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Rational Solutions of CYBE for Simple Compact Real Lie Algebras

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Abstract: In [8,9,10] a theory of rational solutions of the classical Yang-Baxter equation for a simple complex Lie algebra $g$ was presented. We discuss this theory for simple compact real Lie algebras $g$. We prove that all rational solutions have the form

$$X(u, v) = \frac{\Omega}{u - v} + r(u, v),$$

where $r(u, v)$ is a polynomial with coefficients in $g \otimes g$ and $\Omega$ denotes the quadratic Casimir element of $g$. This new type of solutions, which will also be called rational, looks somehow different from those in the Belavin-Drinfeld approach. However, as it turned out in [2], any solution of this type can be transformed into one which depends only on $u - v$, by means of a change of variables and a holomorphic transformation.

In [8,9,10] a correspondence was established between rational solutions of the form (1.1) and so-called {	extit{outer}}s in $g((u^{-1}))$. i.e., subalgebras $W$ of $g((u^{-1}))$ which satisfy the condition

$$u^{-N} g[[u^{-1}]] \subseteq W \subseteq u^{-N} g[[u^{-1}]]$$

for some non-negative integers $N_1$ and $N_2$. The study of rational solutions is essentially based on this correspondence and the description of the maximal orders.

In this present paper, we follow the method developed in [8,9,10] to study rational solutions of the CYBE for a simple compact Lie algebra $g$ over $\mathbb{R}$. We establish a similar correspondence between solutions and orders and are interested in the description of the maximal orders. We obtain that there is only one maximal order, the trivial one. Therefore all rational solutions will have the form

$$X(u, v) = \frac{\Omega}{u - v} + r,$$

where $r \in g \otimes g$ is a constant matrix.

On the other hand, any such $r$ induces a subalgebra $L$ of $g$ together with a non-degenerate $2\times 2$-cycle $B \in Z^2(L, R)$. A subalgebra $L$ for which there exists a non-degenerate $B$ is called quasi Frobenius. Conversely, any pair $(L, B)$ induces a skew symmetric constant $r$ matrix. We prove that any quasi-Frobenius subalgebra of a compact simple Lie algebra is commutative. Consequently, up to gauge equivalence, any rational solution has the form

$$X(u, v) = \frac{\Omega}{u - v} + t_1 \wedge t_2 + \ldots + t_{2n-1} \wedge t_{2n},$$

where $t_1, \ldots, t_{2n}$ are linearly independent elements in a maximal torus of $g$.

Finally we discuss the quantization of the Lie bialgebra structures corresponding to solutions of the form (1.4). The quantization is obtained by twisting the real Yangian $Y_{\mathbb{R}}(g)$.

2. Rational solutions and orders

Let $g$ denote a simple compact Lie algebra over $\mathbb{R}$ and $U(g)$ its universal enveloping algebra. Let $[\ , \ ]$ be the usual Lie bracket on the associative algebra $U(g)^{op}$.

We recall the following notation [1]: $\varphi_{12}, \varphi_{13}, \varphi_{23}: g \otimes g \rightarrow U(g)^{op}$ are the linear maps respectively defined by $\varphi_{12}(a \otimes b) = a \otimes b \otimes 1$. 
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\( \varphi_{12}(a \otimes b) = a \otimes 1 \otimes b, \varphi_{23}(a \otimes b) = 1 \otimes a \otimes b \) and \( \varphi_{23}(a \otimes b) = b \otimes a \otimes 1 \),

for any \( a, b \in g \).

For a function \( X : \mathbb{R}^2 \to g \otimes g \), we consider \( X^{12} : \mathbb{R}^2 \to U(g) \) defined by \( X^{12}(u, v) = \varphi_{12}(X(u, v)) \).

**Definition 2.1.** [\( \oplus \)] A solution of the classical Yang-Baxter equation (CYBE) is a function \( X : \mathbb{R}^2 \to g \otimes g \) such that the following conditions are satisfied:

\[
\begin{align*}
[X^{12}(u, u_2), X^{13}(u_1, u_3)] + [X^{13}(u_1, u_2), X^{23}(u_2, u_3)] + \\
+ [X^{12}(u_1, u_3), X^{23}(u_2, u_3)] &= 0
\end{align*}
\]

**Definition 2.2.** A solution of the CYBE is called rational if it is of the form

\[
X(u, v) = \frac{\Omega}{u-v} + r(u, v),
\]

where \( r(u, v) \) is a polynomial with coefficients in \( g \otimes g \).

**Remark 2.3.** The simplest example of a rational solution is Yang's r-matrix: \( X_0(u, v) = \frac{\Omega_{uv}}{u-v} \). By adding to \( X_0(u, v) \) any skew-symmetric constant r-matrix, we also obtain a rational solution.

We will consider rational solutions up to a certain equivalence relation.

**Definition 2.4.** Two rational solutions \( X_1 \) and \( X_2 \) are said to be gauge equivalent if there exists \( \sigma(u) \in Aut(g[u]) \) such that

\[
X_1(u, v) = (\sigma(u) \otimes \sigma(v))X_2(u, v).
\]

**Remark 2.5.** One can check that gauge transformations applied to rational solutions also give rational solutions.

Let \( \mathbb{R}[[u]] \) be the ring of formal power series in \( u^{-1} \) and \( \mathbb{R}[[u^{-1}]] \) its field of quotients. Set \( g[[u]] = g \otimes_{\mathbb{R}} \mathbb{R}[[u]], g[[u^{-1}]] = g \otimes_{\mathbb{R}} \mathbb{R}[[u^{-1}]] \) and

\[
g((u^{-1})) := g \otimes_{\mathbb{R}} \mathbb{R}((u^{-1})).
\]

There exists a nondegenerate ad-invariant bilinear form on \( g((u^{-1})) \) given by

\[
(2.5) \quad (x(u), y(u)) = \text{Res}_{v \to u} \text{ad} x(v) \cdot \text{ad} y(u).
\]

In [8, Th. 1] a correspondence between rational solutions and a special class of subalgebras of \( g((u^{-1})) \) was presented. The same result holds when \( g \) is real compact.

**Theorem 2.6.** Let \( g \) be a simple compact Lie algebra over \( \mathbb{R} \), there is a natural one-to-one correspondence between rational solutions of the CYBE and subalgebras \( W \leq g((u^{-1})) \) such that

\[
(1) \quad W \supseteq u^{-N}g[[u^{-1}]] \quad \text{for some } N > 0;
\]

\[
(2) \quad W \supseteq g[[u^{-1}]];
\]

\[
(3) \quad W \text{ is a Lagrangian subspace with respect to the bilinear form on } g((u^{-1})) \text{ given by } (2.5), \text{i.e. } W = W^\perp.
\]

**Proof.** We briefly sketch the proof which is similar to that in the complex case. Let \( V := g[[u]] \). Then \( V^* = u^{-1}g[[u^{-1}]] \). If \( f \in V^* \) and \( x \in V \) then \( f(x) := (f, x) \), where \( (, ) \) is the bilinear form given by \( (2.5) \).

Denote by \( Hom_{cont}(V^*, V) \) the set of all maps \( F : V^* \to V \) such that \( Ker(F) \supseteq u^{-N}V \) for some \( N > 0 \).

There exists an isomorphism \( \Phi : V \otimes V \to Hom_{cont}(V^*, V) \) defined by

\[
\Phi(x \otimes y)(f) = f(y)x,
\]

for any \( x, y \in V \) and \( f \in V^* \). The inverse map is given by

\[
\Phi^{-1}(F) = -\sum_{n=1}^{N} \sum_{k=0}^{\infty} F(I_{a_k}^{-k}) \otimes I_{a_k},
\]

for any \( F \in Hom_{cont}(V^*, V) \). We make the remark that \( F(I_{a_k}^{-k}) = 0 \) for \( k \geq N \) so that the sum which appears in \( (2.7) \) is finite.

There is a natural bijection between \( Hom_{cont}(V^*, V) \) and the set of all subspaces \( W \) of \( g((u^{-1})) \) which are complementary to \( V \) and such that \( W \supseteq u^{-N}V \supseteq u^{-N}g[[u^{-1}]] \) for some \( N > 0 \). Indeed, for any \( F \in Hom_{cont}(V^*, V) \), we consider the following subspace of \( g((u^{-1})) \)

\[
W(F) := \{f + F(f) : f \in V^* \}
\]

which satisfies the required properties.

The inverse mapping associates to any \( W \) the linear function \( F_W \) such that for any \( f \in V^* \), \( F_W(f) = -x \), uniquely defined by the decomposition \( f = w + x \) with \( w \in W \) and \( x \in V \).

One can easily see that \( W(\Phi(v)) \) is Lagrangian with respect to the bilinear form \( (\cdot, \cdot) \) if and only if \( r(u, v) = -r^2(u, v) \). Consequently,
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Ω/(u − v) + r(u, v) satisfies the unitarity condition (2.2) if and only if
W(Ψ(r)) is Lagrangian subspace.

Finally, if Ω/(u − v) + r(u, v) is a solution of (2.1)-(2.2), then
(2.9) ([f + Ψ(r)(f), g + Ψ(r)(g)] + Ψ(r)(h)) = 0
for any elements f, g, h in V*. Because W(Ψ(r)) is Lagrangian, (2.9) implies that W(Ψ(r)) is a subalgebra of g((u−1)).

Remark 2.7. One can easily see that if W is contained in g[[u−1]] and satisfies the above properties, then the corresponding rational solution has the form X(u, v) = Ω/(u − v) + r, where r is a constant polynomial.

Definition 2.8. An R-subalgebra W ⊆ g((u−1)) is called an order in g((u−1)) if there exist two non-negative integers N1, N2 such that
(2.10) u−N1g[[u−1]] ⊆ W ⊆ u−N2g[[u−1]].

Obviously g((u−1)) is an order.

Remark 2.9. Let W satisfy conditions (1) and (3) of Theorem 2.6. Then W is an order.

Concerning gauge equivalence, the result of Theorem 2 in [8] remains true:

Theorem 2.10. Let g be simple compact Lie algebra over R. Let X1 and X2 be rational of the CYBE and W1, W2 the corresponding orders in g((u−1)). Let σ(u) ∈ Aut(g[[u]]). Then the following conditions are equivalent:
(1) X1(u, v) = (σ(u) ⊗ σ(u))X2(u, v);
(2) W1 = σ(u)W2.

Definition 2.11. Let V1 and V2 be subalgebras of g((u−1)). We say that V1 and V2 are gauge equivalent if there exists σ(u) ∈ Aut(g[[u]]) such that V1 = σ(u)V2.

3. MAXIMAL ORDERS FOR COMPACT LIE ALGEBRAS

We will prove the following result:

Theorem 3.1. Let g be a simple compact Lie algebra over R. Then any order W in g((u−1)) is gauge equivalent to an order contained in g[[u−1]].

Proof. Let G be a connected compact Lie group whose Lie algebra is g. Then G is embedded into SL(n, C) via any irreducible complex representation. Without any loss of generality, we may suppose the image of a maximal torus T of G is included into the diagonal torus H of SL(n, C),

Let W denote an order of g((u−1)). Since we have the following sequence of embeddings
(3.1) W ⊆ W ⊆ sl(n, C)((u−1)) ⊆ sl(n, C)((u−1)),
we may view any w ∈ W as a matrix in sl(n, C)((u−1)).

Let us prove that for each w ∈ W, the exponential exp(w) defined formally by
(3.2) \exp(w) := \sum_{k=0}^{\infty} \frac{w^k}{k!}

makes sense as an element of SL(n, C)((u−1)).

Without any loss of generality, we may suppose that W is an R[[u−1]]-module of finite rank. We set O := C[[u−1]] and consider the O-module
(3.3) M := O^n + W\cdot O^n + \ldots + W^n\cdot O^n + \ldots.

Let us show that there exists some integer l such that
(3.4) M ⊆ u^lO^n.

If x1, ..., xk is a basis of the R[[u−1]]-module W, then obviously
(3.5) M ⊆ \sum_{k=0}^{\infty} x_1^{a_1} \cdots x_k^{a_k} O^n.

It is well-known that the field K := C((u−1)) may be endowed with the discrete valuation v(\sum_{k=0}^{\infty} a_k u^{-k}) = N. For any f ∈ K, we consider its norm:
(3.6) |f| = 2^{-v(f)}.

On the other hand, one can define a norm on g((u, K)) which is compatible with the norm of K. Given a matrix A of g((u, K)), one sets
(3.7) |A| = 2^{v(g)}
where g := \inf k such that AG ⊆ u^kO^n.

This norm satisfies the properties:
|A| \cdot |A| ≤ |A|, \quad |f(u) \cdot A| = |f(u)||A|, \quad |A_1 + A_2| ≤ \sup |A_1|, |A_2|.

We make the remark that, since W is an order, there exists N ≥ 0 such that |w| ≤ 2N for all w ∈ W.

In order to prove (3.4), it is enough to show that
(3.8) \sup_{(a_1, ..., a_k)} |x_1^{a_1} \cdots x_k^{a_k}| < \infty.

This means that for each 1 ≤ i ≤ r there exists a positive integer M_i such that
(3.9) \sup_{k} |x_i^k| ≤ M_i.
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It suffices to prove that the norms of the eigenvalues of $x_i$ for the action of $x_i$ on $\mathbb{K}^n$ are less or equal to 1, indeed, if this happens, then the coefficients of the characteristic polynomial of $x_i$ have norm less or equal to 1. Because all powers $x_i^k$ are linear combinations of $x_1, x_2$ (where $k$ is the degree of the characteristic polynomial of $x_1$), it follows that
\begin{equation}
|x_i^k| \leq \sup \{1, |x_1|, \ldots, |x_n|\}
\end{equation}
and thus (3.9) will be fulfilled.

Let $w$ be an arbitrary element of $W$. Let $\varepsilon_1(w), \ldots, \varepsilon_n(w)$ be the eigenvalues of $w$ for the action of $w$ on $\mathbb{K}^n$. We will show that $|\varepsilon_i(w)| \leq 1$ for all $i$. Without any loss of generality, we may suppose that $w$ is a diagonalizable element. Consider the eigenvalues $\alpha_1(w), \ldots, \alpha_n(w)$ for the action of $w$ on $(g \otimes \mathbb{R}) \otimes_c \mathbb{C}((u^{-1}))$. Some of them are zero and some behave as roots. For any $\alpha_j(w)$ there exists a corresponding eigenvector which belongs to $W$. Since $W \otimes \mathbb{R}$ is an $\mathbb{A}$-module of finite type, it follows that $|\alpha_j(w)| \leq 1$ for all $j$. On the other hand, because the weights of a representation are linear combinations of simple roots, we have that $\varepsilon_1(w), \ldots, \varepsilon_n(w)$ are linear combinations of some $\alpha_j(w)$ with rational coefficients. This implies that $|\varepsilon_i(w)| \leq 1$ for all $i$.

Thus (3.9) holds and this implies (3.4). Since (3.4) holds for some integer $l$, $\exp(w)$ belongs to $SL(n, \mathbb{C}((u^{-1})))$, for any $w \in W$. We denote by $S$ the connected subgroup generated by $\exp(w)$ for all $w \in W$. Its Lie algebra is $W$.

Recall that $G$ is embedded into $SL(n, \mathbb{C})$ such that the image of a maximal torus $T$ of $G$ is contained in a maximal torus $H$ of $SL(n, \mathbb{C})$, Let $T$ be the affine Bruhat-Tits building associated to $G(\mathbb{R}((u^{-1})))$ and the valuation $v$. Let $T'$ be the affine Bruhat-Tits building associated to $SL(n, \mathbb{C}((u^{-1})))$ and the valuation $v$. According to [4, p. 202-204] there exists an embedding
\begin{equation}
T \hookrightarrow T'
\end{equation}
which is compatible with the preceding embedding $G \hookrightarrow SL(n, \mathbb{C})$.

Since $H'$ is contained in $g((u^{-1}))$, one has that
\begin{equation}
S \subseteq G(\mathbb{R}((u^{-1}))) \hookrightarrow SL(n, \mathbb{C}((u^{-1}))).
\end{equation}

The module $M$ given by (3.3) satisfies the property $SM \subseteq M$, Since $\mathbb{O}^n \subseteq M \subseteq \mathbb{O}^n$, it follows that $S\mathbb{O}^n \subseteq S\mathbb{O}^n$. Therefore $S$ must be a bounded subgroup of $SL(n, \mathbb{C}((u^{-1})))$, i.e., there is an upper bound on the absolute values of the matrix entries of the elements of $S$.

According to [3, p. 161], $S$ is bounded in the sense of Bruhat-Tits homology for the building $T'$ (see [3, p. 161]), Because the embedding $T \hookrightarrow T'$ is compatible with the building metric, it follows that $S$ is a bounded subgroup of $G(\mathbb{R}((u^{-1})))$, in the sense of Bruhat-Tits homology corresponding to the building $T$.

Now the Bruhat-Tits fixed point theorem ([3, p. 157, 161]) implies that $S$ fixes a point $p$ of the building $T$.

It was proved in [7] that the action of $G(\mathbb{R}((u)))$ on the Bruhat-Tits building associated to $G(\mathbb{R}(u))$ and the valuation $\omega$ defined by $\omega(u/v) = deg(v) - deg(u)$ admits a simplicial fundamental domain $\mathcal{D}$ so called "sector". This result remains true when we pass to our building $T$ since, by taking the completion $\mathbb{R}((u^{-1}))$, the building does not change, only the apartment system gets completed. Moreover, the action of $G(\mathbb{R}((u)))$ is continuous. Let $\mathcal{H}$ denote the Cartan subalgebra of $sl(n, \mathbb{C})$ corresponding to $H$ and $\mathcal{H}_R$ its real part. The simplicial fundamental domain for the action of $G(\mathbb{R}((u)))$ on $T$ is contained in the standard apartment of the building $T'$ which is identified with $\mathcal{H}$.

Let $h$ be the point of $\mathcal{H}$ which is equivalent to $p$ via the action of $G(\mathbb{R}((u)))$. There exists $X \in G(\mathbb{R}((u)))$ such that $XP = h$, which implies that $X^{-1}h$ is contained in the stabilizer $P_h$ of $h$ under the action of $G(\mathbb{R}((u)))$ on $T$.

On the other hand, $P_h = P'_h \cap G(\mathbb{R}((u^{-1})))$, where $P'_h$ is the stabilizer of $h$ under the action of $SL(n, \mathbb{C}((u^{-1})))$ on $T'$. It follows that
\begin{equation}
\text{Ad}(X)W \subseteq g \otimes \mathbb{R}((u^{-1})) \cap \text{Lie}(P'_h).
\end{equation}
The stabilizer $P'_h$ was computed in [4, p. 230] and its Lie algebra is
\begin{equation}
\mathcal{O}_h = \{(g_i) \in sl(n, \mathbb{C}((u^{-1}))) : v(g_i) \geq \alpha_j(h)\}.
\end{equation}
Let us prove that
\begin{equation}
g \otimes \mathbb{R}((u^{-1})) \cap \mathcal{O}_h \subseteq g \otimes \mathbb{R}((u^{-1})) \cap \mathcal{O}_h.
\end{equation}

We know that
\begin{equation}
g \otimes \mathbb{R}((u^{-1})) \cap \mathcal{O}_h \subseteq \text{sl}(n) \otimes \mathbb{R}((u^{-1})) \cap \mathcal{O}_h.
\end{equation}

It is enough to show the following:
\begin{equation}
\text{sl}(n) \otimes \mathbb{R}((u^{-1})) \cap \mathcal{O}_h \subseteq \text{sl}(n) \otimes \mathbb{R}((u^{-1})) \cap \mathcal{O}_h.
\end{equation}

If a matrix $(g_j)$ belongs to $\text{sl}(n) \otimes \mathbb{R}((u^{-1})) \cap \mathcal{O}_h$, then $v(g_j) \geq \alpha_j(h)$ for all $i$, $j$ and $g_j + M = 0$. We have $v(g_j) = v(-M) = v(g_j)$. On the other hand, $v(g_j) \geq -\alpha_j(h)$. We conclude that $v(g_j) \geq 0$ and therefore $(g_j)$ belongs to $\text{sl}(n) \otimes \mathbb{R}((u^{-1}))$.

In conclusion, for some $X \in G(\mathbb{R}(u))$, one has that
\begin{equation}
\text{Ad}(X)W \subseteq g \otimes \mathbb{R}[[u^{-1}]]
\end{equation}
which completes the proof. \qed
4. Description of Rational Solutions

Theorem 3.1 has an important consequence:

Corollary 4.1. Let \( g \) be a simple compact Lie algebra over \( \mathbb{R} \). Any rational solution of the CYBE for \( g \) is gauge equivalent to a solution of the form

\[ X(u, v) = \frac{\Omega}{u - v} + r, \]

where \( r \in g \wedge g \) is a constant \( r \)-matrix.

Proof. We know that any order \( W \) of \( g((u^{-1})) \) is gauge equivalent to an order contained in \( g[[u^{-1}]] \). On the other hand, if a rational solution \( X(u, v) \) corresponds to an order \( W \subseteq g[[u^{-1}]] \) then, by Remark 2.7, \( X(u, v) = \frac{\Omega}{u - v} + r \), where \( r \) is a constant polynomial. Because \( X(u, v) \) is a solution of the CYBE, it results that \( r \) itself is a solution of the CYBE.

Let us recall a result which describes constant solutions in a different way. This theorem was formulated for the complex case in \([1]\), but the proof obviously works for any simple compact Lie algebra \( g \) over \( \mathbb{R} \).

Theorem 4.2. Any rational solution of the CYBE of the form (4.1) induces a pair \((L, B)\), where \( L \) is a subalgebra of \( g \) and \( B \) is a non-degenerate 2-cocycle on \( L \). Conversely, any such pair provides a rational solution of the form (4.1).

Remark 4.3. If \( L \) is a commutative subalgebra of \( g \) and \( B \) is a non-degenerate skew-symmetric form on \( L \), let \( r \in L \wedge L \) be the inverse with respect to \( B \), then the corresponding rational solution is \( \frac{\Omega}{u - v} + r \).

Recall that a subalgebra \( L \) of \( g \) is called quasi-Frobenius if there exists a non-degenerate 2-cocycle \( B \in Z^2(L, \mathbb{R}) \).

Theorem 4.4. Let \( g \) be a simple compact Lie algebra over \( \mathbb{R} \). Any quasi-Frobenius subalgebra \( L \) of \( g \) is commutative.

Proof. Any subalgebra of a compact Lie algebra is compact. Therefore \( L \) must be compact as well. Moreover (see for example \([6\), p. 97]) the derived algebra \( L' \) of \( L \) is semisimple and if \( \zeta(L) \) denotes the center of \( L \), then

\[ L = L' \oplus \zeta(L). \]

Let us assume that \( L' \neq 0 \) and there exists a non-degenerate 2-cocycle \( B \) on \( L \). We have the following identity

\[ B([x, y], z) + B([y, z], x) + B([z, x], y) = 0, \]

for any \( x, y \in L' \) and \( z \in \zeta(L) \). This implies \( B([x, y], z) = 0 \), for arbitrary \( x, y \in L' \) and \( z \in \zeta(L) \). Since \( L' \) is semisimple, its derived algebra coincides with \( L' \), we obtain

\[ B(w, z) = 0, \]

for any \( w \in L' \) and \( z \in \zeta(L) \).

On the other hand, since \( L' \) is semisimple, the restriction of \( B \) to \( L' \) is a coboundary; i.e., there exists a non-zero functional \( f \) on \( L' \) such that \( B(w_1, w_2) = f([w_1, w_2]), \) for all \( w_1, w_2 \in L' \). Let \( a_0 \) be the element of \( L' \) which corresponds to \( f \) via the isomorphism \( L' \cong (L')^* \) defined by the Killing form. Then for all \( w \in L' \), one has

\[ B(a_0, w) = K(a_0, [a_0, w]) = 0. \]

Together with (4.6) and (4.4) this implies that

\[ B(a_0, l) = 0, \]

for all elements \( l \) of \( L \). Thus \( B \) is degenerate, which is a contradiction.

Corollary 4.5. Up to gauge equivalence, any rational solution of the CYBE for a simple compact Lie algebra \( g \) over \( \mathbb{R} \) has the form

\[ X(u, v) = \frac{\Omega}{u - v} + t_1 \wedge t_2 + \ldots + t_{2n-1} \wedge t_{2n}, \]

where \( t_1, \ldots, t_{2n} \) are linearly independent elements in a maximal torus \( t \) of \( g \).

Proof. We have seen that rational solutions are determined by pairs \((L, B)\), where \( L \) is a quasi-Frobenius Lie subalgebra and \( B \) is a non-degenerate 2-cocycle on \( L \). By the previous result, \( L \) is a commutative subalgebra and \( B \) is a non-degenerate skew-symmetric form on \( L \). Then \( L \) is contained in a maximal commutative subalgebra \( t \) of \( g \) and the dimension of \( L \) is even, say \( 2n \).

Moreover, it is well known that there exists a basis \( t_1, \ldots, t_{2n} \) in \( L \) such that \( B(t_{2i-1}, t_{2i}) = -B(t_{2i-1}, t_{2i-1}) = 1 \) for \( 1 \leq i \leq n \) and \( B(t_j, t_k) = 0 \) otherwise. The rational solution induced by the pair \((L, B)\) is precisely (4.7).

5. Quantization

Let \( g \) be a simple compact Lie algebra over \( \mathbb{R} \). Let us recall that the rational solution \( X_0(u, v) = \frac{\Omega}{u - v} \) induces a Lie bialgebra structure on \( g[[u]] \) via the 1-cocycle \( \delta_0 \) given by

\[ \delta_0(a(u)) = [a(u) \otimes 1 + 1 \otimes a(v)], X_0(u, v), \]
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for any $a(u) \in g[u]$.

We have seen that, up to gauge equivalence, rational solutions have the form (4.7). To any such solution one can associate a Lie bialgebra structure on $g[u]$ by defining the 1-cocycle

\[
\delta_r(a(u)) = [a(u) \otimes 1 + 1 \otimes a(v), X(u,v)].
\]

(5.2)

Here $r = t_1 \wedge t_2 + \ldots + t_{2n-1} \wedge t_{2n}$. In other words, the Lie bialgebra $(g[u], \delta_r)$ is obtained from the Lie bialgebra $(g[u], \delta_0)$ by so-called twisting via $r$.

Remark 5.1. This notion was introduced by V. G. Drinfeld in a more general setting for Lie quasi-bialgebras.

The purpose of this section is to give a quantization of the Lie bialgebra $(g[u], \delta_r)$.

Let us begin by pointing out that the Lie algebra $(g[u], \delta_0)$ admits a unique quantization which we will denote by $Y_0(g)$ (here $h$ is Planck's constant). The construction is analogous to that of the Yangian introduced in [5]. We recall that if $K$ denotes the Killing form of a simple compact $g$, then $-K$ is a positive definite invariant bilinear form.

Let $\{I_i\}$ be an orthonormal basis in $g$ with respect to $-K$. Then $Y_0(g)$ is the topological Hopf algebra over $\mathbb{R}[h]$ generated by elements $I_1$ and $J_1$ with defining relations

\[
[I_1, J_1] = c_{1,1}^0 J_0
\]

(5.3)

\[
[I_1, I_0] = c_{1,1}^0 J_0
\]

(5.4)

\[
[I_1, J_0] = c_{1,1}^0 J_0
\]

(5.5)

\[
[[I_1, J_0], [I_0, J_1]] + [[I_1, J_0], [J_0, J_1]] + [[I_0, J_0], [J_1, J_0]] = R^2(a_{12} c_{1,1}^0 c_{1,1}^0 c_{1,1}^0) (I_0, I_0, I_0)
\]

(5.6)

where $a_{12} := \frac{1}{2} a_{1,1} c_{1,1}^0 c_{1,1}^0 c_{1,1}^0$ and \( \{x_1, x_2, x_3\} := \sum_{i,j,k} x_i x_j x_k \). The comultiplication, the counit and the antipode are given by the following:

\[
\Delta(I_1) = I_1 \otimes 1 + 1 \otimes I_1
\]

(5.7)

\[
\Delta(J_1) = J_1 \otimes 1 + 1 \otimes J_1 - \frac{h}{2} c_{1,1}^0 J_1 \otimes I_1,
\]

(5.8)

\[
\varepsilon(I_1) = \varepsilon(J_1) = 0, \varepsilon(1) = 1
\]

(5.9)

\[
S(I_1) = -I_1
\]

(5.10)

\[
S(J_1) = -J_1 + \frac{1}{h} I_1
\]

(5.11)

Clearly $Y_0(g)$ contains $U(g)[\hbar]$ as a Hopf subalgebra.

Since the generators of $Y_0(g)$ are simultaneously generators for the complex Yangian and all the structure constants are real, it follows immediately from [5, Th.3] that $Y_0(g)$ is a pseudotriangular Hopf algebra. More precisely, for any real number $a$, define an automorphism $T_a$ of $Y_0(g)$ by the formula

\[
T_a(I) = I_1
\]

(5.12)

\[
T_a(J) = J_1 + a I_1
\]

(5.13)

Then there exists an element $R(u) = 1 + \sum_{i=1}^{\infty} R_i u^{-i}$, where $R_i = \Omega$ and $R_0 \in Y_0(g)[\hbar^2]$, such that the following conditions are satisfied:

\[
(T_{u} \otimes T_{u}) R(u) = R(u + a - b)
\]

(5.14)

\[
(T_{u} \otimes 1) \Delta_0^a(x) = R(u)(T_{u} \otimes 1) \Delta(x) R(u)^{-1}
\]

(5.15)

\[
(\Delta \otimes 1) R(u) = R^1(u) R^{23}(u)
\]

(5.16)

\[
R^{12}(u) R^{23}(-u) = 1 \circ 1
\]

(5.17)

\[
R^{12}(u_1 - u_2) R^{13}(u_1 - u_3) R^{23}(u_2 - u_3)
\]

(5.18)

\[
= R^{23}(u_2 - u_3) R^{13}(u_1 - u_3) R^{12}(u_1 - u_2)
\]

\[
\]

Here $\Delta^a$ denotes the opposite comultiplication.

In order to give a quantization of $(g[u], \delta_0)$, we introduce a deformation of the Yangian $Y_0(g)$ by a so-called quantum twist. The approach is based on [11, Th.3] that we recall below:

**Theorem 5.2.** Let $F \in (U(g)[\hbar])[u]$ such that

\[
F \equiv 1 \mod \hbar
\]

(5.19)

\[
\varepsilon \otimes 1) F = (1 \otimes \varepsilon) F = 1 \otimes 1
\]

(5.20)

\[
(\Delta \otimes 1) F : F^{12} = (1 \otimes \Delta) F : F^{23}
\]

(5.21)

Denote by $\tilde{Y}_0(g)$ the associative unital algebra which has the same multiplication as $Y_0(g)$ but the comultiplication is

\[
\tilde{\Delta} := F^{-1} \Delta F.
\]

Then the following statements hold:

1) $\tilde{Y}_0(g)$ is a Hopf algebra with antipode

\[
\tilde{S} := Q^{-1} S Q,
\]

(5.22)

where $Q = m(S \otimes 1)(F)$,
2) Let \( R(u) := (F^{21})^{-1}R(u)F \). Then the equations (5.14)-(5.18) hold for \( R(u) \) and \( \Delta(u) \).

Remark 5.3. In literature, an element \( F \) satisfying (5.19)-(5.21) is called a quantum twist of \( Y_0(g) \). The Hopf algebra \( Y_0(g) \) is the twisted (or deformed) Yangian by the tensor \( F \).

We can easily construct a quantum twist in the following way:

**Proposition 5.4.** Suppose that \( t_1, \ldots, t_{2n} \) are linearly independent elements in a maximal torus \( t \) of \( g \). Then the two-tensor

\[
F = \exp(h(t_1 \otimes t_2 + \ldots + t_{2n-1} \otimes t_{2n}))
\]

is a quantum twist of \( Y_0(g) \).

**Proof.** Conditions (5.19)-(5.21) can be checked by straightforward computations.

Theorem 5.2 implies the following

**Corollary 5.5.** The deformed Hopf algebra \( Y_0(g) \), obtained by applying the quantum twist \( F \) given by (5.24), is a quantization of \( (g, \Delta_0) \), where \( r = t_1 \wedge t_2 + \ldots + t_{2n-1} \wedge t_{2n} \).

**Proof.** For any \( a \in \hat{Y}_0(g) \), we have to check the following:

\[
h^{-1}(\hat{\Delta}(a) - \hat{\Delta}^0(a)) \mod h = \delta_0(a \mod h).
\]

Since \( \hat{\Delta} = F^{-1} \Delta F \), we obtain

\[
\hat{\Delta}(a) - \hat{\Delta}^0(a) = F^{-1} \Delta(a) F - (F^{21})^{-1} \Delta^0(a) F^{21}.
\]

On the other hand, since \( Y_0(g) \) is a quantization of \( (g, \Delta_0) \), we have that

\[
\Delta(a) - \Delta^0(a) = h\delta_0(a \mod h) + O(h^2).
\]

Using (5.26), (5.27) and \( (F^{21})^{-1}F = \exp(hr) \), we obtain

\[
\hat{\Delta}(a) - \hat{\Delta}^0(a) = h(\Delta(a), r) + \delta_0(a \mod h) + O(h^2).
\]

Finally, we give the explicit formula for the comultiplication and antipode of the twisted Yangian \( \hat{Y}_0(g) \). Let us recall the root system of \( g \) with respect to a torus, according to [6, p. 98-99]. We denote by \( h \) a Cartan subalgebra of \( g \otimes \mathbb{C} \) and let \( \Lambda \) be the root system with respect to \( h \), together with a lexicographic ordering of \( \Lambda \). We choose the root vectors \( e_\alpha \), corresponding to each root \( \alpha \), such that \( K(e_\alpha, e_-\alpha) = -1 \).

Let \( \mathfrak{h}_0 = \{ h \in \mathfrak{h} : \alpha(h) \in \mathbb{R} \text{ for all } \alpha \} \). We put

\[
C_\alpha := \frac{1}{\sqrt{2}}(e_\alpha + e_-\alpha)
\]

\[
S_\alpha := i \frac{1}{\sqrt{2}}(e_\alpha - e_-\alpha)
\]

It is well-known that

\[
\mathfrak{g} = \mathfrak{h}_0 \oplus \bigoplus_{\alpha > 0} (\mathfrak{g} \cap \mathfrak{g}_\alpha \oplus \mathfrak{g}_\alpha)
\]

An orthonormal basis in \( \mathfrak{g}_0 \) with respect to the bilinear form \( -K \), is formed by the elements \( C_\alpha, S_\alpha \) and \( p_j := i h_j \), where \( \{ h_j \} \) is an orthonormal basis in \( \mathfrak{h}_0 \). We choose this basis as our \( \{ I_j \} \). The role of \( \{ J_j \} \) is played correspondingly by some elements denoted by \( U_n, V_n \) and \( P_n \). For any \( h \in \mathfrak{h}_0 \) we have the following:

\[
[ih, C_\alpha] = \alpha(h)S_\alpha
\]

\[
[ih, S_\alpha] = -\alpha(h)C_\alpha
\]

\[
[ih, U_n] = \alpha(h)V_n
\]

\[
[ih, V_n] = -\alpha(h)U_n
\]

Let us consider now a quantum twist \( F \) as in (5.24). Since \( F \) is a product of exponentials, it is enough to perform computations for

\[
F = \exp(h(t_1 \otimes t_2)),
\]

where \( t_1 \) and \( t_2 \) are two linearly independent elements in the torus \( t = \mathfrak{h}_0 \). Let \( t_1 = ih_1 \) and \( t_2 = ih_2 \), where \( h_1 \) and \( h_2 \) are elements of \( \mathfrak{h}_0 \).

**Lemma 5.6.** Let \( T_{1n} := \text{t}_{\text{h}_1}h_2 \) and \( T_{2n} := \text{t}_{\text{h}_2}h_1 \). The following identities hold:

\[
F^{-1}(C_\alpha \otimes 1)F = C_\alpha \otimes \cos(T_{1n}) - S_\alpha \otimes \sin(T_{1n})
\]

\[
F^{-1}(1 \otimes C_\alpha)F = \cos(T_{2n}) \otimes C_\alpha - \sin(T_{2n}) \otimes S_\alpha
\]

\[
F^{-1}(S_\alpha \otimes 1)F = S_\alpha \otimes \cos(T_{2n}) + C_\alpha \otimes \sin(T_{2n})
\]

\[
F^{-1}(1 \otimes S_\alpha)F = \cos(T_{2n}) \otimes S_\alpha + \sin(T_{2n}) \otimes C_\alpha
\]

\[
F^{-1}(U_n \otimes 1)F = U_n \otimes \cos(T_{2n}) - V_n \otimes \sin(T_{2n})
\]

\[
F^{-1}(1 \otimes U_n)F = \cos(T_{2n}) \otimes U_n - \sin(T_{2n}) \otimes V_n
\]
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(5.43) \[ F^{-1}(V_0 \otimes 1)F = V_0 \otimes \cos(T_{1a}) + U_0 \otimes \sin(T_{1a}) \]

(5.44) \[ F^{-1}(1 \otimes V_0)F = \cos(T_{2a}) \otimes V_0 + \sin(T_{2a}) \otimes U_0. \]

Proof. To prove the first identity, we use relations (5.32)-(5.33) and the formula
\[ \exp(\lambda) \mu \exp(-\lambda) = \exp(ad(\lambda)) \mu = \mu + [\lambda, \mu] + \frac{1}{2!} [\lambda, [\lambda, \mu]] + \ldots \]

for \( \lambda := -h(\theta_1 \otimes i \theta_2) \) and \( \mu := C_0 \otimes 1 \).

Identities (5.38)-(5.44) can be proved in a similar way. \( \square \)

Consequently, we obtain the following result:

**Proposition 5.7.** The comultiplication \( \tilde{\Delta} \) of the twisted Yangian \( \tilde{Y}_0(\mathfrak{g}) \) is given on its generators by the following:

\[ \tilde{\Delta}(C_a) = C_a \otimes \cos(T_{1a}) + S_a \otimes \sin(T_{1a}) + \cos(T_{2a}) \otimes C_a - \sin(T_{2a}) \otimes S_a \]

\[ \tilde{\Delta}(S_a) = S_a \otimes \cos(T_{1a}) + C_a \otimes \sin(T_{1a}) + \cos(T_{2a}) \otimes S_a + \sin(T_{2a}) \otimes C_a \]

\[ \tilde{\Delta}(V_a) = U_a \otimes \cos(T_{1a}) - V_a \otimes \sin(T_{1a}) + \cos(T_{2a}) \otimes U_a + \sin(T_{2a}) \otimes V_a \]

\[ \tilde{\Delta}(p_j) = p_j \otimes 1 + 1 \otimes p_j \]

\[ \tilde{\Delta}(p_j) = p_j \otimes 1 + 1 \otimes p_j - \frac{h}{2} [p_j \otimes 1, \tilde{\Omega}], \]

where
\[ \tilde{\Omega} = \sum_{\alpha < \theta} (C_\alpha \cos(T_{2a}) + S_\alpha \sin(T_{2a})) \otimes (C_\alpha \cos(T_{1a}) + S_\alpha \sin(T_{1a})). \]

We conclude by explicating the antipode \( \tilde{S} \) of the twisted Yangian \( \tilde{Y}_0(\mathfrak{g}) \). It is given by \( \tilde{S} = Q^{-1}S \tilde{Q} \), where \( Q = \exp(h(\theta_1 \otimes i \theta_2)) \).

Similarly to Lemma 5.6, one can prove

\[ Q^{-1}C_0Q = \exp(h(\alpha(\theta_2)h_1 + \alpha(\theta_1)h_2)). \]

**Lemma 5.8.** Let \( T_a := ih(\alpha(\theta_2)h_1 + \alpha(\theta_1)h_2) \). The following identities hold:

\[ Q^{-1}C_aQ = \exp(h(\alpha(\theta_2)h_1 + \alpha(\theta_1)h_2))(\cos(T_a)C_a + \sin(T_a)S_a) \]

\[ Q^{-1}S_aQ = \exp(h(\alpha(\theta_2)h_1 + \alpha(\theta_1)h_2))(\cos(T_a)S_a - \sin(T_a)C_a) \]

\[ Q^{-1}U_aQ = \exp(h(\alpha(\theta_2)h_1 + \alpha(\theta_1)h_2))(\cos(T_a)U_a + \sin(T_a)V_a) \]

\[ Q^{-1}V_aQ = \exp(h(\alpha(\theta_2)h_1 + \alpha(\theta_1)h_2))(\cos(T_a)V_a - \sin(T_a)U_a). \]

**Proposition 5.9.** The antipode \( \tilde{S} \) of the deformed Yangian \( \tilde{Y}_0(\mathfrak{g}) \) is given on its generators by

\[ \tilde{S}(C_a) = -\exp(h(\alpha(\theta_2)h_1 + \alpha(\theta_1)h_2))(\cos(T_a)C_a + \sin(T_a)S_a) \]

\[ \tilde{S}(S_a) = -\exp(h(\alpha(\theta_2)h_1 + \alpha(\theta_1)h_2))(\cos(T_a)S_a - \sin(T_a)C_a) \]

\[ \tilde{S}(V_a) = \exp(h(\alpha(\theta_2)h_1 + \alpha(\theta_1)h_2))(\cos(T_a)(-V_a + \frac{h}{4}C_a) + \sin(T_a)(-U_a + \frac{h}{4}S_a)) \]

\[ \tilde{S}(U_a) = \exp(h(\alpha(\theta_2)h_1 + \alpha(\theta_1)h_2))(\cos(T_a)(-V_a + \frac{h}{4}S_a) + \sin(T_a)(U_a - \frac{h}{4}C_a)). \]

\[ \tilde{S}(p_j) = -p_j \]

\[ \tilde{S}(p_j) = -p_j + \frac{h}{2} p_j. \]

REFERENCES


