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# INTEGRAL FORMULAS AND THE OHSAWA-TAKEGOSHI EXTENSION THEOREM.

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ABSTRACT. **We construct a semiexplicit integral representation of the canonical solution to the $\bar\partial$-equation with respect to a plurisubharmonic weight function in a pseudoconvex domain. The construction is based on a construction related to the Ohsawa-Takegoshi extension theorem combined with a method to construct weighted integral representations due to M Andersson.**

## 1. INTRODUCTION

There are basically two different methods to solve the $\bar\partial$-equation in complex analysis: By $L^2$-methods or by explicit integral kernels. In this paper we will show how one can construct a class of semiexplicit kernels using $L^2$-methods. Moreover, the kernels that we find give the canonical solution to the $\bar\partial$-equation in $L^2$-spaces defined by arbitrary plurisubharmonic weight functions.

The prototype for complex analytic integral representations is the one-dimensional Cauchy-Green formula.

$$v(z) = \frac{1}{2\pi i}\int_{\partial\Omega} v(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_\Omega \frac{\bar\partial v(\zeta)\wedge d\zeta}{\zeta - z}.$$

The first term on the right hand side represents the values of holomorphic functions, whereas the second term can be used to find explicit solutions to the inhomogenous $\bar\partial$-equation. The proof follows from the distributional equation for the Cauchy kernel

$$\bar\partial \frac{d\zeta}{\zeta - z} = 2\pi i \mu_z,$$

where $\mu_z$ is a Dirac delta-function at the point $z$.

An analogous formula in higher dimensions is based on the Cauchy-Fantappié kernel,see e g [9],[13] or [4], which generalizes (and reduces to) the Cauchy kernel in one variabel. Given a domain $\Omega$ in $\mathbb{C}^n$ this kernel is obtained by first chosing a form

$$s = \sum s_j(\zeta, z)d\zeta_j$$

with coefficients of class $C^1$ in $\bar\Omega_\zeta \times \bar\Omega_z$. The form $s$ must satisfy the fundamental condition

$$\langle s, z - \zeta\rangle \neq 0$$

for $\zeta \neq z$. The Cauchy-Fantappie kernel associated to $s$ is then

$$u_{n,n-1} = \frac{s \wedge (\bar\partial_\zeta s)^{n-1}}{\langle s, z-\zeta\rangle^n}.$$

Again the kernel satisfies a distributional equation

$$\bar\partial u_{n,n-1} = (2\pi i)^n \mu_z,$$

which leads to a multidimensional Cauchy formula

$$v(z) = \frac{1}{(2\pi i)^n}\int_{\partial\Omega} v(\zeta) u_{n,n-1} - \frac{1}{(2\pi i)^n}\int_\Omega \bar\partial v \wedge u_{n,n-1}.$$

(Similar reprsentation formulas hold for forms of higher degree, but in this paper we shall discuss only the case of functions and 1-forms.)

Of special interest is the case when $s$ can be chosen in such a way that the coefficients $s_j(\zeta, z)$ are holomorphic in $z$ for $\zeta$ fixed on the boundary of $\Omega$. In this case we obtain a representation formula for holomorphic functions with a holomorphic kernel, and we also obtain a formula for solving the inhomogenous $\bar\partial$-equation. The crucial property of the kernel that implies that we get a solution formula for $\bar\partial$ is that $u_{n,n-1}$ then depends holomorphically on the variable $z$ when $\zeta$ is on the boundary of $\Omega$.

In [1], Mats Andersson and the author introduced a generalization of the Cauchy-Fantappie kernel that allow certain *weight factors*. To define such a weighted kernel we need two additional building blocks; one more differential form $q = \sum q_j(\zeta, z) d\zeta_j$ and a holomorphic function of one complex variable $G$, satisfying $G(0) = 1$. The weighted Cauchy-Fantappié kernel is then

$$K = \sum_0^{n-1} \frac{1}{k!} G^{(k)}(\langle q, z-\zeta\rangle) \frac{s \wedge (\bar\partial s)^{n-k} \wedge (\bar\partial q)^k}{\langle s, \zeta - z\rangle^{n-k+1}},$$

One also defines an associated projection kernel

$$P = \frac{1}{n!} G^{(n)}(\langle q, \zeta - z\rangle)(\bar\partial q)^n.$$

Then $\bar\partial K = c_n \mu_z - P$ and, again, one obtains Cauchy formulas of the type above. The boundary integral in the representation formula now gets replaced by

$$\int_{\partial\Omega} vK + \int_\Omega v \wedge P.$$

Again, one is particularily interested in the case when the kernels in this representation formula depend holomorphically on $z$ since in this case we still get solution formulas for the $\bar\partial$-equation. One way to obtain such holomorphic dependence is to choose $G$ and $q$ in a way so that $K$ vanishes for $\zeta$ on the boundary and the coefficients $q_j$ depend holomorphically on $z$. An important model case is when $\Omega = \mathbb{C}^n$, so that the boundary is empty,

$G = \exp$, and $q = \partial|\zeta|^2$, so that $q$ is even independent of $z$. We then get the representation formula for holomorphic functions

$$v(z) = \frac{1}{\pi^n} \int_{\mathbb{C}^n} v(\zeta)\, e^{z\cdot\bar\zeta - |\zeta|^2} d\lambda(\zeta),$$

classical in the theory of Bargmann or Fock space.

Here, the kernel $P$ gives the orthogonal projection of a function in $L^2(\mathbb{C}^n, e^{-|\zeta|^2})$ on the subspace of holomorphic functions. It follows that the corresponding solution formula for $\bar\partial$ gives us the canonical solution in this $L^2$-space, i e the solution of minimal norm. The principal aim of this paper is to show how one can obtain weighted kernels that furnish the canonical solution to the $\bar\partial$-equation in $L^2(\Omega, e^{-\phi})$, where $\Omega$ is a general pseudoconvex domain in $\mathbb{C}^n$, and $\phi$ is a plurisubharmonic function in $\Omega$. Clearly, in this generality one cannot hope for a kernel which is as explicit as in the example above, but we still feel that the construction is sufficiently explicit to be interesting. We define the weight factors starting from the Bergman kernel. In this respect the procedure is somewhat similar to [7], but there the construction does not lead up to the canonical solution.

We find the kernels by combining a recent approach to weighted integral formulas by M Andersson, [2], with an argument that comes from a new proof of the Ohsawa-Takegoshi extension theorem. The plan of the paper is as follows. In section 2 we review Andersson's new construction of weighted integral formulas. In section 3 we give a proof of the Ohsawa-Takegoshi extension theorem for submanifolds of codimension 1. In section 4 we sketch how the argument from section 3 can be modified to treat submanifolds of arbitrary codimension, and in the final section we show how integral kernels can be constructed as a biproduct of that proof.

In section 2 we have also included a sketch of the proof of the Duistermaat-Heckman formula; in particular the complex version of this theorem is intimately related to weighted integral formulas. We stress that this proof is essentially well known (see [10] and [15] for very similar arguments) but we have included it here since it may not be so well known among complex analysts. The reader who is mainly interested in the Ohsawa-Takegoshi theorem can safely go directly to section 3, which can be read independently from the other sections.

I would like to thank Mats Andersson and Robert Berman for valuable discussions.

## 2. Andersson's construction of integral kernels.

The construction is based on properties of a perturbed $\bar\partial$-operator

$$\nabla = \bar\partial - \delta_X,$$

where $\delta_X$ denotes contraction with the holomorphic vector field

$$X = \sum (\zeta_j - z_j)\partial/\partial\zeta_j.$$

Such perturbed $\bar{\partial}$-operators, and their real analogs $d-\delta_X$, have a long history, in particular in connection with equivariant cohomology. We refer to [15] for background and further references. The - easily verified - properties of the $\nabla$-operator that we will use here is that it is an antiderivation that sends a form of bidegree $(p,q)$ to a sum of forms of bidegree $(p,q+1)$ and $(p-1,q)$, and that $\nabla^2=0$. The last property follows since $\bar{\partial}$ and $\delta_X$ (anti)commute, and evidently depends on $X$ being holomorphic. It is not satisfied by the real analog $\nabla_R = d-\delta_X$, for which the corresponding equation is $\nabla_R^2 = \mathcal{L}_X$, the Lie derivative with respect to $X$.

With $s$ defined as in the introduction we next consider the full Cauchy-Fantappié kernel

$$u = \sum_0^{n-1} u_{j+1,j}$$

where

$$u_{j+1,j} = \frac{s \wedge (\bar{\partial} s)^j}{\langle s, \zeta - z\rangle^{j+1}}.$$

The coupling between $u$ and $\nabla$ lies in the equation

(2.1) $$\nabla u = (2\pi i)^n \mu_z - 1.$$

That $\nabla u = -1$ outside of the singular point $\zeta = z$, can be verified by brute computation, or by a slicker argument from [2], that we will come back to shortly. The contribution from the singular point, must be precisely the same as that of $\bar{\partial} u_{n,n-1}$ since all the other terms are of lower order.

Consider next an arbitrary differential form in the $\zeta$-variable, whose coefficients depend on $z$ as a parameter,

$$g = \sum_0^n g_{k,k}$$

with $g_{k,k}$ of bidegree $(k,k)$, and which satisfies

$$\nabla g = 0$$

and

$$g_{0,0}(z) = 1.$$

The weighted kernel associated to $u$ and $g$ is now simply

$$K = u \wedge g.$$

As $\nabla$ is an antiderivation $g_{0,0}(z)=1$ and $\nabla g = 0$ it is immediately clear that

$$\nabla K = ((2\pi i)^n \mu_z - 1) \wedge g = (2\pi i)^n \mu_z - g,$$

which in particular means that the component of $K$ of bidegree $(n,n-1)$, $K_{n,n-1}$, satisfies

(2.2) $$\bar{\partial} K_{n,n-1} = (2\pi i)^n \mu_z - g_{n,n}.$$

We could of course have obtained kernels with similar properties just by multiplying $u_{n,n-1}$ by a $\bar{\partial}$-closed form, but the point is that the possibility of using $\nabla$-closed forms offers a much greater flexibility, as we shall soon see. First, however, we note that equation (2.2) immediately gives general Cauchy formulas

$$(2.3)\quad v(z) = \frac{1}{(2\pi i)^n}\int_{\partial\Omega} v(\zeta)K_{n,n-1} + \int_\Omega vP - \frac{1}{(2\pi i)^n}\int_\Omega \bar\partial v \wedge K_{n,n-1},$$

where $P = g_{n,n}$.

Let us denote by $W$ the space of forms $g = \sum g_{k,k}$, where $g_{k,k}$ is of bidegree $(k,k)$, which satisfy

$$\nabla g = 0.$$

It is clear that $W$ is an algebra ( since $\nabla$ is an antiderivation), which is commutative since all the forms in $W$ are of even degree. A form in $g$ is invertible at a point if and only if $g_{0,0} \neq 0$. It is clear that this condition is necessary since otherwise $g \wedge h$ would always be a form of positive degree. It is also sufficient since, if $g_{0,0} \neq 0$, an inverse to

$$g = g_{0,0}(1 - (g_{0,0} - g)/g_{0,0})$$

can be found by summing a geometric series. If we consider the form $g$ in all of $\Omega$ it follows that the spectrum of the form is equal to the image of $\bar\Omega$ under the map $g_{0,0}$. By the one-dimensional Cauchy formula, functions that are holomorphic in a neighbourhood of the spectrum operate on elements of the algebra. In other words: If $g$ is in $W$, and $G$ is holomorphic in a neighbourhood of $g_{0,0}(\bar\Omega)$, then $G(g)$ is also in $W$.

In the same vein we also give the proof from [2] that $\nabla u = -1$ outside the singular point. First note that

$$-\nabla s = \langle s, \zeta - z\rangle - \bar\partial s,$$

so that by the previous discussion and the hypothesis on $s$, $\nabla s$ is invertible for $\zeta \neq z$. Expanding the inverse in a geometric series we see that

$$u = -s \wedge (\nabla s)^{-1},$$

from where it immediately follows that, indeed, $\nabla u = -1$ for $\zeta \neq z$.

In the sequel a *weight* will be an element in $W$ such that $g_{0,0}(0) = 1$. We next discuss a few ways to find weights. The first way, see [2], starts by noting that if $q$ is a $(1,0)$-form (as always a form in $\zeta$ with coefficients depending on $z$ as a parameter), then, since $\nabla^2 = 0$,

$$\nabla q = \langle q, z - \zeta\rangle + \bar\partial q$$

is a $\nabla$-closed form. If $G$ is a function of one complex variable, holomorphic in a neighbourhood of the image of $\bar\Omega \times \bar\Omega$ under the map $\langle q, z-\zeta\rangle$ and satisfying $G(0) = 1$, then $g = G(\nabla q)$ is a weight. Expanding $G$ in a Taylor series around the point $\langle q, z-\zeta\rangle$ , (the definition of $G(\nabla q)$ by the one-dimensional Cauchy formula shows that this is legitimate), we see that

$$g = \sum_0^n G^{(k)}(\langle q, z-\zeta\rangle)(\bar\partial q)^k/k!,$$

so that $K = u \wedge g$ is just the weighted kernel from [1].

Another class of examples is connected with the circle of ideas around the Duistermaat-Heckman formula, see [8] and [15]. Let $\omega$ be a Kähler form and suppose we can find a function $\phi$ such that $\bar{\partial}\phi = \delta_X \omega$. In case $\phi$ is real valued this is equivalent to $d\phi = \delta_{2\mathrm{Re}\,X}\omega$. Thinking of $\omega$ as a symplectic form it means that $\phi$ is a Hamiltonian function for the real vector field corresponding to $X$. Then $\phi + \omega$ is an element in $W$ and if $G$ is a function of one complex variable such that $G(\phi(z)) = 1$ ($G$ may depend in an arbitrary way on the parameter $z$), then $g = G(\phi + \omega)$ is a weight. The basic example here is $\phi = |z - \zeta|^2$ and $\omega$ the Euclidean metric, which again leads to the representation formula in Bargmann-Fock space.

At this point it is useful to realize that the previous discussion generalizes almost directly to an arbitrary holomorphic vector field, $X$, on a complex manifold having a discrete set of simple zeros (and even, with more work, to more general zero sets). One can then always find a corresponding form, $s$, such that

$$\langle s, X \rangle \neq 0$$

for $z \neq z$. The form dual to $X$ under some Hermitian metric will do. Then, with $u = -s \wedge (\nabla s)^{-1}$

$$\nabla u = \sum_{X(z)=0} a_z \mu_z - 1,$$

where $\mu_z$ is a unit point mass at $z$, and $a_z$ a number depending on the local behaviour of $X$ near $z$. Now consider the form in $W$

$$g = \exp(\phi + \omega).$$

The associated projection kernel $P = g_{n,n}$ is in this case equal to $\exp(\phi)\omega^n/n!$. Suppose the manifold $\Omega$ is compact without boundary, and apply what corresponds to the representation formula (2.3) to the holomorphic function $v = 1$. The result is

$$\int_\Omega \exp(\phi)\omega^n/n! = \sum_{X(z)=0} \exp(\phi(z)) a_z.$$

This is a (nonprecise) version of the Duistermaat-Heckman formula in the complex case.

The real case of the Duistermaat-Heckman formula can be proved in precisely the same way (cf [15]). The $\nabla$-operator is then replaced by $\nabla_R = d - \delta_X$ where $X$ is a real vector field, and $\omega$ is a symplectic form. The main difference between the complex and the real case is that now $\nabla_R^2 = \mathcal{L}_X$, with $\mathcal{L}_X$ being the Lie derivative along $X$. In order to have

$$\nabla_R u = \nabla_R(-s \wedge (\nabla_R s)^{-1}) = -1$$

outside the zeros of $X$, one needs to choose $s$ in such a way that $\nabla_R^2 s = \mathcal{L}_X s = 0$. If we let $s$ be the form dual to $X$ under some metric, this follows from $\mathcal{L}_X X = [X, X] = 0$, if the metric is invariant under the flow of $X$, i e if $X$ is a Killing field. If the field $X$ generates a circle action of the manifold, such an invariant metric can always be found by averaging over the circle. This is the usual setting of the Duistermaat Heckman formula.

So far we have shown two different ways of constructing weights; one starting from a $\nabla$-exact form and another which works for Hamiltonian vector fields and involves the Hamiltonian function. In the last section we will give yet another construction of weights, related to the Ohsawa-Takegoshi extension theorem (see [12]. It is then natural to generalize the previous set-up a little bit.

Up to now we have been dealing with holomorphic vector fields in the sense of sections to the holomorphic tangent bundle (of $\mathbb{C}^n$), but the formalism we have described works just as well for sections to an arbitrary holomorphic bundle, $E$ over $\Omega$, with rank $r$ not necessarily equal to the dimension of the base space. Instead of differential forms of bidegree $(p,q)$ in $\zeta$ we then consider sections to the exterior algebra bundle

$$\bigwedge(E^* \oplus T^*),$$

which are of degree $p$ in $E^*$, $0$ in $d\zeta$ and $q$ in $d\bar{\zeta}$. Given a holomorphic section $X = \sum X_j e_j$ the $\nabla$-operator is defined just as before by $\nabla = \bar{\partial} - \delta_X$. With $s$ a section to $E^*$ such that $\langle s, X\rangle \neq 0$ outside the zero-set of $X$ we again put

$$u = -s \wedge (\nabla s)^{-1} = \sum_0^m u_{j+1,j}, \tag{2.4}$$

where $m = \min(n,r) - 1$. Computing $\nabla u$ in the sense of distributions we find that

$$\nabla u = R_X - 1$$

where $R_X$ is a certain residue term associated to $X$. In case the zero-set, $Z$, of $X$ is a complete intersection, i e codim$(Z) = r$, and moreover $X$ vanishes to first order on $Z$ $R_X$ can be written in terms of local trivializations of $E$ and $E^*$ as

$$R_X = e_1^* \wedge ...e_r^* \wedge \bar{\partial}\frac{1}{X_1} \wedge ...\bar{\partial}\frac{1}{X_r}.$$

Similar formulas also hold in more general situations, see [3] and the references given there.

## 3. THE OHSAWA-TAKEGOSHI THEOREM IN CODIMENSION 1

Let $\Omega$ be a pseudoconvex domain in $\mathbb{C}^n$ and let $V$ be a smooth hypersurface in $\Omega$ defined by $V = \{h = 0\}$ where $h$ is holomorphic in $\Omega$ and such that $\partial h \neq 0$ on $V$. Let $\phi$ be plurisubharmonic in $\Omega$. The celebrated Ohsawa-Takegoshi extension theorem, see [12], asserts that functions holomorphic on $V$ can be extended holomorphically to all of $\Omega$ with good estimates in $L^2(\Omega, e^{-\phi})$.

**Theorem 3.1.** *( Ohsawa-Takegoshi) Assume* $|h| \leq 1$ *in* $\Omega$. *Let* $f$ *be a holomorphic function on* $V$. *Then there is a holomorphic function* $F$ *in* $\Omega$ *such that* $F = f$ *on* $V$ *and*

$$\int_\Omega |F|^2 e^{-\phi} \leq C \int_V |f|^2 e^{-\phi}/|\partial h|^2,$$

*where* $C$ *is a universal constant.*

The main point of the theorem is the $L^2$-estimate with a constant that is independent of the domain and of the plurisubharmonic function. In the original proof one first constructs a local extension of $f$ to a very small neighbourhood of $V$ and then gets a global solution by solving an appropriate $\bar\partial$-equation. What makes the proof tricky is that the standard Hörmander estimate is not quite enough to obtain a good estimate for the extension. Therefore Ohsawa and Takegoshi use a refined version of the Hörmander technique inspired by a result of Donnelly and Fefferman. Many variants of the proof and of the $\bar\partial$-theorem has been developed later. In [5] a proof is given which avoids the local extension and instead derives a $\bar\partial$-theorem where the right hand side is not in $L^2(\Omega)$, but in $L^2(V)$.

We shall now sketch a proof of the theorem which avoids the $\bar\partial$-equation altogether, although the estimate in the end comes from an inequality that arises in the proof of the $\bar\partial$-theorem.

In the proof we will assume that the domain $\Omega$ is bounded with smooth boundary , that $h$ and hence $V$ extend to a neighbourhood of the closure of $\Omega$ and that the weight function $\phi$ is smooth and also extends to a neighbourhood of $\bar\Omega$. We also assume that $f$ extends holomorphically in $V$ to a neighbourhood of $V \cap \bar\Omega$. Such a situation can be obtained by restricting the whole problem to a relatively compact subdomain. Afterwards one can easily pass to the limit using normal families, as long as the constant we obtain in the estimate for the extension only depends on the sup-norm of $h$.

Given all this it is clear that there exists *some* extension of $f$ in $A^2(\Omega, e^{-\phi})$, the space of holomorphic functions in $L^2(\Omega, e^{-\phi})$ . Let $F$ be the extension of minimal norm. Then $F$ is orthogonal in $L^2(\Omega, e^{-\phi})$ to the space of holomorphic functions that vanish on $V$, and in particular to the space $hA^2$. Equivalently, $\bar h F$ is orthogonal to all of $A^2$. Since the domain is pseudoconvex the orthogonal complement of $A^2$ is precisely the range of $\bar\partial^*$, so we can solve

$$\bar\partial^* \alpha = \bar h F,$$

and by taking the minimal solution of this equation we also get $\bar\partial\alpha = 0$. We can now estimate the $L^2$-norm of $F$ as follows.

$$\int_\Omega |F|^2 e^{-\phi} = \int_\Omega \frac{F}{h} \overline{\bar\partial^* \alpha} e^{-\phi} = \int_\Omega F \bar\partial \frac{1}{h} \cdot \bar\alpha e^{-\phi}. \tag{3.1}$$

The expression $\bar\partial(1/h)$ is a residue current with support on $V$ and it can be computed as

$$\bar\partial\frac{1}{h} = 2\pi\frac{\overline{\partial h}}{|\partial h|^2}dV,$$

where $dV$ is surface measure on $V$.

( To see this, first note that by the Poincaré-Lelong formula

$$i\partial\bar\partial\log|h|^2 = 2\pi[V],$$

where $[V]$ is the current of integration on $V$. The left hand side equals

$$i\partial\bar\partial\log|h|^2 = -i\bar\partial\frac{\partial h}{h} = i\partial h\wedge\bar\partial\frac{1}{h},$$

and the right hand side is

$$2\pi[V] = 2\pi\frac{i\partial h\wedge\overline{\partial h}}{|\partial h|^2}dV. \tag{3.2}$$

The claimed formula follows by contracting with $\partial h$.)

Therefore the right hand side in (3.1) equals

$$\int_V f\,\overline{\partial h}\cdot\bar\alpha\frac{dV}{|\partial h|^2}e^{-\phi},$$

which by Cauchy's inequality is dominated by the square root of

$$\int_V |f|^2 e^{-\phi}\frac{dV}{|\partial h|^2}\int_V|\partial h\cdot\alpha|^2 e^{-\phi}\frac{dV}{|\partial h|^2}.$$

To find an estimate for $F$ in terms of $f$ we therefore need to estimate

$$\int_V|\partial h\cdot\alpha|^2 e^{-\phi}\frac{dV}{|\partial h|^2}$$

in terms of $F$. We will obtain such an estimate from an $L^2$-inequality which is based on Siu's so called $\partial\bar\partial$-Bochner-Kodaira technique (see [14]). The explicit form of the inequality for $(0,1)$-forms in $\mathbb{C}^n$ that we need here is taken from [6], where a proof can also be found. In the proposition below $\rho$ is a defining function for $\Omega$, i e a function which is smooth up to the boundary which is negative inside of $\Omega$, vanishes on the boundary of $\Omega$ and has non-vanishing gradient on the boundary.

**Proposition 3.2.** *Assume $\alpha$ is a $(0,1)$-form, smooth in $\Omega$, in the domain of $\bar\partial^*$, such that $\bar\partial\alpha$ and $\bar\partial\bar\partial^*\alpha$ are in $L^2$ . Then*

$$\int\sum\phi_{j\bar k}\alpha_j\bar\alpha_k e^{-\phi}w + \int|\bar\partial^*\alpha|^2e^{-\phi}w - \int\sum w_{j\bar k}\alpha_j\bar\alpha_k e^{-\phi}+$$

$$+\int\sum|\bar\partial_k\alpha_j|^2e^{-\phi}w + \int_\partial w\sum\rho_{j\bar k}\alpha_j\bar\alpha_k e^{-\phi}dS/|\partial\rho| =$$

$$= 2Re\int\bar\partial\bar\partial^*\alpha\cdot\bar\alpha e^{-\phi}w + \int|\bar\partial\alpha|^2e^{-\phi}w.$$

We apply Proposition 3.2 to the $\bar\partial$-closed form $\alpha$ which solves $\bar\partial^*\alpha = \bar h F$, $\bar\partial\alpha = 0$. As $w$ we take

$$w = \log\frac{1}{|h|^2} + (1 - |h|^{2\delta}),$$

with $\delta$ slightly smaller than 1. If $\phi$ is plurisubharmonic and $\partial\Omega$ is pseudoconvex all the terms on the right hand side in Proposition 3.2 are nonnegative. Keeping only the fourth term, we get by the Poincaré-Lelong formula and (3.2) that

$$2\pi\int_V |\partial h\cdot\alpha|^2 e^{-\phi}\frac{dV}{|\partial h|^2} + \delta^2\int_\Omega |\partial h\cdot\alpha|^2|h|^{2\delta-2}e^{-\phi} \leq 2\mathrm{Re}\int_\Omega F\overline{\partial h}\cdot\bar\alpha w e^{-\phi}.$$

A simple application of Cauchy's inequality to the right hand side now gives

$$\int_V |\partial h\cdot\alpha|^2 e^{-\phi}\frac{dV}{|\partial h|^2} \leq C\int_\Omega |F|^2 e^{-\phi},$$

and the proof is complete.

## 4. THE EXTENSION THEOREM FOR HIGHER CODIMENSION.

The Ohsawa-Takegoshi theorem was generalized to submanifolds of higher codimension in certain complex manifolds by Manivel, see [11]. We shall now see how the proof of the codimension 1 case in the previous section can be generalized, but we still discuss only the case when the ambient manifold is a pseudoconvex domain $\Omega$ in $\mathbb{C}^n$. Let $V$ be a submanifold of $\Omega$ of codimension $r$, defined by an equation $h = 0$ where $h$ is a section to a vector bundle of rank $r$, written $h = \sum h_j e_j$ with respect to a local trivialization.We assume that $V$ is smooth and that $\partial h_1 \wedge ...\partial h_r \neq 0$ on $V$. Let $f$ be holomorphic on $V$. After preliminary reductions, as in section 3, we may assume that some holomorphic extension of $f$ in $L^2$ exists, and then estimate the extension $F$ of minimal norm. If $E$ is a trivial bundle (which is by no means necessary, but will be the main case for us), the minimal extension $F$ is orthogonal in $L^2(\Omega, e^{-\phi})$ to any holomorphic function divisible by any of the $h_j$:s. In other words $F\bar h_j$ is ortogonal to $A^2(\Omega, e^{-\phi})$. Hence we can solve

$$F\bar h_j = \bar\partial^*\alpha_j$$

with $\alpha_j$ a $\bar\partial$-closed $(0,1)$-form.

This can be expressed somewhat more elegantly if we choose a hermitian metric on $E$ and introduce the space

$$L^2_{(p,q)}(\Omega, e^{-\phi})$$

of $L^2$-sections to the bundle $\bigwedge(E^*\oplus T^*)$ of bidegree $p$ in $E^*$, $0$ in $d\zeta$ and $q$ in $d\bar\zeta$. Note that the $\bar\partial$-operator extends to a well defined antiderivation on these spaces. We denote by $A^2_{(p,q)}$ the subspace of $\bar\partial$-closed sections, so that in particular $A^2_{(p,0)}$ consists of holomorphic sections to $\bigwedge^p(E^*)$. Contraction with the holomorphic field $h$ , $\delta_h$ , now maps $L^2_{(p,q)}$ to $L^2_{(p-1,q)}$ and $A^2_{(p,q)}$ to $A^2_{(p-1,q)}$. If $(e_j)$ is an orthonormal frame, then the adjoint of $\delta_h$ is

$$\delta_h^* = \sum \bar{h}_j e_j^* \wedge .$$

The previous discussion can now be summarized by saying that since $F$ is orthogonal to $\delta_h A^2_{(1,0)}$ , $\delta_h^* F$ is orthogonal to $A_{(1,0)}$ so, we can solve

$$\delta_h^* F = \sum F \bar{h}_j e_j^* = \bar{\partial}^* \alpha_{1,1} \tag{4.1}$$

with $\alpha_{1,1}$ in $A^2_{(1,1)}$. Next we observe that since $\bar{\partial}\delta_h + \delta_h \bar{\partial} = 0$, we get $\bar{\partial}^* \delta_h^* + \delta_h^* \bar{\partial}^* = 0$ , and since $\delta_h^2 = 0$, we also have $\delta_h^{*2} = 0$. Applying $\delta_h^*$ to (4.1) we thus find that

$$\bar{\partial}^* \delta_h^* \alpha_{1,1} = 0.$$

Since $\Omega$ is pseudoconvex this implies that we can solve

$$\delta_h^* \alpha_{1,1} = \bar{\partial}^* \alpha_{2,2}$$

with $\alpha_{2,2}$ in $A^2_{(2,2)}$. Continuing in this way, we obtain a sequence of forms $\alpha_{j,j}$ in $A^2_{(j,j)}$ such that

$$\delta_h^* \alpha_{j,j} = \bar{\partial}^* \alpha_{j+1,j+1}.$$

Clearly the process stops so that $\alpha_{j,j} = 0$ if $j > \min(r, n)$. If we put $\alpha_{0,0} = F$ , $\alpha = \sum \alpha_{j,j}$ and $\nabla^* = \bar{\partial}^* - \delta_h^*$ we find that

$$\nabla^* \alpha = 0.$$

We could now define $g = *\alpha e^{-\phi}$ to obtain an element in $W$ (see section 2), but it will actually be more convenient to stay with the form $\alpha$ itself. As at the end of section 2 we let $s$ be some arbitrary section of $E^*$ such that $\langle s, h\rangle \neq 0$ when $h \neq 0$ ; e g $s = \delta^* 1$ will do. Put

$$u = -\frac{s}{\nabla s} = \sum \frac{s \wedge (\bar{\partial} s)^{j-1}}{\langle s, h\rangle^j}.$$

Then

$$\nabla u = R_h - 1$$

where $R_h$ is a residue term that can be written as

$$R_h = (2\pi)^r \frac{\overline{\partial h}}{|\partial h|^2} dV e^*,$$

where $\partial h = \partial h_1 \wedge ... \partial h_r$ and $e^* = e_1^* \wedge ... e_r^*$.

With these computations in place we can estimate the $L^2$-norm of $F$. First note that

$$\int_\Omega \langle \nabla(Fu), \alpha\rangle e^{-\phi} = 0 \tag{4.2}$$

since $\nabla^*\alpha = 0$. (Note that there are no boundary terms since $\alpha$ lies in the domain of $\nabla^*$.) Since $\nabla$ is an antiderivation and $\nabla F = 0$ we have $\nabla(Fu) = fR_h - F$. Since $\alpha_{0,0} = F$, (4.2) implies that

$$\int_\Omega |F|^2 e^{-\phi} = \int_V f\langle R_h, \alpha_{r,r}\rangle e^{-\phi}.$$

(For bidegree reasons $\langle F, \alpha\rangle = \langle F, \alpha_{0,0}\rangle$ and $\langle R_h, \alpha\rangle = \langle R_h, \alpha_{r,r}\rangle$.) The estimate of $F$ in terms of $f$ can be achieved by estimating

$$\int_V |\partial h \cdot \alpha_{r,r}|^2 e^{-\phi} \frac{dV}{|\partial h|^2}.$$

The integral on the left hand side is first estimated in terms of an integral involving $\alpha_{r-1,r-1}$ roughly the same way as in section 3, using a variant of Proposition 3.2 for forms of higher degree. One then applies the same procedure to $\alpha_{r-1,r-1}$ and proceeds in the same way until $\alpha_{0,0} = F$ is reached. We will not give the details here. Quite probably the proof can be better organized so that one only has to use one single step by using some version of the Proposition 3.2 for the $\nabla$-operator.

## 5. Integral representations again.

Let us now consider the case of the extension problem when $r = n$ and $V$ is the subvariety of $\Omega$ consisting of one single point, $z$. We choose as our bundle $E$ the holomorphic tangent bundle of $\mathbb{C}^n$ and as $h$ we take $X = \sum(\zeta_j - z_j)\partial/\partial\zeta_j$ . Of course $E^*$ is then the bundle of $(1,0)$-forms and the elements of $L^2_{(p,q)}$ are just $(p,q)$ forms in the usual sense. Moreover, $R_h$ is a point mass of size $(2\pi)^n$ at $z$. The function $f$ to extend from $V$ is a constant. Call the minimal extension $F_z$, and construct the form $\alpha_z$ according to the recipe in the previous section, with $(\alpha_z)_{0,0} = F_z$. Then $\alpha_{n,n}$ is a form of maximal degree which can be written

$$\alpha_{n,n} = \chi(\zeta)d\lambda$$

where $d\lambda$ is the volume element on $\mathbb{C}^n$. We now choose the constant $f$ so that $\chi(z) = e^{\phi(z)}$. If we put $g = *\alpha e^{-\phi}$, then $\nabla g = 0$, and the last normalization means that the function $g_{0,0} = \chi$ equals 1 at $z$. In other words, $g$ is a "weight" in the terminology of section 2. The next theorem says that this weight defines a weighted integral kernel that gives the canonical solution to the $\bar\partial$-equation in $L^2(\Omega, e^{-\phi})$.

**Theorem 5.1.** *Let $\gamma$ be a $\bar\partial$-closed $(0,1)$-form in $C^\infty(\bar\Omega)$. Let*

$$K(\gamma)(z) = -\frac{1}{(2\pi)^n}\int_\Omega \langle \gamma \wedge u, \alpha_z\rangle e^{-\phi}.$$

*Then $\tilde v = K(\gamma)$ is the solution to*

$$\bar\partial v = \gamma$$

*which is of minimal norm in $L^2(\Omega, e^{-\phi})$.*

*Proof.* Let $v$ be an arbitrary function in $\Omega$ smooth up to the boundary. Now consider the formula (4.2) with $F$ replaced by $v$. Then

$$\nabla(vu) = \bar\partial v \wedge u + v(R_h - 1).$$

On the other hand

$$\langle R_h, \alpha\rangle e^{-\phi}$$

is, by the normalization we have made, a point mass at $z$ of size $(2\pi)^n$. Hence

$$v(z) = (2\pi)^{-n}\int_\Omega v\bar F_z e^{-\phi} - (2\pi)^{-n}\int_\Omega \langle\bar\partial v\wedge u, \alpha\rangle e^{-\phi}. \tag{5.1}$$

Applying (5.1) to a holomorphic function $v$, we see that $F_z/(2\pi)^n$ is a holomorphic reproducing kernel for holomorphic functions, and therefore must be equal to the Bergman kernel. (In particular, the first integral on the right hand side of (5.1) depends holomorphically on $z$ for any choice of $v$). Now take $v$ to be an arbitrary solution to $\bar\partial v = \gamma$. Since the first term on the right hand side of (5.1) is the orthogonal projection of $v$ on the holomorphic subspace $A^2(\Omega, e^{-\phi})$ it follows that the second term gives the canonical solution to the $\bar\partial$-equation. □

Notice that in the kernel that we have found, the weight factor $\alpha$ is perfectly smooth (at least if we assume that $\Omega$ is smoothly bounded and strictly pseudoconvex). All the singularities of the kernel come from the kernel $u$ which can be chosen in many different ways and be made completely explicit. A natural question is of course if the weight factor can be estimated in any reasonable way. This remains to be studied. For the moment we can only make a few remarks. Let us first point out that the estimates of $\alpha$ that one can hope for are, like in the Ohsawa-Takegoshi theorem in terms of the $L^2$-norm of $F_z$. This equals (a constant times) $B_\phi(z,z)$, the Bergman kernel for $A^2(\Omega, e^{-\phi})$ on the diagonal, which is a fairly explicit object. The Ohsawa-Takegoshi theorem (for $V$ equal to a point) amounts to an estimate

$$e^\phi \leq C\frac{B_\phi(z,z)}{|\partial k|^2}$$

where $k$ is any holomorphic map from $\Omega$ to $\mathbb{C}^n$ of sup-norm smaller than 1, such that $k(0) = 0$. ( When $\Omega$ is the ball, an optimal choice of $k$ is an automorphism that takes $z$ to the origin, giving $|\partial k|^2 = (1-|z|^2)^{-(n+1)}$.) This is equivalent to an optimal estimate of $|\alpha_{n,n}|^2 e^{-\phi}$ at the point $z$.

As a final remark we note that the kernel $u$ can be chosen in many ways, all giving rise to the canonical solution operator of $\bar\partial$. The kernels that one obtains by changing $u$ are certainly in general different, but they always give the same result when applied to $\bar\partial$-closed forms. One interesting choice of $u$ is when $\Omega$ is the unit ball and we let the point $z$ tend to a boundary point. Chosing $s$ as in [1] we get in the limit, when $z$ is on the boundary that $s = \bar z\cdot d\zeta$ and

$$u = \frac{\bar z\cdot d\zeta}{1-\bar z\cdot\zeta}.$$

Therefore, for reasons of bidegree, the boundary values of $K(\gamma)$ are given by a formula that only involves $\alpha_{1,1}$.

## REFERENCES


[1] B BERNDTSSON AND M ANDERSSON: *Henkin-Ramirez formulas with weight factors.* , Ann. Inst. Fourier (Grenoble) 32 (1982), no. 3, v–vi, 91–110..

[2] M ANDERSSON: *Integral representation with weights I,* Math. Ann. 326 (2003), no. 1, 1–18..

[3] M ANDERSSON: *Residue currents and ideals of holomorphic functions,* Bull. Sci. Math. 128 (2004), no. 6, 481–512.

[4] B BERNDTSSON: *Weighted integral formulas.,* Several complex variables (Stockholm, 1987/1988), 160–187, Math. Notes, 38, Princeton Univ. Press, Princeton, NJ, 1993. .

[5] B BERNDTSSON: *The extension theorem of Ohsawa-Takegoshi and the theorem of Donnelly-Fefferman,* Ann. Inst. Fourier (Grenoble) 46 (1996), no. 4, 1083–1094. .

[6] B BERNDTSSON: *Uniform estimates with weights for the $\overline{\partial}$-equation,* J. Geom. Anal. 7 (1997), no. 2, 195–215.

[7] A CUMENGE: *Sharp estimates for $\overline{\partial}$ on convex domains of finite type,* Ark. Mat. 39 (2001), no. 1, 1–25.

[8] J J DUISTERMAAT AND G HECKMAN: *On the variation in the cohomology of the symplectic form of the reduced phase space,* Invent. Math. 69 (1982), no. 2, 259–268.

[9] G HENKIN AND J LEITERER: *Theory of functions on complex manifolds,* Mathematische Lehrbücher und Monographien, II. Abteilung: Mathematische Monographien [Mathematical Textbooks and Monographs, Part II: Mathematical Monographs], 60. Akademie-Verlag, Berlin, 1984. 226 pp..

[10] K-F LIU: *Holomorphic equivariant cohomology. Math. Ann. 303 (1995),* no. 1, 125–148.

[11] L MANIVEL: *Un théorème de prolongement $L^2$ de sections holomorphes d'un fibré hermitien. (French) [An $L^2$ extension theorem for the holomorphic sections of a Hermitian bundle]* , Math. Z. 212 (1993), no. 1, 107–122.

[12] T OHSAWA AND K TAKEGOSHI: *On the extension of $L^2$ holomorphic functions*, Math. Z. 195 (1987), no. 2, 197–204.

[13] R M RANGE: *Holomorphic functions and integral representations in several complex variables,* Graduate Texts in Mathematics, 108. Springer-Verlag, New York, 1986.

[14] Y-T SIU: *Complex-analyticity of harmonic maps, vanishing and Lefschetz theorems,* J. Differential Geom. 17 (1982), no. 1, 55–138.

[15] W ZHANG: *Lectures on Chern-Weil theory and Witten deformations,* Nankai Tracts in Mathematics, 4. World Scientific Publishing Co., Inc., River Edge, NJ, 2001.



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