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PREPRINT 2005:26

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Göteborg Sweden 2005

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Göteborg, June 2005

Preprint 2005:26
ISSN 1652-9715

Matematiskt centrum
Göteborg 2005

ASYMPTOTICS OF BERGMAN KERNELS

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ABSTRACT. We give an elementary proof of the existence of an asymptotic expansion in powers of k of the Bergman kernel associated to L^k, where L is a positive line bundle. We also give an algoritm for computing the coefficients in the expansion.

1. INTRODUCTION

Let L be a positive hermitian holomorphic line bundle over a complex manifold X. Then $i/2$ times the curvature form $\partial\bar{\partial}\phi$, of L defines a Kähler metric on X, that induces a scalar product on the space of global sections with values in L. The orthogonal projection P from $L^2(X, L)$ onto $H^0(X, L)$, the subspace of holomorphic sections, is the Bergman projection. Its kernel with respect to the scalar product is the Bergman kernel K of $H^0(X, L)$; it is a section of $\overline{L} \otimes L$ over $X \times X$. It can also be characterized as a reproducing kernel for the Hilbert space $H^0(X, L)$, i.e

$$\alpha(x) = (\alpha, K_x) \tag{1.1}$$

[1]for any element α of $H^0(X, L)$, where $K_x = K(\cdot, x)$ is identified with a holomorphic section of $L \otimes \overline{L_x}$, where L_x denotes the fiber of L over x. The restriction of K to the diagonal is a section of $\overline{L} \otimes L$ and we let $B(x) = |K(x, x)|$ be its pointwise norm.

We will study asymptotic properties of the Bergman kernel for L^k. All objects introduced above will be defined with respect to the line bundle L^k.

Our main result is a simple proof of the existence of an asymptotic expansion of the Bergman kernel in powers of k, and an algorithm for computing the coefficients in this expansion. The existence of an expansion is well known, see [14] and [4]. In [14] and [4] it is proved using an asymptotic formula, due to Boutet de Monvel and Sjöstrand, for the boundary behaviour of the Bergman-Szegö kernel for a strictly pseudoconvex domain, [3], extending an earlier result of C Fefferman, [5], to include also the off-diagonal behaviour. The main point of the approach in the present paper is that it is actually simpler to construct an asymptotic formula directly. Even though the inspiration for the construction comes from the calculus of Fourier integral operators with complex phase, the arguments in this paper are elementary. We also believe that our construction gives an efficient method to compute the coefficients in the expansion.

[1]We are abusing notation here: the scalar product $(\cdot, \cdot)$ on $H^0(X, L)$ determines a pairing of K_x with any element of $H^0(X, L)$, yielding an element of L_x.

The method of proof uses localization near an arbitrary point of X. Local holomorphic sections to L^k in a small coordinate neighbourhood , U, are just holomorphic functions on U, and the local norm is a weighted L^2-norm over U with weight function $e^{-k\phi}$ where ϕ is a strictly plurisubharmonic function. Using the ideas from [12] we then compute *local asymptotic Bergman kernels* on U. These are holomorphic kernel functions, and the scalar product with such a kernel function reproduces the values of holomorphic functions on U up to an error that is small as k tends to infinity. Assuming the bundle is globally positive it is then quite easy to see that the global Bergman kernel must be asymptotically equal to the local kernels.

Many essential ideas of our approach were already contained in the book [12] written by the third authour. Here we use them in order to find a short derivation of the Bergman kernel asymptotics. For the closely related problem of finding the Bergman kernel for exponentially weighted spaces of holomorphic functions, this was done by A. Melin and the third author [11], but in the present paper we replace a square root procedure used in that paper by a more direct procedure, which we think is more convenient for the actual computations of the coefficients in the asymptotic expansions. There are also close relations to the subject of weighted integral formulas in complex analysis [2]. We have tried to make the presentation almost self-contained, hoping that it may serve as an elementary introduction to certain micro-local techniques with applications to complex analysis and differential geometry.

2. The local asymptotic Bergman kernel

The local situation is as follows. Fix a point in X. We may choose local holomorhic coordinates x centered at the point and a holomorphic trivialization s of L such that

$$|s|^2 = e^{-\phi(x)}, \tag{2.1}$$

where ϕ is a smooth real valued function. L is positive if and only if all local functions ϕ arising this way are strictly plurisubharmonic. We will call $\phi_0(x) = |x|^2$ the *model fiber metric*, since it may be identified with the fiber metric of a line bundle of constant curvature on $\mathbb{C}^n$. The Kähler form, ω, of the metric on our base manifold X is given by $i/2$ times the curvature form of L,

$$\omega = i\partial\bar{\partial}\phi/2.$$

The induced volume form on X is equal to $\omega_n := \omega^n/n!$. Now any local holomorphic section u of L^k may be written as $us^{\otimes k}$ where u is a holomorphic function. The local expression of the norm of a section to L^k over U is then given by

$$\|u\|^2_{k\phi} := \int_U |u|^2 e^{-k\phi}\omega_n$$

where u is a holomorphic function un U.

We will start with the case when ϕ is analytic. In section 2.6 we will deal with the general case when ϕ is only smooth. First, we will motivate the construction of a local asymptotic Bergman kernel. In the model case the Bergman function B_k is identically equal to $(k/\pi)^n$. In fact, this is also well-known to be true asymptotically for any globally positive L when k tends to infinity. In other words,

$$K(x,\overline{x}) = (k^n\pi^{-n} + ...)e^{k\psi(x,\overline{x})}.$$

Now, if we polarize this formula and add lower order terms we end up with the following ansatz:

$$K(x,\overline{y}) = k^n B(x,\overline{y},k^{-1})e^{k\psi(x,\overline{y})},$$

where

$$B(x,\overline{y}) \sim b_0(x) + b_1(x)k^{-1} + ..., \tag{2.2}$$

and where $b_0(x) = \pi^{-n}$. Writing the reproducing property 1.1 out in terms of the ansatz 2.2 for K suggests that for all locally defined holomorphic functions u

$$u(x) = a_n k^n \int e^{k(\psi(x,\overline{y})-\psi(y,\overline{y}))}B(x,\overline{y},k^{-1})u(y)\det\psi_{y\overline{y}}\,dyd\overline{y} + \\ +O(e^{-\delta k}). \tag{2.3}$$

After this formal motivation we now turn to the construction of local asymptotic Bergman kernels. In the sequel we fix our coordinate neighbourhood to be the unit ball of $\mathbb{C}^n$. Let χ be a smooth function supported in the unit ball and equal to one on the ball of radius 1/2. We will say that K_k is a *reproducing kernel* mod $O(e^{-\delta k})$ for $H_{k\phi}$ if for any fixed x in some neighbourhood of the origin we have that for any local holomorphic function u_k,

$$u_k(x) = (\chi u_k, K_{k,x})_{k\phi} + O(e^{k(\phi(x)/2-\delta)})\,\|u\|_{k\phi}\,, \tag{2.4}$$

uniformly in some neighbourhood of the origin. Furthermore, if $K_{k,x}$ is holomorphic we say that $K_{k,x}$ is a *Bergman kernel* mod $O(e^{-\delta k})$.

Given a positive integer N, Bergman and reproducing kernels mod $O(k^{-N})$ are similarly defined.

2.1. Local reproducing kernels mod $e^{-\delta k}$.

Let ϕ be a strictly plurisubharmonic function in the unit ball.

Let u be a holomorphic function in the ball such that

$$\|u\|^2 := \int_B |u|^2 e^{-k\phi} < \infty.$$

The class of all such function is denoted $H_{k\phi}(B)$. We shall first show that (cf [12]) integrals of the form

$$c_n(k/2\pi)^{-n}\int_\Lambda e^{k\theta\cdot(x-y)}u(y)d\theta\wedge dy, \tag{2.5}$$

define reproducing kernels mod $e^{-\delta k}$ for suitably choosen *contours*

$$\Lambda = \{(y,\theta);\theta=\theta(x,y)\}.$$

Here we think of x as being fixed (close to the origin) and let y range over the unit ball, so that Λ is a $2n$-dimensional submanifold of $B_y \times \mathbb{C}^n_\theta$, and $c_n = i^n(-1)^{n(n+1)/2} = i^{-n^2}$ is a constant of modulus 1 chosen so that $c_n d\bar{y} \wedge dy$ is a positive form. Let us say that such a contour is *good* if uniformly on Λ for x in some neighbourhood of the origin and $|y| \leq 1$

$$2\mathrm{Re}\,\theta \cdot (x-y) \leq -\delta|x-y|^2 - \phi(y) + \phi(x).$$

Thus, e g $\theta = \bar{y}$ defines a good contour if $\phi(x) = |x|^2$.

Proposition 2.1. *For any good contour,*

$$u(x) = (k/2\pi)^n c_n \int_\Lambda e^{k\theta\cdot(x-y)} u(y)\chi(y) d\theta \wedge dy + O(e^{k(\phi(x)/2-\delta)})\|u\|_{k\phi},$$

for x in some fixed neighbourhood of 0 if u is an element of $H_{k\phi}(B)$.

Proof. For λ a real variable between 0 and ∞, we let

$$\Lambda_\lambda = \{(x, y, \theta); \theta + \lambda(\bar{x} - \bar{y}) \in \Lambda\},$$

and denote by $\eta = \eta_k$ the differential form

$$\eta = c_n(k/2\pi)^n e^{k\theta\cdot(x-y)} u(y)\chi(y) d\theta \wedge dy.$$

Our presumptive reproducing formula is the integral, I_0, of η over Λ_0 and it is easy to see that the limit of

$$I_s := \int_{\Lambda_s} \eta$$

as s goes to infinity equals $u(x)$. (This is because $c_n(s/2\pi)^n e^{-s|x-y|^2} d\bar{y} \wedge dy$ tends to a Dirac measure at x as s tends to infinity.) The difference between I_0 and I_s is by Stokes formula

$$I_0 - I_s = \int_{B\times[0,s]} dh^*(\eta)$$

where h is the homotopy map

$$h(y, \lambda) = (y, \theta(x, y) - \lambda(\bar{x} - \bar{y})).$$

Now,

$$d\eta = c_n(k/2\pi)^n e^{k\theta\cdot(x-y)} u d\chi \wedge d\theta \wedge dy.$$

This equals 0 if $|y| < 1/2$, and since θ is good we have the estimate

$$|dh^*(\eta)| \leq Ck^n e^{k(-(\delta/2+\lambda)|x-y|^2 - \phi(y)/2 + \phi(x)/2)} (1+\lambda)^n |u(y)|.$$

If $|x|$ is, say smaller than $1/4$, $|x - y| \geq 1/4$ when $d\eta$ is different from 0, so we get

$$|\int dh^*(\eta)| \leq Ck^n e^{k(\phi(x)/2-\delta)} \int_{|y|>1/2} |u(y)| e^{-k\phi(y)/2} \int_0^s (1+\lambda)^n e^{-k\lambda} d\lambda$$

with a smaller δ. By the Cauchy inequality the first integral in the right hand side is dominated by

$$\|u\|_{k\phi}.$$

Since the last integral is bounded by a constant independent of k we get the desired estimate. □

Thus we have a family of reproducing kernels mod $e^{-\delta k}$. When $\phi = |y|^2$ and $\theta = \bar{y}$ the kernel in the representation is also holomorphic so we even have an asymptotic Bergman kernel mod $e^{-\delta k}$. To achieve the same thing for general weights we need to introduce a bit more flexibility in the construction by allowing a more general class of amplitudes in the integral.

For this we consider differential forms

$$A = A(x, y, \theta, k) = \sum A_j(x, y, \theta, k)\widehat{d\theta_j}$$

of bidegree $(n-1, 0)$. By $\widehat{d\theta_j}$ we mean the wedge product of all the differentials $d\theta_i$ except $d\theta_j$, with a sign chosen so that $d\theta_j \wedge \widehat{d\theta_j} = d\theta$. We assume that A has an asymptotic expansion of order 0

$$A \sim A_0 + k^{-1}A_1 + ...$$

By this we mean that for any $N \geq 0$

$$A - \sum_0^N A_m k^{-m} = O(k^{-N-1})$$

uniformly as k goes to infinity.

We assume also that the coefficients are holomorphic (in the smooth case almost holomorphic) for x, y and θ of norm smaller than 2. Let

$$a d\theta = e^{-k\theta\cdot(x-y)} d_\theta e^{k\theta\cdot(x-y)} A,$$

so that

$$a = D_\theta \cdot A + k(x-y) \cdot A =: \nabla A, \tag{2.6}$$

where $D_\theta = \partial/\partial\theta$. We will say that a function a arising in this way is a negligible amplitude. In the applications we will also need to consider finite order approximations to amplitude functions. Let

$$A^{(N)} = \sum_0^N A_m/k^m$$

and similarily

$$a^{(N)} = \sum_0^N a_m/k^m.$$

Then

$$a^{(N)} = \nabla A^{(N+1)} - D_\theta \cdot A_{N+1}/k^{N+1},$$

so $a^{(N)}$ is a negligible amplitude modulo an error term which is $O(1/k^{N+1})$.

Proposition 2.2. *For any good contour Λ and any negligible amplitude a,*

$$u(x) = c_n(k/2\pi)^n \int_\Lambda e^{k\theta\cdot(x-y)} u(y)\chi(y)(1+a) d\theta \wedge dy + O(e^{k(\phi(x)/2-\delta)})\|u\|_{k\phi},$$

for all x in a sufficiently small neighbourhood of the origin if u is an element of $H_{k\phi}(B)$. Moreover

$$u(x) = c_n(k/2\pi)^n \int_\Lambda e^{k\theta\cdot(x-y)} u(y)\chi(y)(1+a^{(N)})d\theta\wedge dy + O(e^{\phi(x)/2}/k^{N+1-n})\|u\|_{k\phi},$$

Proof. For the first statement we need to verify that the contribution from a is exponentially small as k tends to infinity. But

$$\int_\Lambda e^{k\theta\cdot(x-y)} u(y)\chi(y) a d\theta\wedge dy = \int_\Lambda u(y)\chi(y) d_\theta(e^{k\theta\cdot(x-y)}A)\wedge dy =$$

$$= \int_\Lambda \chi d\left(u(y)e^{k\theta\cdot(x-y)}A\wedge dy\right) = -\int_\Lambda d\chi\wedge u(y)e^{k\theta\cdot(x-y)}A\wedge dy.$$

Again, the last integrand vanishes for $|y| < 1/2$ and is, since Λ is good, dominated by a constant times

$$|u(y)|e^{k(-\delta|x-y|^2-\phi(y)/2+\phi(x)/2)}$$

The last integral is therefore smaller than

$$\|u\|O(e^{k(\phi(x)/2-\delta)})$$

so the first formula is proved. The second formula follows since by the remark immediately preceeding the proposition, $a^{(N)}$ is a good amplitude modulo an error of order $1/k^{N+1}$. □

The condition that an amplitude function a can be written in the form (2.6) can be given in an equivalent very useful way. For this we will use the infinite order differential operator

$$Sa = \sum_0^\infty \frac{1}{(k)^m(m!)}(D_\theta\cdot D_y)^m.$$

This is basically the classical operator that appears in the theory of pseudodifferential operators when we want to replace an amplitude $a(x,y,\theta)$ by an amplitude $b(x,\theta)$ independent of y, see [7]. We let S act on $(n-1)$-forms A as above componentwise. We say that $Sa = b$ for a and b admitting asymptotic expansions if all the coefficients of the powers $(1/k)^m$ in the expansion obtained by applying S to a formally equal the corresponding coefficients in the expansion of b. No convergence of any kind is implied. That Sa equals b to order N means that the same thing holds for $m \leq N$. Note also that since formally

$$S = e^{D_\theta\cdot D_y/k},$$

we have that

$$S^{-1} = e^{-D_\theta\cdot D_y/k} = \sum_0^\infty \frac{1}{(-k)^m(m!)}(D_\theta\cdot D_y)^m.$$

Lemma 2.3. *Let*

$$a \sim \sum (1/k)^m a_m(x,y,\theta)$$

be given. Then there exists an A satisfying (2.6) asymptotically if and only if

$$Sa|_{x=y} = 0.$$

Moreover the last equation holds to order N if and only if $a^{(N)}$ can be written

$$a^{(N)} = \nabla A^{(N+1)} + O(1/k^{N+1}). \tag{2.7}$$

Proof. Note first that S commutes with D_θ and that

$$S\left((x-y)\cdot A\right) = (x-y)\cdot SA - (1/k)D_\theta \cdot SA.$$

Since

$$\nabla A = D_\theta \cdot A + k(x-y)\cdot A.$$

It follows that

$$S\nabla A = k(x-y)\cdot SA, \tag{2.8}$$

so that if a admits a representation $a = \nabla A$, then Sa must vanish for $x = y$. Similarily, if

$$a^{(N)} = \nabla A^{(N+1)} + O(1/k^{N+1}).$$

it follows that $Sa^{(N)}|_{y=x} = 0$ to order N.

Conversely, assume $Sa|_{y=x} = 0$. Then $Sa = (x-y)\cdot B$ for some form B. But (2.8) implies that

$$\nabla S^{-1} = kS^{-1}(x-y)\cdot$$

so

$$a = S^{-1}\left((x-y)\cdot B\right) = (1/k)\nabla S^{-1}B$$

and (2.6) holds with $A = 1/kS^{-1}B$. If the equation $Sa|_{y=x} = 0$ only holds to order N, then

$$Sa^{(N)} = (x-y)\cdot B^{(N)}$$

to order N. Hence

$$a^{(N)} = S^{-1}\left((x-y)\cdot B^{(N)}\right) = (1/k)\nabla S^{-1}B^{(N)} = (1/k)\nabla (S^{-1}B)^{(N)}$$

to order N, so (2.7) holds with $A^{(N+1)} = 1/k(S^{-1}B)^{(N)}$.

□

2.2. **The phase.** Let us now see how to choose the contour Λ to get the phase function, in the ansatz 2.3. In this section we still assume that the plurisubharmonic function ϕ is real analytic and let $\psi(x,y)$ be the unique holomorphic function of $2n$ variables such that

$$\psi(x,\bar{x}) = \phi(x).$$

By looking at the Taylor expansions of ψ and ϕ one can verify that

$$2\mathrm{Re}\,\psi(x,y) - \phi(x) - \phi(y) \le -\delta|x-y|^2 \tag{2.9}$$

for x and y sufficiently small. Following an idea of Kuranishi, see [8], [6], we now find a holomorphic function of $3n$ variables, $\theta(x,y,z)$ that solves the division problem

$$\theta\cdot(x-y) = \psi(x,z) - \psi(y,z), \tag{2.10}$$

This can be done in many ways, but any choice of θ satisfies

$$\theta(x, x, z) = \psi_x(x, z).$$

To fix ideas, we take

$$\theta(x, y, z) = \int_0^1 \partial\psi(tx + (1-t)y, z)dt$$

with ∂ denoting the differential of ψ with respect to the first n variables.

Since $\theta(x, x, z) = \psi_x(x, z)$ it follows that

$$\theta_z(0, 0, 0) = \psi_{xz}(0, 0) = \phi_{x\bar{x}}(0, 0)$$

is a nonsingular matrix. Therefore

$$(x, y, z) \to (x, y, \theta)$$

defines a biholomorphic change of coordinates near the origin. After rescaling we may assume that ψ is defined and satisfies (2.8) and that the above change of coordinates is well defined when $|x|$, $|y|$ and $|z|$ are all smaller than 2. We now define Λ by

$$\Lambda = \{(y, \theta); z = \bar{y}\}.$$

Thus, on Λ, θ is a holomorphic function of x, y and $\bar{y}$ and by (2.9) Λ is a good contour in the sense of the previous section. By Proposition 2.2 we therefore get the following proposition, where we use the notation β for the standard Kähler form in $\mathbb{C}^n$,

$$\beta = i/2 \sum dy_j \wedge d\bar{y}_j.$$

Proposition 2.4. *Suppose that u is in $H_{k\phi}$. If $a(x, y, \theta, 1/k)$ is a negligible amplitude, we have*

$$u(x) = \tag{2.11}$$

$$= (k/\pi)^n \int \chi_x e^{k(\psi(x,\bar{y}) - \psi(y,\bar{y}))}(\det \theta_{\bar{y}})u(y)(1 + a)\beta_n +$$

$$+ O(e^{k(\phi(x)/2+\delta)}) \|u\|_{k\phi},$$

with $a = a(x, y, \theta(x, y, \bar{y})$. Moreover

$$u(x) =$$

$$= (k/\pi)^n \int \chi_x e^{k(\psi(x,\bar{y}) - \psi(y,\bar{y}))}(\det \theta_{\bar{y}})u(y)(1 + a^{(N)})\beta_n +$$

$$+ O(e^{k\phi(x)/2} k^{n-N-1}) \|u\|_{k\phi}$$

2.3. **The amplitude.** In order to get an asymptotic Bergman kernel from (2.10) we need to choose the amplitude a so that

$$\det \theta_{\bar y}(1+a) = B(x,\bar y)\det \psi_{y\bar y},$$

with B analytic. Polarizing in the y-variable, i e replacing $\bar y$ by z, this means that

$$(1+a(x,y,\theta(x,y,z,1/k)) = B(x,z,1/k)\det \psi_{yz}(y,z)/\det \theta_z(x,y,z),$$

where B is a analytic and independent of y. Consider this as an equation between functions of the variables x, y and θ. Let

$$\Delta_0(x,y,\theta) = \det \psi_{yz}(y,z)/\det \theta_z(x,y,z) = \det \partial_\theta \psi_y.$$

Since $\psi_y = \theta$ when $y = x$ we have that $\Delta_0 = 1$ for $y = x$. We need a to be representable in the form (2.6) which by the previous lemma means that $Sa = 0$ for $y = x$. Equivalently, $S(1+a) = 1$ for $y = x$, so we must solve

$$S\left(B(x,z(x,y,\theta),1/k)\Delta_0(x,y,\theta)\right) = 1 \tag{2.12}$$

for $y = x$. This equation should hold in the sense of formal power series which means that the coefficient of $1/k^0$ must equal 1, whereas the coefficient of each power $1/k^m$ must vanish for $m > 0$. In the computations x is held fixed and $z = z(y,\theta)$. The first equation is

$$b_0(x,z(x,y,\theta))\Delta_0(x,x,\theta) = 1. \tag{2.13}$$

This means that $b_0(x,z(x,\theta)) = 1$, for all θ which implies that b_0 is identically equal to 1.

The second condition is

$$(D_\theta \cdot D_y)(b_0\Delta_0) + b_1\Delta_0 = 0 \tag{2.14}$$

for $y = x$. Since we already know that $b_0 = 1$ this means that

$$b_1(z(x,\theta)) = -(D_\theta \cdot D_y)(\Delta_0)\,|_{y=x},$$

which again determines b_1 uniquely. Continuing in this way, using the recursive formula

$$\sum_0^m \frac{(D_\theta \cdot D_y)^l}{l!}(b_{m-l}\Delta_0)\,|_{x=y} = 0 \tag{2.15}$$

for $m > 0$ we can determine all the coefficients b_m, and hence a. Then $Sa|_{y=x} = 0$ so $Sa^{(N)}|_{y=x} = 0$ to order N, and the next proposition follows from Propositions 2.4 and 2.3.

Proposition 2.5. *Suppose that ϕ is analytic. Then there are analytic functions $b_m(x,z)$ defined in a fixed neighbourhood of x so that for each N*

$$(k/\pi)^n(1 + b_1(x,\overline{y})k^{-1} + ... + b_N(x,\overline{y})k^{-N})e^{k\psi(x,\overline{y})}, \tag{2.16}$$

is an asymptotic Bergman kernel mod $O(k^{-(N+1)})$.

2.4. **Computing b_1.** Let us first recall how to express some Riemannian curvature notions in Hermitian geometry. The Hermitian metric two-form $\omega := H_{ij}dy^i \wedge \overline{dy^j}$ determines a connection η on the complex tangent bundle TX with connection matrix (with respect to a holomorphic frame)

$$\eta = H^{-1}\partial H =: \sum \eta_j dy_j. \tag{2.17}$$

The curvature is the matrix valued two-form $\overline{\partial}\eta$ and the scalar curvature s is $\Lambda \mathrm{Tr}\overline{\partial}\eta$ where Λ is contraction with the metric form ω. Hence, in coordinates centered at x where $H(0) = I$ the scalar curvature s at 0 is given by

$$s(0) = -\mathrm{Tr}(\sum \frac{\partial}{\partial \overline{y_j}} \eta_j), \tag{2.18}$$

considering η as matrix. We now turn to the computation of the coefficient b_1 in the expansion 2.16. By the definition of θ we have that

$$\theta_i(x, y, z) = \psi_{y_i}(y, z) + 1/2 \sum_k (\frac{\partial}{\partial y_k} \psi_{y_i})(y, z)(x^k - y^k) + ... \tag{2.19}$$

Differentiating with respect to z gives

$$\theta_z = H + 1/2 \partial_y H(x - y) + ...,$$

where $H = H(y, z)$. Multiplying both sides by H^{-1} and inverting the relation we get

$$\theta_z^{-1} H = I - 1/2(H^{-1}\partial H)(x - y) + ..., \tag{2.20}$$

Taking the determinant of both sides in formula 2.20 gives

$$\Delta_0 = 1\text{-}\mathrm{Tr}\eta/2(x - y) + ..., \tag{2.21}$$

Hence, equation (2.14) now gives, since $-\frac{\partial}{\partial y}(x - y) = 1$, that

$$b_1(0, 0) = (\frac{\partial}{\partial \theta} \cdot (\text{-}\mathrm{Tr}\eta/2)_{x=y} = -\frac{\partial}{\partial \overline{y}} \cdot \mathrm{Tr}\eta/2$$

showing that $b_1(x, \overline{x}) = \frac{s}{2}$, according to 2.18.

2.5. **Twisting with a vector bundle E.** We here indicate how to extend the previous calculation to the case of sections with values in $L^k \otimes E$, where E is a holomorphic vector bundle with a hermitian metric G (see also [9]). First observe that $u(x)$ is now, locally, a holomorphic vector and the Bergman kernel may be identified with a matrix $K(x, \overline{y})$ such that

$$u(x) = \int K(x, \overline{y})G(y, \overline{y})u(y)\psi_{y\overline{y}} e^{-k\psi(y,\overline{y}))} d\overline{y} \wedge dy$$

(compare 2.3). To determine K one now uses the ansatz

$$K(x, \overline{y}) = c_n(k/2\pi)^n e^{k(\psi(x,\overline{y})} B(x, \overline{y}, k^{-1})G(x, \overline{y})^{-1}.$$

Then the condition on the amplitude function becomes

$$(1 + a(x, y, \theta(x, y, z), 1/k) \det(\frac{\partial \theta}{\partial z}(x, y, z)) = \tag{2.22}$$

$$= B(x,z,1/k)G(x,z)^{-1}G(y,z)\det(\psi_{yz}).$$

where a now is a matrix valued form , i.e. Δ_0 in section 2.3 is replaced by the matrix $\Delta_G := \Delta_0 G(x,z)^{-1}G(y,z)$. Note that

$$G(x,z)^{-1}G(y,z) = I - G^{-1}(y,z)\frac{\partial}{\partial y}G(y,z)(x-y) + ... =: I - \eta_E(y,z)(x-y) + ...,$$

where $\eta_E := G^{-1}\frac{\partial}{\partial y}G$ is the connection matrix of E. Hence, the equation 2.21 is replaced by

$$\Delta_G = 1\text{-}(\mathrm{Tr}\eta/2 \otimes I + \eta_E)(x-y) + ...$$

The same calculation as before then shows that the matrix $b_1(0,0)$ is given by

$$b_1(0,0) = -\frac{\partial}{\partial \overline{y}} Tr\eta/2 \otimes I - \frac{\partial}{\partial \overline{y}} \cdot \eta_E = \frac{s}{2} \otimes I + \Lambda\Theta_E,$$

where $\Theta_E := \overline{\partial}\eta_E$ is the curvature matrix of E and Λ denotes contraction with the metric two-form ω.

Remark: Let K_k be the Bergman kernel of $H^0(X, L^k)$, defined with respect a general volume form μ_n. Then the function $G := \mu_n/\omega_n$ defines a hermitian metric on the trivial line bundle E and the asymptotics of K_k can then be obtained as above.

2.6. **Smooth metrics.** Denote by ψ any almost holomorphic extension of ϕ from $\overline{\Delta} = \{y = \bar{x}\}$, i.e. an extension such that the anti-holomorphic derivatives vanish to infinite order on $\overline{\Delta}$. We may also assume that $\overline{\psi(x,\overline{y})} = \psi(y,\overline{x})$. That ψ is almost holomorphic means

$$\left|(\overline{\partial}\psi)(x,y)\right| \leq O(|x-\bar{y}|^N) \tag{2.23}$$

for any N (we will also write the RHS in above as $O(|x-\bar{y}|^\infty)$). Note that by 2.23 the weighted norm of $\overline{\partial}(e^{k\psi})$ may be estimated by k times

$$\left| |x-y|^{2N} e^{k(\psi(x,\overline{y})-\phi(x)/2-\phi(y)/2)} \right| \leq Ck^{-N}(k|x-y|)^2)^N e^{-k\delta|x-y|^2} = O(k^{-N}), \tag{2.24}$$

where we have also used (2.9). Let now

$$\theta = \int_0^1 \psi_x(tx+(1-t)y,z)dt,\ \theta^* = \int_0^1 \psi_{\overline{x}}(tx+(1-t)y,z)dt. \tag{2.25}$$

so that

$$(x-y)\theta + \overline{(x-y)}\theta^* = \psi(x,z) - \psi(y,z) \tag{2.26}$$

Then the smooth map corresponding to $(x,y,z) \mapsto (x,y,\theta)$ is locally smoothly invertible for the same reason as in the analytic case, since $\theta_{\overline{z}} = 0$ when $x = y = \overline{z}$. Define the algebra $\mathcal{A}$ of all functions almost holomorphic when $x = y = \overline{z}$ as the set of smooth functions, f, of x y and z, such that

$$D^\alpha \bar{\partial} f = 0$$

when $x = y = \bar{z}$. We also define the vanishing ideal $\mathcal{I}^\infty$ as the set of smooth functions f such that

$$D^\alpha f = O(|x-y|^\infty),$$

when $z = \bar{y}$. Hence, if f belongs to $\mathcal{A}$ then (the coefficients of) $\bar{\partial} f$ will belong to $\mathcal{I}^\infty$.

Note that $\psi(x + t(x-y), z)$ is in $\mathcal{A}$ for each fixed t. Hence θ is in $\mathcal{A}$ and θ^* is in $\mathcal{I}^\infty$, so that 2.26 gives

$$(x-y)\theta = \psi(x,\overline{y}) - \psi(y,\overline{y}) + O(|x-y|^\infty) \tag{2.27}$$

Proposition 2.6. *Suppose that L is smooth. Then there exists an asymptotic reproducing kernel $K_k^{(N)}$* mod $O(k^{n-N-1})$ *for $H_{k\phi}$, such that*

$$K_k^{(N)}(x,\overline{y}) = e^{k\psi(x,\overline{y})}(b_0 + b_1 k^{-1} + ... + b_N k^{-N}) \tag{2.28}$$

where b_i is a polynomial in the derivatives $\partial_x^\alpha \overline{\partial}_y^\beta \psi(x,\overline{y})$ of the almost holomorphic extension ψ of ϕ. In particular,

$$e^{-k(\phi(x)/2+\phi(y)/2)}(D^\alpha_{x,y}(\overline{\partial}_x, \partial_y))K(x,y) = O(k^{-\infty}) \tag{2.29}$$

uniformly in x and y for any given α.

Proof. We go through the steps in the proof of the analytic case and indicate the necessary modifications.

First we determine the coefficients $b_m(x,z)$ in the same way as in the analytic case, i e by fixing x and solving

$$S(B(z)\Delta_0)|_{y=x} = 1$$

Here S has the same meaning as before and in particular contains only derivatives with respect to θ and no derivatives with respect to $\bar{\theta}$. The difference is that Δ_0 is no longer analytic so B will not be holomorphic, but it will still belong to $\mathcal{A}$ since Δ_0 does.

We next need to consider lemma 2.3 with $a \in \mathcal{A}$. Then we get that

$$a \in \mathcal{A},\ (Sa)_{y=x} = O(k^{-N-1}) \Leftrightarrow \exists A \in \mathcal{A}:\ a = \nabla A + O(k^{-N-1}) \bmod \mathcal{I}^\infty$$

Indeed, this follows from the argument in the analytic case and the fact that if $c \in \mathcal{A}$, then

$$c_{y=x} = 0 \Leftrightarrow \exists d \in \mathcal{A}:\ c = (x-y)d \bmod \mathcal{I}^\infty$$

as can be seen by defining d by

$$d = \int_0^1 c_y(x, ty + (1-t)x, z)dt.$$

Here ∇ also has the same meaning as before and contains only a derivative with respect to θ and no derivative with respect to $\bar{\theta}$. Then Proposition 2.2 holds as before except that there will be one extra contribution in the application of Stokes theorem coming from $\bar{\partial}_\theta A$. Since $\bar{\partial}_\theta A$ vanishes to infinite order when $x = y$, it gives a contribution to the integral which is $O(1/k^N)$ for any N by (2.24).

We therefore get from Proposition 2.2 a reproducing kernel of the form claimed in (2.28) except that the phase function equals

$$k\theta \cdot (x-y) = k(\psi(x,\bar{y}) - \phi(y)) + k\rho$$

with ρ in $\mathcal{A}$. To see that this error is negligible we note that by (2.24)

$$\partial_t (e^{k(\psi(x,\bar{y}) - \phi(x)/2 - \phi(y)/2) + tk\rho}) = O(k^{-\infty}).$$

Integrating this between 0 and 1 we see that the two phase functions are indeed equivalent modulo an error of $O(k^{-\infty})$. □

3. The global Bergman kernel

In this section we will show that, if the curvature of L is positive everywhere on X, then the global Bergman kernel K_k of $H^0(X, L^k)$ is asymptotically equal to the local Bergman kernel $K_k^{(N)}$ of $H_{k\phi}$ (constructed in section 2).

Recall (section 1) that the Bergman kernel K associated to L is a section of $\overline{L} \otimes L$ over $X \times X$. By restriction K_x is identified with a holomorphic section of $\overline{L_x} \otimes L$, where L_x is the fiber of L over x. Given any two vector spaces E and F, the scalar product on L extends uniquely to a pairing

$$(\cdot, \cdot) : \ L \otimes E \times L \otimes F \to E \otimes F, \tag{3.1}$$

linear over E and anti-linear over F. In terms of this pairing K_y has the global reproducing property

$$\alpha(y) = (\alpha, K_y) \tag{3.2}$$

for any element α of $H^0(X, L)$. By taking $\alpha = K_x$ (so that $E = L_x$ and $F = \overline{L_y}$ in 3.1) one gets

$$K(y,x) := K_x(y) = (K_x, K_y). \tag{3.3}$$

This also implies that $\overline{K(x,y)} = K(y,x)$ and that

$$K(x,x) = (K_x, K_x) = \|K_x\|^2 . \tag{3.4}$$

$K(x,x)$ is a section to $\bar{L} \otimes L$. Its norm as a section to this bundle is the Bergman function, which in a local frame with respect to which the metric on L is given by $e^{-\phi}$ equals

$$B(x) = K(x,x) e^{-\phi(x)}.$$

Notice also that by the Cauchy inequality we have an extremal characterization of the Bergman function:

$$B(x) = \sup |s(x)|^2$$

where the supremum is taken over all holomorphic sections to L of norm not greater than 1.

We now denote by K_k the Bergman kernel associated to L^k, and write B_k for the associated Bergman function. It follows from the extremal characterization of the Bergman function and the submeanvalue inequality for a holomorphic section s over a small ball with radius roughly $1/k^{1/2}$ that

$$B_k \leq Ck^n,$$

uniformly on X (see e g [1]).

Let now $K_x^{(N)}(y)$ be the local Bergman kernel of propositions 2.5-2.6, where the coefficients b_m are given by (2.15),

$$\overline{K_x^{(N)}(y)} = (k/\pi)^n(1 + b_1(x,\overline{y})k^{-1} + ... + b_N(x,\overline{y})k^{-N})e^{k\psi(x,\overline{y})}. \tag{3.5}$$

By construction, the coefficients $b_m(x,z)$ are holomorphic if the metric on L - locally represented by ϕ - is real analytic. In case ϕ is only smooth the b_ms are almost holomorphic, meaning that

$$\bar{\partial}_{xz} b_m$$

vanishes to infinite order when $z = \bar{x}$.

Replacing K_y in the relation 3.3 with the local Bergman kernel $K_k^{(N)}$ will now show that $K_k = K_k^{(N)}$ up to a small error term.

Theorem 3.1. *Assume that the smooth line bundle L is globally positive. Let $K_k^{(N)}$ be defined by (3.5), where the coefficients b_m are determined by the recursion (2.15).*

If the distance $d(x,y)$ is sufficiently small, then

$$K_k(x,y) = K_k^{(N)}(x,y) + O(k^{n-N-1})e^{k(\phi(x)/2+\phi(y)/2)}, \tag{3.6}$$

Moreover,

$$D^\alpha(K_k(x,y) - K_k^{(N)}(x,y)) = O(k^{m+n-N-1})e^{k(\phi(x)/2+\phi(y)/2)}$$

if D^α is any differential operator with respect to x and y of order at most m.

Proof. Let us first show that

$$K_k(y,x) = (\chi K_{k,x}, K_{k,y}^{(N)}) + O(k^{n-N-1})e^{k(\phi(x)/2+\phi(y)/2)} \tag{3.7}$$

where χ is a cut-off function equal to 1 in a neighbourhood of x which is large enough to contain y. Fixing x and applying Proposition 2.5 to $u_k = K_{k,x}$ gives 3.7 with the error term

$$e^{\phi(y)/2}O(k^{-N-1})\left\|K_x\right\|.$$

Now, by 3.4 and the estimate for B_k

$$\|K_{k,x}\|^2 = B_k(x)e^{k\phi(x)} \leq Ck^n e^{k\phi(x)},$$

This proves 3.7 with uniform convergence.

Next we estimate the difference

$$u_{k,y}(x) := K_{k,y}^{(N)}(x) - (\chi K_y^{(N)}, K_{k,x}).$$

Since the scalar product in this expression is the Bergman projection,

$$P_k(\chi K_{k,y}^{(N)})(x),$$

$u_{k,y}$ is the L^2-minimal solution to the $\bar\partial$-equation

$$\bar\partial u_{k,y} = \bar\partial(\chi K^{(N)}_{k,y}).$$

The right hand side equals

$$(\bar\partial\chi)K^{(N)}_{k,y} + \chi\bar\partial K^{(N)}_{k,y}.$$

Since χ equals 1 near y it follows from (2.9) and the explicit form of $K^{(N)}_{k,y}$ that the first term is dominated by

$$e^{-\delta k}e^{k(\phi(\cdot)/2+\phi(y)/2)}.$$

The second term vanishes identically in the analytic case. In the smooth case $\bar\partial K^{(N)}_k$ can by (2.24) be estimated by

$$O(1/k^\infty)e^{k(\phi(\cdot)/2+\phi(y)/2)}. \tag{3.8}$$

Altogether $\bar\partial u_{k,y}$ is therefore bounded by (3.7), so by the Hörmander L^2-estimate we get that

$$\|u_{k,y}\|^2 \leq O(1/k^\infty)e^{k\phi(y)/2}.$$

But, since the estimate on $\bar\partial u_{k,y}$ is even uniform, we get by an argument involving the Cauchy integral formula in a ball around x of radius roughly $1/k^{1/2}$ that $u_{k,y}$ satisfies a pointwise estimate

$$|u_{k,y}(x)|^2 \leq O(1/k^\infty)e^{k(\phi(y)/2+\phi(x)/2)}.$$

Combining this estimate for $u_{k,y}(x)$ with (3.6) we finally get

$$|K^{(N)}_{k,y}(x) - \overline{K_k(y,x)}|e^{-k\phi(x)/2-k\phi(y)/2} \leq O(1/k^{N+1}). \tag{3.9}$$

Since K_k is hermitian (i e $K_k(x,y) = \overline{K_k(y,x)}$) this proves the proposition except for the statement on convergence of derivatives.

In the analytic case the convergence of derivatives is, by the Cauchy estimates, an automatic consequence of the uniform convergence, since the kernels are holomorphic in x and $\bar y$. In the smooth case, we have that

$$\bar\partial K^{(N)}_k(x,\bar z) = O(1/k^\infty)e^{k(\phi(\cdot)/2+\phi(y)/2)}.$$

This implies that the Cauchy estimates still hold for the difference between K_k and $K^{(N)}_k$, up to an error which is $O(1/k^\infty)$, and so we get the convergence of derivatives even in the smooth case.

□

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