, it is its geometry that is of interest. Thus, the concept of geometric regularity or geometric continuity of a specific order is the relevant concept. In this paper we discuss necessary and sufficient conditions for a surface to be geometrically continuous of order one and two or, in other words, being tangent plane continuous and curvature continuous. Particularly, we consider a 4-patch surface, where the patches have a general representation. The focus is on the regularity of the point where four patches meet and the compatibility conditions that must appear.

Keyword: 4-patch surface, tangent plane continuity, curvature continuity, compatibility conditions

1 Introduction

In many applications in Computer-Aided Geometric Design (CAGD) and Computer Graphics a surface is composed of several patches, where a patch usually is representated by a Bezier polynomial, B-spline or NURB. In particular, each patch is as regular as is needed. Thus, when considering regularity of a surface such as tangent plane continuity or curvature continuity, the lack of regularity only occurs somewhere at a common boundary curve between two or more patches.

Regularity for a surface constituting of two adjacent patches intersecting in a common boundary curve, see Figure 1, is a well studied problem. A general approach for G^1 as well as G^2 continuity, i e tangent plane continuity and curvature continuity respectively, was given by Juergen Kahmann in a paper from 1983, see [7]. In the same paper he also applies his result to the case of Bezier patches. Other authors such as Degen [3], Liu and Hoschek [10], Liu [9], DeRose [4], have also treated tangent plane continuity in the 2-patch case. In the case of curvature continuity for this type of 2-patch surface we refer to articles by Kiciak [8], Ye, Liang and Nowacki [14].

A more complicated situation is where four patches intersect at a common point, where every pair of adjacent patches meet at a common boundary, see Figure 2. Among the many authors that has treated regularity problem in this 4-patch surface case are Bézier [1], Sarraga [12] and [13], Ye and Nowacki [15]. For further references and an overview, look at the book by Hoschek and Lasser [6].

Most often the approach has been to study the regularity problem in the case of a specific patch representation. In this paper we consider a 4-patch surface with a general patch representation. We give necessary and sufficient compatibility condition in order to have tangent plane continuity and curvature continuity respectively for such a surface. The conditions are general and independent of the patch representation.

2 Geometric continuity of order 1

When discussing regularity of a surface our focus is on the geometry of the surface and not its representation. Consequently, the notation geometric continuity of a certain order is the proper concept in this context. The lowest order of regularity is G^0, which means that the surface is connected. Another way to put it is to require that its representation is continuous. The next level of regularity is tangent plane continuity, denoted by G^1, which will be defined next.

Definition 1 *A continuous surface is said to be tangent plane continuous, denoted by G^1, if every point on the surface has a unique tangential plan, which varies continuously on the surface. Such a surface is also said to be geometrically continuous of order one.*

Let us look at Definition 1 in a specific situation. In this connection we use the notation $r \in C^1_{\#}$ for a differential function $(u, v) \mapsto r(u, v) \in \mathbb{R}^3$, with $r_u \times r_v \neq 0$ for $0 \leq u, v \leq 1$. Suppose that a surface S constitutes of two patches with a common boundary. Let the patches be described as $(u, v) \mapsto r^{(1)}(u, v)$ with $0 \leq u, v \leq 1$ and $(s, t) \mapsto r^{(2)}(s, t)$ with $0 \leq s, t \leq 1$. Supposing further that each patch is regular enough, i e that $r^{(1)}, r^{(2)} \in C^1_{\#}$. In order for the surface S to be tangent plane continuous the only points that not automatically fulfill the G^1-condition are those along the common boundary of the two patches, see Figure 1. On this boundary curve we must particularly have $v \mapsto r^{(1)}(1, v) = r^{(2)}(0, t(v))$ for $0 \leq v \leq 1$, where $v \mapsto t(v)$ maps $[0, 1]$ continuously and uniquely onto $[0, 1]$, i e that the two patches together constitute a continuous surface. Moreover, at a common boundary point the tangent planes considered from patch $r^{(1)}$ and $r^{(2)}$ respectively must coinside.

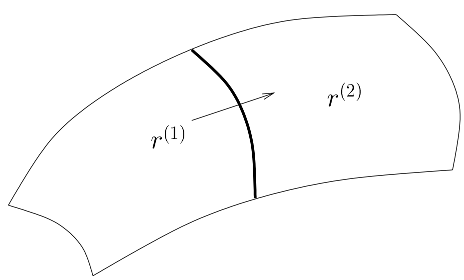

Figure 1: Two patches connected by a common boundary

At a particular boundary point with parameter value v this can be formulated as

$$\mathit{span}\{r_u^{(1)}(1, v), r_v^{(1)}(1, v)\} = \mathit{span}\{r_s^{(2)}(0, t(v)), r_t^{(2)}(0, t(v))\},$$

which must not be degenerated. An equivalent way to put it is that

$$r_u^{(2)}(0, t(v)) = \lambda_{12}(v)\, r_u^{(1)}(1, v) + \kappa_{12}(v)\, r_v^{(1)}(1, v), \quad 0 \leq v \leq 1, \tag{2.1}$$

where λ_{12} and κ_{12} are continuous functions along the boundary curve, and the tangent vectors $r_u^{(1)}(1, v)$ and $r_v^{(1)}(1, v)$ are linearly independent for every $v \in [0, 1]$. Without loss of generality, from now on we identify the parameters v and t on the common boundary.

The formula description in (2.1) of two adjacent patches is very well studied. In this paper we will rather consider the problem with four adjacent patches, i e a surface constituting of four patches where every two adjacent patches have a common boundary, see Figure 2. Moreover, the four patches intersect in a common point V. The difference between the 2-patch case compared to the 4-patch case is that in the later case exists compatibility conditions that must be satisfied in order for every two patches with a common boundary to fulfill equation (2.1). Thus, we want to find the correct conditions for a 4-patch surface to be tangentially continuous at the intersection point V.

The compatibility conditions can be rephrased in such a way that we formulate necessary and sufficient conditions on the functions λ_{ij} and κ_{ij} at the intersection point V.

First, by using the relation (2.1) we get the next four relations between the patches (1)–(2), (2)–(3), (4)–(3) and (1)–(4). For practical purposes we use the same parameters u and v for all the patches, where $0 \leq u, v \leq 1$. Thus

$$\begin{aligned} r_u^{(2)}(0,v) &= \lambda_{12}(v) r_u^{(1)}(1,v) + \kappa_{12}(v) r_v^{(1)}(1,v) \\ r_u^{(3)}(0,v) &= \lambda_{43}(v) r_u^{(4)}(1,v) + \kappa_{43}(v) r_v^{(4)}(1,v) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} r_v^{(4)}(u,0) &= \lambda_{14}(u) r_v^{(1)}(u,1) + \kappa_{14}(u) r_u^{(1)}(u,1) \\ r_v^{(3)}(u,0) &= \lambda_{23}(u) r_v^{(2)}(u,1) + \kappa_{23}(u) r_u^{(2)}(u,1). \end{aligned} \tag{2.3}$$

We have here used a patch numbering as is indicated in Figure 2.

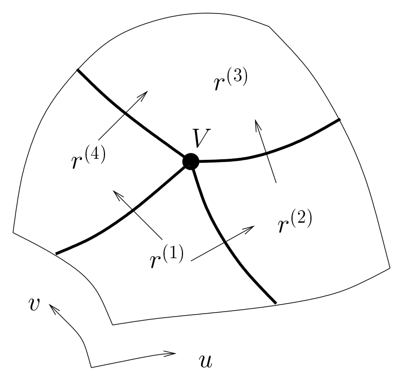

Figure 2: Four patches connected in a common vertex V

Besides the above relations we also have the following natural identities for a 4-patch surface in order to satisfy G^0 on the common boundary between two nearby patches

$$\begin{aligned} r_v^{(1)}(1,v) &= r_v^{(2)}(0,v) \\ r_v^{(4)}(1,v) &= r_v^{(3)}(0,v) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} r_u^{(1)}(u,1) &= r_u^{(4)}(u,0) \\ r_u^{(2)}(u,1) &= r_u^{(3)}(u,0). \end{aligned} \tag{2.5}$$

Using the equations (2.2)–(2.5), we prove the next theorem, which is the main result in this section. In the case where the patches are described by polynomials, this result was already published by Bézier in 1986. See [1], p 44-46.

Theorem 1 *Let S be a surface consisting of four $C^1_\#$-patches, where each pair of adjacent patches has a common boundary. Let V be the intersection point of the four patches. See Figure 2. Then, necessary and sufficient (compatibility) conditions in order for the surface S to be tangent plane continuous, G^1, are that the continuous functions λ_{ij} and κ_{ij}, defined in the equations (2.2) and (2.3), fulfill the relations*

$$\begin{cases} \kappa_{12} = \lambda_{14}\kappa_{43} \\ \kappa_{14} = \lambda_{12}\kappa_{23} \end{cases} \tag{2.6}$$

and

$$\begin{cases} \lambda_{12} - \lambda_{43} = \kappa_{14}\kappa_{43} \\ \lambda_{14} - \lambda_{23} = \kappa_{12}\kappa_{23} \end{cases} \tag{2.7}$$

at the point V.

Remark. The notation κ_{12}, λ_{14}, κ_{43}, etc, are to be interpreted as $\kappa_{12}(1)$, $\lambda_{14}(1)$, $\kappa_{43}(0)$, etc.

Proof. In order to prove the above statement, we must see under what condition the equations (2.2) and (2.3) together are satisfied. We start by eliminating $r_u^{(2)}$, $r_v^{(2)}$, $r_u^{(4)}$ and $r_v^{(4)}$ in the equations (2.2) and (2.3) by using (2.4) and (2.5) to get

$$\begin{array}{lllll} \lambda_{12}r_u^{(1)} & +\kappa_{12}r_v^{(1)} & -r_u^{(3)} & & =0 \\ \lambda_{43}r_u^{(1)} & & -r_u^{(3)} & +\kappa_{43}r_v^{(3)} & =0 \\ \kappa_{14}r_u^{(1)} & +\lambda_{14}r_v^{(1)} & & -r_v^{(3)} & =0 \\ & \lambda_{23}r_v^{(1)} & +\kappa_{23}r_u^{(3)} & -r_v^{(3)} & =0. \end{array} \tag{2.8}$$

The equation system (2.8) can easily be changed to the equivalent system that follows

$$\begin{array}{rrrrl} \lambda_{12}\, r_u^{(1)} & +\kappa_{12}\, r_v^{(1)} & -r_u^{(3)} & & =0 \\ (\lambda_{43}-\lambda_{12}+\kappa_{14}\kappa_{43})\, r_u^{(1)} & +(\lambda_{14}\kappa_{43}-\kappa_{12})\, r_v^{(1)} & & & =0 \\ \kappa_{14}\, r_u^{(1)} & +\lambda_{14}\, r_v^{(1)} & & -r_v^{(3)} & =0 \\ (\lambda_{12}\kappa_{23}-\kappa_{14})\, r_u^{(1)} & +(\lambda_{23}-\lambda_{14}+\kappa_{12}\kappa_{23})\, r_v^{(1)} & & & =0. \end{array} \tag{2.9}$$

From the second and fourth equations in (2.9) we get

$$(\lambda_{43}-\lambda_{12}+\kappa_{14}\kappa_{43})r_u^{(1)} + (\lambda_{14}\kappa_{43}-\kappa_{12})r_v^{(1)} = 0$$

and

$$(\lambda_{12}\kappa_{23}-\kappa_{14})r_u^{(1)} + (\lambda_{23}-\lambda_{14}+\kappa_{12}\kappa_{23})r_v^{(1)} = 0.$$

Since the vectors $r_u^{(1)}, r_v^{(1)}$ span the tangent plane, it must hold that

$$\kappa_{12} = \lambda_{14}\kappa_{43}$$
$$\kappa_{14} = \lambda_{12}\kappa_{23}$$

and

$$\lambda_{12} - \lambda_{43} = \kappa_{14}\kappa_{43}$$
$$\lambda_{14} - \lambda_{23} = \kappa_{12}\kappa_{23}.$$

Thus, it is obvious that a necessity and sufficiency for simultanies satisfication of the equations (2.2) and (2.3) to be fulfilled is that the equalities (2.6) and (2.7) are true. This observation concludes the proof. □

Let us look at some simple consequences of Theorem 1. Obviously, the functions λ_{ij} are not allowed to be zero if tangential continuity is to be satisfied. Thus, if e g $\kappa_{43}(0) \neq 0$ then $\kappa_{12}(0) \neq 0$, which follows from 2.6. On the other hand, if $\kappa_{14}(0) = 0$ then also $\kappa_{23}(0) = 0$. This situation is examplified in Figure 3. In general, it follows from equation (2.6) that the pair of κ_{ij}'s in each equality must both be zero or non-zero. A simplier case is, of course, when all κ_{ij} are zero.

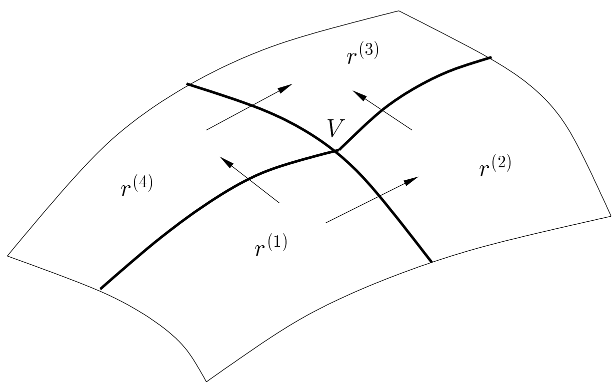

Figure 3: Four patches connected in a common vertex V with $\kappa_{14}(0) = \kappa_{23}(0) = 0$ and $\kappa_{12}(0)\kappa_{43}(0) \neq 0$

Another observation that can made from (2.6) and (2.7) is that the next relations are true

$$\begin{aligned}\lambda_{14}\lambda_{43} &= \lambda_{14}(\lambda_{12} - \kappa_{14}\kappa_{43}) = \lambda_{12}\lambda_{14} - \lambda_{14}\kappa_{43}\kappa_{14} \\ &= \lambda_{12}\lambda_{14} - \kappa_{12}\kappa_{14}\end{aligned}$$

and

$$\begin{aligned}\lambda_{12}\lambda_{23} &= \lambda_{12}(\lambda_{14} - \kappa_{12}\kappa_{23}) = \lambda_{12}\lambda_{14} - \kappa_{12}\lambda_{12}\kappa_{23} \\ &= \lambda_{12}\lambda_{14} - \kappa_{12}\kappa_{14}.\end{aligned}$$

In particular, we have

$$\lambda_{12}\lambda_{23} = \lambda_{14}\lambda_{43}, \tag{2.10}$$

which will be usefull in the next section.

3 Geometric continuity of order 2

As mentioned previously, it is the regularity of the surface and not its representation that is of interest. Therefore we introduce, similarly as before, the following concept.

Definition 2 *A tangent plane continuous, G^1, surface is said to be curvature continuous, denoted by G^2, if every point on the surface has a unique Dupin indicatrix, which varies continuously on the surface. Such a surface is also said to be geometrically continuous of order two.*

Another equivalent way to describe the notation of curvature continuity is to say that the normal curvature at each point and in each tangential direction[1] has to be unique, or the principal curvatures are unique. These differential geometric notations are introduced and explained in any book about differential geometry, e g [2]. First we consider a 2-patch surface, where the patches are of regularity $C^2_\# = C^1_\# \cap C^2$. Thus, lack of G^2-regularity for the surface S can only occur at the common boundary of the two patches. Next, we give a necessary and sufficient condition for a 2-patch surface to be curvature continuous. This result was proved in a paper by Juergen Kahmann [7]. We have

Lemma 1 *Let S be a 2-patch surface consisting of the two $C^2_\#$-patches $(u,v) \mapsto r^{(1)}(u,v)$ and $(u,v) \mapsto r^{(2)}(u,v)$, where $0 \le u,v \le 1$, satisfying equation (2.1) on their common boundary. A necessary and sufficient condition for the surface S to be curvature continuous is that the following relation is fulfilled*

$$\begin{aligned} r^{(2)}_{uu}(0,v) = \lambda^2_{12}(v) r^{(1)}_{uu}(1,v) + 2\lambda_{12}(v)\kappa_{12}(v) r^{(1)}_{uv}(1,v) + \kappa^2_{12}(v) r^{(1)}_{vv}(1,v) \\ + \mu_{12}(v) r^{(1)}_u(1,v) + \nu_{12}(v) r^{(1)}_v(1,v), \quad 0 \le v \le 1, \end{aligned} \tag{3.11}$$

where the functions λ_{12}, κ_{12}, μ_{12} and ν_{12} are continuous.

Proof Let P be any point on the common boundary and t any tangential vector on the G^1-surface at the point P. Let $s \mapsto \gamma(s)$ be a curve on S crossing the common boundary between the two patches at the point $P = \gamma(s_o)$, with its tangential vector t at that point. Let $\gamma_i = \gamma|_{r^{(i)}}$ for $i = 1,2$. Since the surface is tangent plane continuous, the tangent vector t must satisfy

$$\begin{aligned} \gamma_2'(s_o) = t = \alpha_2 r^{(2)}_u + \beta_2 r^{(2)}_v &= \alpha_2(\lambda_{12} r^{(1)}_u + \kappa_{12} r^{(1)}_v) + \beta_2 r^{(1)}_v \\ &= \alpha_2 \lambda_{12} r^{(1)}_u + (\alpha_2\kappa_{12} + \beta_2) r^{(1)}_v \\ &= \alpha_1 r^{(1)}_u + \beta_1 r^{(1)}_v = \gamma_1'(s_o), \end{aligned} \tag{3.12}$$

where we have used formula (2.1) and the next relations

$$\begin{aligned} \alpha_1 &= \alpha_2 \lambda_{12} \\ \beta_1 &= \alpha_2 \kappa_{12} + \beta_2. \end{aligned}$$

A necessary and sufficient condition in order to have curvature continuity is that the normal curvature k in the direction t at the point P is independent of coordinate system or representation, i e

$$(k_1\, N_1)|_P = (k_2\, N_2)|_P, \tag{3.13}$$

[1] In fact, it is enough that it holds for 3 pairwise linearly independent tangential directions by the 3-Tangent Theorem, see Pegna & Wolter [11] or Hoschek & Lasser [6], p 333.

where

$$(k_i\, N_i)|_P = (\frac{\gamma_i''}{|\gamma_i'|^2} - \frac{\gamma_i''\cdot\gamma_i'}{|\gamma_i'|^4}\,\gamma_i')|_P.$$

Since $\gamma_1' = \gamma_2' = t$ at the point P, it follows from the above formula that equation (3.13) is equivalent to

$$(\gamma_1'' - \gamma_2'')|_P = 0$$

modulo a tangent vector.

Combining the fact that $s \mapsto \gamma_i(s) = r^{(i)}(u_i(s), v_i(s))$ with the relations in (3.12), we get

$$\begin{aligned}
(\gamma_1'' - \gamma_2'')|_P &= u_1'' r_u^{(1)} + v_1'' r_v^{(1)} + (u_1')^2 r_{uu}^{(1)} + 2u_1'v_1' r_{uv}^{(1)} + (v_1')^2 r_{vv}^{(1)} \\
&\quad - u_2'' r_u^{(2)} - v_2'' r_v^{(2)} - (u_2')^2 r_{uu}^{(2)} - 2u_2'v_2' r_{uv}^{(2)} - (v_2')^2 r_{vv}^{(2)} \\
&= u_1'' r_u^{(1)} + v_1'' r_v^{(1)} + \alpha_1^2 r_{uu}^{(1)} + 2\alpha_1\beta_1 r_{uv}^{(1)} + \beta_1^2 r_{vv}^{(1)} \\
&\quad - u_2'' r_u^{(2)} - v_2'' r_v^{(2)} - \alpha_2^2 r_{uu}^{(2)} - 2\alpha_2\beta_2 r_{uv}^{(2)} - \beta_2^2 r_{vv}^{(2)} \\
&= u_1'' r_u^{(1)} + v_1'' r_v^{(1)} + (\alpha_2\lambda_{12})^2 r_{uu}^{(1)} + 2\alpha_2\lambda_{12}(\alpha_2\kappa_{12} + \beta_2) r_{uv}^{(1)} + (\alpha_2\kappa_{12} + \beta_2)^2 r_{vv}^{(1)} \\
&\quad - u_2'' r_u^{(2)} - v_2'' r_v^{(2)} - \alpha_2^2 r_{uu}^{(2)} - 2\alpha_2\beta_2 r_{uv}^{(2)} - \beta_2^2 r_{vv}^{(2)} \\
&= \alpha_2^2(\lambda_{12}^2 r_{uu}^{(1)} + 2\lambda_{12}\kappa_{12} r_{uv}^{(1)} + \kappa_{12}^2 r_{vv}^{(1)} - r_{uu}^{(2)}) + 2\alpha_2\beta_2(\lambda_{12} r_{uv}^{(1)} + \kappa_{12} r_{vv}^{(1)} - r_{uv}^{(2)}) \\
&\quad + \beta_2^2(r_{vv}^{(1)} - r_{vv}^{(2)}) + u_1'' r_u^{(1)} + v_1'' r_v^{(1)} - u_2'' r_u^{(2)} - v_2'' r_v^{(2)}.
\end{aligned}$$

From this follows the equivalent formulation in equation (3.11). $\square$

We now continue to consider a 4-patch surface as in Figure 2. The first and most obvious conditions to have geometric continuity of order two are that the equalities in (2.2) and (2.3) are satisfied after a differentiation with respect to the parameters v and u respectively. We then get

$$\begin{aligned}
r_{uv}^{(2)}(0,v) &= \lambda_{12}'(v)\, r_u^{(1)}(1,v) + \lambda_{12}(v)\, r_{uv}^{(1)}(1,v) + \kappa_{12}'(v)\, r_v^{(1)}(1,v) + \kappa_{12}(v)\, r_{vv}^{(1)}(1,v) \\
r_{uv}^{(3)}(0,v) &= \lambda_{43}'(v)\, r_u^{(4)}(1,v) + \lambda_{43}(v)\, r_{uv}^{(4)}(1,v) + \kappa_{43}'(v)\, r_v^{(4)}(1,v) + \kappa_{43}(v)\, r_{vv}^{(4)}(1,v)
\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}
r_{vu}^{(4)}(u,0) &= \lambda_{14}'(u)\, r_v^{(1)}(u,1) + \lambda_{14}(u)\, r_{vu}^{(1)}(u,1) + \kappa_{14}'(u)\, r_u^{(1)}(u,1) + \kappa_{14}(u)\, r_{uu}^{(1)}(u,1) \\
r_{vu}^{(3)}(u,0) &= \lambda_{23}'(u)\, r_v^{(2)}(u,1) + \lambda_{23}(u)\, r_{vu}^{(2)}(u,1) + \kappa_{23}'(u)\, r_u^{(2)}(u,1) + \kappa_{23}(u)\, r_{uu}^{(2)}(u,1).
\end{aligned} \tag{3.15}$$

From Lemma 1,we know that curvature continuity implies that the next four relations must be fulfilled, i e

$$\begin{aligned}
r_{uu}^{(2)}(0,v) &= \lambda_{12}^2(v) r_{uu}^{(1)}(1,v) + 2\lambda_{12}(v)\kappa_{12}(v) r_{uv}^{(1)}(1,v) + \kappa_{12}^2(v) r_{vv}^{(1)}(1,v) \\
&\quad + \mu_{12}(v) r_u^{(1)}(1,v) + \nu_{12}(v) r_v^{(1)}(1,v)
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
r_{uu}^{(3)}(0,v) &= \lambda_{43}^2(v) r_{uu}^{(4)}(1,v) + 2\lambda_{43}(v)\kappa_{43}(v) r_{uv}^{(4)}(1,v) + \kappa_{43}^2(v) r_{vv}^{(4)}(1,v) \\
&\quad + \mu_{43}(v) r_u^{(4)}(1,v) + \nu_{43}(v) r_v^{(4)}(1,v)
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
r_{vv}^{(3)}(u,0) &= \lambda_{23}^2(u) r_{vv}^{(2)}(u,1) + 2\lambda_{23}(u)\kappa_{23}(u) r_{uv}^{(2)}(u,1) + \kappa_{23}^2(u) r_{uu}^{(2)}(u,1) \\
&\quad + \mu_{23}(u) r_v^{(2)}(u,1) + \nu_{23}(u) r_u^{(2)}(u,1)
\end{aligned} \tag{3.18}$$

$$r_{vv}^{(4)}(u,0) = \lambda_{14}^2(u) r_{vv}^{(1)}(u,1) + 2\lambda_{14}(u)\kappa_{14}(u) r_{uv}^{(1)}(u,1) + \kappa_{14}^2(u) r_{uu}^{(1)}(u,1) + \mu_{14}(u) r_v^{(1)}(u,1) + \nu_{14}(u) r_u^{(1)}(u,1). \tag{3.19}$$

We are now in a position to prove the main result in this section.

Theorem 2 *Let S be a G^1-surface consisting of four $C^2_{\#}$-patches, where each pair of adjacent patches has a common boundary. Let V be the intersection point of the four patches. See Figure 2. Then, necessary and sufficient (compatibility) conditions in order for the surface S to be curvature continuous, G^2, are that the differentiable functions λ_{ij} and κ_{ij} and the continuous functions μ_{ij} and ν_{ij}, defined in the equations (3.14)–(3.19), satisfy the relations*

$$\begin{cases} 2\lambda_{43}\lambda'_{14}\kappa_{43} - \nu_{12} + \nu_{43}\lambda_{14} + \mu_{14}\kappa_{43}^2 = 0 \\ 2\lambda_{23}\lambda'_{12}\kappa_{23} - \nu_{14} + \nu_{23}\lambda_{12} + \mu_{12}\kappa_{23}^2 = 0 \\ 2\lambda_{43}\kappa_{43}\kappa'_{14} - \mu_{12} + \mu_{43} + \nu_{43}\kappa_{14} + \nu_{14}\kappa_{43}^2 = 0 \\ 2\lambda_{23}\kappa_{23}\kappa'_{12} - \mu_{14} + \mu_{23} + \nu_{23}\kappa_{12} + \nu_{12}\kappa_{23}^2 = 0 \\ \lambda'_{43} - \lambda_{23}\lambda'_{12} + \lambda_{43}\kappa'_{14} - \lambda_{12}\kappa'_{23} + \kappa_{14}\kappa'_{43} - \mu_{12}\kappa_{23} + \nu_{14}\kappa_{43} = 0 \\ \lambda'_{23} - \lambda_{43}\lambda'_{14} + \lambda_{23}\kappa'_{12} - \lambda_{14}\kappa'_{43} + \kappa_{12}\kappa'_{23} - \mu_{14}\kappa_{43} + \nu_{12}\kappa_{23} = 0 \end{cases} \tag{3.20}$$

at the point V.

Proof First, we know that the equations (3.14) and (3.15) must hold. Using the relations (2.4) and (2.5) in order to eliminate the vectors $r_u^{(2)}$ and $r_v^{(2)}$ together with $r_u^{(4)}$ and $r_v^{(4)}$ in those equations, we get

$$\begin{aligned} &\lambda_{12} r_{uv}^{(1)} - r_{uv}^{(2)} + \lambda'_{12} r_u^{(1)} + \kappa'_{12} r_v^{(1)} + \kappa_{12} r_{vv}^{(1)} = 0 \\ &\lambda_{43} r_{uv}^{(4)} - r_{uv}^{(3)} + \lambda'_{43} r_u^{(1)} + \kappa'_{43} r_v^{(4)} + \kappa_{43} r_{vv}^{(4)} = 0 \\ &\lambda_{14} r_{vu}^{(1)} - r_{vu}^{(4)} + \kappa'_{14} r_u^{(1)} + \lambda'_{14} r_v^{(1)} + \kappa_{14} r_{uu}^{(1)} = 0 \\ &\lambda_{23} r_{vu}^{(2)} - r_{vu}^{(3)} + \kappa'_{23} r_u^{(2)} + \lambda'_{23} r_v^{(2)} + \kappa_{23} r_{uu}^{(2)} = 0. \end{aligned}$$

After some minor rearrangements, we get the following equivalent formulation of the above equation system

$$\begin{aligned} &\lambda_{12} r_{uv}^{(1)} - r_{uv}^{(2)} + \lambda'_{12} r_u^{(1)} + \kappa'_{12} r_v^{(1)} + \kappa_{12} r_{vv}^{(1)} = 0 \\ &\lambda_{43} r_{uv}^{(4)} - r_{uv}^{(3)} + \lambda'_{43} r_u^{(1)} + \kappa'_{43} r_v^{(4)} + \kappa_{43} r_{vv}^{(4)} = 0 \\ &\lambda_{14} r_{vu}^{(1)} - r_{vu}^{(4)} + \kappa'_{14} r_u^{(1)} + \lambda'_{14} r_v^{(1)} + \kappa_{14} r_{uu}^{(1)} = 0 \\ &(-\lambda'_{43} + \lambda_{23}\lambda'_{12} - \lambda_{43}\kappa'_{14} + \lambda_{12}\kappa'_{23} - \kappa_{14}\kappa'_{43}) r_u^{(1)} + (\lambda'_{23} + \lambda_{23}\kappa'_{12} - \lambda_{43}\lambda'_{14} \\ &\quad + \kappa_{12}\kappa'_{23} - \lambda_{14}\kappa'_{43}) r_v^{(1)} - \lambda_{43}\kappa_{14} r_{uu}^{(1)} + \lambda_{23}\kappa_{12} r_{vv}^{(1)} + \kappa_{23} r_{uu}^{(3)} - \kappa_{43} r_{vv}^{(3)} = 0. \end{aligned} \tag{3.21}$$

Considering the last part in the last equation in system (3.21) and using equations (3.16) and (3.19) combined with (2.4) and (2.5) we get

$$\begin{aligned}
&-\lambda_{43}\kappa_{14}r^{(1)}_{uu}+\lambda_{23}\kappa_{12}r^{(1)}_{vv}+\kappa_{23}r^{(3)}_{uu}-\kappa_{43}r^{(3)}_{vv}\\
&\quad = \quad -\lambda_{43}\kappa_{14}r^{(1)}_{uu}+\lambda_{23}\kappa_{12}r^{(1)}_{vv}+\kappa_{23}r^{(2)}_{uu}-\kappa_{43}r^{(4)}_{vv}\\
&\quad = \quad -\lambda_{43}\kappa_{14}r^{(1)}_{uu}+\lambda_{23}\kappa_{12}r^{(1)}_{vv}\\
&\qquad\qquad +\kappa_{23}(\lambda_{12}^2r^{(1)}_{uu}+2\lambda_{12}\kappa_{12}r^{(1)}_{uv}+\kappa_{12}^2r^{(1)}_{uu}+\mu_{12}r^{(1)}_{u}+\nu_{12}r^{(1)}_{v})\\
&\qquad\qquad -\kappa_{43}(\lambda_{14}^2r^{(1)}_{vv}+2\lambda_{14}\kappa_{14}r^{(1)}_{uv}+\kappa_{14}^2r^{(1)}_{uu}+\mu_{14}r^{(1)}_{v}+\nu_{14}r^{(1)}_{u})\\
&\quad = \quad \kappa_{23}(\mu_{12}r^{(1)}_{u}+\nu_{12}r^{(1)}_{v})-\kappa_{43}(\mu_{14}r^{(1)}_{v}+\nu_{14}r^{(1)}_{u})\\
&\qquad\qquad +(\kappa_{23}\lambda_{12}^2-\kappa_{43}\kappa_{14}^2-\lambda_{43}\kappa_{14})r^{(1)}_{uu}\\
&\qquad\qquad +(\kappa_{23}\kappa_{12}^2-\kappa_{43}\lambda_{14}^2+\lambda_{23}\kappa_{12})r^{(1)}_{vv}\\
&\qquad\qquad +2(\kappa_{23}\lambda_{12}\kappa_{12}-\kappa_{43}\lambda_{14}\kappa_{14})r^{(1)}_{uv}.
\end{aligned}$$

In order to further reduce the above formula, we use the relations (2.6) and (2.7). We get

$$\begin{aligned}
\kappa_{23}\lambda_{12}^2-\kappa_{43}\kappa_{14}^2-\lambda_{43}\kappa_{14} &= \kappa_{14}\lambda_{12}-\kappa_{43}\kappa_{14}^2-\lambda_{43}\kappa_{14}\\
&= \kappa_{14}(\lambda_{12}-\lambda_{43}-\kappa_{14}\kappa_{43})=0,\\
\kappa_{23}\kappa_{12}^2-\kappa_{43}\lambda_{14}^2+\lambda_{23}\kappa_{12} &= \kappa_{23}\kappa_{12}^2-\kappa_{12}\lambda_{14}+\lambda_{23}\kappa_{12}\\
&= \kappa_{12}(\lambda_{23}-\lambda_{14}+\kappa_{12}\kappa_{23})=0
\end{aligned}$$

and

$$\kappa_{23}\lambda_{12}\kappa_{12}-\kappa_{43}\lambda_{14}\kappa_{14} = \kappa_{14}\kappa_{12}-\kappa_{12}\kappa_{14}=0.$$

Thus, it follows that

$$\begin{aligned}
&-\lambda_{43}\kappa_{14}r^{(1)}_{uu}+\lambda_{23}\kappa_{12}r^{(1)}_{vv}+\kappa_{23}r^{(3)}_{uu}-\kappa_{43}r^{(3)}_{vv}\\
&\qquad =(\kappa_{23}\mu_{12}-\kappa_{43}\nu_{14})r^{(1)}_{u}+(\kappa_{23}\nu_{12}-\kappa_{43}\mu_{14})r^{(1)}_{v}.
\end{aligned}$$

Input the above equality in the last equation in formula (3.21). The independence of the two vectors $r^{(1)}_u$ and $r^{(1)}_v$ together with (2.6) gives

$$\begin{aligned}
-\lambda'_{43}-\lambda_{43}\kappa'_{14}+\lambda_{23}\lambda'_{12}+\lambda_{12}\kappa'_{23}-\kappa_{14}\kappa'_{43}+\kappa_{23}\mu_{12}-\kappa_{43}\nu_{14}=0\\
\lambda'_{23}-\lambda_{43}\lambda'_{14}+\lambda_{23}\kappa'_{12}+\kappa_{12}\kappa'_{23}-\lambda_{14}\kappa'_{43}+\kappa_{23}\nu_{12}-\kappa_{43}\mu_{14}=0.
\end{aligned}\tag{3.22}$$

The equations (3.22) are necessary in order to fulfill the condition of geometric continuity of order 2.

Let us continue to study the equations (3.16)–(3.19) more closely. After a reformulation of this equation system we have

$$\begin{aligned}
&2\lambda_{12}\kappa_{12}r^{(1)}_{uv}+\mu_{12}r^{(1)}_{u}+\nu_{12}r^{(1)}_{v}+\lambda_{12}^2r^{(1)}_{uu}+\kappa_{12}^2r^{(1)}_{vv}-r^{(3)}_{uu}=0\\
&(\mu_{43}+2\lambda_{43}\kappa_{43}\kappa'_{14}-\mu_{12}\frac{\lambda_{43}}{\lambda_{12}})r^{(1)}_{u}+(2\lambda_{43}\kappa_{43}\lambda'_{14}-\nu_{12}\frac{\lambda_{43}}{\lambda_{12}})r^{(1)}_{v}+\nu_{43}r^{(3)}_{v}\\
&\quad+(\lambda_{43}^2+2\lambda_{43}\kappa_{43}\kappa_{14}-\lambda_{12}\lambda_{43})r^{(1)}_{uu}-\kappa_{12}^2\frac{\lambda_{43}}{\lambda_{12}}r^{(1)}_{vv}-(1-\frac{\lambda_{43}}{\lambda_{12}})r^{(3)}_{uu}+\kappa_{43}^2r^{(3)}_{vv}=0\\
&(2\lambda'_{12}\lambda_{23}\kappa_{23}-\nu_{14}\frac{\lambda_{23}}{\lambda_{14}})r^{(1)}_{u}+(\mu_{23}+2\kappa'_{12}\lambda_{23}\kappa_{23}-\mu_{14}\frac{\lambda_{23}}{\lambda_{14}})r^{(1)}_{v}+\nu_{23}r^{(3)}_{u}\\
&\quad-\kappa_{14}^2\frac{\lambda_{23}}{\lambda_{14}}r^{(1)}_{uu}+(\lambda_{23}^2+2\kappa_{12}\lambda_{23}\kappa_{23}-\lambda_{14}\lambda_{23})r^{(1)}_{vv}+\kappa_{23}^2r^{(3)}_{uu}-(1-\frac{\lambda_{23}}{\lambda_{14}})r^{(3)}_{vv}=0\\
&2\lambda_{14}\kappa_{14}r^{(1)}_{uv}+\nu_{14}r^{(1)}_{u}+\mu_{14}r^{(1)}_{v}+\kappa_{14}^2r^{(1)}_{uu}+\lambda_{14}^2(u)r^{(1)}_{vv}-r^{(3)}_{vv}=0.
\end{aligned}\tag{3.23}$$

Let us start to consider the second equation in the above system. We want to rewrite this equation in order to make it easier to handle. Using the relations (2.4)–(2.9) combined with (3.16) and (3.19), we get

$$
\begin{aligned}
&(\mu_{43}\lambda_{12}+2\lambda_{12}\lambda_{43}\kappa_{43}\kappa'_{14}-\mu_{12}\lambda_{43})r_u^{(1)}+\lambda_{43}(2\lambda_{12}\kappa_{43}\lambda'_{14}-\nu_{12})r_v^{(1)}+\nu_{43}\lambda_{12}r_v^{(3)}+\lambda_{12}\lambda_{43}(\lambda_{43}\\
&\quad+2\kappa_{14}\kappa_{43}-\lambda_{12})r_{uu}^{(1)}-\lambda_{43}\kappa_{12}^2r_{vv}^{(1)}-(\lambda_{12}-\lambda_{43})r_{uu}^{(3)}+\lambda_{12}\kappa_{43}^2r_{vv}^{(3)}\\
&=(\mu_{43}\lambda_{12}+2\lambda_{12}\lambda_{43}\kappa_{43}\kappa'_{14}-\mu_{12}\lambda_{43})r_u^{(1)}+\lambda_{43}(2\lambda_{12}\kappa_{43}\lambda'_{14}-\nu_{12})r_v^{(1)}+\nu_{43}\lambda_{12}(\kappa_{14}r_u^{(1)}\\
&\quad+\lambda_{14}r_v^{(1)})+\lambda_{12}\lambda_{43}\kappa_{14}\kappa_{43}r_{uu}^{(1)}-\lambda_{43}\kappa_{12}^2r_{vv}^{(1)}-\kappa_{14}\kappa_{43}r_{uu}^{(2)}+\lambda_{12}\kappa_{43}^2r_{vv}^{(4)}\\
&=(\mu_{43}\lambda_{12}+2\lambda_{12}\lambda_{43}\kappa_{43}\kappa'_{14}-\mu_{12}\lambda_{43}+\nu_{43}\lambda_{12}\kappa_{14})r_u^{(1)}+(2\lambda_{12}\lambda_{43}\kappa_{43}\lambda'_{14}-\nu_{12}\lambda_{43}+\nu_{43}\lambda_{12}\lambda_{14})r_v^{(1)}\\
&\quad+\lambda_{12}\lambda_{43}\kappa_{14}\kappa_{43}r_{uu}^{(1)}-\lambda_{43}\kappa_{12}^2r_{vv}^{(1)}-\kappa_{14}\kappa_{43}(\lambda_{12}^2r_{uu}^{(1)}+2\lambda_{12}\kappa_{12}r_{uv}^{(1)}+\kappa_{12}^2r_{vv}^{(1)}+\mu_{12}r_u^{(1)}+\nu_{12}r_v^{(1)})\\
&\quad+\lambda_{12}\kappa_{43}^2(\lambda_{14}^2r_{vv}^{(1)}+2\lambda_{14}\kappa_{14}r_{uv}^{(1)}+\kappa_{14}^2r_{uu}^{(1)}+\mu_{14}r_v^{(1)}+\nu_{14}r_u^{(1)})\\
&=(\mu_{43}\lambda_{12}+2\lambda_{12}\lambda_{43}\kappa_{43}\kappa'_{14}-\mu_{12}\lambda_{43}+\nu_{43}\lambda_{12}\kappa_{14}-\kappa_{14}\kappa_{43}\mu_{12}+\lambda_{12}\kappa_{43}^2\nu_{14})r_u^{(1)}+(2\lambda_{12}\lambda_{43}\kappa_{43}\lambda'_{14}\\
&\quad-\nu_{12}\lambda_{43}+\nu_{43}\lambda_{12}\lambda_{14}-\kappa_{14}\kappa_{43}\nu_{12}+\lambda_{12}\kappa_{43}^2\mu_{14})r_v^{(1)}+(\lambda_{12}\lambda_{43}\kappa_{14}\kappa_{43}-\kappa_{14}\kappa_{43}\lambda_{12}^2+\lambda_{12}\kappa_{43}^2\kappa_{14}^2)r_{uu}^{(1)}\\
&\quad+(-\lambda_{43}\kappa_{12}^2-\kappa_{14}\kappa_{43}\kappa_{12}^2+\lambda_{12}\kappa_{43}^2\lambda_{14}^2)r_{vv}^{(1)}+2(-\kappa_{14}\kappa_{43}\lambda_{12}\kappa_{12}+\lambda_{12}\kappa_{43}^2\lambda_{14}\kappa_{14})r_{uv}^{(1)}\\
&=(\mu_{43}\lambda_{12}+2\lambda_{12}\lambda_{43}\kappa_{43}\kappa'_{14}-\mu_{12}\lambda_{43}+\nu_{43}\lambda_{12}\kappa_{14}-\kappa_{14}\kappa_{43}\mu_{12}+\lambda_{12}\kappa_{43}^2\nu_{14})r_u^{(1)}+(2\lambda_{12}\lambda_{43}\kappa_{43}\lambda'_{14}\\
&\quad-\nu_{12}\lambda_{43}+\nu_{43}\lambda_{12}\lambda_{14}-\kappa_{14}\kappa_{43}\nu_{12}+\lambda_{12}\kappa_{43}^2\mu_{14})r_v^{(1)}+\lambda_{12}\kappa_{14}\kappa_{43}(\lambda_{43}-\lambda_{12}+\kappa_{14}\kappa_{43})r_{uu}^{(1)}\\
&\quad-\kappa_{12}^2(\lambda_{43}+\kappa_{14}\kappa_{43}-\lambda_{12})r_{vv}^{(1)}-2\lambda_{12}\kappa_{14}\kappa_{43}(\kappa_{12}-\lambda_{14}\kappa_{43})r_{uv}^{(1)}\\
&=(\mu_{43}\lambda_{12}+2\lambda_{12}\lambda_{43}\kappa_{43}\kappa'_{14}-\mu_{12}\lambda_{43}+\nu_{43}\lambda_{12}\kappa_{14}-\kappa_{14}\kappa_{43}\mu_{12}+\lambda_{12}\kappa_{43}^2\nu_{14})r_u^{(1)}+(2\lambda_{12}\lambda_{43}\kappa_{43}\lambda'_{14}\\
&\quad-\nu_{12}\lambda_{43}+\nu_{43}\lambda_{12}\lambda_{14}-\kappa_{14}\kappa_{43}\nu_{12}+\lambda_{12}\kappa_{43}^2\mu_{14})r_v^{(1)}=0.
\end{aligned}
$$

The independence of the tangential vectors $r_u^{(1)}$ and $r_v^{(1)}$ gives

$$
\begin{aligned}
\mu_{43}\lambda_{12}+2\lambda_{12}\lambda_{43}\kappa_{43}\kappa'_{14}-\mu_{12}\lambda_{43}+\nu_{43}\lambda_{12}\kappa_{14}-\mu_{12}\kappa_{14}\kappa_{43}+\nu_{14}\lambda_{12}\kappa_{43}^2&=0\\
2\lambda_{12}\lambda_{43}\kappa_{43}\lambda'_{14}-\nu_{12}\lambda_{43}+\nu_{43}\lambda_{12}\lambda_{14}-\nu_{12}\kappa_{14}\kappa_{43}+\mu_{14}\lambda_{12}\kappa_{43}^2&=0
\end{aligned}
$$

or equivalently, by the use of relation (2.7), we get

$$
\begin{aligned}
\lambda_{12}(\mu_{43}+2\lambda_{43}\kappa_{43}\kappa'_{14}-\mu_{12}+\nu_{43}\kappa_{14}+\nu_{14}\kappa_{43}^2)&=0\\
\lambda_{12}(2\lambda_{43}\kappa_{43}\lambda'_{14}-\nu_{12}+\nu_{43}\lambda_{14}+\mu_{14}\kappa_{43}^2)&=0.
\end{aligned}
\tag{3.24}
$$

We now continue our examination of the equation system (3.23) by considering its third equation. As above, we use (2.4)–(2.9) combined with (3.16) and (3.19). We have

$$
\begin{aligned}
&\lambda_{23}(2\lambda'_{12}\lambda_{14}\kappa_{23}-\nu_{14})r_u^{(1)}+(\mu_{23}\lambda_{14}+2\kappa'_{12}\lambda_{14}\lambda_{23}\kappa_{23}-\mu_{14}\lambda_{23})r_v^{(1)}+\nu_{23}\lambda_{14}r_u^{(3)}-\lambda_{23}\kappa_{14}^2r_{uu}^{(1)}\\
&\quad+\lambda_{14}\lambda_{23}(\lambda_{23}+2\kappa_{12}\kappa_{23}-\lambda_{14})r_{vv}^{(1)}+\lambda_{14}\kappa_{23}^2r_{uu}^{(3)}-(\lambda_{14}-\lambda_{23})r_{vv}^{(3)}
\end{aligned}
$$

$$
\begin{aligned}
=&\lambda_{23}(2\lambda'_{12}\lambda_{14}\kappa_{23}-\nu_{14})r_u^{(1)}+(\mu_{23}\lambda_{14}+2\kappa'_{12}\lambda_{14}\lambda_{23}\kappa_{23}-\mu_{14}\lambda_{23})r_v^{(1)}+\nu_{23}\lambda_{14}(\lambda_{12}r_u^{(1)}\\
&+\kappa_{12}r_v^{(1)})-\lambda_{23}\kappa_{14}^2r_{uu}^{(1)}+\lambda_{14}\lambda_{23}\kappa_{12}\kappa_{23}r_{vv}^{(1)}+\lambda_{14}\kappa_{23}^2r_{uu}^{(2)}-\kappa_{12}\kappa_{23}r_{vv}^{(4)}
\end{aligned}
$$

$$
\begin{aligned}
=&(2\lambda_{23}\lambda'_{12}\lambda_{14}\kappa_{23}-\nu_{14}\lambda_{23}+\nu_{23}\lambda_{14}\lambda_{12})r_u^{(1)}+(\mu_{23}\lambda_{14}+2\kappa'_{12}\lambda_{14}\lambda_{23}\kappa_{23}-\mu_{14}\lambda_{23}+\nu_{23}\lambda_{14}\kappa_{12})r_v^{(1)}\\
&-\lambda_{23}\kappa_{14}^2r_{uu}^{(1)}+\lambda_{14}\lambda_{23}\kappa_{12}\kappa_{23}r_{vv}^{(1)}+\lambda_{14}\kappa_{23}^2(\lambda_{12}^2r_{uu}^{(1)}+2\lambda_{12}\kappa_{12}r_{uv}^{(1)}+\kappa_{12}^2r_{vv}^{(1)}+\mu_{12}r_u^{(1)}+\nu_{12}r_v^{(1)})\\
&-\kappa_{12}\kappa_{23}(\lambda_{14}^2r_{vv}^{(1)}+2\lambda_{14}\kappa_{14}r_{uv}^{(1)}+\kappa_{14}^2r_{uu}^{(1)}+\mu_{14}r_v^{(1)}+\nu_{14}r_u^{(1)})
\end{aligned}
$$

$$
\begin{aligned}
=&(2\lambda_{23}\lambda'_{12}\lambda_{14}\kappa_{23}-\nu_{14}\lambda_{23}+\nu_{23}\lambda_{14}\lambda_{12}+\mu_{12}\lambda_{14}\kappa_{23}^2-\nu_{14}\kappa_{12}\kappa_{23})r_u^{(1)}+(\mu_{23}\lambda_{14}+2\kappa'_{12}\lambda_{14}\lambda_{23}\kappa_{23}\\
&-\mu_{14}\lambda_{23}+\nu_{23}\lambda_{14}\kappa_{12}+\nu_{12}\lambda_{14}\kappa_{23}^2-\mu_{14}\kappa_{12}\kappa_{23})r_v^{(1)}+(-\lambda_{23}\kappa_{14}^2+\lambda_{14}\kappa_{23}^2\lambda_{12}^2-\kappa_{12}\kappa_{23}\kappa_{14}^2)r_{uu}^{(1)}\\
&+(\lambda_{14}\lambda_{23}\kappa_{12}\kappa_{23}+\lambda_{14}\kappa_{23}^2\kappa_{12}^2-\lambda_{14}^2\kappa_{12}\kappa_{23})r_{vv}^{(1)}+2(\lambda_{14}\kappa_{23}^2\lambda_{12}\kappa_{12}-\kappa_{12}\kappa_{23}\lambda_{14}\kappa_{14})r_{uv}^{(1)}
\end{aligned}
$$

$$
\begin{aligned}
=&(2\lambda_{23}\lambda'_{12}\lambda_{14}\kappa_{23}-\nu_{14}\lambda_{23}+\nu_{23}\lambda_{14}\lambda_{12}+\mu_{12}\lambda_{14}\kappa_{23}^2-\nu_{14}\kappa_{12}\kappa_{23})r_u^{(1)}+(\mu_{23}\lambda_{14}+2\kappa'_{12}\lambda_{14}\lambda_{23}\kappa_{23}\\
&-\mu_{14}\lambda_{23}+\nu_{23}\lambda_{14}\kappa_{12}+\nu_{12}\lambda_{14}\kappa_{23}^2-\mu_{14}\kappa_{12}\kappa_{23})r_v^{(1)}+\kappa_{14}^2(\lambda_{14}-\lambda_{23}-\kappa_{12}\kappa_{23})r_{uu}^{(1)}\\
&+\lambda_{14}\kappa_{12}\kappa_{23}(\lambda_{23}+\kappa_{12}\kappa_{23}-\lambda_{14})r_{vv}^{(1)}+2\lambda_{14}\kappa_{12}\kappa_{23}(\kappa_{23}\lambda_{12}-\kappa_{14})r_{uv}^{(1)}
\end{aligned}
$$

$$
\begin{aligned}
=&(2\lambda_{23}\lambda'_{12}\lambda_{14}\kappa_{23}-\nu_{14}\lambda_{23}+\nu_{23}\lambda_{14}\lambda_{12}+\mu_{12}\lambda_{14}\kappa_{23}^2-\nu_{14}\kappa_{12}\kappa_{23})r_u^{(1)}+(\mu_{23}\lambda_{14}+2\kappa'_{12}\lambda_{14}\lambda_{23}\kappa_{23}\\
&-\mu_{14}\lambda_{23}+\nu_{23}\lambda_{14}\kappa_{12}+\nu_{12}\lambda_{14}\kappa_{23}^2-\mu_{14}\kappa_{12}\kappa_{23})r_v^{(1)}=0.
\end{aligned}
$$

The same argument as before, i e the independence of the tangential vectors $r_u^{(1)}$ and $r_v^{(1)}$, implies that

$$
\begin{aligned}
2\lambda_{23}\lambda'_{12}\lambda_{14}\kappa_{23}-\nu_{14}\lambda_{23}+\nu_{23}\lambda_{14}\lambda_{12}+\mu_{12}\lambda_{14}\kappa_{23}^2-\nu_{14}\kappa_{12}\kappa_{23}&=0\\
\mu_{23}\lambda_{14}+2\kappa'_{12}\lambda_{14}\lambda_{23}\kappa_{23}-\mu_{14}\lambda_{23}+\nu_{23}\lambda_{14}\kappa_{12}+\nu_{12}\lambda_{14}\kappa_{23}^2-\mu_{14}\kappa_{12}\kappa_{23}&=0.
\end{aligned}
$$

Using the relation (2.7) we can simplify the above written equalities into the next two ones

$$
\begin{aligned}
\lambda_{14}(2\lambda_{23}\lambda'_{12}\kappa_{23}-\nu_{14}+\nu_{23}\lambda_{12}+\mu_{12}\kappa_{23}^2)&=0\\
\lambda_{14}(\mu_{23}+2\kappa'_{12}\lambda_{23}\kappa_{23}-\mu_{14}+\nu_{23}\kappa_{12}+\nu_{12}\kappa_{23}^2)&=0.
\end{aligned}
\tag{3.25}
$$

Combining the results in (3.22), (3.24) and (3.25) with the fact that $\lambda_{ij}\neq 0$, we get the compatibility conditions (3.20) for G^2, which are necessary and sufficient for having a simultaneous satisfication of the equations (3.14)–(3.19). This ends the proof. □

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