On Global Existence for the Spherically Symmetric Einstein-Vlasov System in Schwarzschild Coordinates

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Abstract

The spherically symmetric Einstein-Vlasov system in Schwarzschild coordinates (i.e. polar slicing and areal radial coordinate) is considered. An improved continuation criterion for global existence of classical solutions is given. Two other types of criteria which prevent finite time blow-up are also given.

1 Introduction

A central issue in general relativity is the problem of spherically symmetric gravitational collapse and to determine whether or not the weak- and strong cosmic censorship conjectures (CCC) hold true in this case. These conjectures were raised by Penrose in the late '60s. By now they have precise mathematical formulations [8] and are considered to be among the most important problems in classical general relativity. In brief the meaning of the weak CCC is as follows: The Penrose singularity theorem says that there are initial data that will lead to singular spacetimes, i.e. geodesically incomplete spacetimes. The weak cosmic censorship states that generically, singularities will always be covered by an event horizon and distant observers will never be able to "see" the singularities. Singularities that can be "seen" are called naked. To answer the CCC a detailed picture of the global properties of the solutions to the Cauchy problem for the Einstein matter equations are necessary.

In 1999 Demetrios Christodoulou finished his long term project on the spherically symmetric Einstein-Scalar Field system (ESF) by proving both the weak- and strong cosmic censorship conjecture [6] by a careful analysis of the Cauchy problem. This important work is the only case where

such a study has been completed in an affirmative sense. On the contrary, Christodoulou has shown that when the matter model is dust (i.e. a pressureless perfect fluid) then there are naked singularities generically [9]. It is however a rather common belief that a dust model is not an appropriate matter model for studying cosmic censorship since the pressure is assumed to be zero. A study of a more general fluid model for gravitational collapse suffers at present from a lack of satisfying mathematical results of such equations in the absence of gravity and in the case of Newtonian gravity. A successful treatment in these cases seems necessary before it will be mathematically meaningful to couple such a model to the Einstein equations. A matter model that is known to have satisfying mathematical features is the Vlasov model which gives a kinetic description of the matter. In particular it was shown independently by Pfaffelmoser [21] and Lions and Perthame [17] that in the case of Newtonian gravity, where a kinetic description gives rise to the Vlasov-Poisson system, classical solutions exist globally in time. This system can now be said to be well-understood mathematically and it is therefore natural to choose the Vlasov model for investigating gravitational collapse in general relativity. In particular this matter model is phenomenological in contrast to a scalar field which is said to be field theoretical. It would certainly be desirable to establish similar results for a phenomenological model as Christodoulou has done for a scalar field. One way to attack the cosmic censorship conjecture is to prove global existence in a singularity avoiding coordinate system. It is then often conjectured that cosmic censorship would follow straightfowardly. The first attempt by Christodoulou on the ESF system was indeed of this nature. However, in that case it turned out that such an approach was doomed to failure [8] by finding initial data that lead to naked singularities [5]. In the case of Vlasov matter it is still an open question whether or not naked singularities form for any initial data. (Note that cosmic censorship holds true if the set of initial data which generates naked singularities is a null set). Thus the approach of proving global existence in a certain coordinate system is yet a possile path to weak CCC. The aim of this paper is to present some new results on the problem of global existence of the spherically symmetric Einstein-Vlasov system in Schwarzschild coordinates. In particular we give an improved continuation criterion. We also consider two other types of criteria, one in which we study the case when all matter is assumed to fall inwards. A continuation criterion for this system was first given by Rein and Rendall in [23]. To discuss the content of this criterion we need to introduce the main concept in kinetic theory, the distribution function. Consider a collection of particles, where the particles are stars, galaxies or even clusters of galaxies. A characteristic

feature of kinetic theory is that the model is statistical and that the particle system is described by a distribution function $f = f(t, x, p)$, which represents the density of particles with given space-time position $(t, x) \in \mathbb{R} \times \mathbb{R}^3$ and momentum $p \in \mathbb{R}^3$. Let us define the function $Q(t)$ by

$$Q(t) := \sup\{|p| : \exists (s, x) \in [0, t] \times \mathbb{R}^3 \text{ such that } f(s, x, p) \neq 0\}. \quad (1)$$

Hence, there are no particles at time t having larger momentum than $Q(t)$ in the system. If we now consider a solution of the spherically symmetric Einstein-Vlasov system on a time interval $[0, T[$, then it is proved in [23] that if $Q(t)$ is bounded on $[0, T[$ then the solution can be extended beyond T. This is called a continuation criterion. This is analogous to the situation for the Vlasov-Maxwell system where the same condition ensures that the solution can be extended. This was proved in a classical work by Glassey and Strauss [15] but recent results by Pallard [19], [20] improve this criterion.

The outline of the paper is as follows. The Einstein-Vlasov system in Schwarzschild coordinates is introduced in section 2. A brief review of previous results is also given here. In section 3 we present some general a priori bounds not previously in the literature. In section 4 we make a comparison between the Einstein-Vlasov- and the Vlasov-Poisson system to point out the main differencies. An improved continuation criterion is proved in section 5 using a mixed L^{3+} and weighted L^∞ approach. In section 6 an approach based on hypotheses along characteristics is given. A discussion about what is expected to happen if these bounds are not satisfied is also included. In section 7 the case when all matter is falling inwards is studied. It is seen that the well-known result that a Schwarzschild spacetime with mass M has closed null rays when $r = 3M$, also plays an essential role for the Einstein-Vlasov system.

2   The Einstein-Vlasov system

For a derivation of the system given below we refer to [22] and [25]. In Schwarzschild coordinates the spherically symmetric metric takes the form

$$ds^2 = -e^{2\mu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (2)$$

The Einstein equations are

$$e^{-2\lambda}(2r\lambda_r - 1) + 1 = 8\pi r^2 \rho, \quad (3)$$
$$e^{-2\lambda}(2r\mu_r + 1) - 1 = 8\pi r^2 p, \quad (4)$$

3

$$\lambda_t = -4\pi r e^{\lambda+\mu} j, \tag{5}$$

$$e^{-2\lambda}(\mu_{rr} + (\mu_r - \lambda_r)(\mu_r + \frac{1}{r})) - e^{-2\mu}(\lambda_{tt} + \lambda_t(\lambda_t - \mu_t)) = 4\pi q. \tag{6}$$

The Vlasov equation for the density distribution function $f = f(t, r, w, F)$ reads

$$\partial_t f + e^{\mu-\lambda}\frac{w}{E}\partial_r f - (\lambda_t w + e^{\mu-\lambda}\mu_r E - e^{\mu-\lambda}\frac{F}{r^3 E})\partial_w f = 0, \tag{7}$$

where

$$E = E(r, w, F) = \sqrt{1 + w^2 + F/r^2}. \tag{8}$$

The matter quantities are defined by

$$\rho(t, r) = \frac{\pi}{r^2}\int_{-\infty}^{\infty}\int_{0}^{\infty} E f(t, r, w, F) \, dw dF, \tag{9}$$

$$p(t, r) = \frac{\pi}{r^2}\int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{w^2}{E} f(t, r, w, F) \, dw dF, \tag{10}$$

$$j(t, r) = \frac{\pi}{r^2}\int_{-\infty}^{\infty}\int_{0}^{\infty} w f(t, r, w, F), \, dw dF \tag{11}$$

$$q(t, r) = \frac{\pi}{r^4}\int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{F}{E} f(t, r, w, F) \, dw dF. \tag{12}$$

For technical reasons we also introduce the quantity

$$k(t, r) = \frac{\pi}{r^2}\int_{-\infty}^{\infty}\int_{0}^{\infty} |w| f(t, r, w, F) \, dw dF, \tag{13}$$

so that k is nonnegative and dominates p and $|j|$ but is dominated by ρ. The variables $w \in (-\infty, \infty)$ and $F \in [0, \infty)$ are the radial component of the momentum and the square of the angular momentum respectively. The following boundary conditions are imposed

$$\lim_{r\to\infty} \lambda(t, r) = \lim_{r\to\infty} \mu(t, r) = \lambda(t, 0) = 0, \; \forall t \geq 0. \tag{14}$$

The last condition follows by requiring a regular center. We point out that the Einstein equations are not independent and that e.g. the equations (5) and (6) follow by (3), (4) and (7).

For an extensive review of previous results of this system see [1]. Here we give a short summary of the results concerning the Cauchy problem. The issue of global existence for this system was first investigated by Rein and Rendall in [23] where they obtain global existence for small (compactly

4

supported) initial data and in addition they show that the resulting space-time is geodesically complete. They also prove the continuation criterion mentioned in the introduction. In [27] Rendall studies the same system using a different set of coordinates (maximal slicing and isotropic radial coordinate). Global existence for small initial data is obtained as well as an equivalent continuation criterion. In [28] Rendall shows that there exists initial data leading to trapped surfaces and therefore to singular spacetimes by the Penrose singularity theorem. The natural question is then if the weak CCC holds. To investigate this it is necessary to consider large initial data. This is done in [25] and [27]. In the former case Schwarzschild coordinates are used and it is proved that if matter stays uniformly away from the center of symmetry then global existence follows. In the latter case where the maximal and isotropic gauge is used a similar result is shown assuming an additional condition on one of the metric coefficients. The advantage being that the result is obtained in a more direct way and the analysis of the Vlasov equation is not necessary. Recently, Dafermos and Rendall [11] have shown a similar result for the Einstein-Vlasov system in double-null coordinates which has a very interesting and important consequence. In the proof of the CCC for the ESF system Christodoulou analyzes the formation of trapped surfaces and their presence play an important role in his proof. Dafermos [10] then strengthens the relation between trapped surfaces and weak CCC; it is shown that a single trapped surface or marginally trapped surface in the maximal development implies weak CCC. The theorem in [10] requires certain hypotheses on the solutions of the Einstein matter system. The main purpose of [11] is to show that these hypotheses are satisfied for the spherically symmetric Einstein-Vlasov system, which is indeed shown to be the case. This result might thus be useful in the search for a proof of the weak CCC. Two numerical investigations have been carried out where the primarily goal was to study critical collapse and determine if Vlasov matter is of type I or type II, i.e. if arbitrary small black holes can form (type II) or if there is a mass gap (type I). In the first study [26] Schwarzschild coordinates were used and in the second [18] maximal slicing and an areal radial coordinate was used. Both of these studies indicate that Vlasov matter is of type I, and moreover, no signs of singularity formation could be seen in either of these two cases which support the idea that global existence holds in these two coordinate systems. The investigation in this paper concerns the Einstein-Vlasov system in Schwarzschild coordinates but we plan to investigate other choices of gauge conditions both analytically and numerically in the future.

Let us now specify the sets of initial data that will be considered. Regular

5

initial data is defined as the set of nonnegative compactly supported smooth functions $f_0 = f_0(r, w, F)$, such that

$$4\pi^2 \int_0^r \int_{-\infty}^{\infty} \int_0^{\infty} E f_0(r, w, F) dw dF dr < \frac{r}{2}.$$

The last condition says that no trapped surfaces are present initially. Given $R_- > 0$, we define the class of regular initial data with radial cut-off, I_{R_-}, as the subset of regular initial data such that $f_0 = 0$ when $r \leq R_-$. Likewise, given $F_- > 0$, the subset I_{F_-} such that $f_0 = 0$ when $F \leq F_-$, is called regular initial data with angular momentum cut-off, and consequently, I_{R_-, F_-} is the subset of regular initial data with radial and angular momentum cut-off. Thus, for initial data in I_{R_-, F_-} there is no matter in a ball around the centre and there is a lower bound on the angular momentum F. By conservation of angular momentum the latter property is then true for all times. (Angular momentum cut-off has shown to be essential also in previous works, see [14] and [3].) We also make the definitions

$$F_+ := \sup\{F : \exists (r, w) \in \mathbb{R}^2 \text{ such that } f_0(r, w, F) \neq 0\}, \qquad (15)$$

and

$$R_+ := \inf\{R : f_0(r, \cdot, \cdot) = 0 \text{ for all } r \geq R\}. \qquad (16)$$

Let us write down a couple of known facts about the system (3)-(14). A solution to the Vlasov equation can be written

$$f(t, r, w, F) = f_0(R(0, t, r, w, F), W(0, t, r, w, F), F), \qquad (17)$$

where R and W are solutions of the characteristic system

$$\frac{dR}{ds} = e^{(\mu - \lambda)(s, R)} \frac{W}{E(R, W, F)}, \qquad (18)$$

$$\frac{dW}{ds} = -\lambda_t(s, R) W - e^{(\mu - \lambda)(s, R)} \mu_r(s, R) E(R, W, F)$$

$$+ e^{(\mu - \lambda)(s, R)} \frac{F}{R^3 E(R, W, F)}, \qquad (19)$$

such that the characteristic $(R(s, t, r, w, F), W(s, t, r, w, F), F)$ goes through the point (r, w, F) when $s = t$. This representation shows that f is nonnegative for all $t \geq 0$, $\|f\|_\infty = \|f_0\|_\infty$, and that for regular initial data with angular momentum cut-off, $f(t, r, w, F) = 0$ if $F \leq F_-$. There are two

6

known conserved quantities for the Einstein-Vlasov system, conservation of the number of particles

$$4\pi^2 \int_0^\infty e^{\lambda(t,r)} \left(\int_{-\infty}^\infty \int_0^\infty f(t,r,w,F) dF dw \right) dr,$$

and conservation of the ADM mass

$$M := 4\pi \int_0^\infty r^2 \rho(t,r) dr. \tag{20}$$

These conservation laws follow from general argments but can easily be obtained by taking the time derivative of the integral expressions and make use of the Einstein equations and the Vlasov equation. The mass function m is defined by

$$m(t,r) := 4\pi \int_0^r \eta^2 \rho(t,\eta) d\eta, \tag{21}$$

and by integrating (4) we find

$$e^{-2\lambda(t,r)} = 1 - \frac{2m(t,r)}{r}. \tag{22}$$

Thus, as long as the solution exists we have

$$\frac{m(t,r)}{r} < 1/2. \tag{23}$$

A fact that will be used frequently is that

$$\mu + \lambda \leq 0.$$

This is easily seen by adding the equations (3) and (4), which gives

$$\lambda_r + \mu_r \geq 0,$$

and then using the boundary conditions on λ and μ. From (22) we see that $\lambda \geq 0$, and it follows that $\mu \leq 0$. Finally, we note that in [23] a local existence theorem is proved and it will be used below that a classical (or regular) solution exists on some time interval $[0,T[$.

3   General a priori bounds

In this section we show a few a priori bounds which are easy to derive but cannot be found in the literature. Together with the conserved quantities mentioned in section 2, and the inequality (23), these results constitute the only known a priori bounds for the spherically symmetric Einstein-Vlasov system.

Lemma 1 *Let (f, μ, λ) be a regular solution to the Einstein-Vlasov system. Then*

$$\int_0^\infty 4\pi(\rho + p)re^{2\lambda}e^{\mu+\lambda}dr \leq 1, \tag{24}$$

$$\int_0^\infty (\frac{m}{r^2} + 4\pi p)re^{2\lambda}e^{\mu}dr \leq 1. \tag{25}$$

Proof. Using the boundary conditions (14) and that $\mu + \lambda \leq 0$ we get

$$
\begin{aligned}
1 \geq 1 - e^{\mu+\lambda}(t,0) &= \int_0^\infty \frac{d}{dr}e^{\mu+\lambda}dr \\
&= \int_0^\infty (\mu_r + \lambda_r)e^{\mu+\lambda}dr.
\end{aligned}
$$

The right hand side equals (24) by equations (3) and (4) which completes the first part of the lemma. The second part follows by studying e^μ instead of $e^{\mu+\lambda}$.

$\square$

Next we show that not only $\rho(t, \cdot) \in L^1$, which follows from the conservation of the ADM mass, but that also $e^{2\lambda}\rho(t, \cdot) \in L^1$.

Lemma 2 *Let (f, μ, λ) be a regular solution to the Einstein-Vlasov system. Then*

$$\int_0^\infty r^2 e^{2\lambda}\rho(t,r)dr \leq \int_0^\infty r^2 e^{2\lambda}\rho(0,r)dr + \frac{t}{8\pi}. \tag{26}$$

Proof. Using the Vlasov equation we obtain

$$
\begin{aligned}
\partial_t \left(r^2 e^{2\lambda}\rho(t,r)\right) &= -\partial_r \left(r^2 e^{\mu+\lambda}j\right) - re^{\mu+\lambda}2je^{2\lambda}\frac{m}{r} \\
&\leq -\partial_r \left(r^2 e^{\mu+\lambda}j\right) + \frac{1}{2}re^{\mu+\lambda}(\rho + p)e^{2\lambda}.
\end{aligned}
$$

Here we used that $m/r \leq 1/2$ together with the elementary inequality $2|j| \leq \rho + p$, which follows from the expressions (9)-(11). In view of (18) we see that $\lim_{r\to\infty} r^2 j(t,r) = 0$, since the initial data has compact support. Since the solution is regular and hence bounded the boundary term at $r = 0$ also vanishes (as a matter of fact spherical symmetry even implies that $j(t,0) = 0$). Thus, by lemma 1 we get

$$\partial_t \int_0^\infty r^2 e^{2\lambda}\rho(t,r)dr \leq \frac{1}{8\pi},$$

which completes the proof of the lemma.

□

A natural question is if the inequalies (24) and (25) can bound any matter quantity in an L^p−space with $p > 1$, in view of the fact that merely r and not r^2 is present in the integrand. For the Vlasov-Poisson- and the Vlasov-Maxwell system such L^p estimates (with $p > 1$) play an essential role in the global existence proofs (although the problem in $1+3$ dimensions is open for the Vlasov-Maxwell system affirmative results exist in lower dimensions [13]) and such results are thus desirable. We have the following result (where $L^p(R^3, \omega)$ means the L^p space with weight ω):

Lemma 3 *Let (f, μ, λ) be a regular solution to the Einstein-Vlasov system for initial data with angular momentum cut-off. Then*

$$q(t, \cdot) \in L^{5/4}(R^3, e^{\mu+\lambda}e^{2\lambda}).$$

Proof. Here we use the momentum variable v instead of (w, F). These are related as follows

$$w = \frac{x \cdot v}{r}, \ F = x^2 v^2 - (x \cdot v)^2 = |x \times v|^2,$$

where $r = |x|$. From [22] it follows that q now takes the form

$$q = \int_{R^3} \frac{Ff}{r^2\sqrt{1+v^2}}dv.$$

By the assumption on the initial data we have that $r\sqrt{1+v^2} \geq \sqrt{F_-}$ on the support of f and we get

$$
\begin{align}
q &= \int_{R^3} \frac{Ff}{r^2\sqrt{1+v^2}}dv \tag{27} \\
&\leq \int_{|v| \leq K} \frac{Ff}{r^2\sqrt{1+v^2}}dv + \int_{|v| \geq K} \frac{Ff}{r^2\sqrt{1+v^2}}dv \tag{28} \\
&\leq \int_{|v| \leq K} \frac{Ff\sqrt{1+v^2}}{r^2(1+v^2)}dv + \frac{1}{K}\int_{|v| \geq K} \frac{Ff}{r^2}dv \tag{29} \\
&\leq C(F_-, F_+, \|f\|_\infty)K^4 + \frac{C}{K}\int_{R^3} \frac{Ff\sqrt{1+v^2}}{r}dv \tag{30} \\
&\leq C\left(\int_{R^3} \frac{Ff\sqrt{1+v^2}}{r}dv\right)^{4/5}. \tag{31}
\end{align}
$$

9

Here we took

$$K = \left(\int_{R^3} \frac{Ff\sqrt{1+v^2}}{r} dv \right)^{1/5}.$$

Now since $F \leq F_+$ we get that

$$q^{5/4} \leq C \int_{R^3} \frac{f\sqrt{1+v^2}}{r} dv \leq C\frac{\rho}{r}, \qquad (32)$$

and from (24) we obtain

$$\int_0^\infty r^2 q^{5/4} e^{\mu+\lambda} e^{2\lambda} dr \leq C \int_0^\infty \rho r e^{2\lambda} e^{\mu+\lambda} dr \leq C,$$

and the lemma follows.

$\square$

Remark. The first factor $e^{\mu+\lambda}$ in the weight $e^{\mu+\lambda} e^{2\lambda}$ is the square root of the determinant of the metric and can thus be viewed as a part of the integration measure. The second factor $e^{2\lambda}$ is always greater than one which then sharpens the bound for q, in analogy with lemma 2.

4 A comparison between the EV- and the VP system

It is instructive to compare the structure of the spherically symmetric Vlasov-Poisson system, where the gravitational field is Newtonian, with the spherically symmetric Einstein-Vlasov system in Schwarzshild coordinates (in other coordinates such a comparison can be slightly different). The former system is the limit of the latter as the speed of light goes to infinity, as was proved in [24], but the mathematical understanding of these two systems are nevertheless very different. In this section we try to point out the main differencies between these systems from a mathematical point of view. A Newtonian gravitational system is modelled in kinetic theory by the Vlasov-Poisson system which reads

$$\partial_t f + p \cdot \nabla_x f - E(t,x) \cdot \nabla_p f = 0, \qquad (33)$$
$$E = \nabla\phi, \quad \Delta\phi = \rho := \int_{R^3} f(t,x,p)dp. \qquad (34)$$

In the case of spherical symmetry Batt [4] gave a global existence proof in 1977, but later on this was also obtained for general data, independently

by Pfaffelmoser [21] and Lions and Perthame [17]. The method of proof by Pfaffelmoser (see also [29] and [16]) is based on a study of the characteristic system associated with the Vlasov equation. The quantity that plays a major role is $Q(t)$, which was defined in the introduction. If $Q(t)$ can be bounded by a continuous function in t, then global existence follows. In [16] the sharpest bound of $Q(t)$ has been given for the system (33)-(34), where it is proved that $Q(t)$ grows at most as $C(1+t)\log(C+t)$. As have already been pointed out, the continuation criterion in [23] shows that a bound of $Q(t)$ is sufficient for global existence also in the case of the spherically symmetric Einstein-Vlasov system. It is therefore natural to compare the characteristic systems in the two cases, since the natural way to control $Q(t)$ is to obtain bounds on the solutions to this system. In the case of the spherically symmetric Einstein-Vlasov system this means to bound $E(t) = \sqrt{1 + W^2(t) + F/R^2(t)}$. It follows from the characteristic equations (18) and (19) that E satisfies

$$\frac{d}{ds}E(s) = (-W)\left[e^{\mu+\lambda}\left(\frac{m}{R^2} + 4\pi R\left(\frac{(-W)}{E}j + p\right)\right)\right]. \tag{35}$$

Here

$$m(t,r) = \int_0^r \eta^2 \rho(t,\eta)\, d\eta.$$

A comparison with the analog equation for the Vlasov-Poisson system leads to the following observations:

- the factor (-W) is only present in EV not VP

- only the m-term is present in VP

- the dependence on j and p is pointwise

Having a Grönvall estimate in mind the first point is a severe difficulty for the Einstein-Vlasov system since the magnitude of $|W|$ and E might be of the same order, and thus gives much less room for estimating the right-hand side. The second and third point concern the regularity of the right hand side. The quasi local mass m is an average of the energy density ρ and is thus more regular than ρ itself, and this fact is important in both the Pfaffelmoser- and the Lions-Perthame methods of proof. In the EV case there is a pointwise dependence on the matter terms j and p which indeed is a difficulty but as will be seen in section 5 this dependence is in a sense mild. Other additional difficulties are the following facts:

- Q^4 instead of Q^3 bound the matter terms $\rho, |j|$, and p

- conservation of ADM mass does not lead to an L^p bound of ρ with $p > 1$

The densities in the two cases are (in Minkowski space) given by

$$\rho^{VP} = \int_{R^3} f \, dp, \text{ and } \rho^{EV} = \int_{R^3} \sqrt{1 + |p|^2} f \, dp.$$

Again, having a Grönvall estimate in mind it is a disadvantage that ρ^{EV} (and the other matter terms) are bounded in terms of Q^4 instead of Q^3 as is the case for ρ^{VP}. Conservation of energy in the VP case means that

$$\int_{R^3} \int_{R^3} |p|^2 f \, dp \, dx,$$

is bounded and this in turn implies that $\rho^{VP}(t, \cdot) \in L^{5/3}$, whereas conservation of the ADM mass does not qualify ρ^{EV} into any L^p space with $p > 1$. In both methods of proof in the VP case such an a priori bound plays a fundamental role. We conclude this section by a comparison with the relativistic Vlasov-Poisson system. By starting from the relativistic Vlasov-Maxwell system, which describes a charged plasma, and assuming spherical symmetry the following system is obtained

$$\partial_t f + \hat{p} \cdot \nabla_x f + \beta E(t, x) \cdot \nabla_p f = 0, \tag{36}$$

$$E = \nabla \phi, \quad \Delta \phi = \rho := \int_{R^3} f(t, x, p) dp. \tag{37}$$

Here $\hat{p} = p/\sqrt{1 + p^2}$ and $\beta = 1$. This system makes sense also without the assumption of spherical symmetry as well as with $\beta = -1$ and is called the relativistic Vlasov-Poisson system with a repulsive ($\beta = 1$) or attractive ($\beta = -1$) potential. This system is not Lorentz invariant and is not a physically correct equation (see [2] for a discussion of this point) but from a formalistic point of view *the only difference* from the VP system is that the p in front of the $\nabla_x f$−term has been changed to $\hat{p}$, and in this sense resembles the EV system. Nevertheless, the mathematical results in the two cases are very different. The most relevant results in this context can be summarized as follows (we refer to [12], [14] for details):

- $\beta = 1$. A general global existence result is still missing for the RVP system but the spherically- and the cylindrically symmetric cases have been settled.

- $\beta = -1$. Solutions blow up in finite time for a large class of initial data.

We see that even in the repulsive case much less is known than for the Vlasov-Poisson system, and apparently, in the more relevant attractive case, the situation is very different. In conclusion, it is far from clear that the well-developed machinery for the classical Vlasov-Poisson system can contribute much to the understanding of the spherically symmetric Einstein-Vlasov system. Furthermore, the blow-up result for the relativistic Vlasov-Poisson system with $\beta = -1$ indicates that if the Einstein-Vlasov system prevents blow-up in finite time, the method for proving it might be quite subtle.

5 A mixed $L^{3+}-$ and weighted L^∞ approach

In this section we make assumptions on the matter quantities and show that these are sufficient for bounding $Q(t)$. Since these assumptions are satisfied when $Q(t)$ is bounded we obtain an improved continuation criterion. From the proof it will be clear that the pointwise dependence on j and p in the characteristic equation, discussed in the previous section, is in a sense mild, i.e. as long as j and p do not contain too sharp and narrow peaks then their contributions can be controlled.

Assume that there exist constants A and B such that for $t \in [0, T[$,

$$r^2 k(t, r) \leq A, \tag{38}$$

$$\|\rho(t, \cdot)\|_{L^{3+}(B_T^3)} \leq B. \tag{39}$$

Here B_T^3 is the ball $\{x : |x| \leq R_+ + T\}$ and L^{3+} means that it is sufficient that $\rho \in L^{3+\epsilon}$ for any given $\epsilon > 0$. Recall also the definition of k in equation (13). First note that these assumptions are weaker than the assumption that $Q(t)$ is bounded on $[0, T[$. Indeed, from (9) it follows that $\rho \leq CQ(t)^4$. Therefore a bound on $Q(t)$ implies that

$$\rho(t, r) \leq C, \text{ on } [0, T[, \tag{40}$$

which in view of the fact that the severe difficulties are concentrated at $r = 0$, is significantly stronger than (38) (recall $k \leq \rho$). That (39) follows from (40) is immediate. It is also instructive to see that (38) is a natural hypothesis in view of the fact that

$$\frac{m}{r} \leq \frac{1}{2} \text{ on } [0, T[. \tag{41}$$

From the boundary condition $\lambda(t, 0) = 0$ it also follows that

$$\frac{m}{r} \to 0, \text{ as } r \to 0.$$

13

If we assume that there exists a constant C and a positive number α such that

$$\rho(t, r) \leq \frac{C}{r^\alpha},$$

and then estimate m/r we get

$$\frac{m}{r} = \frac{\int_0^r \eta^2 \rho(t, \eta) d\eta}{r} \leq C r^{2-\alpha}. \tag{42}$$

Thus, if such an estimate for ρ exists then $\alpha \leq 2$, which shows that (38) is natural. However, j or p might have sharp, and then by (41) necessarily narrow, peaks so that such an estimate is not valid at all. We have not been able to exlude that such peaks exist which is the reason for imposing assumption (38). This assumption still allows for sharp peaks in the matter quantity q, which is the part of ρ which is associated to the tangential momenta. The potential difficulty of sharp peaks in j and p is of major interest to understand and it might be that this difficulty is a particular feature of the Einstein-Vlasov system in Schwarzschild coordinates. We have the following theorem.

Theorem 1 *Consider a solution of the Einstein-Vlasov system for regular initial data with radial cut-off on its maximal time interval $[0, T[$ of existence. Assume that (38) and (39) hold. Then $Q(t)$ is bounded on $[0, T[$, and $T = \infty$.*

Proof. The quantity that will be considered is E, and the equation along a characteristic $(R(t), W(t), F)$ for E reads

$$\frac{d}{ds} E(R(s), W(s), F) = \frac{(-W)}{E} \left[e^{\mu+\lambda} \frac{m}{R^2} + 4\pi R e^{\mu+\lambda} \left(\frac{(-W)}{E} j + p \right) \right] E. \tag{43}$$

Note that we often choose to write $(-W)$ since the major case of interest is when the characteristic is ingoing, i.e. $W < 0$. It follows that

$$\log E(T) = \log E(0) + \int_0^T \frac{(-W)(t)}{E(t)} \left[e^{\mu+\lambda} \frac{m}{R^2} + 4\pi R e^{\mu+\lambda} \left(\frac{(-W)}{E} j + p \right) \right] dt. \tag{44}$$

The maximal time interval of existence is $[0, T[$ and we can assume that $T < \infty$. Then there exists a characteristic $(R(t), W(t), F)$ and a $T_0 \in [R_-, T]$ such that

$$\lim_{t \to T_0} R(t) = 0,$$

since otherwise the finiteness of T would contradict the result in [25]. (It is not necessary for our arguments that the limit do exist when $T_0 = T$,

but due to the causal character of spacetime it exists.) We consider the set of all characteristics with this property and we pick a characteristic $(R^*(t), W^*(t), F^*)$ in this set with an associated time T_0^* such that there is no other characteristic in this set with an associated T_0 strictly less than T_0^*. Hence we have the following properties of $(R^*(t), W^*(t), F^*)$ and T_0^*,

$$\lim_{t \to T_0^*} R^*(t) = 0, \tag{45}$$

and

$$f(t, 0, w, F) = 0 \text{ for all } t < T_0^*. \tag{46}$$

From now on we drop the superscript $*$ and consider a characteristic $(R(t), W(t), F)$ with the properties above. From the equations (4) and (22) we have

$$\mu_r(t, r) = (\frac{m}{r^2} + 4\pi r p)e^{2\lambda}, \tag{47}$$

and by using (5) the time integral in (44) can be written

$$\int_0^T \frac{(-W(t))}{E(t)} \left[e^{\mu - \lambda} \mu_r(t, R(t)) - \lambda_t(t, R(t)) \frac{(-W(t))}{E(t)} \right] dt. \tag{48}$$

Let us denote the curve $(t, R(t))$, $0 \le t \le T_0$, by γ. By using that

$$\frac{dR}{dt} = -e^{\mu - \lambda} \frac{(-W)}{E},$$

the integral above can be written as a curve integral

$$\int_\gamma \frac{(-W(t))}{E(t)} e^{(-\mu + \lambda)(t, r)} \lambda_t(t, r) dr + \frac{(-W(t))}{E(t)} e^{(\mu - \lambda)(t, r)} \mu_r(t, r) dt. \tag{49}$$

Let C denote the closed curve $\gamma + C_2 + C_3$, oriented counter clockwise, where

$$C_2 = \{(t, 0) : T_0 \ge t \ge 0\},$$

and

$$C_3 = \{(0, r) : 0 \le r \le R(0)\}.$$

Let Ω denote the domain enclosed by C. We get by Green's formula in the plane

$$\oint_C \frac{(-W(t))}{E(t)} e^{-\mu + \lambda} \lambda_t dr + \frac{(-W(t))}{E(t)} e^{\mu - \lambda} \mu_r dt$$
$$= \int \int_\Omega \frac{(-W(t))}{E(t)} \left[\partial_r \left(e^{\mu - \lambda} \mu_r - \partial_t \left(e^{-\mu + \lambda} \lambda_t \right) \right) \right] dt dr$$
$$- \int \int_\Omega \frac{d}{dt} \left(\frac{(-W(t))}{E(t)} \right) \left(e^{-\mu + \lambda} \lambda_t \right) dt dr. \tag{50}$$

Note that the factor $W(t)/E(t)$, which really only is defined along the characteristic, has been considered as a function of t alone, i.e. for given t it is defined to be constant in r with the value it has at $(t, R(t))$. Alternatively, E could have been viewed as a function of t and r, by taking $E = \sqrt{1 + W^2(t) + F/r^2}$. The final conclusion would of course be the same. An interesting fact is now that the first term in the right hand side which contains a combination of second order derivatives of μ and λ can be substituted by one of the Einstein equations (6). Indeed, we get

$$\int\int_\Omega \frac{(-W(t))}{E(t)} \left(\partial_r \left(e^{\mu-\lambda} \mu_r \right) - \partial_t \left(e^{-\mu+\lambda} \lambda_t \right) \right) dt dr$$

$$= \int\int_\Omega \frac{(-W(t))}{E(t)} \left((\mu_{rr} + (\mu_r - \lambda_r))e^{\mu-\lambda} - \lambda_{tt} - (\mu_t - \lambda_t) \right) e^{-\mu+\lambda} dt dr$$

$$= \int\int_\Omega \frac{(-W(t))}{E(t)} \left(4\pi q e^{\mu+\lambda} - (\mu_r - \lambda_r)\frac{e^{\mu-\lambda}}{r} \right) dt dr$$

$$= \int\int_\Omega \frac{(-W(t))}{E(t)} e^{\mu+\lambda} \left(4\pi(\rho - p) + 4\pi q - \frac{2m}{r^3} \right) dt dr. \qquad (51)$$

Here we used (47) for μ_r and that

$$\lambda_r = (4\pi\rho - \frac{m}{r^2})e^{2\lambda},$$

which follows from (3). By using the characteristic equations (18) and (19) we get that the second term in (50) is given by

$$-\int\int_\Omega \left(\left(\mu_r e^{\mu-\lambda} + \lambda_t \frac{(-W(t))}{E(t)} \right) \left(1 - \frac{W^2}{E^2} \right) - \frac{Fe^{\mu-\lambda}}{R^3 E^2} \right) e^{-\mu+\lambda} \lambda_t dt dr.$$

$$(52)$$

To summarize we have obtained the following identity

$$\oint_C \frac{(-W(t))}{E(t)} e^{-\mu+\lambda} \lambda_t dr + \frac{(-W(t))}{E(t)} e^{\mu-\lambda} \mu_r dt$$

$$= \int\int_\Omega \frac{(-W(t))}{E(t)} e^{\mu+\lambda} \left(4\pi(\rho - p) + 4\pi q - \frac{2m}{r^3} \right) dt dr$$

$$- \int\int_\Omega \left[\left(\mu_r e^{\mu-\lambda} - \lambda_t \frac{(-W(t))}{E(t)} \right) \left(1 - \frac{W^2}{E^2} \right) - \frac{Fe^{\mu-\lambda}}{R^3 E^2} \right] e^{-\mu+\lambda} \lambda_t dt dr.$$

$$=: T_1 + T_2. \qquad (53)$$

Note that the function in square brackets in T_2 only depends on t and its argument is $(t, R(t))$, whereas it is (t, r) for $e^{-\mu+\lambda}\lambda_t$. The term T_1 must

16

be considered as the main term. If the integrand in (48) is nonnegative we can drop the factor $(-W(t))/E(t)$ and estimate the time integral by T_1 alone (as in section 7). More importantly, T_1 has a geometrical meaning; ignoring the factor $(-W)/E$ it is the integral of the Gauss curvature of the two dimensional manifold $R_t^+ \times R_r^+$ with metric $ds^2 = -e^{2\mu}dt^2 + e^{2\lambda}dr^2$. (We really have in mind the formula in (51) obtained before we used Einstein's equations to get an expression in terms of matter quantities.) It is interesting to note that the principal term in the work on BV solutions of the ESF system by Christodoulou [7] is of the same form, and in complete analogy with our result (see p. 1172 in [7]).

By our assumption that $\rho \in L^{3+}$, it is an easy matter to estimate T_1 as it stands. However, we may first note that in the case of initial data with angular momentum cut-off, T_1 can be simplified since the following relation holds

$$\rho(t,r) - p(t,r) = q(t,r) + \int_{-\infty}^{\infty} \int_0^{\infty} \frac{\pi}{r^2 E} f(t,r,w,F)dFdw,$$

where the last term is harmless. Indeed, note that

$$\int_{-\infty}^{\infty} \int_0^{\infty} \frac{\pi}{r^2 E} f(t,r,w,F)dFdw \leq \frac{1}{F_-} r^2 \rho,$$

where we used the property that $F \geq F_-$. This term is bounded in terms of the ADM mass. Therefore, the remaining and the principal part, T_1^{prin}, of T_1 reads

$$T_1^{prin} := \int \int_{\Omega} \frac{(-W(t))}{E(t)} e^{\mu+\lambda} \left(8\pi q - \frac{2m}{r^3} \right) dtdr,$$

which has the advantage that only q and not ρ is explicitly in the integrand (m depends of course on ρ). This will have significance in section 7.

We now estimate T_1 in (53). By using $|W|/E < 1$ and $e^{\mu+\lambda} \leq 1$, we have

$$|T_1| \ \leq \ \int \int_{\Omega} \left(\frac{2m}{r^3} + 8\pi\rho \right) dtdr = \int_0^T \int_0^{R(t)} \left(\frac{2m}{r^3} + 8\pi\rho \right) drdt. \tag{54}$$

It should be noted that it is our assumption (39) on ρ which allows for this rough estimate. In a more careful treatment the major difficulty should be associated with an ingoing trajectoria, i.e. $W < 0$, and therefore the signs of the terms in the integrand should be important (see section 7). By our

assumption (39) we get by Hölder's inequality $(1/p + 1/p' = 1, \ p > 3)$

$$m = \int_0^r \eta^2 \rho(t, \eta) \, d\eta \le \left(\int_0^r \eta^2 d\eta \right)^{1/p'} \left(\int_0^r \eta^2 \rho^p d\eta \right)^{1/p}$$

$$\le C r^{2+} \left(\int_0^r \eta^2 \rho^p d\eta \right)^{1/p}, \tag{55}$$

and

$$\int_0^{R(t)} \rho(t, r) \, dr \le \left(\int_0^{R(t)} r^{-2p'/p} dr \right)^{1/p'} \left(\int_0^{R(t)} r^2 \rho^p dr \right)^{1/p}$$

$$\le C \left(\int_0^{R(t)} r^2 \rho^p dr \right)^{1/p}, \tag{56}$$

where we have used that $p'/p < 1/2$. In view of these estimates we obtain

$$|T_1| \le CBT. \tag{57}$$

Now we consider the term T_2. We begin with a lemma showing that our hypotheses imply that λ is bounded.

Lemma 4 *Assume that (38) and (39) hold on $[0, T[$. Then λ is uniformly bounded on $[0, T[$.*

Proof. From the relation (22) we see that λ is bounded as long as m/r is strictly less than $1/2$. From (39) it follows that

$$\frac{m}{r} \le \frac{r}{3^{2/3}} \|\rho\|_{L^3} \le \frac{rB}{3^{2/3}}.$$

If $r \le 3^{2/3}/4B$ it follows that $m/r \le 1/4$ and λ is bounded by $\log 2/2$. For $r \ge 3^{2/3}/4B$ we consider equation (5) and in view of (38) we get

$$|\lambda_t| \le \frac{4\pi A}{r} \le \frac{16\pi AB}{3^{2/3}},$$

and since λ is bounded initially it follows that λ is bounded on $[0, T[$ when $r \ge 3^{2/3}/4B$. By combining these estimates the lemma follows.

$\square$

18

To estimate T_2, note again that the argument of the part of the integrand in square brackets is $(t, R(t))$ while the argument of $e^{-\mu+\lambda}\lambda_t$ is (t, r). By using

$$\mu_r(t, r) = (m/r^2 + 4\pi r p)e^{2\lambda},$$

we obtain four terms in the integrand of T_2. Since

$$\lambda_t(t, r) = -4\pi r e^{\mu+\lambda} j,$$

and the sign of j is indefinite we cannot drop any of these terms. For the first term we have

$$\left| \int \int_\Omega \left[\frac{m(t, R(t))}{R(t)^2} e^{(\mu+\lambda)(t,R(t))} \right] e^{(-\mu+\lambda)(t,r)} e^{(\mu+\lambda)(t,r)} 4\pi r(-j(t,r)) \, dtdr \right|$$

$$\leq \int \int_\Omega \left[\frac{m}{R(t)} e^{\mu+\lambda} \right] e^{2\lambda} 4\pi |j| \, dtdr. \tag{58}$$

Here we used that $r \leq R(t)$. The expression in square brackets is bounded by $1/2$ and since λ is bounded in view of Lemma 5, this term is bounded by

$$C \int \int_\Omega |j| drdt \leq C\|\rho\|_{L^{3+}} T \leq CBT.$$

The remaining terms are estimated in a similar fashion by again using that $r \leq R(t)$ together with (38) to estimate the terms $rR(t)p(t, R(t))$, and $rR(t)|j|$. In conclusion we have obtained that

$$T_1 + T_2 \leq CBT.$$

Therefore, a bound on $E(t)$ follows from (44) if we can bound the curve integrals over C_2 and C_3. The curve integral over C_3 is obviously bounded in terms of the initial data and the curve integral over C_2 is zero by (46). This completes the proof of Theorem 1.

$\square$

Remark. We have seen that our assumption on ρ makes the estimate easy and we have only to consider the $r-$integration in the T_1 term. It is however clear that a more careful treatment must benefit from the time integration. Indeed, if ρ behaves like $r^{-3/2}$, then an approach that only focus on the $r-$integration in the T_1 term will fail while such a behaviour of ρ is perfectly allowed by (23) and the a priori bound in Lemma 1. The methods of proof of global existence in the case of the Vlasov-Poisson system make use of such a time integration in a cruscial way.

6 An approach along the characteristics

In this section we give conditions on μ_r and λ_r along a characteristic which guarantee the boundedness of $Q(t)$. These conditions constitute a borderline; we will show that when these conditions are satisfied, $W(t)$ must become 0 before $R(t) = 0$, and thus the characteristic "turns" from ingoing to outgoing, and if they are not satisfied, we show under additional hypotheses that $e^{\mu-\lambda}$ becomes sufficiently small to make $R(t)$ stay uniformly away from zero. These results clearly support the idea that there is a mechanism which prevents blow-up of the solutions.

Define
$$g(r) = \frac{1}{r}(1 + \frac{1}{\log r}), \ 0 < r < 1/3.$$

We make the following hypotheses. Along a characteristic $(t, R(t), W(t), F)$ we assume that the following conditions hold when $R(t) < 1/3$,

$$\mu_r(t, R(t)) \le g(R(t)) \tag{59}$$

$$\mu_r(t, R(t)) - \lambda_r(t, R(t)) \le \frac{1}{R(t)}. \tag{60}$$

The condition $R < 1/3$ is only technical and we could have taken any $R < C$ by minor changes of the presentation. We are going to show that $Q(t)$ is bounded under these hypotheses but first let us discuss these assumptions and put them into a context. In spirit these assumptions are similar to the hypothesis (38) in the previous section. Observe that

$$\mu_r = (\frac{m}{r^2} + 4\pi r p)e^{2\lambda},$$

and we see that (59) implies that

$$4\pi e^{2\lambda} p \le \frac{g(R(t))}{R(t)}.$$

Thus, since $g(r)$ is roughly $1/r$ this is of the same order as (38) with the main differencies that the constant A is now precisely specified and that the factor $e^{2\lambda}$ is included. In this situation the factor $e^{2\lambda}$ is actually bounded by 3, since by (59) (for $R < 1/3$)

$$\frac{m}{R^2}e^{2\lambda} \le 1/R,$$

and using (22) it follows that $m/R \le 1/3$ so that $e^{2\lambda} \le 3$.

A natural question is what can happen if these assumptions are not satisfied and the following lemma gives a partial answer.

Lemma 5 *Let $(R(t), W(t), F)$ be any solution to the characteristic system on a time interval $I = [t_1, t_2]$ with $1/3 > R(t_1) \geq \delta > 0$ and $W(t) \leq 0$ on I, and assume either that*

$$\mu_r(t, R(t)) \geq g(R(t)) \text{ and } \mu_t(t, R(t)) \leq 0,$$

or that

$$\mu_r(t, R(t)) - \lambda_r(t, R(t)) \geq \frac{1}{R(t)} \text{ and } \mu_t(t, R(t)) - \lambda_t(t, R(t)) \leq 0,$$

on I. Then there exists an $\epsilon > 0$, which only depends on δ such that

$$R(s) \geq \epsilon, \ s \in [t_1, t_2].$$

It should be noted that by making assumptions on the signs of both μ_t and λ_t along the *entire* characteristic (note that in Lemma 5 these assumptions are only made on that part where the spatial derivatives are large) one can easily get control of ingoing characteristics with $F > 0$. We formulate this in a lemma.

Lemma 6 *Let $(R(t), W(t), F)$ be any solution to the characteristic system on a time interval $I = [t_1, t_2]$ with $R(t_1) \geq \delta > 0$, $F > 0$, and $W(t) \leq 0$ on I. Assume that $\mu_t(t, R(t)) \leq 0$ and that $\lambda_t(t, R(t)) \geq 0$. Then there exists an $\epsilon > 0$, which only depends on δ such that*

$$R(s) \geq \epsilon, \ s \in [t_1, t_2].$$

Before giving the proofs of these lemmas let us discuss the sign assumptions in Lemma 5 and compare the content of these two lemmas. The assumption that $\mu_t \leq 0$ when $\mu_r \geq g$ (or $\mu_t - \lambda_t \leq 0$ when $\mu_r - \lambda_r \geq 1/R$) is of course the weak part of lemma 5. However, note that it always holds that $\mu \leq 0$ (or $\mu - \lambda \leq 0$) and that for collapsing matter where the density is increasing, μ (or $\mu - \lambda$) would decrease and in this sense the case $\mu_t(t, R(t)) \leq 0$ (or $\mu_t(t, R(t)) - \lambda(t, R(t)) \leq 0$) should correspond to a regime where most matter collapse and little disperse. At least intuitively, this should be the worst case. Since the sign assumption in Lemma 5 is only made on the part of the characteristic where the spatial derivative is large it strengthens this picture since it is in the collapsing regime that these derivatives are expected to be large. This is indeed supported by the numerical simulation in [26]. Moreover, in Lemma 6 we have taken sign assumptions on both μ_t and λ_t instead of only on μ_t (or on $\mu_t - \lambda_t$). Lemma 6 also requires initial data with cut-off ($F > 0$), whereas the proof of Lemma 5 holds generally. From this

discussion we therefore consider Lemma 5 as being of more interest than Lemma 6.

Proof of Lemma 5. Let us first consider the case when $\mu_r \geq g(R)$. Writing the characteristic equation for $R(s)$ as

$$\frac{dR^{-1}(s)}{ds} = \left[\frac{e^{\mu-\lambda}}{R(s)} \frac{(-W(s))}{E(R, W, F)}\right] R^{-1}(s),$$

and noting that

$$(-W)/E < 1,$$

we see that $R^{-1}(s)$ can be controlled if we can show that

$$\frac{e^{(\mu-\lambda)(t,R(t))}}{R(t)} \leq C \log \frac{1}{R(t)}.$$

Now

$$\frac{e^{(\mu-\lambda)}(s, R(s))}{R(s)} \leq \frac{e^{\mu}(s, R(s))}{R(s)},$$

since $\lambda \geq 0$, and we have

$$\begin{aligned}
\frac{d}{ds} \frac{e^{\mu(s,R(s))}}{R(s)} &= \left(\mu_t + \mu_r \frac{dR}{ds}\right) \frac{e^{\mu}}{R} - \left(\frac{W e^{\mu-\lambda}}{E R}\right) \frac{e^{\mu}}{R} \\
&= \left(\mu_r(t, R(t)) - \frac{1}{R(t)}\right) \frac{e^{\mu-\lambda}W e^{\mu}}{E R} \\
&\leq \frac{1}{R(t) \log(1/R(t))} (-\frac{dR}{ds}) \frac{e^{\mu}}{R}.
\end{aligned}$$

Here the assumptions on μ_r and μ_t were used. Hence we get for $t \in [t_1, t_2]$,

$$\frac{e^{\mu(t,R(t))}}{R(t)} \leq \frac{e^{\mu(t_1,R(t_1))}}{R(t_1)} \frac{\log R(t)}{\log R(t_1)},$$

which completes the first part of the lemma. In the second case the proof is analogous but easier since it follows that

$$\frac{d}{ds} \left(\frac{e^{\mu(s,R(s))-\lambda(s,R(s))}}{R(s)}\right) \leq 0,$$

so that

$$\frac{e^{(\mu-\lambda)(t,R(t))}}{R(t)}$$

is nonincreasing on $[t_1, t_2]$.

$\square$

Proof of Lemma 6. The spirit is the same as in the previous proof. Since $F > 0$ we have

$$\frac{d}{ds}R^{-1}(s) = \left[\frac{e^{(\mu-\lambda)(s,R(s))}}{R(s)} \frac{|W(s)|}{\sqrt{1+W^2+F/R^2}} \right] R^{-1}(s)$$

$$\leq \frac{1}{\sqrt{F}} e^{(\mu-\lambda)(s,R(s))} |W(s)| R^{-1}(s)$$

$$\leq \frac{1}{\sqrt{F}} e^{(\mu-\lambda)(s,R(s))} E(R,W,F) R^{-1}(s).$$

This inequality shows that it is sufficient to bound $e^\mu E$. By using the characteristic equation for E in (43) and inserting the expressions for μ_r and λ_t and using the assumptions of the lemma it is an immediate fact that

$$\frac{d}{ds}(e^{\mu(s,R(s))} E) \leq 0.$$

$\square$

We now state and prove the main result of this section.

Theorem 2 *Let* $f_0 \in I_{R_-,F_-}$ *and let* $(R(s,0,r,w,F), W(s,0,r,w,F), F)$ *be any solution to the characteristic system (18)-(19) on* $[0,T[$ *such that* $f_0(r,w,F) \neq 0$, *and assume that (59) and (60) hold along this characteristic. Then* $Q(t)$ *is bounded on* $[0,T[$.

Proof. We can without loss of generality assume that $F_- \geq 1$. The argument gets more transparent with this assumption but it is clear from below that it is easy to modify the argument for any $F_- > 0$. Let $t_0 = T - 1/6$. Since the solution is regular on $[0,T[$ we have that $C_0 := Q(t_0) < \infty$, in particular $R(t_0) \geq \frac{\sqrt{F_-}}{C_0}$, and $|W| \leq C_0$, for all characteristics $(R(t),W(t),F)$. We now consider the time interval $[t_0,T[$ and we show that all matter stays uniformly away from the centre on $[t_0,T[$. The class of characteristics with $R(t_0) \geq 1/4$ will have $R(t) \geq 1/12$ for all $t \in [t_0,T[$ by the fact that $|dR/ds| \leq 1$. Hence we consider the complementary class of characteristics with $\frac{\sqrt{F_-}}{C_0} \leq R(t_0) \leq 1/4$. First we note that if $dR/ds > 0$ then $dR/ds \neq 0$, i.e. $W \neq 0$, as long as $R \leq 1/3$ since we have for any t with $W(t) = 0$ and $R(t) \leq 1/3$,

$$\frac{dW}{ds} = -e^{\mu-\lambda}\mu_r(1+F/R^2) + \frac{Fe^{\mu-\lambda}}{R^3 E}$$

$$\geq \left(\frac{-1}{R} + \frac{1}{R\log(1/R)} + \frac{F}{R^3\log(1/R)} \right) e^{\mu-\lambda} > 0,$$

23

by (59) and since $F \geq 1$. Thus by continuity, an outgoing characteristic cannot turn around and become ingoing as long as $R \leq 1/3$. If it turns back when $R \geq 1/3$ it can never get closer than $R = 1/4$ for $t \leq T$. We have thus found that the case of interest is when $dR/ds < 0$ and $\frac{\sqrt{F_-}}{C_0} \leq R(t_0) \leq 1/4$. We now show that for any such characteristic, W will become zero before R becomes zero. By (19) we get

$$
\frac{d}{ds}(e^{2\lambda}W^2) = -2\lambda_r \frac{W}{E} e^{\mu+\lambda} W^2 - 2\frac{W}{E} e^{\mu+\lambda} \mu_r (1 + W^2 + F/R^2) + \frac{2WFe^{\mu+\lambda}}{R^3 E}
$$
$$
= -2(\mu_r - \lambda_r)\frac{dR}{ds} e^{2\lambda} W^2 - \frac{2F}{R^2}(\mu_r - \frac{1}{R})\frac{dR}{ds} e^{2\lambda} - 2\frac{dR}{ds} e^{2\lambda} \mu_r.
$$

Hence, by (59), and the fact that $\log R < -1$ and $F \geq 1$ we have

$$
\frac{d}{ds}\left(e^{\int_0^s 2(\mu_r - \lambda_r)\frac{dR}{ds}d\tilde{s}} e^{2\lambda} W^2\right)
$$
$$
= \left(-\frac{2F}{R^2}(\mu_r - \frac{1}{R})\frac{dR}{ds} e^{2\lambda} - 2\frac{dR}{ds} e^{2\lambda} \mu_r\right) e^{\int_0^s 2(\mu_r - \lambda_r)\frac{dR}{ds}d\tilde{s}}
$$
$$
\leq \left(\frac{-2F}{R^3 \log R} - 2\mu_r\right) e^{2\lambda} \frac{dR}{ds} e^{\int_0^s 2(\mu_r - \lambda_r)\frac{dR}{ds}d\tilde{s}}
$$
$$
\leq \frac{-F}{R^3 \log R} \frac{dR}{ds} e^{\int_0^s 2(\mu_r - \lambda_r)\frac{dR}{ds}d\tilde{s}}
$$
$$
\leq \frac{-F}{R^3 \log R} \frac{dR}{ds} R^2 \leq \frac{-F}{R \log R} \frac{dR}{ds}. \tag{61}
$$

Integrating this inequality gives

$$
e^{\int_{t_0}^t 2(\mu_r - \lambda_r)\frac{dR}{ds}ds} e^{2\lambda(t,R(t))} W^2(t) - e^{2\lambda(t_0,R(t_0))} W^2(t_0) \leq
$$
$$
\leq -F \log|\log R(t)| + F \log|\log R(t_0)|, \tag{62}
$$

so that

$$
e^{2\lambda(t,R(t))} W^2(t) \leq
$$
$$
\leq e^{-\int_{t_0}^t 2(\mu_r - \lambda_r)\frac{dR}{ds}ds} \left(e^{2\lambda(t_0,R(t_0))} W^2(t_0) - F \log\frac{|\log R(t)|}{|\log R(t_0)|}\right). \tag{63}
$$

This relation implies that W will vanish before R becomes 0. From the arguments of the first part of the proof we have seen that $dW/ds > 0$ when $W = 0$, so that the characteristic necessarily becomes outgoing and cannot turn back in again as long as $R \leq 1/3$.

$\square$

24

7 The case of purely ingoing matter

A special case of interest is when all matter is falling inwards. This is for example the situation when dust is used as matter model; if the matter is initially falling inwards, there will be no outgoing matter at any time. Moreover, in the numerical simulation [26] most matter is seen to be falling inwards in the supercritical case when black holes (presumably) form. We believe that a good understanding of this case will also have impact on the general case since control of ingoing matter give (in a certain sense) control of the outgoing part. In this section we will take initial data such that all particles are initially falling inwards and we will assume that all matter continue to fall inwards at least on some time interval $[0, T[$. The other assumption is connected to the relation between the enclosed mass $m(t, r)$ and the areal radius r. It is well-known that there are closed null geodesics when $r = 3M$ in the Schwarzschild spacetime with mass M. Therefore it is not surprising that the relation $r = 3m$ turns out to have significance also in the Einstein-Vlasov system. We will show that when all matter is falling inwards, control of ingoing characteristics can be obtained either if $r \leq 3m(t, r)$ everywhere, or if $r \geq 3m$ along the characteristics. Before stating our hypotheses we define the outgoing and ingoing part of j,

$$
j^+(t, r) = \frac{\pi}{r^2} \int_0^\infty dw \int_0^\infty dF \, w f(t, r, w, F), \tag{64}
$$

$$
j^-(t, r) = \frac{\pi}{r^2} \int_{-\infty}^0 dw \int_0^\infty dF \, w f(t, r, w, F). \tag{65}
$$

On the time interval $[0, T[$ we assume that

$$
j^+ = 0, \tag{66}
$$

$$
r \geq 3m(t, r). \tag{67}
$$

We remark that the first hypothesis can be replaced by assuming that j^+ is bounded but we find it more transparent to think of the results in this section as the case of purely ingoing matter. The first question we ask is what can happen if the second hypothesis is not satisfied and the following lemma gives a partial answer.

Lemma 7 *Let $(R(s), W(s), F)$ be any solution to the characteristic system on a time interval $I = [t_1, t_2]$ such that $R(t_1) \geq \delta > 0$. Assume that (66) holds and that*

$$
R(s) \leq 3m(s, R(s)),
$$

on I. Then there exists an $\epsilon > 0$, which only depends on δ such that

$$R(s) \geq \epsilon, \ s \in [t_1, t_2].$$

Proof. The proof is similar to the proof of Lemma 5. Indeed, recall that

$$\mu(t, r) = -\int_r^\infty e^{2\lambda}\left(\frac{m(t, \eta)}{\eta^2} + 4\pi\eta p(t, \eta)\right)d\eta. \qquad (68)$$

Let us write

$$\mu = \hat{\mu} + \check{\mu}, \qquad (69)$$

where

$$\hat{\mu}(t, r) = -\int_r^\infty e^{2\lambda}\frac{m(t, \eta)}{\eta^2}d\eta.$$

Since $e^{\hat{\mu}} \leq e^\mu$ we proceed as in Lemma 5 and consider $\frac{d}{ds}(e^{\hat{\mu}}/R)$. From the equations (5) and (22) it follows that

$$\hat{\mu}_t(t, r) = \int_r^\infty 4\pi j e^{\lambda + \mu}(1 + 2e^{2\lambda}\frac{m(t, \eta)}{\eta})d\eta, \qquad (70)$$

so $\mu_t \leq 0$ by (66). It is then easy to show that $e^{\hat{\mu}}/R$ is nonincreasing by a similar calculation as in Lemma 5.

$\square$

The main result in this section concerns an ingoing characteristic for which $d(e^{\hat{\mu}}E)/ds \geq 0$. We have seen above that control of this quantity is sufficient for controlling $R(t)^{-1}$. The condition that the time derivative is nonnegative should then be the relevant case. One can think of dividing the time interval into subintervals where $e^{\hat{\mu}}E$ does increase and where it decreases and then add up the contributions where $d(e^{\hat{\mu}}E)/ds \geq 0$. However, if the sign of $d(e^{\hat{\mu}}E)/ds$ changes an infinite number of times on $[0, T[$, then it is not clear that the infinite sum of the contributions converge due to the nature of the estimate we obtain below. This is the reason for including this assumption.

Theorem 3 *Let $f_0 \in I_{R_-, F_-}$ and such that $f_0(\cdot, w, \cdot) = 0$ if $w > 0$. Let $(R(s, 0, r, w, F), W(s, 0, r, w, F), F)$ be any solution to the characteristic system (18)-(19) on $[0, T[$ such that $f_0(r, w, F) \neq 0$. Assume that $d(e^{\hat{\mu}}E)/ds \geq 0$ on $[0, T[$, and assume that (66) and (67) hold. Then there exists an $\epsilon > 0$ such that*

$$R(s, 0, r, w, F) > \epsilon, \ \text{for all } s \in [0, T[. \qquad (71)$$

Proof. We will only sketch the proof since it is to some extent similar to the proof of Theorem 1. Again we consider $e^{\hat{\mu}}E$ and we show that this quantity can be controlled. We have

$$\frac{d}{ds}\left(E(R(s),W(s),F)e^{\hat{\mu}(s,R(s))}\right) =$$

$$= 4\pi\left[Re^{\mu+\lambda}\left(\frac{W^2}{E^2}j + \frac{|W|}{E}p\right) + \int_{R(s)}^{\infty}e^{\mu+\lambda}j\left(1 + \frac{2e^{2\lambda}m}{\eta}\right)d\eta\right]Ee^{\hat{\mu}}.$$

Here we have inserted the formula (70) for μ_t. Note also that the terms involving m/R^2 have cancelled. From the assumption that $e^{\hat{\mu}}E$ is nondecreasing we have (note that the integral term is nonpositive by (66))

$$\frac{W^2}{E^2}j + \frac{|W|}{E}p \geq 0,$$

and thus

$$\frac{W^2}{E^2}j + \frac{|W|}{E}p \leq \frac{|W|}{E}j + p.$$

We get

$$\frac{d}{ds}\left(E(R(s),W(s),F)e^{\hat{\mu}(s,R(s))}\right)$$

$$\leq 4\pi\left[Re^{\mu+\lambda}\left(\frac{|W|}{E}j + p\right) + \int_{R(s)}^{\infty}e^{\mu+\lambda}j(s,\eta)\left(1 + \frac{2e^{2\lambda}m(s,\eta)}{\eta}\right)d\eta\right]Ee^{\hat{\mu}}.$$

$$(72)$$

This is a similar to equation (43) and we are again going to apply Green's formula in the $(t,r)-$plane. This time we will choose the *outer* domain Ω, which is enclosed by the curve $(t, R(t))$, $t \in [0,T[$, which we denote by γ, together with the curves

$$
\begin{aligned}
C_2 &= \{(T,r) : R(T) \leq r \leq R^{\infty}\}, \\
C_3 &= \{(t,R^{\infty}) : T \geq t \geq 0\}, \\
C_4 &= \{(0,r) : R^{\infty} \geq r \geq R(0)\}.
\end{aligned}
$$

Here $R^{\infty} \geq R_+ + T$, so that $f = 0$ when $r \geq R^{\infty}$. The closed curve $\gamma + C_2 + C_3 + C_4$, we denote by C and it is oriented clockwise. We integrate (72) in time and obtain a curve integral over γ, analogous to (49), and we note that the part containing $|W|j/E + p$ can be written

$$\int_{\gamma}e^{-\mu+\lambda}\lambda_t dr + e^{\mu-\lambda}\check{\mu}_r ds,$$

27

and we then have by Green's formula

$$\oint_C e^{-\mu+\lambda}\lambda_t dr + e^{\mu-\lambda}\breve{\mu}_r ds$$

$$= \int\int_\Omega \partial_t \left(e^{-\mu+\lambda}\lambda_t\right) - \partial_r \left(e^{\mu-\lambda}\breve{\mu}_r\right) dsdr$$

$$= \int\int_\Omega \partial_t \left(e^{-\mu+\lambda}\lambda_t\right) - \partial_r \left(e^{\mu-\lambda}\mu_r\right) dsdr + \int\int_\Omega \partial_r \left(e^{\mu+\lambda}\frac{m}{r^2}\right) dsdr.$$

$$(73)$$

Using the calculations that led to equation (51) we obtain the identity

$$\oint_C e^{-\mu+\lambda}\lambda_t dr + e^{\mu-\lambda}\breve{\mu}_r ds$$

$$= \int\int_\Omega 4\pi e^{\mu+\lambda}\left[(\rho+p)e^{2\lambda}\frac{m}{r} + 2p - \rho\right] drds$$

$$+ \int\int_\Omega \int_{-\infty}^\infty \int_0^\infty \frac{4\pi^2 e^{\mu+\lambda}}{r^2 E} f(t,r,w,F)dFdw\, drds.$$

$$(74)$$

Recall from (72) that we wish to compute the curve integral

$$\int_0^T 4\pi Re^{\mu+\lambda}\left(\frac{|W|}{E}j + p\right) + \int_0^T \int_{R(s)}^\infty 4\pi j e^{\lambda+\mu}(1 + 2e^{2\lambda}\frac{m}{\eta})d\eta ds$$

$$= \oint_C e^{-\mu+\lambda}\lambda_t dr + e^{\mu-\lambda}\breve{\mu}_r ds + \int_0^T \int_{R(s)}^\infty 4\pi j e^{\lambda+\mu}(1 + 2e^{2\lambda}\frac{m}{\eta})d\eta ds$$

$$- \int_{C_2+C_3+C_4} e^{-\mu+\lambda}\lambda_t dr + e^{\mu-\lambda}\breve{\mu}_r ds.$$

$$(75)$$

Since $j = 0$ when $r \geq R^\infty$ we get

$$\oint_C e^{-\mu+\lambda}\lambda_t dr + e^{\mu-\lambda}\breve{\mu}_r ds + \int_0^T \int_{R(s)}^\infty 4\pi j e^{\lambda+\mu}(1 + 2e^{2\lambda}\frac{m}{\eta})d\eta ds$$

$$= \int\int_\Omega 4\pi e^{\mu+\lambda}\left[(\rho+p)e^{2\lambda}\frac{m}{r} + 2p - \rho + j(1 + 2e^{2\lambda}\frac{m}{r})\right] drds$$

$$+ \int\int_\Omega \int_{-\infty}^\infty \int_0^\infty \frac{4\pi^2 e^{\mu+\lambda}}{r^2 E} f(t,r,w,F)dFdw\, drds.$$

$$(76)$$

The definitions of the matter terms and assumption (66) imply that

$$p \leq -j \leq \rho, \qquad (77)$$

and the first term in (76) can now be estimated by

$$(\rho + p)e^{2\lambda}\frac{m}{r} + 2p - \rho + j(1 + 2e^{2\lambda}\frac{m}{r})$$
$$= -(\rho - p)(1 - e^{2\lambda}\frac{m}{r}) + (p + j)(1 + 2e^{2\lambda}\frac{m}{r}) \le 0,$$

where we used

$$e^{2\lambda}\frac{m}{r} = \frac{m/r}{1 - 2m/r} \le 1,$$

by assumption (67). A bound for the second term in (76) follows as in the proof of Theorem 1 where T_1^{prin} was introduced. The two first terms in (75) have now been estimated and we are left with the boundary term

$$-\int_{C_2 + C_3 + C_4} e^{-\mu + \lambda}\lambda_t \, dr + e^{\mu - \lambda}\check{\mu}_r \, ds.$$

The integral over C_2 is nonpositive by assumption (66) since

$$-\int_{C_2} e^{-\mu + \lambda}\lambda_t \, dr + e^{\mu - \lambda}\check{\mu}_r \, ds = -\int_{R(T)}^{R^\infty} e^{-\mu + \lambda}\lambda_t \, dr = \int_{R(T)}^{R^\infty} 4\pi e^{2\lambda}rj \le 0.$$

The integral over C_3 is zero by letting R^∞ tend to infinity since $m \le M$. Finally, the integral over C_4 depends only on the initial data and is thus bounded (we remark that it is at this point it is troublesome to add up an infinite sum of contributions as was previuosly discussed) which completes the proof of the theorem.

$\square$

Remark. An alternative way of presenting a similar result would have been to assume that the sign of $d(e^{\hat{\mu}}E)/ds$, and of $R(t) - 3m(t, R(t))$ only change a finite number of times on $[0, T[$. These assumptions together with (66) would be sufficient for global existence. Such a proof would rest on the results in this section together with a more involved analysis of the boundary terms.

Acknowledgement

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