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SUBHARMONICITY PROPERTIES OF THE BERGMAN KERNEL AND SOME OTHER FUNCTIONS ASSOCIATED TO PSEUDOCONVEX DOMAINS.

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ABSTRACT. Let D be a pseudoconvex domain in $\mathbb{C}^k_t \times \mathbb{C}^n_z$ and let ϕ be a plurisubharmonic function in D. For each t we consider the n-dimensional slice of D, $D_t = \{z; (t,z) \in D\}$, let ϕ^t be the restriction of ϕ to D_t and denote by $K_t(z,\zeta)$ the Bergman kernel of D_t with the weight function ϕ^t. Generalizing a recent result of Maitani and Yamaguchi (corresponding to $n=1$ and $\phi=0$) we prove that $\log K_t(z,z)$ is a plurisubharmonic function in D. We also generalize an earlier results of Yamaguchi concerning the Robin function and discuss similar results in the setting of $\mathbb{R}^n$.

1. INTRODUCTION

Let D be a pseudoconvex domain in $\mathbb{C}^k_t \times \mathbb{C}^n_z$ and let ϕ be a plurisubharmonic function in D. For each t we consider the n-dimensional slice of D, $D_t = \{z; (t,z) \in D\}$ and the restriction, ϕ^t, of ϕ to D_t. Denote by $A^2_t = A^2(D_t, e^{-\phi^t})$ the Bergman space of holomorphic functions in D_t satisfying

$$\int_{D_t} |h|^2 e^{-\phi^t} < \infty.$$

The Bergman kernel $K_t(\zeta, z)$ of A^2_t for a point z in D_t is the unique holomorphic function of ζ satisfying

$$\int_{D_t} h(\zeta)\overline{K_t(\zeta,z)} e^{-\phi(t,\zeta)} = h(z)$$

for all functions h in A^2_t. We shall prove the following theorem.

Theorem 1.1. *With the notation above, the function* $\log K_t(z,z)$ *is plurisubharmonic, or identically equal to* $-\infty$ *in* D.

In particular $\log K_t$ is plurisubharmonic in t for z fixed. Theorem 1.1 was previously obtained in [15] in the case $n=1$ and $\phi=0$.

Theorem 1.1 may be seen as a complex version of Prekopa's theorem (see [16]) from convex analysis. This theorem says that if $\phi(x,y)$ is a convex function in $\mathbb{R}^m_x \times \mathbb{R}^n_y$ and we define the function $\tilde{\phi}$ in $\mathbb{R}^m_x$ by

$$e^{-\tilde{\phi}(x)} = \int_{\mathbb{R}^n} e^{-\phi(x,y)} dy, \tag{1.1}$$

then $\tilde{\phi}$ is also convex. Equivalently, we may define

$$\tilde{\phi}(x) = \log k(x),$$

where

$$k(x) = \left(\int_{\mathbb{R}^n} e^{-\phi(x,y)} dy \right)^{-1}.$$

For each x fixed, $k(x)$ can be seen as the "Bergman kernel" for the space $\mathrm{Ker}\,(d)$ of constant functions in $\mathbb{R}^n$, since the scalar product in

$$L^2(\mathbb{R}^n, e^{-\phi(x,\cdot)})$$

of a function, u, with $k(x)$ equals the mean value of u, i e the orthogonal projection of u on the space of constants. Thus Theorem 1.1 is what we get by replacing the convexity hypothesis in Prekopa's theorem by plurisubharmonicity, and the kernel of d by the kernel of $\bar{\partial}$. (In the complex setting we also need to pay attention to the domains involved, since a general pseudoconvex domain cannot be defined by an inequality involving global plurisubharmonic functions.)

One interesting case of the theorem , where the analogy to Prekopa's theorem is more evident, is when $(t, 0)$ lies in D (for t in some open set), and D_t and ϕ^t are both for fixed t invariant under rotations $r_\theta(z) = e^{i\theta} z$. It then follows from the mean value property for holomorphic functions that $K_t(\zeta, 0)$ is for each fixed t a constant independent of ζ,

$$K_t = \left(\int_{D_t} e^{-\phi^t} \right)^{-1}.$$

The following theorem from [3] is therefore a corollary of Theorem 1.1.

Theorem 1.2. *Assume that for each fixed t, D_t and ϕ^t are invariant under rotations $r_\theta(z) = e^{i\theta} z$. Define the function $\tilde{\phi}$ by*

$$e^{-\tilde{\phi}(t)} = \int_{D_t} e^{-\phi(t,\xi)}.$$

Then $\tilde{\phi}$ is plurisubharmonic.

In particular, taking $\phi = 0$ it follows that under the hypothesis of Theorem 1.2, the function

$$-\log |D_t|,$$

where $|V|$ stands for the volume of a set, is plurisubharmonic. This has recently been used by Cordero-Erausquin (see [7]) to give a proof of the Santaló inequality.

Still under the hypotheses of Theorem 1.2 we can also introduce a large parameter, p, and define a function $\tilde{\phi}_p$ by

$$e^{-p\tilde{\phi}_p(t)} = \int_{D_t} e^{-p\phi(t,\xi)}.$$

Thus $e^{-\tilde\phi_p(x)}$ is the L^p-norm of $e^{-\phi(x,\cdot)}$. From the plurisubharmonicity of $\tilde\phi_p$ it is not hard to deduce that

$$\tilde\phi_\infty = \inf_\xi \phi$$

is also plurisubharmonic. This is one version of Kiselman's *minimum principle* for plurisubharmonic functions, [10].

One main application of Kiselman's minimum principle, combined with a use of the Legendre transform, was to give a procedure to "attenuate the singularities" of a given plurisubharmonic function: Given an arbitrary plurisubharmonic function ϕ, and a number $c > 0$, Kiselman constructed a new plurisubharmonic function which is finite at all points where the Lelong number of ϕ is smaller that c and still has a logarithmic singularity at points where the Lelong number of ϕ exceeds c. This was in turn used to give an easy proof of Siu's theorem on the analyticity of sets defined by Lelong numbers (see [11]).

It is a consequence of the Hörmander L^2-estimates for the $\bar\partial$-equation that if a is a point in a bounded domain Ω and ϕ is plurisubharmonic in Ω, then there is some holomorphic function in $L^2(\Omega, e^{-\phi})$ which does not vanish at a, if and only if the function $e^{-\phi}$ is locally integrable in some neighbourhood of a. Using this we can prove the following theorem, which can be seen as an alternative way of attenuating the singularities of plurisubharmonic functions.

Theorem 1.3. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ and let ϕ be plurisubharmonic in Ω. Let ψ be the plurisubharmonic function in $\Omega \times \Omega$ defined by*

$$\psi(a, z) = \phi(z) + (n-1)\log|z-a|.$$

Put

$$\chi(a) = \log K_a(a, a),$$

where K_a is the Bergman kernel for $A^2(\Omega, e^{-2\psi^a})$. Then χ is plurisubharmonic in Ω, is finite at any point where the Lelong number of ϕ is smaller than 1 and has a logarithmic singularity at any point where the Lelong number of ϕ is larger than 1. The singularity set of χ, $\{a; \chi(a) = -\infty\}$ is equal to (the analytic) set where the Lelong number of ϕ is at least 1.

Theorem 1.3 suggests the introduction of a family of Lelong numbers,

$$\gamma_s(\phi, a)$$

by replacing the function ψ by

$$\phi(z) + s\log|z-a|$$

for $0 \le s < n$, and looking at points where the corresponding function χ is singular. We would then get the so called integrability index (see e g [12]) for $s = 0$ and the classical Lelong numbers for $s = n-1$.

Theorem 1.1 is also intimately connected with another result concerning curvature of vector bundles. We explain this in the simplest case, when D

is the product $U \times \Omega$ of two domains in $\mathbb{C}^k_t$ and $\mathbb{C}^n_z$ respectively. Let us also here assume that ϕ is a bounded function, so that all the Bergman spaces $A^2(\Omega, e^{-\phi^t})$ are equal as vector spaces, but the norm varies with t. We can then define a vector bundle, E, over U by taking $E_t = A^2(\Omega, e^{-\phi^t})$. This is then a trivial vector bundle, of infinite rank, with an hermitian metric defined by the Hilbert space norm. Our claim is that this vector bundle is positive in the sense of Nakano. This can be proved by methods very similar to the proof of Theorem 1.1. Such a result however seems to be more natural in the setting of complex fibrations with compact fibers (so that the Bergman spaces are of finite dimension) and we will come back to it in a future publication.

We shall give two proofs of Theorem 1.1. The first, and simplest, one is modeled on one proof of Prekopa's theorem given by Brascamp and Lieb, [1]. Brascamp and Lieb used in their proof a version of Hörmander's L^2-estimates for the d-operator instead of $\bar\partial$. They also proved directly this L^2-estimate by an inductive procedure, using a version of Prekopa's theorem in smaller dimensions. Our first proof adapts this proof to the complex case but starts from Hörmander's theorem.

The second proof does not use Hörmander's theorem, but rather the a priori estimates behind it. (It is somewhat similar to a recent proof of Theorem 1.2 given by Cordero-Erausquin, [6], which is in turn inspired by [2].) Our proof is based on a representation of the Bergman kernel as the pushforward of a subharmonic form. We have included that proof since it seems to us that it will be useful in other similar situations. As an example of that we give a generalization of a rather remarkable result of Yamaguchi on the plurisubharmonicity of the Robin function, [19]. We finish the paper with a short discussion of what a real variable version of a subharmonic form should be and how this notion can be used to prove Prekopa's theorem and real variable versions of Yamaguchi's result, [5].

I would like to thank Christer Borell for several interesting discussions on the material of this paper.

CONTENTS

2. A SPECIAL CASE OF THEOREM 1.1

Let V be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$, with defining function ρ so that $V = \{\zeta, \rho(\zeta) < 0\}$. Let U be a domain in $\mathbb{C}$ and let ϕ be a smooth strictly plurisubharmonic function in a neighbourhood of $U \times V$. Fix a point z in V and let $K_t(\cdot, z)$ be the Bergman kernel for V with the weight function ϕ^t.The main step in proving Theorem 1.1 is to prove that in this situation, $K_t(z,z)$ is a subharmonic function of t.

For any square integrable holomorphic function h in V

$$h(z) = \int_V h(\zeta)\overline{K_t(\zeta,z)}e^{-\phi^t} \tag{2.1}$$

is independent of t. We shall differentiate this relation with respect to t and will then have use for the following lemma.

Lemma 2.1. *Let V be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$, and let ϕ be a function in $\Delta \times V$ which is smooth up to the boundary. Let $K_t(\zeta, z)$ be the Bergman kernel for the domain V with weight function ϕ^t. Then K_t is for z fixed in V smooth up to the boundary of V as a function of ζ, and moreover depends smoothly on t.*

Proof. Let v_t be a smooth function in V supported in a small neighbourhood of z, depending smoothly on t, and put $f_t = \bar\partial v_t$. Let α_t be the solution of the $\bar\partial$-Neumann problem

$$\Box_t \alpha_t = (\bar\partial\bar\partial_t^* + \bar\partial_t^*\bar\partial)\alpha_t = f_t,$$

where $\bar\partial_t^*$ is the adjoint of $\bar\partial$ with resepect to the weight ϕ^t. Since $u_t = \bar\partial_t^*\alpha_t$ is the minimal solution in $L^2(V, e^{-\phi^t})$ to the equation $\bar\partial u = f_t$ we have

$$u_t(\zeta) = v_t(\zeta) - \int_{\chi \in V} v_t(\xi)K_t(\zeta,\xi)e^{-\phi^t}.$$

Choosing v_t appropriately (i e so that $v_t e^{-\phi^t}$ is a radial function of integral one in a small ball with center z) we get that the last term on the right hand side is equal to $K_t(\zeta, z)$. It is therefore enough to prove that α has the smoothness properties stated. To see this, note that if t is close to 0

$$\Box_t = \Box_0 - S_t,$$

with S_t an operator of order 1 with smooth coefficients which vanishes for $t = 0$. Hence

$$(I - R_t)\alpha_t := (I - \Box_0^{-1}S_t)\alpha_t = \Box_0^{-1}f_t.$$

For t sufficiently close to 0 we can invert the operator $I - R_t$ and the lemma follows from basic regularity properties of the $\bar\partial$-Neumann problem in strictly pseudoconvex domains. $\Box$

We now differentiate the relation 2.1 with respect to $\bar t$, using the lemma. Let us denote by ∂_t^ϕ the differential operator

$$e^\phi \frac{\partial}{\partial t} e^{-\phi} = \frac{\partial}{\partial t} - \frac{\partial \phi}{\partial t}.$$

It follows that the function

$$u = \partial_t^\phi K_t$$

is for fixed t orthogonal to the space of holomorphic functions in A_t^2. By the reproducing property of the Bergman kernel we have

$$\Phi(t) := K_t(z,z) = \int_V K_t(\zeta,z)\overline{K_t(\zeta,z)}e^{-\phi^t}.$$

We shall use this formula to compute $\partial^2\Phi/\partial t\partial\bar{t}$. We first get, using the notation $\bar\partial_t = \partial/\partial\bar{t}$

$$\frac{\partial\Phi}{\partial\bar{t}} = \int_V \bar\partial_t K_t\overline{K_t}e^{-\phi^t} + \int_V K_t\overline{\partial_t^\phi K_t}e^{-\phi^t}.$$

Since K_t is holomorphic and u is orthogonal to the space of holomorphic functions, the second term vanishes. We next differentiate once more.

$$\frac{\partial^2\Phi}{\partial t\partial\bar{t}} = \int_V |\bar\partial_t K_t|^2 e^{-\phi^t} + \int_V \partial_t^\phi\bar\partial_t K_t\overline{K_t}e^{-\phi^t}.$$

Using the commutation rule

$$\partial_t^\phi\bar\partial_t = \bar\partial_t\partial_t^\phi + \phi_{t\bar{t}} \tag{2.2}$$

in the second term we get

$$\frac{\partial^2}{\partial t\partial\bar{t}}\Phi = \int_V |\bar\partial_t K_t|^2 e^{-\phi^t} + \int_V \phi_{t\bar{t}}|K_t|^2 e^{-\phi^t} + \int_V \bar\partial_t\partial_t^\phi K_t\overline{K_t}e^{-\phi^t}.$$

Moreover, by differentiating the relation

$$0 = \int_V \partial_t^\phi K_t\,\overline{K_t}e^{-\phi^t}$$

we find that

$$\int_V \bar\partial_t\partial_t^\phi\,K_t\overline{K_t}e_t^{-\phi} = -\int_V |\partial_t^\phi K_t|^2 e^{-\phi^t} = -\int_V |u|^2 e^{-\phi^t}.$$

All in all we therefore have that

$$\frac{\partial^2\Phi}{\partial t\partial\bar{t}} = \int_V |\bar\partial_t K_t|^2 e^{-\phi^t} + \int_V \phi_{t\bar{t}}|K_t|^2 e^{-\phi^t} - \int_V |u|^2 e^{-\phi^t}. \tag{2.3}$$

To estimate the last term we note that u solves the $\bar\partial$-equation

$$\bar\partial u := f = \bar\partial\partial_t^\phi K_t = K_t\bar\partial\frac{\partial\phi}{\partial t},$$

(the last equation follows from a commutation rule similar to 2.2 since K_t is holomorphic). Moreover, u is the minimal solution to this equation, since u is orthogonal to the space of holomorphic functions. By Hörmander's theorem (see [8] for an appropriate formulation) we therefore get that

$$\int_V |u|^2 e^{-\phi^t} \le \int_V \sum(\phi^t)^{j\bar{k}} f_j\bar{f}_k e^{-\phi^t},$$

where $(\phi^t)^{j\bar{k}}$ is the inverse of the complex Hessian of ϕ^t. Inserting this into 2.3 and discarding the first (nonnegative) term we have

$$\frac{\partial^2 \Phi}{\partial t \partial \bar{t}} \geq \int_V |K_t|^2 D e^{-\phi^t},$$

where

$$D = \phi_{t\bar{t}} - \sum (\phi_{z_j \bar{z}_k})^{-1} \phi_{t\bar{z}_j} \overline{\phi_{t\bar{z}_k}}.$$

D equals precisely the determinant of the full complex Hessian of ϕ divided by the determinant of the Hessian of ϕ^t. Since ϕ is strictly plurisubharmonic, this quantity is positive, and it follows that Φ is subharmonic.

To see that in fact even $\log K_t$ is subharmonic we change the weight function ϕ to $\phi(t, \zeta) + \psi(t)$ where ψ is an arbitrary smooth subharmonic function. The Bergman kernel for the new weight $\phi + \psi$ is $e^\psi K_t$, where K_t is the Bergman kernel for ϕ. Therefore $e^\psi K_t$ is subharmonic for any choice of subharmonic function ψ. This implies that $\log K_t$ is subharmonic.

3. The general case of Theorem 1.1

In the previous section we have proved Theorem 1.1 when the domains D_t are smoothly bounded and do not depend on t, under the extra assumption that ϕ is smooth up to the boundary. The general case is in principle a rather straightforward consequence of this special case. There is however one subtility, arising from the fact that some of the fiber domains D_t may not be smoothly bounded. This happens at points where the topology of the fiber changes, something which is not at all excluded by our hypotheses. (The simplest such example is when $D_t = \{\psi(z) < \mathrm{Re}\, t\}$ where ψ is a subharmonic function of one variable with two logarithmic poles.When $\mathrm{Re}\, t$ is large negative, D_t is a union of two disjoint islands around the poles. The two islands come closer as $\mathrm{Re}\, t$ increases and eventually touch in a figure eight, after which they join to one single domain.)

Lemma 3.1. *Let Ω_0 and Ω_1 be bounded domains in $\mathbb{C}^n$, with Ω_0 compactly included in Ω_1. Let ϕ_j be a sequence of continuous weight functions in Ω_1 such that*

$$\phi_j = \phi$$

in $\bar{\Omega}_0$ and that ϕ_j increases and tends to to infinity almost everywhere in $\Omega_1 \setminus \Omega_0$. Assume that the space of holomorphic functions in $L^2(\Omega_1, e^{-\phi_0})$ is dense in the space of holomorphic functions in $L^2(\Omega_0, e^{-\phi_0})$. Fix a point z in Ω_0 and let K_j be the Bergman kernel for z in $L^2(\Omega_1, \phi_j)$. Let K be the Bergman kernel for z in $L^2(\Omega_0, \phi)$.

Then $K_j(z, z)$ increases to $K(z, z)$.

Proof. The extremal characterisation of Bergman kernels,

$$K(z, z) = \sup |h(z)|^2,$$

where the supremum is taken over all holomorphic functions of L^2-norm at most 1 makes it clear that $K_j(z,z)$ is an increasing sequence and that each $K_j(z,z)$ is smaller than $K(z,z)$. Since

$$K_j(z,z) = \int_{\Omega_1} |K_j|^2 e^{-\phi_j}$$

it follows in particular that K_j has uniformly bounded norm in $L^2(\Omega_1, e^{-\phi_j})$. The sequence K_j therefore has a weakly convergent subsequence in $L^2(\Omega_0, e^{-\phi})$. Let k be the limit of some weakly convergent subsequence. If h lies in $L^2(\Omega_1, e^{-\phi_0})$ we have that

$$|\int_{\Omega_1\setminus\Omega_0} h\overline{K_j} e^{-\phi_j}|^2 \leq \int_{\Omega_1\setminus\Omega_0} |h|^2 e^{-\phi_j} \|K_j\|^2_{\phi_j}$$

tends to zero. It follows that any weak limit k satisfies

$$h(z) = \int_{\Omega_0} h\bar{k} e^{-\phi}.$$

Since holomorphic functions in $L^2(\Omega_1, e^{-\phi_0})$ are dense in $L^2(\Omega_0, e^{-\phi_0})$, the same relation holds for any h in $L^2(\Omega_0, e^{-\phi_0})$. Since k is necessarily also holomorphic, $k = K$ and the limit is in fact uniform on compact subsets of Ω_0. In particular

$$\lim K_j(z) = K(z).$$

□

The proofs of the next two lemmas is similar but simpler and is therefore omitted.

Lemma 3.2. *Let Ω be a bounded domain and ϕ a plurisubharmonic weight function . Let Ω_j be an increasing family of subdomains with union equal to Ω. Let z be a fixed point in Ω_0 and let K_j and K be the Bergman kernels for Ω_j and Ω (with weight function ϕ) respectively. Then $K_j(z,z)$ decreases to $K(z,z)$.*

Lemma 3.3. *Let Ω be a bounded domain and ϕ_j a decreasing sequence of plurisubharmonic weight functions. Let z be a fixed point in Ω and let K_j and K be the Bergman kernels for the weight functions ϕ_j and ϕ respectively. Then $K_j(z,z)$ decreases to $K(z,z)$.*

To verify one of the hypotheses in Lemma 4.1 we need an approximation result.

Lemma 3.4. *Let Ω_0 and Ω_1 be smoothly bounded pseudoconvex domains in $\mathbb{C}^n$ with Ω_0 compactly included in Ω_1. Assume there is a smooth plurisubharmonic function ρ in $\bar{\Omega}_1$ such that $\Omega_0 = \{z \in \Omega_1, \rho(z) < 0\}$. Then holomorphic functions in $L^2(\Omega_1)$ are dense in the space of holomorphic functions in $L^2(\Omega_0)$.*

Proof. Let h be a square integrable holomorphic function in Ω_0. The crux of the proof is to approximate h by functions holomorphic in a neighbourhood of the set $X = \{\rho \leq 0\}$. This can be done by standard L^2-theory if 0 is a regular value of ρ so that the boundary of Ω_0 is smooth. In the non-smooth case, the possibility to approximate with function holomorphic near X follows from a result by Bruna and Burgues, cf Theorem B in [4].

Next, we let h be holomorphic near X and show how to approximate h with functions holomorphic in Ω_1. Let H be an arbitrary extension of h from a neighbourhood of X to a smooth function with compact support in Ω_1 and put $f = \bar{\partial} H$. Let $k_j(s)$ be a sequence of increasing convex functions that vanish for $s < 0$ and tend to infinity for $s > 0$ and set $\phi_j = k_j \circ \rho$. By Hörmander's theorem, [9], we can solve the equation $\bar{\partial} v_j = f$ with estimates in $L^2(\Omega_1, e^{-\phi_j})$. Since f is supported in the complement of Ω_0 it follows that v_j tends to zero in $L^2(\Omega_0)$. Hence $H - v_j$ is an approximating sequence. □

The final lemma gives the semicontinuity of K_t.

Lemma 3.5. *Let $D = \{(t, z); \rho(t, z) < 0\}$ where ρ is smooth and strictly plurisubharmonic near the closure of D and moreover has non-vanishing gradient on ∂D. Assume ϕ is smooth and plurisubharmonic near the closure of D. Then $K_t(z, z)$ is for fixed z upper semicontinuous as a function of t.*

Proof. Consider a point t and let s be nearby points tending to t. We may choose $\epsilon > 0$ so that all fibers D_s are contained in the open set V where $\rho(t, z) < \epsilon$. Note that the set-valued function $t \to D_t$ is lower semicontinuous, in the sense that if D_t contains a compact set K, the K is contained in all D_s for s sufficiently close to t. Let $K_s(\zeta, z)$ be the Bergman kernel of D_s for a fixed point z. Since the domains D_s all contain a fixed open neighbourhood of z the L^2-norms of K_s are bounded. Any sequence of K_s therefore has a subsequence weakly convergent on any compact subset of D_t. The L^2-norm of any weak limit k can not exceed the liminf of the L^2-norms of K_s over D_s. By the extremal characterization of Bergman kernels it follows that

$$\limsup K_s(z, z) \leq K_t(z, z),$$

so we are done. □

We can now complete the proof of Theorem 1.1, and start by proving that $\log K_t$ is plurisubharmonic in t for z fixed. We first assume that D is smoothly bounded, defined as

$$D = \{(t, z); \rho(t, z) < 0\}$$

where ρ is smooth and strictly plurisubharmonic near the closure of D. We also assume that ϕ is smooth and plurisubharmonic near the closure of D. Assume first $k = 1$ and fix a point t in $\mathbb{C}$, say $t = 0$. If U is a sufficiently small neighbourhood of 0 all the fibers D_t are contained in a fixed pseudoconvex domain $V = \{\rho(0,\zeta) < \epsilon\}$. In $U \times V$ we can compose ρ with an increasing sequence of smooth convex functions k_j that tend to infinity when ρ is positive. We can now apply the result from section 3 to $U \times V$ with ϕ replaced by $\phi_j = \phi + k_j \circ \rho$ and let j tend to infinity. Since the set where a smooth strictly subharmonic function equals zero has zero measure, ϕ_j tends to infinity a e in $\Omega_1 \setminus \Omega_0$ By Lemma 4.1 it follows that $\log K_t$ can be written as an increasing limit of functions subharmonic with respect to t. Since, by the last lemma, $\log K_t$ is also upper semicontinuous it follows that it is subharmonic. Again by the upper semicontinuity we get that $\log K_t$ is plurisubharmonic if $k \geq 1$ since its restriction to any line is subharmonic.

It is now easy to remove the extra hypothesis on D and ϕ. If D is an arbitrary pseudoconvex open set it has a smooth strictly plurisubharmonic exhaustion function, and so can be written as an increasing union of domains of the type satisfying the extra hypotheses. Near each such domain we can regularize ϕ by convolution. From lemmas 4.2 and 4.3 we get that $\log K_t$ is a decreasing limit of plurisubharmonic functions, and so is plurisubharmonic, or identically equal to minus infinity.

We have thus proved that, under the hypotheses of Theorem 1.1, $\log K_t$ is subharmonic as a function of t for z fixed. To see that it is plurisubharmonic in t and z jointly we use, as in [19], the Oka trick of variation of the domain. We need to prove that, for any choice of a in $\mathbb{C}^n$, the function

$$\log K_t(z + ta, z + ta)$$

is subharmonic in t. But, this is precisely the Bergman kernel at z for the domain

$$D_t - ta$$

with the weight function translated similarily. Since the translated domains are also pseudoconvex, and the translated weight function is still plurisubharmonic, it follows that $\log K_t(z + ta, z + ta)$ is subharmonic in t and we are done.

4. Subharmonic currents

We shall next give an alternate proof of Theorem 1.1 which is based on a representation of the Bergman kernel as the pushforward of a subharmonic form. To prepare for this we give in this section some general facts on subharmonic forms or currents.

Let T be a current of bidimension $(1,1)$, i e of bidegree (n,n) in $U \times \mathbb{C}^n$ where U is an open set in $\mathbb{C}$. We say that T is subharmonic if

$$i\partial\bar{\partial}T \geq 0.$$

Let π be the projection from $\mathbb{C}_t \times \mathbb{C}^n_z$ to $\mathbb{C}_t$. If T is compactly supported in the fiber direction, so that the support of T is included in $U \times K$ with K a compact subset of $\mathbb{C}^n$ the pushforward $\pi_*(T)$ of T to U is the distribution in U defined by

$$\pi_*(T).\chi = T.\pi^*\chi$$

for any smooth compactly supported $(1,1)$ form χ in U. Similarily, if T is a current of bidegree $(n+1, n+1)$ we define the pushforward of T by the same formula, but taking χ to be a function. Since

$$i\partial\bar{\partial}\pi_*(T) = \pi_*(i\partial\bar{\partial}T)$$

it is clear that $\pi_*(T)$ is subharmonic if T is a subharmonic current of bidegree (n,n).

If T is an (n,n)-differential form with, say, bounded coefficients, the pushforward of T is a function whose value at a point t equals

$$\int_{\{t\}\times\mathbb{C}^n_z} T.$$

Clearly, the pushforward only depends on the component of T of bidegree (n,n) in z. Conversely, let κ be a form of bidegree (n,n) in z, with coefficients depending on t. It follows from the above that to prove that the function

$$\int_{\{t\}\times\mathbb{C}^n_z} \kappa$$

is subharmonic it suffices to find a subharmonic form T of bidimension $(1,1)$ which is compactly supported in the fiber direction and whose component of bidegree (n,n) in z equals κ.

In order for this argument to work it is crucial that T be globally defined and compactly supported in the fiber direction (or at least satisfies integrablility conditions). The currents that we will encounter later are however only defined in some pseudoconvex domain. To get globally defined currents we extend by 0 in the complement of the pseudoconvex domain. This of course introduces a discontinuity which gives an extra contribution to take into account when computing $i\partial\bar{\partial}T$ in the sense of distributions. The local calculations needed are summarized in the following lemma, which is a variant of a by now standard method to prove L^2-estimates for the $\bar{\partial}$-equation, see [9] p 103 .

Lemma 4.1. *Let ρ be a smooth real valued function in an open set U in $\mathbb{C}^n$. Assume that $\partial\rho \neq 0$ on $S = \{z, \rho(z) = 0\}$, so that S is a smooth real hypersurface. Let T be a real differential form of bidimension $(1,1)$ defined where $\rho < 0$, with coefficients extending smoothly up to S. Assume*

$$\partial\rho \wedge T$$

vanishes on S, and that

$$\partial\rho \wedge \bar{\partial}\rho \wedge T$$

vansihes to second order on S. Extend T to a current $\tilde{T}$ in U by putting $\tilde{T} = 0$ where $\rho > 0$. Then

$$i\partial\bar{\partial}\tilde{T} = \chi_{\rho<0} \quad i\partial\bar{\partial}T + \frac{i\partial\bar{\partial}\rho \wedge T dS}{|\partial\rho|}, \tag{4.1}$$

where dS is surface measure on S and χ is a characteristic function.

In particular, even though it is not assumed that all of T, but only certain components of T, vanish on S, the contribution coming from the discontinuity is a measure, and not, as might be expected, a current of order 1.

Proof. The hypotheses on T mean that

$$\sum \rho_j T_{j\bar{k}} = \rho c_k, \tag{4.2}$$

where $\sum c_k \rho_{\bar{k}}$ vanishes on S. Therefore, on S,

$$0 = \sum \frac{\partial}{\partial \bar{z}_k}(\rho c_k) = \sum \rho_j \frac{\partial T_{j\bar{k}}}{\partial \bar{z}_k} + \sum \rho_{j\bar{k}} T_{j\bar{k}}$$

so

$$-\sum \rho_j \frac{\partial T_{j\bar{k}}}{\partial \bar{z}_k} = \sum \rho_{j\bar{k}} T_{j\bar{k}}. \tag{4.3}$$

Let w be a smooth function of compact support in U. Then, using the divergence theorem and writing $T_{j\bar{k}}$ for the components of T, we find that

$$\int_{\rho<0} i\partial\bar{\partial}w \wedge T = \int_{\rho<0} \sum w_{j\bar{k}} T_{j\bar{k}} =$$

$$= \int_{\rho=0} \sum \rho_j w_{\bar{k}} T_{j\bar{k}} dS/|\partial\rho| - \int_{\rho<0} \sum \frac{\partial w}{\partial z_j} \frac{\partial T_{j\bar{k}}}{\partial \bar{z}_k}.$$

By equation (4.2) the boundary integral vanishes. Applying the divergence theorem once more to the second integral we get

$$\int_{\rho<0} w \sum \frac{\partial^2 T_{j\bar{k}}}{\partial z_j \partial \bar{z}_k} - \int_{\rho=0} w \sum \rho_j \frac{\partial T_{j\bar{k}}}{\partial \bar{z}_k} dS/|\partial\rho|.$$

We then use (4.3) in the new boundary integral and find

$$\int_{\rho<0} i\partial\bar{\partial}w \wedge T = \int_{\rho<0} w i\partial\bar{\partial}T + \int_{\rho=0} \sum \rho_{j\bar{k}} T_{j\bar{k}} dS/|\partial\rho|.$$

This completes the proof of the lemma. □

5. Second proof of Theorem 1.1

Again, we first consider the situation described at the beginning of section 2. As before, our starting point is the fact that the function

$$u = \partial_t^\phi K_t$$

is for fixed t orthogonal to the space of holomorphic functions in A_t^2. We now put

$$k_t = K_t d\zeta_1 \wedge ... d\zeta_n,$$

so that k_t is K_t interpreted as an $(n,0)$-form and, slightly abusively, define

$$\partial_t^\phi k_t = \partial_t^\phi K_t \, d\zeta_1 \wedge ... d\zeta_n.$$

Since $\bar\partial$ has closed range, the orthogonal complement of the kernel of $\bar\partial$ equals the range of $\bar\partial^*$. Therefore $\partial_t^\phi k_t = \bar\partial^* \alpha$ for some form α (in ζ) of bidegree $(n,1)$ which can also be taken to be $\bar\partial$-closed (and is then uniquely determined). By an argument similar to Lemma 2.1, α depends smoothly on t. Write $\alpha = \sum \alpha_j d\bar\zeta_j \wedge d\zeta$. Since α lies in the domain of $\bar\partial^*$, α satisfies the $\bar\partial$-Neumann boundary condition $\sum \alpha_j \rho_j = 0$ on the boundary of V.

Put $\gamma = \sum \alpha_j d\hat{\zeta}_j$, where $d\hat{\zeta}_j$ stands for the wedge product of all $d\zeta_k$:s except $d\zeta_j$, with a sign so that

$$d\zeta_j \wedge d\hat{\zeta}_j = d\zeta_1 \wedge ... d\zeta_n.$$

For later reference we note that the $\bar\partial$-Neumann boundary condition on α translates to $\partial\rho \wedge \gamma = 0$ on ∂V. Put $g = dt \wedge \gamma + k_t$ and let $\partial^\phi = e^\phi \partial e^{-\phi}$ be a twisted ∂-operator. The equation

$$\partial_t^\phi k_t = \bar\partial^* \alpha$$

is equivalent to

$$\partial^\phi g = 0.$$

We claim that the form T defined as

$$T = c_n g \wedge \bar g e^{-\phi},$$

where c_n is a constant of modulus 1 chosen so that T is positive, for ζ in V and $T = 0$ for ζ outside of V is a subharmonic form. Since the component of T of bidegree (n,n) in ζ equals

$$\kappa_t = c_n k_t \wedge \bar k_t$$

it then follows that

$$K_t(z,z) = \int \kappa_t$$

is a subharmonic function of t.

To prove the subharmonicity of T we first compute $i\partial\bar\partial T$ for ζ inside of V. We use the product rule

$$\partial(a \wedge \bar b \, e^{-\phi}) = \partial^\phi a \wedge \bar b \, e^{-\phi} + (-1)^{\deg a} a \wedge \overline{\bar\partial b} \, e^{-\phi},$$

and a similar rule for applying $\bar\partial$. Remembering that $\partial^\phi g = 0$ we get

$$i\partial\bar\partial T = c_n i \partial^\phi \bar\partial g \wedge \bar g \, e^{-\phi} + c_n i \bar\partial g \wedge \overline{\bar\partial g} \, e^{-\phi}. \tag{5.1}$$

From the commutation rule

$$(\partial^\phi \bar\partial + \bar\partial \partial^\phi) g = \partial\bar\partial\phi \wedge g,$$

together with $\partial^\phi g = 0$ it follows that the first term on the right hand side can be written

$$i\partial\bar\partial\phi \wedge T.$$

This term is therefore nonnegative since ϕ is plurisubharmonic. To analyse the second term we introduce the notation ζ_0 for t and α_0 for $-K_t$, so that g can be written

$$-\sum_0^n \alpha_j d\hat{\zeta}_j.$$

The second term equals

$$\sum_{jk} \frac{\partial \alpha_j}{\partial \bar{\zeta}_k} \overline{\frac{\partial \alpha_k}{\partial \bar{\zeta}_j}}.$$

Here the indices run from 0 to n. Consider first the part of the sum where both indices are greater than 0. Since the form α is $\bar{\partial}$-closed for fixed t this part equals

$$\sum_1^n |\frac{\partial \alpha_j}{\partial \bar{\zeta}_k}|^2$$

multiplied by the volume form $d\lambda$. Evidently, the part of the sum where both indices are 0 equals

$$|\frac{\partial \alpha_0}{\partial \bar{\zeta}_0}|^2 d\lambda.$$

Finally, the terms in the sum when precisely one of the indices are 0 vanish since $\alpha_0 = K_t$ is a holomorphic function of ζ.

In conclusion, $i\partial\bar{\partial}T \geq 0$ for ζ in V. It now remains to compute the contribution to $i\partial\bar{\partial}T$ which comes from cutting off T outside of V. We apply Lemma 4.1 to our current $T = c_n g \wedge \bar{g} e^{-\phi}$ and ρ equal to the defining function of V. Then ρ is independent of $t = \zeta_0$, so

$$\partial\rho \wedge g = \partial\rho \wedge dt \wedge \gamma = 0$$

on $U \times \partial V$ since $\partial\rho \wedge \gamma = 0$ on ∂V. Hence the hypotheses of Lemma 4.1 are fulfilled. Since V is pseudoconvex it follows that $ic_n\partial\bar{\partial}\rho \wedge g \wedge \bar{g}$ is non-negative on ∂V so

$$i\partial\bar{\partial}T \geq 0.$$

In conclusion, T is a subharmonic current and it follows that K_t is a subharmonic function of t for z fixed. The rest of the proof of Theorem 1.1 runs as before.

6. Singularities of plurisubharmonic functions

We first recall the definitions and basic properties of Lelong numbers (our basic reference for these matters is [12]). If ϕ is a plurisubharmonic function in an open set U in $\mathbb{C}^n$ and a is a point in U, the *Lelong number* of ϕ at a is

$$\gamma(\phi, a) = \lim_{r \to 0} (\log r)^{-1} \sup_{|z-a|=r} \phi(z). \tag{6.1}$$

Equivalently (see [12] p 176) we may introduce the mean value of ϕ over the sphere centered at a with radius r, $M(\phi, a, r)$ and put

(6.2) $$\gamma(\phi, a) = \lim_{r\to 0} (\log r)^{-1} M(\phi, a, r).$$

The Lelong number measures the strength of the singularity of ϕ at a. If $\gamma(\phi, a) > \tau$ then

$$\phi(z) \leq \tau \log |z - a|$$

for z close to a.

In the one variable case we can decompose a subharmonic function locally as a sum of a harmonic part and a potential

$$p(z) = \int \log |z - \zeta| d\mu(\zeta)$$

where $\mu = 1/(2\pi)\Delta\phi$. It is easy to verify that the Lelong number is then equal to $\mu(\{a\})$. Using the potential it is also easy to see that, in the one variable case, the Lelong number at a is greater than or equal to one if and only if $e^{-2\phi}$ is not integrable over any neighbourhood of a.

In any dimension one defines $\iota(\phi, a)$, the *integrability index* of ϕ at a, as the infimum of all positive numbers t such that

$$e^{-2\phi/t}$$

is locally integrable in some neighbourhood of a. By a theorem of Skoda ([18]), the inequality

$$\iota(\phi, a) \leq \gamma(\phi, a) \leq n\iota(\phi, a)$$

holds in any dimension. The left inequality here (which is the hard part) says that if the Lelong number of ϕ at a is strictly smaller than 1, then $e^{-2\phi}$ is locally integrable near a.

Let Ω be a domain in $\mathbb{C}^n$ and let ϕ be a plurisubharmonic function in Ω. We consider the Bergman kernel $K(z, z)$ for $A^2(\Omega, \phi)$. It is clear that if a is a point Ω and $e^{-\phi}$ is not integrable in any neighbourhood of a, then any holomorphic function in $A^2(\Omega, \phi)$ must vanish at a, so in particular $K(a, a) = 0$. Conversely, if Ω is bounded and $e^{-\phi}$ is integrable in some neighbourhood of a then a standard application of Hörmander's L^2-estimates shows that there exists some function in $A^2(\Omega, \phi)$ which does not vanish at a. Since $K(a, a)$ equals the supremum of the modulus squared of all functions in $A^2(\Omega, \phi)$ of norm 1, it follows that $K(a, a) > 0$ in that case. Thus, at least if Ω is bounded, the set where $\log K = -\infty$ is precisely equal to the nonintegrability locus of $e^{-\phi}$.

For z in Ω and w in $\mathbb{C}^n$ we now consider the restriction of ϕ to the complex line through z determined by w

$$\phi_{z,w}(\lambda) = \phi(z + \lambda w).$$

For any fixed z in Ω $\phi_{z,w}$ is defined for λ in the unit disk, if w is small enough. Let $K_{z,w}(0, 0)$ be the Bergman kernel for the unit disk, with Lebesgue measure normalized so that the total area is one, equipped with the weight function $2\phi_{z,w}$. By the above, $K_{z,w}(0,0) = 0$ if and only if the Lelong number of $\phi_{z,w}$ at the origin is at least 1. By Theorem 1.1, $\log K_{z,w}$ is a plurisubharmonic function, so for fixed z the set of w where it equals $-\infty$ is either pluripolar or contains a neighbourhood of the origin. Thus, if the Lelong number at the origin of one single slice function is smaller than 1, it must be smaller than 1 for all slices outside a pluripolar set.

It follows that the Lelong numbers of all slices outside a pluripolar set are equal. This common value also equals the Lelong number of ϕ at z. To see this, first note that by the first definition of Lelong number in terms of supremum over spheres, it follows that the Lelong number for the restriction of ϕ to any line through z must be at least as big as the n-dimensional Lelong number at z. The converse inequality follows if we use the second definition of Lelong numbers in terms of mean values over spheres, and apply Fatou's lemma. To avoid the consideration of exceptional lines we now introduce the function

$$\phi_\epsilon(z) = \frac{1}{2}\int_{|w|=\epsilon} \log K_{z,w}(0,0) dS(w),$$

where the surface measure dS is normalized so that the sphere has total measure equal to 1.

Theorem 6.1. *The function ϕ_ϵ is well defined and plurisubharmonic in the open set Ω_ϵ of points of Ω whose distance to the boundary is greater than ϵ. The sequence ϕ_ϵ decreases to ϕ as ϵ decreases to 0. The singularity set S where $\phi_\epsilon = -\infty$ is for any $\epsilon > 0$ equal to the analytic set where the Lelong number of ϕ is at least 1. If the Lelong number of ϕ at z equals $\tau > 1$, the Lelong number of ϕ_ϵ at z is at least equal to $\tau - 1$.*

Proof. Since $\log K_{z,w}$ is subharmonic with respect to w t is clear that ϕ_ϵ decreases with ϵ to $\log K_{z,0}$. But $K_{z,0}$ is the Bergman kernel at the origin for a normalized disk with a constant weight, $e^{-2\phi(z)}$, and so equals $e^{2\phi(z)}$. Hence the limit of ϕ_ϵ is equal to ϕ. If the Lelong number of ϕ at z is smaller than 1 we have seen above that $\log K_{z,w}(0,0)$ is not identically equal to $-\infty$ so its mean value over a sphere, ϕ_ϵ is not equal to $-\infty$. On the other hand we have also seen above that if $\gamma(\phi, z) \geq 1$, then $\log K_{z,w} = -\infty$ for w in a full neighbourhod of 0, so $\phi_\epsilon(z) = -\infty$. Hence S is equal to the set where $\gamma(\phi, z) \geq 1$, which by Siu's analyticity theorem, [17], is analytic.

It remains only to prove the last statement of the theorem, so assume 0 lies in Ω and that $\gamma(\phi, 0) = \tau > 1$. Then, if $\tau' < \tau$,

$$e^{-\phi(z)} \geq 1/|z|^{\tau'}$$

if $|z|$ is small enough. For w fixed and $h(\lambda)$ holomorphic we get

$$\int_{|\lambda|<1} |h|^2 e^{-2\phi(z+\lambda w)} dm(\lambda) \geq \int_{|\lambda w|<|z|} |h|^2 e^{-2\phi(z+\lambda w)} dm(\lambda) \geq$$

$$\geq \int_{|\lambda|<1} |h|^2/(|2z|^{2\tau'}) dm(\lambda) \geq C|h(0)|^2/|z|^{2(\tau'-1)}.$$

Hence

$$K_{z,w}(0,0) \leq C_1|z|^{2(\tau'-1)}$$

where the constant can be taken uniform for all w of fixed modulus equal to $\epsilon > 0$. It follows that the Lelong number of ϕ_ϵ at z is at least $\tau' - 1$, and therefore at least $\tau - 1$ since τ' is an arbitrary number smaller than τ. $\square$

The function ϕ_ϵ thus "attenuates the singularities" of ϕ in much the same way as Kiselman's construction in [12]. (Kiselman even gets that the Lelong number of the constructed function *equals* $\tau - 1$.) In precisely the same way as in Kiselman, [11], this construction can be used to prove the Siu analyticity theorem. Let

$$E_\tau = \{z; \gamma(\phi, z) \geq \tau\}.$$

First, it follows from the Hörmander-Bombieri theorem that the nonintegrability locus of any plurisubharmonic function is always analytic. For a given plurisubharmonic function, ϕ, and $\delta > 0$ we put, for some choice of $\epsilon > 0$

$$\psi = 3n\phi_\epsilon/\delta.$$

By Theorem 6.1, ψ is finite at any point where $\gamma(\phi, z) < 1$, and therefore (see [9]) $e^{-\psi}$ is locally integrable near any such point. On the other hand e^{ψ} is not locally integrable near a point where $\gamma(\phi, z) \geq (1 + \delta)$ since the Lelong number of ψ at such a point is at least $3n$. Therefore we have, if Z denotes the nonintegrability locus of $e^{-\psi}$, that

$$E_{1+\delta} \subset Z \subset E_1.$$

Rescaling, we may of course for any $\tau > 0$ and $\delta > 0$ in a similar way find an analytic set $Z_{\tau,\delta}$ such that

$$E_\tau \subset Z_{\tau,\delta} \subset E_{\tau-\delta}.$$

Hence E_τ equals the intersection of the analytic sets $Z_{\tau,\delta}$ for $\delta > 0$ and is therefore analytic.

In a similar way we can consider, instead of restrictions of ϕ to lines, the restriction of ϕ to k-dimensional subspaces. This will give us a scale of "Lelong numbers" for $k = 1, ...n$ that starts with the classical Lelong number and ends with the integrability index.

We close this section by sketching an alternative way of relating Lelong numbers to Bergman kernels, leading up to Theorem 1.3 of the introduction. In [18] it is proved that if the Lelong number of ϕ at a is strictly smaller than 1, then $e^{-2\phi}$ is locally integrable in some neighbourhood of a. Actually, Skoda's proof of this fact gives a bit more, namely that

$$I(a) := \int_{|z-a|<\delta} e^{-2\phi(z)}/|z-a|^{2n-2} dm(z)$$

is also finite, if δ is small enough. (The same argument as in section 7 of [18] gives, with $d\sigma = \Delta\phi$ that

$$\int_{|z|<r} e^{-2\phi(z)}/|z|^{2n-2} \leq C \int_{|z|<r,|x|<R} |z|^{-2n-2}|z-x|^{-2n+\epsilon} d\sigma(x)dm(z),$$

which is finite since

$$\int d\sigma(x)/|x|^{2n-2-\epsilon}$$

is finite.) On the other hand, $I(a)$ is comparable to the average of

$$\int e^{-2\phi(a+\lambda w)} dm(\lambda)$$

over all w on a sphere, so $I(a)$ must be infinite if the Lelong number of ϕ at a is larger than or equal to 1. In conclusion

$$\{a; I(a) = \infty\} = \{a, \gamma(\phi, a) \geq 1\}.$$

We now introduce the plurisubharmonic function

$$\psi(z,a) = \phi(z) + (n-1)\log|z-a|$$

and let K_a be the Bergman kernel for Ω with weight $2\psi^a(z) = 2\psi(z,a)$. It then follows that $\chi(a) = \log K_a(a,a)$ is plurisubharmonic and equal to $-\infty$ precisely where $\gamma(\phi,\cdot) \geq 1$, so we have proved the first part of Theorem 1.3 from the introduction. The last part of Theorem 1.3 follows from an argument similar to the last part of the proof of Theorem 6.1.

7. PLURISUBHARMONICITY OF POTENTIALS.

In this section we shall prove a generalization of an earlier result of Yamaguchi on the Robin function. Let D be a smoothly bounded pseudoconvex set in $\mathbb{C}^k_t \times \mathbb{C}^n_\zeta$ and let as before D_t be the n-dimensional slices of D. In this section we assume D has a smooth defining function $\rho(t,\zeta)$ such that $\partial_\zeta \rho \neq 0$ on the boundary of D. In particular all the fiber domains are smoothly bounded and have the same topology.

Theorem 7.1. *Let K be a compact subset of $\mathbb{C}^n$ that is contained in D_t for all t in an open set U. Let μ be a positive measure with support in K. Let $u(t)$ be the negative of the energy of μ with respect to the Green function G_t of D_t*

$$u(t) = \int_{D_t} G_t(z,\zeta)d\mu(z)d\mu(\zeta).$$

Then u is plurisubharmonic in U.

Here we mean by the Green function the unique function vanishing on the boundary and satisfying that $\Delta_\zeta G$ is a unit point mass at z .

The Green function G of a domain Ω with pole at z can be written as the Newton kernel plus a smooth term

$$G(z,\zeta) = -\frac{c_n}{|z-\zeta|^{2n-2}} + \psi(z,\zeta)$$

where ψ is harmonic in z and in ζ. The function

$$\Lambda(z) = \psi(z, z)$$

is called the Robin function of the domain Ω. Let the measure μ in Theorem 7.1 be a uniform mass distribution on a small ball centered at the point z. u is then equal to (the negative of) the energy of μ with respect to the Newton kernel plus the Robin function at z of the domain D_t. Since the Newtonian energy is independent of t it follows that Λ is plurisubharmonic as a function of t. Just like in the case of Bergman kernels this implies that even $\log \Lambda$ is subharmonic, if $n > 1$. To see this, let a be some complex number and consider the domain $D(a)$ with fibers

$$D(a)_t = e^{at} D_t,$$

which, being a biholomorphic image if D, is still pseudoconvex. The Robin function of $D(a)$ equals $e^{-(2n-2)\mathrm{Re}\,(at)}\Lambda$, so these functions are subharmonic for any choice of a. It follows that $\log \Lambda$ is subharmonic if $2n - 2 \neq 0$, i e if $n > 1$. Finally, we can again apply the Oka technique of variation of the domain (cf the end of section 3) to conclude that if Λ is the Robin function of a fixed domain Ω, $\log \Lambda(z)$ is plurisubharmonic as a function of z in Ω.

Proof. We consider the Green function G_t of D_t and let $g(t, z) = g_t(z)$ be the Green potential of μ in D_t. We may assume that μ is given by a smooth density and it is then not hard to see that g is smooth up to the boundary in D. Let β be the standard Euclidean Kähler form in $\mathbb{C}^n$ and set

$$T = i\partial g \wedge \bar\partial g \wedge \beta_{n-1}$$

in D and $T = 0$ outside of D. (Here we use the notation $\omega_p = \omega^p/p!$ for $(1, 1)$-forms ω.) Notice that T is a nonnegative form and that $\pi_*(T)$ is given by

$$\pi_*(T)(t) = \int_{D_t} |\partial g_t|^2 = -\int_{D_t} \Delta g_t\, g_t = -u(t),$$

where, as in Theorem 7.1, u is the energy of μ. Since g vanishes on the boundary of D, T satisfies the hypotheses of Lemma 4.1. By Lemma 4.1

$$i\partial\bar\partial T \geq \chi_D i\partial\bar\partial T \tag{7.1}$$

if D is pseudoconvex. In D we get

$$i\partial\bar\partial T = -(i\partial\bar\partial g)^2 \wedge \beta_{n-1}. \tag{7.2}$$

Write

$$i\partial\bar\partial g = i\partial_z\bar\partial_z g + i\partial_t\bar\partial_t g + i\partial_t\bar\partial_z g + i\partial_z\bar\partial_t g.$$

Hence

$$(i\partial\bar\partial g)^2 \wedge \beta_{n-1} =$$

$$= 2\mathrm{Re}\,(i\partial_t\bar\partial_t g \wedge i\partial_z\bar\partial_z g) \wedge \beta_{n-1} + 2\mathrm{Re}\,(i\partial_t\bar\partial_z g \wedge i\partial_z\bar\partial_t g) \wedge \beta_{n-1} =$$

$$= (2\Delta_t g \Delta_z g - 2\sum |\frac{\partial^2 g}{\partial z_j \partial \bar t}|^2) d\lambda.$$

We thus find

$$-i\partial\bar\partial u = \pi_*(i\partial\bar\partial T) \geq$$

$$\geq 2\left(\int_{D_t} -\Delta_t g\,\Delta_z g + 2\int_{D_t}\sum |\frac{\partial^2 g}{\partial z_j \partial \bar t}|^2\right) idt\wedge d\bar t \geq$$

$$\geq 2(-\Delta_t \int_{D_t} g_t d\mu) idt\wedge d\bar t = -2i\partial\bar\partial u.$$

(Notice that we may move the Laplacian with respect to t outside the integral sign since μ is independent of t and compactly supported inside D_t.) Thus $i\partial\bar\partial u \geq 0$, so u is subharmonic and the proof of Theorem 7.1 is complete. □

Notice that the statement in Theorem 7.1 may be generalized to Green functions for other elliptic equations, besides the Euclidean Laplacian (see also Yamaguchi and Levenberg [13]). First, we may replace the Euclidean metric by an arbitrary Kähler metric, with Kähler form ω, on $\mathbb{C}^n$, and consider the Laplacian with respect to this metric. The same proof as above applies if we only replace the Euclidean Kähler form β by ω. We may even go one step further and consider elliptic operators of the form

$$Lu = i\partial\bar\partial u \wedge \Omega$$

where Ω is a closed positive form of bidegree $(n-1, n-1)$.

It is also worth pointing out that the assumption on pseudoconvexity in Theorem 7.1 can be relaxed. In the proof, convexity properties of the boundary of D only intervene in the application of Lemma 4.1, to conclude that the form

$$F = i\partial\bar\partial\rho \wedge i\partial g \wedge \bar\partial g \wedge \beta_{n-1}$$

is nonnegative on the boundary of D. Therefore we may replace the hypothesis of pseudoconvexity in Theorem 7.1 by the hypothesis "$F \geq 0$". This is of course rather implicit, but to get an idea of how that condition relates to pseudoconvexity we can consider domains D in $\mathbb{C}\times\mathbb{C}^n$ of a special form. Let us assume e g that the slices D_t only depend on $\mathrm{Re}\, t$ and form an increasing family with respect to $\mathrm{Re}\, t$, so that they are defined by inequalities

$$D_t = \{z; v(z) < \mathrm{Re}\, t\}.$$

When checking the positivity of the form F above one may then replace both ρ and g by $r = v - \mathrm{Re}\, t$, since ρ and g are positive multiples of r. We then see that, whereas the pseudoconvexity of D is equivalent to v being *plurisubharmonic*, F is positive if and only if v is *subharmonic*. In particular this is a condition that also makes sense in $\mathbb{R}^n$. In the next section we shall briefly discuss analogs of the formalism of the last four sections in $\mathbb{R}^n$.

8. CONVEXITY PROPERTIES OF FIBER INTEGRALS IN $\mathbb{R}^n$

We consider $\mathbb{R}^{n+1}$ with the coordinates $(x_0, ...x_n)$. When κ is a function with compact support or satisfying suitable integrability conditions, we want to study convexity properties of the fiber integral

$$\Phi(t) = \int_{x_0=t} \kappa dx_1...dx_n = \int_{x_0=t} \kappa.$$

Just like in section 4 we shall arrange things so that

$$\kappa = T_{00}$$

where (T_{jk}) is a matrix of functions. The basic fact of section 4, that the operation of pushforward of a form commutes with the $i\partial\bar\partial$-operator, is now replaced by the following lemma.

Lemma 8.1. *Let $T = (T_{jk})$ be a matrix of L^∞ functions in $\mathbb{R}^{n+1}$. Suppose that for some $R > 0$, T vanishes when $|(x_1, ...x_n)| > R$. Put*

$$\Phi(t) = \int_{x_0=t} T_{00}.$$

If T is smooth then

$$\Phi''(t) = \int_{x_0=t} \sum_0^n \frac{\partial^2 T_{jk}}{\partial x_j \partial x_k}.$$

If T is not smooth the same formula holds in the sense of distributions, if the right hand side is interpreted as the distribution, S, whose action on a test function α is

$$S.\alpha = \int_{\mathbb{R}^{n+1}} \sum_0^n \frac{\partial^2 T_{jk}}{\partial x_j \partial x_k} \alpha(x_0).$$

Proof. If T is smooth the first formula is clear since the integral of any term involving a derivative with respect to a variable different from x_0 vanishes. Hence

$$\int_{x_0=t} \sum_0^n \frac{\partial^2 T_{jk}}{\partial x_j \partial x_k} = \int_{x_0=t} \frac{\partial^2 T_{00}}{\partial x_0 \partial x_0} = \Phi''(t).$$

The non-smooth case follows from the definition of distributional derivatives. □

It is clear that the lemma holds even if T does not necessarily have compact support. It suffices that first and second order derivatives of the coefficients of T are integrable. Later on we will also have use for a generalization of Lemma 4.1

Lemma 8.2. *Let $T = (T_{jk})$ be a matrix of functions that are smooth up to the boundary in a smoothly bounded domain $\Omega = \{\rho < 0\}$ in $\mathbb{R}^N$. Assume that*

$$\sum_j T_{jk}\rho_j = O(\rho)$$

and

$$\sum_j T_{jk}\rho_j\rho_k = O((\rho)^2)$$

at the boundary of D. Extend the definition of T to a matrix $\tilde{T}$ in all of $\mathbb{R}^N$ by putting $\tilde{T}$ equal to 0 in the complement of D. Then we have in the sense of distributions

$$\sum \frac{\partial^2 \tilde{T}_{jk}}{\partial x_j \partial x_k} = \chi_D \sum \frac{\partial^2 T_{jk}}{\partial x_j \partial x_k} + \sum T_{jk}\rho_{jk}\frac{dS}{|d\rho|}.$$

The proof is essentially the same as the proof of lemma 4.1. Let us now consider in particular matrices of the form

$$T_{jk} = \gamma_j\gamma_k e^{-\phi}.$$

To compute derivatives of T_{jk} we use the notation

$$d_j = \partial/\partial x_j$$

and

$$d_j^\phi = e^\phi d_j e^{-\phi}.$$

We get

(8.1)

$$\frac{\partial^2 T_{jk}}{\partial x_j \partial x_k} = \left(d_k\gamma_j d_j\gamma_k + d_j^\phi d_k(\gamma_j)\gamma_k + d_j^\phi\gamma_j d_k^\phi\gamma_k + \gamma_j d_j d_k^\phi\gamma_k\right)e^{-\phi} =$$

$$= \left(d_k\gamma_j d_j\gamma_k + d_k d_j^\phi(\gamma_j)\gamma_k + d_j^\phi\gamma_j d_k^\phi\gamma_k + \gamma_j d_j d_k^\phi\gamma_k + \phi_{jk}\gamma_j\gamma_k\right)e^{-\phi}$$

(where in the last line we have used the commutation relation

$$d_j^\phi d_k = d_k d_j^\phi + \phi_{jk}.)$$

It follows that if we assume

$$\sum d_k^\phi \gamma_k = 0$$

then

$$\sum_0^n \frac{\partial^2 T_{jk}}{\partial x_j \partial x_k} = \left(\sum d_k\gamma_j d_j\gamma_k + |\sum d_k^\phi\gamma_k|^2 + \phi_{jk}\gamma_j\gamma_k\right)e^{-\phi}.$$

This identity can be used exactly as in section 5 to prove the real Prekopa theorem. Let ϕ be a convex function and put

$$k(t) = \gamma_0(t) = \left(\int_{x_0=t} e^{-\phi}\right)^{-1}.$$

Since

$$\int_{x_0=t} \gamma_0(x_0)e^{-\phi} = 1$$

it follows that

$$\int_{x_0=t} d_0^\phi(\gamma_0)e^{-\phi} = 0$$

for any t. This implies that we can solve

$$d_0^\phi \gamma_0 = -\sum_1^n d_j^\phi \gamma_j e^{-\phi}$$

and

$$d_j \gamma_k = d_k \gamma_j$$

with γ and its first derivatives going rapidly to zero at infinity (this is easiest to see for $n = 1$, which is all that is needed for the Prekopa theorem). If we then define $T_{jk} = \gamma_j \gamma_k e^{-\phi}$ as above it follows by Lemma 8.1 that

$$k''(t) = \frac{d^2}{dt^2} \int_{x_0=t} k^2(x_0) e^{-\phi} = \int_{x_0=t} \left(\sum d_k \gamma_j d_j \gamma_k + \phi_{jk} \gamma_j \gamma_k \right) e^{-\phi}.$$

But, since γ_0 only depends on x_0, it follows just as in the complex case that

$$\sum d_k \gamma_j d_j \gamma_k = |d_0 \gamma_0|^2 + \sum_1^n |d_k \gamma_j|^2 \geq 0.$$

Hence $k(t)$ is convex and it follows just like in section 3 that even $\log c$ is convex, since replacing ϕ by $\phi + a x_0$ we see that $k(t) e^{at}$ is convex for any choice of a.

In the same way we can adapt the argument of section 7 to prove convexity of Green potentials (and hence the Robin function, see also [5] who prove a stronger convexity property of the Robin function), but in that case it is a little bit less evident what the choice of T should be. To explain this we shall first discuss a general notion of subharmonic form in $\mathbb{R}^n$.

Consider the space, F, of differential forms on $\mathbb{R}_x^N \times \mathbb{R}_y^N$ whose coefficients depend only on x. The usual exterior derivative, d, preserves this space of forms. We introduce a new exterior derivative, $d^\#$ on F as

$$d^\# = \sum dy_j \wedge \partial / \partial x_j,$$

where the partial derivative acts on the coefficients of a form (this operator and the space F are not invariantly defined under changes of coordinates). If we introduce the operator τ on F by letting it change dx_j to dy_j and vice versa, then $d^\# = \tau d \tau$ and it is clear that $(d^\#)^2 = 0$. We say that a form in F is of bidegree (p, q) if its respective degrees in dx and dy are p and q. A (p, p)-form $\eta = \sum \eta_{IJ} dx_I \wedge dy_J$ is symmetric if $\eta_{IJ} = \eta_{JI}$, or equivalently $\tau \eta = (-1)^p \eta$. Put

$$\omega = \sum dx_j \wedge dy_j.$$

A form of bidegree (N, N) is *positive* if it is a nonnegative multiple of $\omega_N := \omega^N / N!$, and a general symmetric form, η, of bidegree (p, p) is positive if

$$a_1 \wedge \tau a_1 \wedge ... a_{N-p} \wedge \tau a_{N-p} \wedge \eta$$

is positive for any choice of forms a_j of bidegree $(1, 0)$. It is not hard to check that a form

$$\sum a_{ij} dx_i \wedge dy_j$$

is positive if and only if the matrix (a_{ij}) is positively semidefinite. A smooth function ϕ is therefore convex precisely when $dd^{\#}\phi$ is a positive form. It also follows that a positive $(1,1)$-form can be written as a sum of forms of the type

$$a \wedge \tau a,$$

with a of type $(1,0)$. Therefore the wedge product of a positive form with a positive $(1,1)$-form is again positive. Similarily if we define dv_{jk} as the wedge product of all differentials except dx_j and dy_k, ordered so that $dx_j \wedge dy_k \wedge dv_{jk} = \omega_N$, then

$$\mu = \sum a_{jk} dv_{jk}$$

is also positive exactly when (a_{jk}) is nonnegative as a matrix. A form $T = \sum T_{jk} dv_{jk}$ in F of bidegree $(N-1, N-1)$ is *subharmonic* if

$$dd^{\#}T = \sum_0^n \frac{\partial^2 T_{jk}}{\partial x_j \partial x_k} \omega_N$$

is positive.

With these definitions it is clear that to apply Lemma 8.1 to prove convexity of fiber integrals, we must look for subharmonic forms of bidegree (n,n) in $\mathbb{R}^{n+1}$. Let D be a smoothly bounded domain in $\mathbb{R}^{n+1}$ defined by an inequality $D = \{\rho < 0\}$ where the gradient of ρ does not vanish on the boundary of D and let D_t be the n-dimensional slices. We say that such a domain satisfies condition (C) if

$$d\rho \wedge d^{\#}\rho \wedge dd^{\#}\rho \wedge \omega'_{n-1}$$

is positive for x on the boundary of D. This condition is clearly satisfied if D is convex and it also holds if the fibers D_t are of the form

$$D_t = \{x' = (x_1, ...x_n); v(x') < x_0\}$$

where v is subharmonic. As in the case of Theorem 7.1 we assume that the gradient of ρ with respect to x' is never 0 for x_0 in an open set U , so that all the slices are smoothly bounded and have the same topology. Let G_t be the Green function of D_t. We then have.

Theorem 8.3. *Assume that D satisfies condition (C). Let K be a compact subset of $\mathbb{R}^n$ that is contained in D_t for all t in U. Let μ be a positive measure with support in K. Let $u(t)$ be the negative of the energy of μ with respect to the Green function G_t of D_t*

$$u(t) = \int_{D_t} G_t(x,\xi) d\mu(x) d\mu(\xi).$$

Then u is convex in U.

The proof of Theorem 8.3 runs in much the same way as the proof of Theorem 7.1. Let g_t be the Green potential of μ in D_t and put

$$g(x_0, x') = g_{x_0}(x').$$

Let

$$\omega' = \sum_1^n dx_j \wedge dy_j,$$

and put

$$T = dg \wedge d^{\#} g \wedge \omega'_{n-1} = \sum_0^n T_{jk} dv_{jk},$$

for x in D, and $T = 0$ outside of D. Then $T_{00} = |dg_{x_0}|^2$, so that

$$\int_{x_0=t} T_{00} = -u(t).$$

By Lemma 4.1 the contribution we get from the discontinuity at the boundary of D when we compute $dd^{\#}T$ equals

$$dd^{\#}\rho \wedge T dS/|d\rho|.$$

If D satisfies condition (C), this expression is nonnegative (since dg is a positive multiple of $d\rho$ at the boundary of D). By Lemma 8.2 we therefore have (using $d\omega' = d^{\#}\omega' = 0$) that

$$\sum \frac{\partial^2 T_{jk}}{\partial x_j \partial x_k} \omega_{n+1} = dd^{\#} T \geq -(dd^{\#} g)^2 \wedge \omega'_{n-1}.$$

Applying Lemma 8.1 we now get as in the complex case

$$-u''(t) \geq \int_{x_0=t} \sum_1^n |\frac{\partial^2 g}{\partial x_j \partial x_0}|^2 - 2u''(t),$$

and it follows that u is convex.

Let us finally consider the implications of Theorem 8.3 for the Robin function. Again as in section 7 we take μ to be a positive measure of total mass 1 which is given by a constant density on a small ball centered at a fixed point x that we assume to be contained in all the fibers D_t, for t in some open set U. The energy integral $u(t)$ then equals $\Lambda_t(x) - c$ where Λ is the Robin function for D_t and c is a constant. It follows that the Robin function is a convex function of t if D satisfies condition (C). Moreover, Λ is strictly convex at any point t such that the expression

$$dd^{\#}\rho \wedge T$$

is strictly positive at some point of the boundary of D_t. Consider now the situation when all the fibers are translates of one fixed domain Ω in $\mathbb{R}^n$

$$D_t = \Omega + ta$$

with a a fixed direction in $\mathbb{R}^n$. Then $\rho(x_0, x) = r(x - x_0 a)$ where r is a defining function for Ω. It follows from the Hopf lemma that dg is a strictly positive multiple of $d\rho$ at the boundary of D, so

$$dd^{\#}\rho \wedge T$$

is a strictly positive multiple of

$$\nu = dd^{\#}\rho \wedge d\rho \wedge d^{\#}\rho \wedge \omega'_{n-1}.$$

To check the positivity of this $(n+1, n+1)$-form, we pull it back under the map $(x_0, x, y_0, y) \to (x_0, x - x_0 a, y_0, y - y_0 a)$. It is then not hard to see ν is positive for any choice of a if ρ is convex and that moreover ν is strictly positive at any point where the Hessian of r restricted to the null space of dr is strictly positive. If Ω is smoothly bounded and convex there will always be at least some such point at the boundary and the Robin function is therefore strictly convex. We have therefore proved a special case of a result from [5]:

Theorem 8.4. *Let Ω be a smoothly bounded convex domain in $\mathbb{R}^n$ and let Λ be the Robin function of Ω. Then Λ is strictly convex.*

In [5] a stronger convexity of the Robin function is proved (namely the *harmonic radius*, $\Lambda^{-1/(n-2)}$, is strongly concave), but Theorem 8.4 is already sufficient to prove the unicity of the *harmonic center* of Ω, i e the point where Λ attains its minimum.

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