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*PREPRINT 2005:19*

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Göteborg Sweden 2005



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Göteborg, May 2005

Preprint 2005:19  
ISSN 1652-9715

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Matematiskt centrum  
Göteborg 2005

# CURVATURE OF VECTOR BUNDLES AND SUBHARMONICITY OF BERGMAN KERNELS.

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ABSTRACT. In a previous paper, [1], we have studied a property of subharmonic dependence on a parameter of Bergman kernels for a family of weighted  $L^2$ -spaces of holomorphic functions. Here we prove a result on the curvature of a vector bundle defined by this family of  $L^2$ -spaces itself, which has the earlier results on Bergman kernels as a corollary. Applying the same arguments to spaces of holomorphic sections to line bundles over a locally trivial fibration we also prove that if a holomorphic vector bundle,  $V$ , over a complex manifold is ample in the sense of Hartshorne, then  $V \otimes \det V$  has an Hermitian metric with curvature strictly positive in the sense of Nakano.

## 1. INTRODUCTION

Let us consider a domain  $D = U \times \Omega$  and a function  $\phi$  plurisubharmonic in  $D$ . For the moment we also assume for simplicity that  $\phi$  is smooth up to the boundary and strictly plurisubharmonic in  $D$ . Then, for each  $t$  in  $U$ ,  $\phi^t(\cdot) := \phi(t, \cdot)$  is plurisubharmonic in  $\Omega$  and we denote by  $A_t^2$  the Bergman spaces of holomorphic functions in  $\Omega$  with norm

$$\|h\|^2 = \|h\|_t^2 = \int_{\Omega} |h|^2 e^{-\phi^t}.$$

The spaces  $A_t^2$  are then all equal as vector spaces but have norms that vary with  $t$ . The - infinite rank - vector bundle  $E$  over  $U$  with fiber  $E_t = A_t^2$  is therefore trivial as a bundle but is equipped with a nontrivial metric. The main result of this paper is the following theorem.

**Theorem 1.1.** *The hermitian bundle  $(E, \|\cdot\|_t)$  is strictly positive in the sense of Nakano.*

Of the two main differential geometric notions of positivity (see section 2, where these matters will be reviewed in the slightly non standard setting of bundles of infinite rank), positivity in the sense of Nakano is the stronger one and implies the weaker property of positivity in the sense of Griffiths. On the other hand the Griffiths notion of positivity has nicer functorial properties and implies in particular that the dual bundle is negative (in the sense of Griffiths). This latter property is in turn equivalent to the condition that if  $\xi$  is any nonvanishing local holomorphic section to the dual bundle, then the function

$$\log \|\xi\|^2$$

is strictly plurisubharmonic. In our case, we can obtain such holomorphic sections to the dual bundle from point evaluations. More precisely, let  $f$  be a holomorphic map from  $U$  to  $\Omega$  and define  $\xi_t$  by its action on a local section to  $E$

$$\langle \xi_t, h_t \rangle = h_t(f(t)).$$

Since the right hand side here is a holomorphic function of  $t$ ,  $\xi$  is indeed a holomorphic section to  $E^*$ . The norm of  $\xi$  at a point is given by

$$\|\xi_t\|^2 = \sup_{\|h_t\| \leq 1} |h_t(f(t))| = K_t(f(t), f(t)),$$

where  $K_t(z, z)$  is the Bergman kernel function for  $A_t^2$ . It therefore follows from Theorem 1.1 that  $K_t(z, z)$  is plurisubharmonic in  $D$ , which is essentially the main result of [1].

The proof of Theorem 1.1 is based on a formula of Griffiths, see e.g [7], for the curvature of a subbundle,  $E$ , of a holomorphic bundle,  $F$ . If  $\Theta^E$  and  $\Theta^F$  denote the respective curvature operators we have, for any sections  $u$  and  $v$  of the subbundle  $E$  that

$$(\Theta_{jk}^F u, v) = (\Theta_{jk}^E u, v) + (\pi_{E^\perp} D_j^F u, \pi_{E^\perp} D_k^F v).$$

In this formula, the last term equals the *second fundamental form* of  $E$  acting on  $u$  and  $v$ , and is here expressed by the Chern connection of  $F$  acting on sections of  $E$ , projected on the orthogonal complement of  $E$  in  $F$ .

We shall apply this formula with  $E$  being the bundle introduced above and  $F$  the bundle of all  $L^2$  functions on  $\Omega$  equipped with the same norm. The curvature of  $F$  is easily seen to be the operator of multiplication with the Hessian of  $\phi$  with respect to  $t$ . This is therefore a positive operator, and to prove Theorem 1.1, we must control the second fundamental form with this operator. For this we note that the second fundamental form is, by definition, given by elements in the orthogonal complement of  $A_t^2$ . These elements are the  $L^2$ -minimal solutions of certain  $\bar{\partial}$ -equations. An estimate for these  $L^2$ -minimal solutions is furnished by Hörmander's  $L^2$ -estimate for the  $\bar{\partial}$ -equation. The basic idea behind the proof comes from a proof of *Prekopa's theorem* by Brascamp and Lieb, and is explained more closely in [1].

There is also a natural analog of Theorem 1.1 for locally trivial fibrations. We consider a complex manifold  $X$  which is fibered over another complex manifold  $Y$ . We then have a holomorphic map,  $p$ , from  $X$  to  $Y$  with surjective differential, and all the fibers  $X_t = p^{-1}(t)$  are diffeomorphic. We shall even assume that the fibration is locally trivial holomorphically, so that every point in the base has a neighbourhood,  $U$ , such that  $p^{-1}(U)$  is biholomorphic to  $U \times Z$ , where  $Z$  is a fixed complex manifold. Moreover, under this biholomorphism, the projection  $p$  goes over into the natural projection from  $U \times Z$  to  $U$ . Let  $L$  be a positive line bundle over  $X$  and assume that under the local trivializations discussed above,  $L$  restricted to

$p^{-1}(U)$  is isomorphic to the pullback of a line bundle on  $Z$  under the natural projection from  $U \times Z$  to  $Z$ . For each  $t$  in  $Y$  we can now consider the space

$$\tilde{E}_t = \Gamma(X_t, L|X_t)$$

of global holomorphic sections to  $L$  over  $X_t$ . By the local triviality of the fibration, and the assumption on  $L$ , the vector spaces  $\tilde{E}_t$  define a holomorphic vector bundle over  $Y$ . We would now like to define a hermitian norm on the bundle  $\tilde{E}$  by taking the  $L^2$ -norm over each fiber, but we can not do so directly since we have no canonically defined measure on the fibers to integrate against. We therefore consider instead the bundle  $E$  with fibers

$$E_t = \Gamma(X_t, L|X_t \otimes K_{X_t}),$$

where  $K_{X_t}$  is the canonical bundle of each fiber. Elements of  $E_t$  can be naturally integrated over the fiber and we obtain in this way a metric,  $\|\cdot\|$  on  $E$  in complete analogy with the plane case. We then get the same conclusion as before:

**Theorem 1.2.** *( $E, \|\cdot\|$ ) is positive in the sense of Nakano.*

One example of this situation that arises naturally is obtained if we start with a (finite rank) holomorphic vector bundle  $V$  over  $Y$  and let  $\mathbb{P}(V)$  be the associated bundle of projective spaces of the dual bundle  $V^*$ . This is then clearly a locally trivial holomorphic fibration as before and a line bundle  $L$  satisfying the conditions we have discussed is obtained by taking

$$L = O_{\mathbb{P}(V)}(1),$$

the hyperplane section bundle over each fiber. The global holomorphic sections of this bundle over each fiber are now the linear forms on  $V^*$ , i.e. the elements of  $V$ . In other words,  $\tilde{E}$  is isomorphic to  $V$ . As explained above, we are not able to produce a metric on  $\tilde{E}$  by integrating over the fibers, so instead we take as  $L$

$$L = O_{\mathbb{P}(V)}(r+1)$$

(with  $r$  being the rank of  $V$ ) and define  $E$  as before

$$E = \Gamma(X_t, L|X_t \otimes K_{X_t}).$$

One can then verify that  $E$  is isomorphic to  $V \otimes \det V$ . The condition that  $L$  is positive is now equivalent to  $O_{\mathbb{P}(V)}(1)$  being positive which is the same as saying that  $V$  is *ample* in the sense of Hartshorne, [8]. We therefore obtain the following result as a corollary of Theorem 1.2.

**Theorem 1.3.** *Let  $V$  be a (finite rank) holomorphic vector bundle over a complex manifold which is ample in the sense of Hartshorne. Then  $V \otimes \det V$  has a smooth hermitian metric which is strictly positive in the sense of Nakano.*

It is a well known conjecture of Griffiths, [6], that an ample vector bundle is positive in the sense of Griffiths. Theorem 1.3 would follow from this conjecture, since by a theorem of Demailly, [5],  $V \otimes \det V$  is Nakano

positive if  $V$  itself is Griffiths positive. It seems however that not so much is known about Griffiths' conjecture in general, except that it does hold when  $Y$  is a compact curve (see [10], [3]).

Finally, I would like to thank Sebastien Boucksom for pointing out the relation between Theorem 1.1 and the Griffiths conjecture.

## 2. CURVATURE OF FINITE AND INFINITE RANK BUNDLES

Let  $E$  be a holomorphic vector bundle with a hermitian metric over a complex manifold  $Y$ . By definition this means that there is a holomorphic projection map  $p$  from  $E$  to  $Y$  and that every point in  $Y$  has a neighbourhood  $U$  such that  $p^{-1}(U)$  is isomorphic to  $U \times W$ , where  $W$  is a vector space equipped with a smoothly varying hermitian metric. In our applications it is important to be able to allow this vector space to have infinite dimension, in which case we assume that the metrics are also complete, so that the fibers are Hilbert spaces.

Let  $t = (t_1, \dots, t_m)$  be a system of local coordinates on  $Y$ . The Chern connection,  $D_{t_j}$  is now given by a collection of differential operators acting on smooth sections to  $U \times W$  and satisfying

$$\partial_{t_j}(u, v) = (D_{t_j}u, v) + (u, \bar{\partial}_{t_j}v),$$

with  $\partial_{t_j} = \partial/\partial t_j$  and  $\bar{\partial}_{t_j} = \partial/\partial \bar{t}_j$ . The curvature of the Chern connection is a  $(1, 1)$ -form of operators

$$\Theta = \sum \Theta_{jk} dt_j \wedge d\bar{t}_k,$$

where the coefficients  $\Theta_{jk}$  are densely defined operators on  $W$ . By definition these coefficients are the commutators

$$\Theta_{jk} = [D_{t_j}, \bar{\partial}_{t_k}].$$

The vector bundle is said to be positive in the sense of Griffiths if for any section  $u$  to  $W$  and any vector  $v$  in  $\mathbb{C}^m$

$$\sum (\Theta_{jk}u, u)v_j \bar{v}_k \geq \delta \|u\|^2 |v|^2$$

for some positive  $\delta$ .  $E$  is said to be positive in the sense of Nakano if for any  $m$ -tuple  $(u_1, \dots, u_m)$  of sections to  $W$

$$\sum (\Theta_{jk}u_j, u_k) \geq \delta \sum \|u_j\|^2$$

Taking  $u_j = uv_j$  we see that Nakano positivity implies positivity in the sense of Griffiths.

The dual bundle of  $E$  is the vector bundle  $E^*$  whose fiber at a point  $t$  in  $Y$  is the Hilbert space dual of  $E_t$ . There is therefore a natural antilinear isometry between  $E$  and  $E^*$ , which we will denote by  $J$ . If  $u$  is a local section to  $E$ ,  $\xi$  is a local section to  $E^*$ , and  $\langle \cdot, \cdot \rangle$  denotes the pairing between  $E^*$  and  $E$  we have

$$\langle \xi, u \rangle = (u, J\xi).$$



Under the natural holomorphic structure on  $E^*$  we then have

$$\bar{\partial}_{t_j} \xi = J^{-1} D_{t_j} J \xi,$$

and the Chern connection on  $E^*$  is given by

$$D_{t_j}^* \xi = J^{-1} \bar{\partial}_{t_j} J \xi.$$

It follows that

$$\bar{\partial}_{t_j} \langle \xi, u \rangle = \langle \bar{\partial}_{t_j} \xi, u \rangle + \langle \xi, \bar{\partial}_{t_j} u \rangle,$$

and

$$\partial_{t_j} \langle \xi, u \rangle = \langle D_{t_j}^* \xi, u \rangle + \langle \xi, D_{t_j} u \rangle,$$

and hence

$$0 = [\partial_{t_j}, \bar{\partial}_{t_j}] \langle \xi, u \rangle = \langle \Theta_{jk}^* \xi, u \rangle + \langle \xi, \Theta_{jk} u \rangle,$$

if we let  $\Theta^*$  be the curvature of  $E^*$ . If  $\xi_j$  is an  $r$ -tuple of sections to  $E^*$ , and  $u_j = J \xi_j$ , we thus see that

$$\sum (\Theta_{jk}^* \xi_j, \xi_k) = - \sum (\Theta_{jk} u_k, u_j).$$

Notice that the order between  $u_k$  and  $u_j$  in the right hand side is opposite to the order between the  $\xi$ s in the left hand side. Therefore  $E^*$  is negative in the sense of Griffiths iff  $E$  is positive in the sense of Griffiths, but we can not draw the same conclusion in the the case of Nakano positivity.

If  $u$  is a holomorphic section to  $E$  we also find that

$$\frac{\partial^2}{\partial t_j \partial \bar{t}_k} (u, u) = (D_{t_j} u, D_{t_k} u) - (\Theta_{jk} u, u)$$

and it follows after a short computation that  $E$  is (strictly) negative in the sense of Griffiths if and only if  $\log \|u\|^2$  is (strictly) plurisubharmonic for any nonvanishing holomorphic section  $u$ .

We next briefly recapitulate the Griffiths formula for the curvature of a subbundle. Assume  $E$  is a holomorphic subbundle of the bundle  $F$ , and let  $\pi$  be the fiberwise orthogonal projection from  $F$  to  $E$ . We also let  $\pi_\perp$  be the orthogonal projection on the orthogonal complement of  $E$ . By the definition of Chern connection we have

$$D^E = \pi D^F.$$

Let  $\bar{\partial}_{t_j} \pi$  be defined by

$$(2.1) \quad \bar{\partial}_{t_j} (\pi u) = (\bar{\partial}_{t_j} \pi) u + \pi (\bar{\partial}_{t_j} u).$$

Computing the commutators occuring in the definition of curvature we see that

$$(2.2) \quad \Theta_{jk}^E u = -(\bar{\partial}_{t_k} \pi) D_{t_j}^F u + \pi \Theta_{jk}^F u,$$

if  $u$  is a section to  $E$ . By (2.1)  $(\bar{\partial} \pi) v = 0$  if  $v$  is a section to  $E$ , so

$$(2.3) \quad (\bar{\partial} \pi) D^F u = (\bar{\partial} \pi) \pi_\perp D^F u.$$

Since  $\pi\pi_\perp = 0$  it also follows that

$$(\bar{\partial}\pi)\pi_\perp D^F u = -\pi\bar{\partial}(\pi_\perp D^E u),$$

so if  $v$  is also a section to  $E$ ,

$$\begin{aligned} ((\bar{\partial}_{t_k}\pi)D_{t_j}^F u, v) &= -(\bar{\partial}_{t_k}(\pi_\perp D_{t_j}^F u), v) = \\ &= ((\pi_\perp D_{t_j}^F u), D_{t_k}^F v) = (\pi_\perp(D_{t_j}^F u), \pi_\perp(D_{t_k}^F v)). \end{aligned}$$

Combining with (2.2) we finally get that if  $u$  and  $v$  are both sections to  $E$  then

$$(2.4) \quad (\Theta_{jk}^F u, v) = (\pi_\perp(D_{t_j}^F u), \pi_\perp(D_{t_k}^F v)) + (\Theta_{jk}^E u, v),$$

which is the starting point for the proof in the next section.

### 3. THE PROOF OF THEOREM 1.1

We consider the set up described before the statement of Theorem 1.1 in the introduction. Thus  $E$  is the vector bundle over  $U$  whose fibers are the Bergman spaces  $A_t^2$  equipped with the weighted  $L^2$  metrics induced by  $L^2(\Omega, e^{-\phi^t})$ . We also let  $F$  be the vector bundle with fiber  $L^2(\Omega, e^{-\phi^t})$ , so that  $E$  is a trivial subbundle of the trivial bundle  $F$  with a metric induced from a nontrivial metric on  $F$ . From the definition of the Chern connection we see that

$$D_{t_j}^F = \partial_{t_j} - \phi_j,$$

where the last term in the right hand side should be interpreted as the operator of multiplication by the (smooth) function  $-\phi_j = -\partial_{t_j}\phi^t$ . (In the sequel we use the letters  $j, k$  for indices of the  $t$ -variables, and the letters  $\lambda, \mu$  for indices of the  $z$ -variables.) For the curvature of  $F$  we therefore get

$$\Theta_{jk}^F = \phi_{jk},$$

the operator of multiplication with the Hessian of  $\phi$  with respect to the  $t$ -variables. We shall now apply formula (2.4), so let  $u_j$  be smooth sections to  $E$ . This means that  $u_j$  are functions that depend smoothly on  $t$  and holomorphically on  $z$ . To verify the positivity of  $E$  in the sense of Nakano we need to estimate from below the curvature of  $E$  acting on the  $k$ -tuple  $u$ ,

$$\sum (\Theta_{jk}^E u_j, u_k).$$

By (2.4) this means that we need to estimate from above

$$\sum (\pi_\perp(\phi_j u_j), \pi_\perp(\phi_k u_k)) = \|\pi_\perp(\sum \phi_j u_j)\|^2.$$

Put  $w = \pi_\perp(\sum \phi_j u_j)$ . For fixed  $t$ ,  $w$  solves the  $\bar{\partial}_z$ -equation

$$\bar{\partial}w = \sum u_j \phi_{j\lambda} d\bar{z}_\lambda,$$

since the  $u_j$ s are holomorphic in  $z$ . Moreover, since  $w$  lies in the orthogonal complement of  $A^2$ ,  $w$  is the minimal solution to this equation.

We shall next apply Hörmander's weighted  $L^2$ -estimates for the  $\bar{\partial}$ -equation. The precise form of these estimates that we need says that if  $f$  is a  $\bar{\partial}$ -closed form in a pseudoconvex domain  $\Omega$ , and if  $\psi$  is a smooth strictly plurisubharmonic weight function, then the minimal solution  $w$  to the equation  $\bar{\partial}v = f$  satisfies

$$\int_{\Omega} |w|^2 e^{-\psi} \leq \int_{\Omega} \sum \psi^{\lambda\mu} f_{\lambda} \bar{f}_{\mu} e^{-\psi},$$

where  $(\psi^{\lambda\mu})$  is the inverse of the complex Hessian of  $\psi$  (see [4]).

In our case this means that

$$\int_{\Omega} |w|^2 e^{-\phi^t} \leq \int_{\Omega} \sum \phi^{\lambda\mu} \phi_{j\lambda} u_j \bar{\phi}_{k\mu} u_k e^{-\phi^t}.$$

Inserting this estimate in formula (2.4) together with the formula for the curvature of  $F$  we find

$$(3.1) \quad \sum (\Theta_{jk}^E u_j, u_k) \geq \int_{\Omega} \sum_{jk} \left( \phi_{jk} - \sum_{\lambda\mu} \phi^{\lambda\mu} \phi_{j\lambda} \bar{\phi}_{k\mu} \right) u_j \bar{u}_k e^{-\phi^t}.$$

We claim that the expression

$$D_{jk} =: \sum_{jk} \left( \phi_{jk} - \sum_{\lambda\mu} \phi^{\lambda\mu} \phi_{j\lambda} \bar{\phi}_{k\mu} \right),$$

in the integrand is a positive definite matrix at any fixed point. By a linear change of variables in  $t$  we may of course assume that the vector  $u$  that  $D$  acts on equals  $(1, 0, \dots, 0)$ . Let  $\Phi = i\partial\bar{\partial}\phi$  where the  $\partial\bar{\partial}$ -operator acts on  $t_1$  and the  $z$ -variables, the remaining  $t$ -variables being fixed. Then

$$\Phi = \Phi_{11} + i\alpha \wedge d\bar{t}_1 + idt_1 \wedge \bar{\alpha} + \Phi',$$

where  $\Phi_{11}$  is of bidegree  $(1, 1)$  in  $t_1$ ,  $\alpha$  is of bidegree  $(1, 0)$  in  $z$ , and  $\Phi'$  is of bidegree  $(1, 1)$  in  $z$ . Then

$$\Phi_{m+1} = \Phi^{m+1}/(m+1)! = \Phi_{1,1} \wedge \Phi'_m - i\alpha \wedge \bar{\alpha} \wedge \Phi'_{m-1} \wedge idt_1 \wedge d\bar{t}_1.$$

Both sides of this equation are forms of maximal degree that can be written as certain coefficients multiplied by the Euclidean volume form of  $\mathbb{C}^{m+1}$ . The coefficient of the left hand side is the hessian of  $\phi$  with respect to  $t_1$  and  $z$  together. Similarly, the coefficient of the first term on the right hand side is  $\phi_{11}$  times the hessian of  $\phi$  with respect to the  $z$ -variables only. Finally, the coefficient of the last term on the right hand side is the norm of the  $(0, 1)$  form in  $z$

$$\bar{\partial}_z \partial_{t_1} \phi$$

measured in the metric defined by  $\Phi'$ , multiplied by the volume form of the same metric. Dividing by the coefficient of  $\Phi'_m$  we thus see that the matrix  $D$  acting on a vector  $u$  as above equals the hessian of  $\phi$  with respect to  $t_1$  and  $z$  divided by the hessian of  $\phi$  with respect to the  $z$ -variables only. This expression is therefore positive so the proof of Theorem 1.1 is complete.

#### 4. LOCALLY TRIVIAL FIBRATIONS

The proof of Theorem 1.2 is essentially the same as the proof in the preceding paragraph. As the statement is local we may assume that the total space  $X$  is a product  $X = U \times Z$  where  $U$  is open in  $\mathbb{C}^k$  and  $Z$  is a compact complex manifold of dimension  $m$ . Over  $X$  we have a holomorphic line bundle  $L$  which is the pull back of a bundle on  $Z$ . The line bundle  $L$  is given a metric with strictly positive curvature on  $X$ . Slightly abusively, we will denote by  $K_Z$  the pull back of the canonical bundle (i.e. the bundle of  $(m, 0)$ -forms) on  $Z$  to  $X$  and also the restriction of this bundle to each fiber  $X_t = \{t\} \times Z$ . On each fiber there is a natural pairing of sections to  $L \otimes K_Z$  with values in the space of forms of maximal degree on the fiber. Locally, for a decomposable section,  $u = s \otimes dz$  it is given by

$$[u, u] = c_k |s|^2 dz \wedge d\bar{z},$$

where  $c_k$  is a constant of modulus 1 chosen so that the expression is always nonnegative and  $|s|$  is the norm given by the metric on  $L$ . For each  $t$  in  $U$  this induces a metric on the space of smooth section to  $L \otimes K_Z$  over  $X_t$ ,

$$\|u\|^2 = \int_{X_t} [u, u].$$

We let  $F$  be the (infinite rank) vector bundle over  $U$  whose fiber over a point  $t$  in  $U$  is the  $L^2$ -space defined by this metric. Similarly  $E$  is the (finite rank) vector bundle of holomorphic sections. Note that, while  $E$  naturally extends as a bundle over all of  $Y$ , there seems to be no canonical way of extending the definition of  $F$  globally. Since our computation are local, this is however not needed. Let  $\theta$  be the connection form of the Chern connection on  $L$  over  $X$ . Locally, in terms of a local trivialisation where the metric is given by a weight function  $\phi$ ,  $\theta = -\partial\phi$ . The Chern connection on the bundle  $F$  is now

$$D_{t_j}^F = \partial_{t_j} + \theta_{t_j},$$

where  $\theta_{t_j}$  is the coefficient of  $dt_j$  in  $\theta$ . For the curvature of  $F$  we get

$$\Theta_{jk}^F = c_{jk},$$

the operator of multiplication with the  $t$ -part of the curvature of  $L$ . The proof of Theorem 1.2 now follows the same lines as the proof of Theorem 1.1, using the Kodaira-Nakano-Hörmander estimate for line bundles over compact manifolds, see [4].

#### 5. BUNDLES OF PROJECTIVE SPACES

Let  $V$  be a holomorphic line bundle of finite rank  $r$  over a complex manifold  $Y$ , and let  $V^*$  be its dual bundle. We let  $\mathbb{P}(V)$  be the fiber bundle over  $Y$  whose fiber at each point  $t$  of the base is the projective space of lines in  $V_t^*$ ,  $\mathbb{P}(V_t^*)$ . Then  $\mathbb{P}(V)$  is a holomorphically locally trivial fibration. There is a naturally defined line bundle  $O_{\mathbb{P}(V)}(1)$  over  $\mathbb{P}(V)$  whose restriction to any fiber  $\mathbb{P}(V_t^*)$  is the hyperplane section bundle. One way to define this

bundle is to first consider the tautological line bundle  $O_{\mathbb{P}(V)}(-1)$ . The total space of this line bundle is just the total space of  $V^*$  with the zero section removed, and the projection to  $\mathbb{P}(V)$  is the map that sends a nonzero point in  $V_t^*$  to its image in  $\mathbb{P}(V_t^*)$ . The bundle  $O_{\mathbb{P}(V)}(1)$  is then defined as the dual of  $O_{\mathbb{P}(V)}(-1)$ . The global holomorphic sections of this bundle over any fiber are in one to one correspondence with the linear forms on  $V_t^*$ , i.e. the elements of  $V$ . More generally,  $O_{\mathbb{P}(V)}(1)^l = O_{\mathbb{P}(V)}(l)$  has as global holomorphic sections over each fiber the homogeneous polynomials on  $V_t^*$  of degree  $l$ , i.e. the elements of the  $l$ :th symmetric power of  $V$ . We shall apply Theorem 1.2 to the line bundles

$$L(l) =: O_{\mathbb{P}(V)}(l).$$

Let  $E(l)$  be the vector bundle whose fiber over a point  $t$  in  $Y$  is the space of global holomorphic sections of  $L(l) \otimes K_{\mathbb{P}(V_t^*)}$ . If  $l < r$  there is only the zero section, so we assume from now on that  $l$  is greater than or equal to  $r$ .

We claim that

$$E(r) = \det V,$$

the determinant bundle of  $V$ . To see this, note that  $L(r) \otimes K_{\mathbb{P}(V_t^*)}$  is trivial on each fiber, since the canonical bundle of  $(r-1)$ -dimensional projective space is  $O(-r)$ . The space of global sections is therefore one dimensional. A convenient basis element is

$$\sum_1^r z_j \widehat{dz}_j,$$

if  $z_j$  are coordinates on  $V_t^*$ . Here  $\widehat{dz}_j$  is the wedge product of all differentials  $dz_k$  except  $dz_j$  with a sign chosen so that  $dz_j \wedge \widehat{dz}_j = dz_1 \wedge \dots \wedge dz_r$ . If we make a linear change of coordinates on  $V_t^*$ , this basis element gets multiplied with the determinant of the matrix giving the change of coordinates, so the bundle of sections must transform as the determinant of  $V$ . Since

$$L(r+1) \otimes K_{\mathbb{P}(V_t^*)} = O_{\mathbb{P}(V)}(1) \otimes L(r) \otimes K_{\mathbb{P}(V_t^*)},$$

it also follows that

$$E(r+1) = V \otimes \det V.$$

In the same way

$$E(r+m) = S^m(V) \otimes \det V,$$

where  $S^m(V)$  is the  $m$ th symmetric power of  $V$ .

Let us now assume that  $V$  is ample in the sense of Hartshorne, see [8]. By a theorem of Hartshorne, [8],  $V$  is ample if and only if  $L(1)$  is ample, i.e. has a metric with strictly positive curvature. Theorem 1.2 then implies that the  $L^2$ -metric on each of the bundles  $E(r+m)$  for  $m \geq 0$  has curvature which is strictly positive in the sense of Nakano, so we obtain:

**Theorem 5.1.** *Let  $V$  be a vector bundle (of finite rank) over a complex manifold. Assume  $V$  is ample in the sense of Hartshorne. Then for any  $m \geq 0$  the bundle*

$$S^m(V) \otimes \det V$$

*has an hermitian metric with curvature which is (strictly) positive in the sense of Nakano.*

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