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ABSTRACT. We give new a proof of the general Briançon-Skoda theorem about ideals of holomorphic functions by means of multivariable residue calculus. The method gives new variants of this theorem for products of ideals. Moreover, we obtain a related result for the ideal generated by the the subdeterminants of a matrix-valued generically surjective holomorphic function, generalizing the duality theorem for a complete intersection. We also provide explicit versions of the various results, including the general Briançon-Skoda theorem, with integral representation formulas.

1. Introduction

Let $\phi, f_1, \ldots, f_m$ be holomorphic functions in a neighborhood of the origin in $\mathbb{C}^n$. The Briançon-Skoda theorem, [8], states that $\phi^{\min(n,m)}$ belongs to the ideal (f) generated by f_j if $|\phi| \le C|f|$. This condition is equivalent to that ϕ belongs to the integral closure of the ideal (f). The original proof is based on Skoda's L^2-estimates in [16], see Remark 1 below, and actually gives the stronger statement that $\phi \in (f)$ if $|\phi| \le C|f|^{\min(n,m)}$. There are generalizations to more arbitrary rings, see, e.g., [13].

In general this result cannot be improved but for certain tuples f_j a much weaker size condition on ϕ is enough to guarantee that ϕ belongs to (f). For instance, the ideal $(f)^2$ is generated by the $m(m+1)/2$ functions $g_{jk} = f_j f_k$, and $|f|^2 \sim |g|$, so if we apply the previous result we get that $\phi \in (f)^2$ if $|\phi| \le C|f|^{\min(2n, m(m+1))}$. However, in this case actually the power $\min(n,m)+1$ is enough. In general we have

Theorem 1.1 (Briançon-Skoda)**.** *If $f = (f_1, \ldots, f_m)$ and ϕ are holomorphic at 0 in $\mathbb{C}^n$ and $|\phi| \le C|f|^{\min(m,n)+r-1}$, then $\phi \in (f)^r$.*

In [2] we gave a new proof of the case $r = 1$ by means of multivariable residue calculus. In this note we extend this method to cover the general case of Theorem 1.1, and as a by-product we get various related results. In the first one we consider several possibly different tuples.

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Theorem 1.2. *Let f_j, $j = 1, \dots, r$, be m_j-tuples of holomorphic functions at $0 \in \mathbb{C}^n$ and assume that*

$$|\phi| \le C|f_1|^{s_1} \cdots |f_r|^{s_r}$$

for all s such that $s_1 + \cdots + s_r \le n + r - 1$ and $1 \le s_j \le m_j$. Then $\phi \in (f_1) \cdots (f_r)$.

Notice that this immediately implies Theorem 1.1 in the case $m \ge n$ by just choosing all $f_j = f$. In certain cases Theorem 1.2 can be improved, as one can see by taking $f_j = f$ and $m < n$ and compare with Theorem 1.1. Another case is when all the functions in the various tuples f_j together form a regular sequence.

Theorem 1.3. *Let f_j, $j = 1, \dots, m$, be m_j-tuples of holomorphic functions at $0 \in \mathbb{C}^n$ and assume that the codimension of $\{f_1 = \cdots f_r = 0\}$ is $m_1 + \cdots + m_r$. If*

$$|\phi| \le C \min(|f_1|^{m_1}, \dots, |f_r|^{m_r}),$$

then $\phi \in (f_1) \cdots (f_r)$.

Remark 1. As was mentioned above the Briançon-Skoda theorem follows by direct applications of Skoda's L^2-estimate if $m \le n$. In fact, if ψ is any plurisubharmonic function, the L^2-estimate guarantees a holomorphic solution to $f \cdot u = \phi$ such that

$$\int_{X \setminus Z} \frac{|u|^2}{|f|^{2(\min(m,n+1)-1+\epsilon)}} e^{-\psi} dV < \infty$$

provided that

$$\int_{X \setminus Z} \frac{|\phi|^2}{|f|^{2(\min(m,n+1)+\epsilon)}} e^{-\psi} dV < \infty.$$

If $|\phi| \le C|f|^m$, the second integral is finite (taking $\psi = 0$) if ϵ is small enough, and thus Skoda's theorem provides the desired solution. The case when $r > 1$ is obtained by iteration. If $m > n$ a direct use of the L^2-estimate will not give the desired result. However, see [10], in this case one can find an n-tuple $\tilde{f}$ such that $(\tilde{f}) \subset (f)$ and $|\tilde{f}| \sim |f|$, and the theorem then follows by applying the L^2-estimate to the tuple f'.

In the same way, Theorem 1.2 can easily be proved from the L^2-estimate if $m_1 + \cdots + m_r \le n + r - 1$. To see this, assume for simplicity that $r = 2$, and that $|\phi| \le C|f_1|^{m_1}|f_2|^{m_2}$. Choosing $\psi = 2(m_1 + \epsilon) \log |f_1|$, Skoda's theorem give a solution to $f_2 \cdot u = \phi$ such that

$$\int_{X \setminus Z} \frac{|u|^2}{|f_1|^{2(m_1+\epsilon)}} dV < \infty.$$

Another application then gives v_j such that $f_1 \cdot v_j = u_j$. This means that ϕ belongs to $(f_1)(f_2)$. However, we do not know whether one can derive Theorem 1.3 from the L^2-estimate when $m_1 + \cdots + m_r > n + r - 1$. $\square$

Now consider a $r \times m$ matrix f_j^k of holomorphic functions, $r \leq m$, with rows $f_1, \ldots, f_r$. We let F be the $m!/(m-r)!r!$ tuple of functions $\det(f_j^{I_k})$ for increasing multiindices I of length r. We will refer to F as the determinant of f. If f_j are the rows of the matrix, considered as sections of the trivial bundle E^*, then F is just the section $f_r \wedge \ldots f_1$ of the bundle $\Lambda^r E^*$. Our next result is a Briançon-Skoda type result for the tuple F. It turns out that it is enough with a much less power than $m!/(m-r)!r!$. Let Z be the zero set of F and notice that $\operatorname{codim} Z \leq m - r + 1$; this is easily seen by Gauss elimination.

Theorem 1.4. *Let F be the determinant of the holomorphic matrix f as above. If*

$$|\phi| \leq C|F|^{\min(n, m-r+1)},$$

then $\phi \in (F)$.

Remark 2. This result is closely related to the following statement which was proved in [3]. *Suppose that ϕ is an r-tuple of holomorphic functions and let $\|\phi\|$ be the pointwise norm induced by f, i.e., $\|\phi\| = \det(ff^*)\langle (ff^*)^{-1}\phi, \phi\rangle$. If*

$$\|\phi\| \lesssim |F|^{\min(n, m-r+1)},$$

then $f\psi = \phi$ has a local holomorphic solution. □

Remark 3. Another related situation is when f is a section of a bundle E^*, ϕ takes values in $\Lambda^\ell E$, and we ask for a holomorphic section ψ of $\Lambda^{\ell+1} E$ such that $\delta_f \psi = \phi$, provided that the necessary compatibility condition $\delta_f \phi = 0$ is fulfilled. Let $p = \operatorname{codim}\{f = 0\}$. Then a sufficient condition is that

$$|\phi| \leq C|f|^{\min(n, m-\ell)}$$

if $\ell \leq m - p$, whereas there is no condition at all if $\ell > m - p$, see Theorems 1.2 and 1.4 and Corollary 1.5 in [2]. □

Theorem 1.4 is proved by constructing a certain residue current R with support on the analytic set Z, such that $R\phi = 0$ implies that ϕ belongs to the ideal (F) locally. The size conditions of ϕ then implies that $R\phi = 0$ by brutal force, see Theorem 2.3 below. There may be more subtle reasons for annihilation. For instance, in the generic case, i.e., when $\operatorname{codim} Z = m - r + 1$, even the converse statement holds; if ϕ is in the ideal (F) then actually $R\phi = 0$, see Theorem 2.3 (iv). The analogous statement also holds for the equation $f\psi = \phi$ in Remark 2, see [3]. These results are therefore extensions of the well-known duality theorem of Dickenstein-Sessa and Passare, [11] and [14], stating that if f is a tuple that defines a complete intersection, i.e., $\operatorname{codim}\{f = 0\} = m$, then $\phi \in (f)$ if and only if ϕ annihilates the Coleff-Herrera current defined by f. Theorems 1.2 and 1.3 (as well as Theorem 1.1) are obtained along the same lines, by an appropriate choice of matrix f.

It has been discussed for several years, see, e.g., [6] and [19], whether one can prove the Briançon-Skoda theorem with an explicit integral formula. In [2] we discovered such a formula for the case $r=1$ of Theorem 1.1. In the second part of this paper we construct new completely explicit integral representations of holomorphic functions that provide effective proofs of Theorems 1.1 to 1.4. In fact, for any holomorphic function ϕ we construct a holomorphic decomposition

$$\phi = T\phi + S\phi, \tag{1.1}$$

such that $T\phi$ belongs to the ideal in question and $S\phi$ vanishes as soon as ϕ annihilates the residue current R.

2. The ideal generated by the determinant section

Although we are mainly interested in the local results in this paper it is convenient to adopt an invariant perspective. We therefore assume that we have Hermitian vector bundles E and Q of ranks m and $r \leq m$, respectively, over a complex n-dimensional manifold X, and a holomorphic morphism $f\colon E \to Q$. We also assume that f is generically surjective, i.e., that the analytic set Z where f is not surjective has at least codimension 1. If ϵ_j is a local holomorphic frame for Q, then $f = f_1 \otimes \epsilon_1 + \cdots + f_r \otimes \epsilon_r$, where f_j are sections of the dual bundle E^*. Moreover, $F = f_r \wedge \ldots \wedge f_1 \otimes \epsilon_1 \wedge \ldots \wedge \epsilon_r$ is an invariantly defined section of $\Lambda^r E^* \otimes \det Q^*$ that we will call the determinant section associated with f. Notice that if e_j is a local frame for E with dual frame e_j^* for E^*, then $f_j = \sum_1^m f_j^k e_k^*$, and

$$F = \sum_{|I|=r}' F_I e_{I_1}^* \wedge \ldots \wedge e_{I_r}^*,$$

where the sum runs over increasing multiindices I and $F_I = \det(F_j^{I_k})$. Let $S^\ell Q^*$ be the subbundle of $(Q^*)^{\otimes \ell}$ consisting of symmetric tensors. We introduce the complex

$$\begin{aligned} \cdots \stackrel{\delta_f}{\to} \Lambda^{r+k-1} E \otimes S^{k-1} Q^* \otimes \det Q^* \stackrel{\delta_f}{\to} \cdots \\ \stackrel{\delta_f}{\to} \Lambda^{r+1} E \otimes Q^* \otimes \det Q^* \stackrel{\delta_f}{\to} \Lambda^r E \otimes \det Q^* \stackrel{\delta_F}{\to} \mathbb{C} \to 0, \end{aligned} \tag{2.1}$$

where

$$\delta_f = \sum_j \delta_{f_j} \otimes \delta_{\epsilon_j},$$

δ_{f_j} and δ_{ϵ_j} denote interior multiplication on ΛE and from the left on $SQ^* \otimes \det Q^*$, respectively, and

$$\delta_F = \delta_f^r / r! = \delta_{f_r} \cdots \delta_{f_1} \otimes \delta_{\epsilon_1} \cdots \delta_{\epsilon_r}.$$

It is readily checked that (2.1) actually is a complex. Notice that if $r = 1$, then (2.1) is the usual Koszul complex and therefore exact whenever f is pointwise surjective. This is, however, not true when $r > 1$.

In $X \setminus Z$ we let σ_j be the sections of E with minimal norms such that $f_k\sigma_j = \delta_{jk}$. Then $\sigma = \sigma_1 \otimes \epsilon_1^* + \ldots + \sigma_r \otimes \epsilon_r^*$ is the section of $\mathrm{Hom}\,(Q, E)$ such that, for each section ϕ of Q, $v = \sigma\phi$ is the solution to $fv = \phi$ with pointwise minimal norm. We also have the invariantly defined section

$$\boldsymbol{\sigma} = \sigma_1\wedge\ldots\wedge\sigma_r \otimes \epsilon_r^*\wedge\ldots\wedge\epsilon_1^*$$

of $\Lambda E \otimes = \det Q^*$, and it is in fact the section with minimal norm such that $F\boldsymbol{\sigma} = 1$, see, e.g., [3].

Example 1. Assume that E and Q are trivial and let ϵ_j be an ON-frame for Q and e_j an ON-frame for E, with dual frame e_j^*. If $F = \sum'_{|I|=r} F_I e_{I_1}^*\wedge\ldots\wedge e_{I_r}^*$ as above, then

$$\boldsymbol{\sigma} = \sum_{|I|=r}{}' \frac{\bar{F}_I}{|F|^2} e_{I_1}\wedge\ldots\wedge e_{I_r}.$$

□

We will consider $(0, q)$-forms with values in $\Lambda^{r+k-1}E \otimes S^{k-1}Q^* \otimes \det Q^*$, and it is convenient to consider them as sections of $\Lambda^{r+k+q-1}(E \oplus T_{0,1}^*(X)) \otimes S^{k-1}Q^* \otimes \det Q^*$, so that δ_f anti-commutes with $\bar\partial$, and $\delta_F\bar\partial = (-1)^r\delta_F\bar\partial$. In what follows we let $\otimes$ denote usual tensor product all Q^*-factors, and wedge product of $\Lambda(E \oplus T_{0,1}^*(X))$-factors. Thus for instance

$$\sigma \otimes \boldsymbol{\sigma} = (\sum_1^r \sigma_j \otimes \epsilon_j^*) \otimes (\sigma_1\wedge\ldots\wedge\sigma_r \otimes \epsilon_1^*\wedge\ldots\wedge\epsilon_r^*) = 0.$$

Moreover, for each $k \geq 1$, $(\bar\partial\sigma)^{\otimes(k-1)}$ is a symmetric tensor; more precisely,

$$(\bar\partial\sigma)^{\otimes(k-1)} = \sum_{|\alpha|=k-1} (\bar\partial\sigma_1)^{\alpha_1}\wedge\ldots\wedge(\bar\partial\sigma_r)^{\alpha_r} \otimes \epsilon_\alpha^*, \tag{2.2}$$

where

$$\epsilon_\alpha^* = \frac{(\epsilon_1^*)^{\alpha_1}\dot\otimes\cdots\dot\otimes(\epsilon_r^*)^{\alpha_r}}{\alpha_1!\cdots\alpha_r!},$$

and $\dot\otimes$ denotes symmetric tensor product. For each $k \geq 1$ we define in $X \setminus Z$ the $(0, k-1)$-forms

$$u_k = (\bar\partial\sigma)^{\otimes(k-1)} \otimes \boldsymbol{\sigma} = \sigma_1\wedge\ldots\wedge\sigma_r\wedge(\bar\partial\sigma)^{\otimes(k-1)} \otimes \epsilon^* \tag{2.3}$$

(where $\epsilon^* = \epsilon_r^*\wedge\ldots\wedge\epsilon_1^*$), with values in $\Lambda^{r+k-1}E \otimes S^{k-1}Q^* \otimes \det Q^*$.

Proposition 2.1. *In $X \setminus Z$ we have that*

$$\delta_F u_1 = 1, \quad \delta_f u_{k+1} = \bar\partial u_k, \; k \geq 1. \tag{2.4}$$

Proof. Since δ_{ϵ_j} act from the left, and $\delta_{f_j}\bar\partial\sigma_\ell = 0$ for all ℓ, we have that

$$\delta_f u_{k+1} = \delta_f\big[\sigma_1\wedge\ldots\wedge\sigma_r\wedge(\bar\partial\sigma)^{\otimes k}\otimes\epsilon^*\big] =$$
$$\delta_f\big[\sigma_1\wedge\ldots\wedge\sigma_r\wedge\bar\partial\sigma\big]\otimes(\bar\partial\sigma)^{\otimes(k-1)}\otimes\epsilon^* =$$
$$\sum_{j=1}^r \delta_{f_j}(\sigma_1\wedge\ldots\wedge\sigma_r)\wedge\bar\partial\sigma_j\otimes(\bar\partial\sigma)^{\otimes(k-1)}\otimes\epsilon^* =$$
$$\bar\partial(\sigma_1\wedge\ldots\wedge\sigma_r)\wedge(\bar\partial\sigma)^{\otimes(k-1)}\otimes\epsilon^* = \bar\partial u_k.$$

Since $\delta_F u_1 = F\boldsymbol{\sigma} = 1$, the proposition is proved. □

If we let $u = u_1 + u_2 + \cdots$, and let δ denote either δ_f or δ_F, then (2.4) can be written as $(\delta - \bar\partial)u = 1$. To analyze the singularities of u at Z we will use the following lemma (Lemma 4.1) from [3].

Lemma 2.2. *If $F = F_0 F'$ for some holomorphic function F_0 and non-vanishing holomorphic section F', then*

$$s' = F_0\sigma, \quad S' = F_0\boldsymbol{\sigma}$$

are smooth across Z.

Notice that $|F|^{2\lambda}u$ and $\bar\partial|F|^{2\lambda}\wedge u$ are well-defined forms in X for $\mathrm{Re}\,\lambda >> 0$.

Theorem 2.3. *(i) The forms $|F|^{2\lambda}u$ and $\bar\partial|F|^{2\lambda}\wedge u$ have analytic continuations as currents in X to $\mathrm{Re}\,\lambda > -\epsilon$. If $U = |F|^{2\lambda}u|_{\lambda=0}$ and $R = \bar\partial|F|^{2\lambda}\wedge u|_{\lambda=0}$, then*

$$(\delta - \bar\partial)U = 1 - R.$$

(ii) The current R has support on Z and $R = R_p + \cdots + R_\mu$, where $p = \mathrm{codim}\,Z$ and $\mu = \min(n, m - r + 1)$.

(iii) If ϕ is a holomorphic function and $R\phi = 0$, then locally $F\Psi = \phi$ has holomorphic solutions.

(iv) If $\mathrm{codim}\,Z = m - r + 1$ and $F\Psi = \phi$ has a holomorphic solution, then $R\phi = R_{m-r+1}\phi = 0$.

(v) If $|\phi| \le C|F|^\mu$, then $R\phi = 0$.

Here, of course, $R_k = \bar\partial|F|^{2\lambda}\wedge u_k|_{\lambda=0}$ is the component of R which is a $(0,k)$-current with values in $\Lambda^{r+k-1}E\otimes S^{k-1}Q^*\otimes\det Q^*$.

Proof. In the case $r = 1$, this theorem is contained in Theorems 1.1 to 1.4 in [2], and most parts of the proof are completely analogous. Therefore we just point out the necessary modifications. By Hironaka's theorem and a further toric resolution, following the technique developed in [5] and [15], we may assume that locally $F = F_0F'$ as in Lemma 2.2. Since moreover $\sigma\otimes\boldsymbol{\sigma} = 0$, we have then that locally in the resolution

$$u_k = \frac{(\bar\partial s')^{\otimes(k-1)}\otimes S'}{F_0^k}.$$

It is then easy to see that the proposed analytic extensions exist and we have that

$$U_k = \Big[\frac{1}{F_0^k}\Big](\bar\partial s')^{\otimes(k-1)} \otimes S', \tag{2.5}$$

and

$$R_k = \bar\partial\Big[\frac{1}{F_0^k}\Big]\wedge(\bar\partial s')^{\otimes(k-1)} \otimes S', \tag{2.6}$$

where $[1/F_0^k]$ is the usual principal value current. If $R\phi = 0$, then $(\delta - \bar\partial)U\phi = \phi$, and hence by successively solving the $\bar\partial$-equations

$$\bar\partial w_k = U_k\phi + \delta w_{k+1},$$

we finally get the holomorphic solution $\Psi = U_1\phi + \delta w_2$. All parts but *(iv)* now follow in a similar way as in [2]. Notice in particular, that $k \le \min(n, m-r+1)$ in (2.6) for degree reasons, so that $R\phi = 0$ if the hypothesis in (v) is satisfied. As for *(iv)*, let us assume that we have a holomorphic section Ψ of $\Lambda^r E \otimes \det Q^*$ such that $F\Psi = \phi$. If $\Psi = \psi \otimes \epsilon_*$, then $F\Psi = \delta_{f_r}\cdots\delta_{f_1}\psi$. Since u_{m-r+1} has full degree in e_j we have that

$$\begin{aligned} u_{m-r+1}\phi = \phi\sigma_1\wedge\ldots\wedge\sigma_r\wedge(\bar\partial\sigma)^{\otimes(m-r)} \otimes \epsilon^* = \\ (\delta_{f_r}\cdots\delta_{f_1}\psi)\sigma_1\wedge\ldots\wedge\sigma_r\wedge(\bar\partial\sigma)^{\otimes(m-r)} \otimes \epsilon^* = \\ \psi\wedge(\bar\partial\sigma)^{\otimes(m-r)} \otimes \epsilon^*|_{\lambda=0} = \\ (\bar\partial\sigma)^{\otimes(m-r)} \otimes \Psi = \bar\partial\big(\sigma \otimes (\bar\partial\sigma)^{\otimes(m-r+1)}\big) \otimes \Psi = \bar\partial u'_{m-r} \otimes \Psi. \end{aligned}$$

Since $\operatorname{codim} Z = m - r + 1$ we have that $R = R_{m-r+1}$ according to part (ii), so

$$R\phi = R_{m-r+1}\phi = \bar\partial|F|^{2\lambda}\wedge u_{m-r+1}\phi|_{\lambda=0} = -\bar\partial\big(\bar\partial|F|^{2\lambda}\wedge u'_{m-r} \otimes \Psi|_{\lambda=0}\big).$$

However,

$$\bar\partial|F|^{2\lambda}\wedge u'_{m-r} \otimes \Psi|_{\lambda=0}$$

vanishes for degree reasons, precisely in the same way as R_k vanishes for $k \le m - r$. □

Proof of Theorem 1.4. If we consider the matrix f as a morphism $E\colon Q$, for trivial bundles E and Q, the theorem immediately follows from parts (v) and (iii) of Theorem 2.3. □

Remark 4. As we have seen, the reason for the power $m - r + 1$ in Theorem 1.4 (and in part (v) of Theorem 2.3) when n is large, is that the complex (2.1) terminates at $k = m - r + 1$. If one tries to analyse the section F by means of the usual Koszul complex with respect to the basis $(e_I)'_{|I|=r}$, then one could hope that for some miraculous reason the corresponding forms u_k would vanish when $k > m-r+1$, although one has $m!/(m-r)!r!$ dimensions (basis elements). However, this is not the case in general. Take for instance the simplest non-trivial case,

$m = 3$ and $r = 2$, and choose $f_1 = (1, 0, \xi_1)$, $f_2 = (0, 1, \xi_2)$ and choose the trivial metric. Then

$$F_{12} = 1, \quad F_{13} = \xi_2, \qquad F_{23} = \xi_1,$$

and $\sigma = \bar{F}/|F|^2$, so that

$$\sigma_{12} = \frac{1}{1 + |\xi_1|^2 + |\xi_2|^2}, \sigma_{13} = \frac{\bar{\xi}_2}{1 + |\xi_1|^2 + |\xi_2|^2}, \quad \sigma_{23} = \frac{\bar{\xi}_1}{1 + |\xi_1|^2 + |\xi_2|^2}.$$

Now $m - r + 1 = 2$, but if we form the usual Koszul complex, with say that basis $\epsilon_1, \epsilon_2, \epsilon_3$, so that

$$\sigma = \sigma_{12}\epsilon_1 + \sigma_{13}\epsilon_2 + \sigma_{23}\epsilon_3 = \frac{1}{|F|^2}(\epsilon_1 + \bar{\xi}_2\epsilon_2 + \bar{\xi}_1\epsilon_3),$$

we have

$$\sigma \wedge (\bar{\partial}\sigma)^2 = \frac{2}{|F|^6} d\bar{\xi}_1 \wedge d\bar{\xi}_2 \wedge \epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3,$$

and this form is not zero. To get an example where Z is non-empty, one can multiply f with a function f_0. □

3. Products of ideals

For $j = 1, \ldots, r$, let $E_j \to X$ be a Hermitian vector bundle of rank m_j and let f_j be a section of E_j^*. Moreover, let $E = \oplus_1^r E_j$ and let $Q \simeq \mathbb{C}^r$ with ON-basis $\epsilon_1, \ldots, \epsilon_r$. If we consider f_j as sections of E, then $f = \sum_1^r f_j \otimes \epsilon_j$ is a morphism $E \to Q$. Moreover, $F\Psi = \phi$ with $\Psi = \psi \otimes \epsilon^*$ as before, means that $\delta_{f_r} \cdots \delta_{f_1}\psi = \phi$, and hence that ϕ belongs to the product ideal $(f_1) \cdots (f_r)$. To obtain such a solution Ψ we proceed as in the previous section. Notice that now σ_j can be identified with the section of E_j with minimal norm such that $f_j\sigma_j = 1$. Moreover, $|F| = |f_1| \cdots |f_r|$. In this case we therefore have

$$R_k = \bar{\partial}|F|^{2\lambda} \wedge u_k = \bar{\partial}(|f_1|^{2\lambda} \cdots |f_r|^{2\lambda}) \wedge \sigma_1 \wedge \ldots \wedge \sigma_r \wedge \sum_{|\alpha| = k-1} (\bar{\partial}\sigma_1)^{\alpha_1} \wedge \ldots \wedge (\bar{\partial}\sigma_r)^{\alpha_r} \otimes \epsilon_\alpha^* \otimes \epsilon^* |_{\lambda=0}.$$

For degree reasons R_k will vanish unless

$$0 \le \alpha_j \le m_j - 1 \quad \text{and} \quad \alpha_1 + \cdots + \alpha_r \le n - 1. \tag{3.1}$$

Proof of Theorem 1.2. Consider the tuples f_j as sections of E_j. For each j, let e_{ji}, $i = 1, \ldots, m_j$, be a local frame for E_j so that $f_j = \sum_{i=1}^{m_j} f_j^i e_{ji}^*$. After a suitable resolution we may assume that for each j, $f_j = f_j^0 f_j'$, where f_j^0 is holomorphic, and f_j' is a non-vanishing section of E_j^*. Therefore, R_k is a sum of terms like

$$\bar{\partial}(|f_1^0|^{2\lambda} \cdots |f_r^0|^{2\lambda} v^\lambda) \wedge \frac{\beta}{(f_1^0)^{\alpha_1+1} \cdots (f_r^0)^{\alpha_r+1}}\Big|_{\lambda=0},$$

where v is smooth and non-vanishing. By the same argument as before this current is annihilated by ϕ if $|\phi| \le C|f_1|^{\alpha_1+1} \cdots |f_r|^{\alpha_r+1}$, and in view of (3.1) and the hypothesis in the theorem, taking $s_j = \alpha_j + 1$, therefore ϕ annihilates R. It now follows from Theorem 2.3 (iii) that $F\Psi = \phi$ has a holomorphic solution, and thus $\psi \in (f_1)\cdots(f_r)$. □

We can also easily obtain the Briançon-Skoda theorem.

Proof of Theorem 1.1. Assume that the tuple $f = (f^1, \dots, f^m)$ is given. Choose disjoint isomorphic bundles $E_j \simeq \mathbb{C}^m$ with isomorphic bases e_{ji}, and let $f_j = \sum_{i=1}^m f^i e^*_{ji}$. Outside $Z = \{f = 0\}$ we have $\sigma_j = \sum_1^m \sigma^i e_{ji}$. Now $\bar\partial\sigma^i$ are linearly dependent, since $\sum_1^m f^i\bar\partial\sigma^i = \bar\partial\sum_1^m f^i\sigma^i = \bar\partial 1 = 0$. Thus the form u_k must vanish if $k-1 > m-1$, and therefore R_k vanishes unless $k \le \min(n, m)$. Since $|f_j| = |f|$, locally in the resolution, we have

$$R_k = \bar\partial|f|^{2r\lambda}\wedge\frac{\beta}{(f^0)^{k+r-1}}\Big|_{\lambda=0},$$

and hence it is annihilated by ϕ if $|\phi| \le C|f|^{\min(m,n)+r-1}$. □

It remains to consider the case when the f_j together define a complete intersection. The proof is very much inspired by similar proofs in [20].

Proof of Theorem 1.4. We now assume that $\operatorname{codim}\{f_1 = \cdots = f_r = 0\} = m_1 + \cdots + m_r$. In particular, $m_1 + \cdots + m_r \le n$. Let ξ be a test form times ϕ. If the support is small enough, after a resolution of singularities and further localization, $R.\xi$ becomes a sum of terms, the worst of which are like

$$\int \bar\partial\big(|f_1^0|^{2\lambda}\cdots|f_r^0|^{2\lambda}\big)\wedge\frac{s'_1\wedge\dots\wedge s'_r\wedge(\bar\partial s'_1)^{m_1-1}\wedge\dots\wedge(\bar\partial s'_r)^{m_r-1}\wedge\tilde\xi\rho}{(f_1^0)^{m_1}\cdots(f_r^0)^{m_r}}\Big|_{\lambda=0},$$

where $\tilde\xi$ is the pull-back of ξ and ρ is a cut-off function in the resolution. We may assume that each f_j^0 is a monomial times a non-vanishing factor in a local coordinate system τ_k. Let τ be one of the coordinate factors in, say, f_1 (with order ℓ), and consider the integral that appears when $\bar\partial$ falls on $|\tau^\ell|^{2\lambda}$. If τ does not occur in any other f_j^0, then the assumption $|\phi| \le C|f_1|^{m_1}$ implies that $\tilde\phi$ is divisible by $\tau^{\ell m_1}$. Hence $\tilde\phi$ and therefore also $\tilde\xi$ annihilates the singularity as before, so that the integral vanishes. We now claim that if, on the other hand, τ occurs in some of the other factors, then the integral vanishes because of the complete intersection assumption. Thus let us assume that τ occurs in $f_2^0, \dots, f_k^0$ but not in $f_{k+1}^0, \dots, f_r^0$. The forms $s_j = |f_j|^2\sigma_j$ are smooth and, moreover,

$$\tilde\gamma = \frac{s'_{k+1}\wedge\dots\wedge s'_r\wedge(\bar\partial s'_{k+1})^{m_{k+1}-1}\wedge\dots\wedge(\bar\partial s'_r)^{m_r-1}\wedge\tilde\xi}{(f_{k+1}^0)^{m_{k+1}}\cdots(f_r^0)^{m_r}}$$

is the pull-back of

$$\gamma = \frac{s_{k+1}\wedge\dots\wedge s_r\wedge(\bar\partial s_{k+1})^{m_{k+1}-1}\wedge\dots\wedge(\bar\partial s_r)^{m_r-1}\wedge\xi}{|f_{k+1}|^{2m_{k+1}}\cdots|f_r^0|^{2m_r}}.$$

Since the form γ has codegree $1+(m_1-1)+\cdots+(m_k-1)$ in $d\bar{z}$, which is strictly less than $m_1+\cdots+m_k = \operatorname{codim}\{f_1=\cdots=f_k=0\}$, the anti-holomorphic factor of the denominator vanishes on $\{f_1=\cdots=f_k=0\}$. Therefore, each term of its pull-back vanishes where $\tau=0$, so it must contain either a factor $\bar\tau$ or $d\bar\tau$. However, because of the assumption, the (pull-back) of the denominator contains no factor $\bar\tau$, so each term of $\tilde\gamma$ will contain $\bar\tau$ or $d\bar\tau$. Therefore, the integral that appears when $\bar\partial$ falls on $|\tau|^{2\lambda\ell}$ will vanish when $\lambda=0$. □

4. Explicit integral representation

We are now going to supply explicit proofs of Theorems 1.1 to 1.4. Since all of them are local, we assume that the functions f and ϕ are defined in a convex neighborhood X of the closure of the unit ball $\mathbb{B}$ in $\mathbb{C}^n$. We first recall the construction of weighted representation formulas for holomorphic functions from [1]. For fixed $z\in X$, let δ_η denote interior multiplication with the vector field

$$2\pi i\sum(\zeta_j-z_j)\frac{\partial}{\partial\zeta_j},$$

and let $\nabla_\eta=\delta_\eta-\bar\partial$. Then we have (lower indices denote bidegree), see [1],

Proposition 4.1. *Assume that z is a fixed point in X and $g=g_{0,0}+\ldots+g_{n,n}$ is a smooth form in X with compact support such that $\nabla_\eta g=0$ and $g_{0,0}(z)=1$. Then*

$$\phi(z)=\int g\phi=\int g_{n,n}\phi \tag{4.1}$$

for each holomorphic function ϕ in X.

For further reference we also notice, see [4], that:

(i) if g^1 and g^2 satisfy the assumptions in the proposition (it is enough that one of them has compact support), then also $g=g^1\wedge g^2$ does.

(ii) it is enough that g us smooth in a neighborhood of the point z.

Example 2. Let χ be a cutoff function in X that is identically 1 in a neighborhood of $\overline{\mathbb{B}}$. Moreover, let

$$s(\zeta,z)=\frac{1}{2\pi i}\frac{\partial|\zeta|^2}{|\zeta|^2-\bar\zeta\cdot z}.$$

Then for each $z\in\mathbb{B}$,

$$g=\chi-\bar\partial\chi\wedge\frac{s}{\nabla_\eta s}=\chi-\bar\partial\chi\wedge[s+s\wedge\bar\partial s+s\wedge(\bar\partial s)^2+\cdots+s\wedge(\bar\partial s)^{n-1}]=$$

$$\chi-\bar\partial\chi\wedge\sum_{k=1}^{n}\frac{1}{(2\pi i)^k}\frac{\partial|\zeta|^2\wedge(\bar\partial\partial|\zeta|^2)^{k-1}}{(|\zeta|^2-\bar\zeta\cdot z)^k}$$

a compactly supported form such that ∇_η-closed and $g_{0,0}(z)=1$. Moreover, g depends holomorphically on z. □

Example 3. Another possible choice is

$$g=\Big(1+\nabla_{\zeta-z}\frac{\bar\zeta\cdot d\zeta}{2\pi i(1-|\zeta|^2)}\Big)^{-\nu}=\Big(\frac{1-\bar\zeta\cdot z}{1-|\zeta|^2}-\omega\Big)^{-\nu} \tag{4.2}$$

for positive ν, where $\omega=(i/2\pi)\partial\bar\partial\log(1/(1-|\zeta|^2))$. It is $\mathcal{O}((1-|\zeta|^2)^\nu)$ near the boundary and therefore at least of class $C^{\nu-1}$. □

Now assume that f is a holomorphic $r\times m$-matrix, that we consider as a holomorphic morphism $f\colon E\to Q$ with respect to some fixed ON-bases for the trivial bundles $E\simeq\mathbb{C}^m$ and $Q\simeq\mathbb{C}^r$. We will construct the decomposition (1.1) from the currents U and R in Section 2, following an idea in [4]. First we choose holomorphic $(1,0)$-forms h_j in X, Hefer forms, such that

$$\delta_\eta h_j=f_j(\zeta)-f_j(z),$$

and let $h=\sum_1^m h_j\otimes\epsilon_j^*$. We may also assume that h_j, and hence h, depend holomorphically on the parameter z. Now $\delta_h\colon E_{k+1}\to E_k$, for $k\geq 1$, and hence

$$(\delta_h)_k\colon E_{k+1}\to E_1,\quad k\geq 0,$$

if $(\delta_h)_\ell=\delta_h^\ell/\ell!$. It is easily seen that

$$\delta_\eta(\delta_h)_k=(\delta_h)_{k-1}\delta_f-\delta_{f(z)}(\delta_h)_{k-1}. \tag{4.3}$$

So far δ_F has only acted on $(0,0)$-forms with values in $\Lambda^r E$. We now extend it to general (p,q)-forms, with the convention that one insert a minus sign when $p+q$ is odd. Thus we let

$$\delta_F\alpha=(-1)^{(r+1)(\deg\alpha+1)}\delta_{f_r}\cdots\delta_{f_1}\otimes\delta_{\epsilon_1}\cdots\delta_{\epsilon_r},$$

where $\deg\alpha$ is the degree of α in $\Lambda(E\otimes T^*(X))$. With this convention δ_F, as well as δ_f, will anti-commute with $\bar\partial$ and δ_η.

It is possible to find $(1,0)$-form-valued mappings $H_k^0\colon E_k\to\mathbb{C}$, such that

$$\delta_\eta H_1^0=\delta_F(\zeta)-\delta_{F(z)},\quad \delta_\eta H_k^0=H_{k-1}^0\delta_{f(\zeta)}-\delta_{F(z)}(\delta_h)_{k-1},\ k\geq 2. \tag{4.4}$$

The form H_1^0 is a usual Hefer form. The right hand side of the second equation for $k=2$ is now holomorphic and δ_η-closed, and it is well-known then that there exists a holomorphic solution H_2^0. We may as well assume that it depends holomorphically on the parameter z in X. The existence of H_k^0 in general follows by induction. For explicit choices of solutions in X, see [4]. We now define

$$H^1U=\sum_{k=1}^{\min(n+1,m-r+1)}(\delta_h)_{k-1}U_k,$$

and

$$H^0R=\sum_{k=1}^{\min(n,m-r+1)} H^0_kR_k.$$

Theorem 4.2. *If ϕ is holomorphic in X and g is the smooth form in Example 2, then we have the holomorphic decomposition*

$$\phi(z)=\delta_{F(z)}\int H^1U\wedge g\phi+\int H^0R\wedge g\phi,\quad z\in\mathbb{B}. \tag{4.5}$$

Proof. First we assume that $z\in\mathbb{B}\setminus Z$, and consider the form

$$g'=\delta_{F(z)}H^1U+H^0R.$$

We have that $g_{0,0}=\delta_F U_1$, and hence $g_{0,0}(z)=1$. Moreover, using (4.3) and (4.4) it is readily verified that $\nabla_\eta g'=0$. Since $g'\wedge g$ is smooth in a neighborhood of z it follows form Proposition 4.1 and the subsequent remarks that (4.5) holds for this z. However, since both sides of (4.5) are holomorphic in $\mathbb{B}$ the theorem is proved. □

In particular, $\Psi(z)=\int H^1U\wedge g\phi$ is an explicit solution to $\delta_{F(z)}\Psi=\phi$ if $R\phi=0$. We now consider this solution in more detail. In view of (2.2) and (2.3) we have, outside Z, that

$$(\delta_h)_{k-1}u_k=\sum_{|\alpha|=k-1}(\delta_{h_1})_{\alpha_1}\cdots(\delta_{h_r})_{\alpha_r}\big[\sigma_1\wedge\ldots\wedge\sigma_r\wedge(\bar\partial\sigma_1)^{\alpha_1}\wedge\ldots\wedge(\bar\partial\sigma_r)^{\alpha_r}\big]\otimes\epsilon^*.$$

Moreover, since we have the trivial metric,

$$\sigma_j=\sum_{i=1}^m\sigma_{ij}e_j,\quad j=1,\ldots,r,$$

are just the columns in the matrix $f^*(ff^*)^{-1}$. Suppressing the non-vanishing section ϵ^*, we have

Corollary 4.3. *Let f be a generically surjective holomorphic $r\times m$-matrix in X with rows f_j, considered as sections of the trivial bundle E^*, and assume that the hypothesis of Theorem 1.4 is fulfilled. Then*

$$\psi(z)=\int H^1U\wedge g\phi$$

is an explicit solution to $\delta_{F(z)}\psi(z)=\delta_{f_1(z)}\cdots\delta_{f_r(z)}\psi(z)=\phi(z)$ in $\mathbb{B}$, where $H^1U\phi$ is the value at $\lambda=0$ of (the analytic continuation of)

$$|f|^{2\lambda}\sum_{k=1}^{\min(n+1,m-r+1)}\sum_{|\alpha|=k-1}(\delta_{h_1})_{\alpha_1}\cdots(\delta_{h_r})_{\alpha_r}\big[\sigma_1\wedge\ldots\wedge\sigma_r\wedge(\bar\partial\sigma_1)^{\alpha_1}\wedge\ldots\wedge(\bar\partial\sigma_r)^{\alpha_r}\big]\phi \tag{4.6}$$

and g is the form in Example 2. If $m-r+1 \le n$, then $H^1U\phi$ is locally integrable, and the value at $\lambda = 0$ exists in the ordinary sense.

Proof. It remains to verify the claim about the local integrability. In fact, after a resolution of singularities, cf., (2.5), it follows that $U_k\phi$ is locally integrable if $|\phi| \lesssim |F|^k$. If $m-r+1 \le n$, then the sum terminates at $k = m-r+1$, and therefore the current is locally integrable; otherwise the worst term is like $U_{n+1}\phi$, and it will not we locally integrable in general. □

If all the f_j takes values in different bundles E_j^* and $E = \oplus E_j$, then we can simplify the expression for H^1U further. In this case, cf., Section 3,

$$\sigma_j = \sum_{i=1}^{m_j} \frac{\bar{f}_j^i}{|f_j|^2} e_{ij}, \quad j = 1, \ldots, r.$$

Moreover, with natural choices of Hefer forms h_j, δ_{h_j} will vanish on forms with values in E_k for $k \neq j$, and hence we get

Corollary 4.4. *Let f_j be m_j-tuples of functions, considered as sections of the trivial bundles E_j^* over X. If the conditions of Theorem 1.2 or 1.3 are fulfilled, or if all f_j are equal to some fixed m-tuple f, and the condition in Theorem 1.1 is fulfilled, then*

$$\psi(z) = \int H^1U\phi \wedge g$$

is an explicit solution to $\delta_{f_1(z)} \cdots \delta_{f_r(z)} \psi(z) = \phi(z)$ in $\mathbb{B}$, where $H^1U\phi$ is the value at $\lambda = 0$ of (the analytic continuation of)

$$(4.7) \quad |f|^{2\lambda} \sum_{k=1}^{n+1} \sum_{|\alpha|=k-1} (\delta_{h_1})_{\alpha_1}[\sigma_1 \wedge (\bar\partial\sigma_1)^{\alpha_1}] \wedge \ldots \wedge (\delta_{h_r})_{\alpha_r}[\sigma_r \wedge (\bar\partial\sigma_r)^{\alpha_r}]\phi.$$

$N = \min(n+1, m-r+1)$, and g is the form in Example 2.

In the case of Theorems 1.2 and 1.3, only terms such that $\alpha_j \le m_j$ actually occur. In the case of Theorem 1.1 we have only terms such that $k \le m$.

We conclude this paper with some brief comments on Berndtsson's classical division formula from [7]. As mentioned in the introduction, the first known explicit formula for the Briançon-Skoda theorem ($r = 1$) was in Theorem 9.5 in [4]; in fact, it is identical to the formula above in the case $r = 1$, and it is different from Berndtsson's formula. Surprisingly enough it was recently discovered, [12], that the general case of the Briançon-Skoda theorem, i.e., Theorem 1.1, actually can be obtained from Berndtsson's classical formula, and we will sketch the proof below; for more details, see [12]. However, we see no way of proving any of the variations discussed in this paper by Berndtsson type formulas.

As before we consider the given m-tuple f as a section of the trivial bundle E^* over X. Let s be the section of E with minimal norm such that $fs = |f|$. Then $s = |f|^2\sigma$ in the previous notation. If the metric is trivial and $f = \sum f_j e_j^*$, then $s = \sum \bar{f}_j e_j$. For $\epsilon > 0$, let

$$\sigma^\epsilon = \frac{s}{|f|^2 + \epsilon},$$

let $h = h_j e_j^*$ be a Hefer form as before, and let

$$g' = 1 - \nabla_\eta h \cdot \sigma^\epsilon = \frac{\epsilon}{|f|^2 + \epsilon} + f(z) \cdot \sigma^\epsilon + h \cdot \bar{\partial}\sigma^\epsilon.$$

By Proposition 4.1 we have the representation formula

$$\phi(z) = \int \Big(\frac{\epsilon}{|f|^2 + \epsilon} + f(z) \cdot \sigma^\epsilon + h \cdot \bar{\partial}\sigma^\epsilon \Big)^{\min(n+1,m)+r-1} \phi \wedge g. \tag{4.8}$$

At least if the form g from Example 3 is used, the resulting formula is precisely of the type in [7] though derived in a somewhat different way.

Proposition 4.5. *Assume that f, ϕ are holomorphic in X and that*

$$|\phi| \lesssim |f|^{\min(n,m)+r-1}, \tag{4.9}$$

holds. When $\epsilon \to 0$ the formula (4.8) *converges to an explicit representation of ϕ in $\mathbb{B}$ as an element in the ideal $(f)^r$.*

Sketch of proof. To begin with we assume that $m > n$. Then the power of g' is $n + r$, and expanding we get that

$$(g')^{n+r}\phi = \sum_{\ell=1}^{r} c_\ell \Big(\frac{\epsilon}{|f|^2 + \epsilon} + h \cdot \bar{\partial}\sigma^\epsilon \Big)^{n+\ell} (f(z) \cdot \sigma^\epsilon)^{r-\ell}\phi + \cdots ,$$

where $\cdots$ denote terms in $(f)^r$. Taking for granted that these latter terms actually converge to currents with values in $(f)^r$ when $\epsilon \to 0$, we have to prove that the first terms tend to zero. When expanding further, for degree reasons, the worst term that appears is

$$\frac{\epsilon}{|f|^2 + \epsilon} (h \cdot \bar{\partial}\sigma^\epsilon)^n (f(z) \cdot \sigma^\epsilon)^{r-1}\phi.$$

Using the technique in Section 2 we may assume that $f = f_0 f'$ where $f' \neq 0$, and then this term is dominated by

$$\epsilon \frac{|df_0||f_0|}{(|f_0|^2|f'|^2 + \epsilon)^2}. \tag{4.10}$$

Assuming furthermore, as we may, that f_0 is a monomial, it is readily checked that the expression (4.10) tends to 0 in L^1_{loc} when $\epsilon \to 0$. We then consider the case when $m \leq n$. When expanding $(g')^{m+r-1}\phi$, besides terms in $(f)^r$, the worst term that appears is

$$(h \cdot \bar{\partial}\sigma^\epsilon)^m (f(z) \cdot \sigma^\epsilon)^{r-1}\phi. \tag{4.11}$$

Using that

$$\Big(h\cdot\bar\partial\frac{s}{|f|^2+\epsilon}\Big)^m = h_1\wedge\ldots\wedge h_m\wedge\epsilon\frac{\bar\partial s_m\wedge\ldots\wedge\bar\partial s_1}{(|f|^2+\epsilon)^{m+1}},$$

it follows that also (4.11) is dominated by (4.10), and so the proposition is proved. □

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