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Abstract. We derive necessary and sufficient conditions for an ambiskew polynomial ring to have a Hopf algebra structure of a certain type. This construction generalizes many known Hopf algebras, for example $U(\mathfrak{sl}_2)$, $U_q(\mathfrak{sl}_2)$ and the enveloping algebra of the 3-dimensional Heisenberg Lie algebra. In a torsion-free case we describe the finite-dimensional simple modules, in particular their dimensions and prove a Clebsch-Gordan decomposition theorem for the tensor product of two simple modules. We construct a Casimir type operator and prove that any finite-dimensional weight module is semisimple.

1. Introduction

In [4], the authors define a four parameter deformation of the Heisenberg (oscillator) Lie algebra $\mathcal{W}^{\gamma}_{\alpha,\beta}(q)$ and study its representations. Moreover by requiring this algebra to be invariant under $q \to q^{-1}$, they define a Hopf algebra structure on $\mathcal{W}^{\gamma}_{\alpha,\beta}(q)$ generalizing several previous results.

The quantum group $U_q(\mathfrak{sl}_2(\mathbb{C}))$ has by definition the structure of a Hopf algebra. In [6], an extension of this quantum group to an associative algebra denoted by $U_q(f(H,K))$ (where f is a Laurent polynomial in two variables) is defined and finite-dimensional representations are studied. The authors show that under certain conditions on f, a Hopf algebra structure can be introduced. Among these Hopf algebras is for example the Drinfeld double $\mathcal{D}(\mathfrak{sl}_2)$.

All of the mentioned algebras fall (after suitable mathematical formalization in the case of $\mathcal{W}^{\gamma}_{\alpha,\beta}(q)$) into the class of so called ambiskew polynomial rings (see Section 2 for the definition). Motivated by these examples of similar classes of algebras, all of which can be equipped with Hopf algebra structures, we consider a certain type of Hopf structures on a class of ambiskew polynomial rings.

In Section 2, we recall some definitions and fix notation. We present the conditions for a certain Hopf structure on an ambiskew polynomial ring in Section 3, while Section 4 is devoted to examples. In Section 5 we introduce some convenient notation and state some useful formulas for viewing R as an algebra of functions on its set of maximal ideals. Finite-dimensional simple modules are studied in Section 6. Those have already been classified in [7], but we focus on describing the dimensions in terms of the highest weights. The main result is stated in Theorem 6.17. The classical Clebsch-Gordan theorem for $U(\mathfrak{sl}_2)$ is generalized in Section 7 to the present more general setting, using the results of the previous section. Finally, in Section 8 we first construct a kind of Casimir operator and prove that it can be used to distinguish non-isomorphic simple modules. This is then used to prove that any weight module is semisimple.

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2. Preliminaries

Throughout, $\mathbb{K}$ will be an algebraically closed field of characteristic zero. All algebras are associative and unital $\mathbb{K}$-algebras.

By a *Hopf structure* on an algebra A we mean a triple (Δ, ε, S) where the *coproduct* $\Delta : A \to A \otimes A$ is a homomorphism, ($A \otimes A$ is given the tensor product algebra structure) the *counit* $\varepsilon : A \to \mathbb{K}$ is a homomorphism, and the *antipode* $S : A \to A$ is an anti-homomorphism such that

$$(\mathrm{Id} \otimes \Delta)(\Delta(x)) = (\Delta \otimes \mathrm{Id})(\Delta(x)), \qquad \text{(Coassociativity)} \tag{2.1}$$

$$(\mathrm{Id} \otimes \varepsilon)(\Delta(x)) = x = (\varepsilon \otimes \mathrm{Id})(\Delta(x)), \qquad \text{(Counit axiom)} \tag{2.2}$$

$$m\Big((S \otimes \mathrm{Id})(\Delta(x))\Big) = \varepsilon(x) = m\Big((\mathrm{Id} \otimes S)(\Delta(x))\Big), \qquad \text{(Antipode axiom)} \tag{2.3}$$

for all $x \in A$. Here $m : A \otimes A \to A$ denotes the multiplication map of A. A *Hopf algebra* is an algebra equipped with a Hopf structure. An element $x \in A$ of a Hopf algebra A is called *grouplike* if $\Delta(x) = x \otimes x$ and *primitive* if $\Delta(x) = x \otimes 1 + 1 \otimes x$. In the former case it follows from the axioms that $\varepsilon(x) = 1$, x is invertible and $S(x) = x^{-1}$ while in the latter $\varepsilon(x) = 0$ and $S(x) = -x$.

If V_i $(i = 1, 2)$ are two modules over a Hopf algebra H, then $V_1 \otimes V_2$ becomes an H-module in the following way

$$a(v_1 \otimes v_2) = \sum_i (a_i' v_1) \otimes (a_i'' v_2) \tag{2.4}$$

for $v_i \in V_i$ $(i = 1, 2)$ if $a \in H$ with $\Delta(a) = \sum_i a_i' \otimes a_i''$. From (2.1) it follows that if V_i $(i = 1, 2, 3)$ are modules over H then the natural vector space isomorphism $V_1 \otimes (V_2 \otimes V_3) \simeq (V_1 \otimes V_2) \otimes V_3$ is an isomorphism of H-modules. From (2.2) follows that the one-dimensional module $\mathbb{K}_\varepsilon$ associated to the representation ε of H is a tensor unit, i.e. $\mathbb{K}_\varepsilon \otimes V \simeq V \simeq V \otimes \mathbb{K}_\varepsilon$ as H-modules for any H-module V.

Let R be a finitely generated commutative algebra over $\mathbb{K}$. Let σ be an automorphism of R, $\mathsf{h} \in R$ and $\xi \in \mathbb{K} \backslash \{0\}$. Then we define the algebra $A = A(R, \sigma, \mathsf{h}, \xi)$ as the associative $\mathbb{K}$-algebra formed by adjoining to R two symbols X_+, X_- subject to the relations

$$X_\pm a = \sigma^{\pm 1}(a) X_\pm \text{ for } a \in R, \tag{2.5}$$

$$X_+ X_- = \mathsf{h} + \xi X_- X_+. \tag{2.6}$$

This algebra is called an *ambiskew polynomial ring*. Its structure and representations were studied by Jordan [8] (see also references therein).

We recall the definition of a *generalized Weyl algebra (GWA)* (see [1] and references therein). If B is a ring, σ an automorphism of B, and $t \in B$ a central element, then the generalized Weyl algebra $B(\sigma, t)$ is the ring extension of B generated by two elements x_+, x_- subject to the relations

$$\begin{aligned} x_\pm a &= \sigma^{\pm 1}(a) x_\pm, \quad \text{for } a \in B, \\ x_- x_+ &= t, \quad \text{and} \quad x_+ x_- = \sigma(t). \end{aligned} \tag{2.7}$$

The relation between these two constructions is the following. Let $A = A(R, \sigma, \mathsf{h}, \xi)$ be an ambiskew polynomial ring. Denote by $R[t]$ be the polynomial ring in one variable t with coefficients in R and let us extend the automorphism σ of R to a $\mathbb{K}$-algebra automorphism of $R[t]$ satisfying

$$\sigma(t) = \mathsf{h} + \xi t. \tag{2.8}$$

Then A is isomorphic to the GWA $R[t](\sigma, t)$.

3. The Hopf structure

Let $A = A(R, \sigma, \mathsf{h}, \xi)$ be a skew polynomial ring and assume that R has been equipped with a Hopf structure. In this section we will extend the Hopf structure on R to A. We make the following anzats, guided by [4] and [6]:

$$\Delta(X_\pm) = X_\pm \otimes r_\pm + l_\pm \otimes X_\pm, \tag{3.1}$$

$$\varepsilon(X_\pm) = 0, \tag{3.2}$$

$$S(X_\pm) = s_\pm X_\pm. \tag{3.3}$$

The elements $r_\pm, l_\pm$ and $s_\pm$ will be assumed to belong to R.

Theorem 3.1. *Formulas (3.1)-(3.3) define a Hopf algebra structure on A which extends that of R iff*

$$(\sigma \otimes \mathrm{Id}) \circ \Delta|_R = \Delta \circ \sigma|_R = (\mathrm{Id} \otimes \sigma) \circ \Delta|_R, \tag{3.4a}$$

$$S \circ \sigma|_R = \sigma^{-1} \circ S|_R, \tag{3.4b}$$

$$\Delta(\mathsf{h}) = \mathsf{h} \otimes r_+ r_- + l_+ l_- \otimes \mathsf{h}, \tag{3.5a}$$

$$\varepsilon(\mathsf{h}) = 0, \tag{3.5b}$$

$$S(\mathsf{h}) = -(l_+ l_- r_+ r_-)^{-1} \mathsf{h}, \tag{3.5c}$$

$$r_\pm \textit{ and } l_\pm \textit{ are grouplike, i.e. } \Delta(x) = x \otimes x \textit{ for } x \in \{r_\pm, l_\pm\}, \tag{3.6a}$$

$$\sigma(l_\pm) \otimes \sigma(r_\mp) = \xi l_\pm \otimes r_\mp, \tag{3.6b}$$

$$(s_\pm)^{-1} = -l_\pm \sigma^{\pm 1}(r_\pm). \tag{3.7}$$

Proof. From (2.5)-(2.6) we see that ε extends to a homomorphism $A \to \mathbb{K}$ satisfying (3.2) if and only if (3.5b) holds. Assume for a moment that Δ extends to a homomorphism $A \to A \otimes A$. From (3.1)-(3.2) it follows that ε is a counit iff

$$\varepsilon(r_+) = \varepsilon(r_-) = \varepsilon(l_+) = \varepsilon(l_-) = 1. \tag{3.8}$$

Δ is coassociative iff (dropping the $\pm$)

$$(\mathrm{Id} \otimes \Delta)(\Delta(X)) = (\Delta \otimes \mathrm{Id})(\Delta(X))$$

which is equivalent to

$$X \otimes \Delta(r) + l \otimes X \otimes r + l \otimes l \otimes X = X \otimes r \otimes r + l \otimes X \otimes r + \Delta(l) \otimes X,$$

or

$$X \otimes (\Delta(r) - r \otimes r) = (\Delta(l) - l \otimes l) \otimes X. \tag{3.9}$$

From (2.5)-(2.6) follows that A has a $\mathbb{Z}$-gradation defined by requiring that $\deg r = 0$ for $r \in R$, $\deg X_\pm = \pm 1$. This also induces a $\mathbb{Z}^2$-gradation on $A \otimes A$ in a natural way. The left and right hand sides of equation (3.9) are homogenous of different $\mathbb{Z}^2$-degrees, namely $(\pm 1, 0)$ and $(0, \pm 1)$ respectively. Hence, since homogenous elements of different degrees must be linearly independent, (3.9) is equivalent to both sides being zero which holds iff $r_\pm$ and $l_\pm$ are grouplike.

Δ respects (2.5) iff (again dropping $\pm$)

$$\begin{gathered}
\Delta(X)\Delta(a) = \Delta(\sigma(a))\Delta(X),\\
(X\otimes r + l\otimes X)\Delta(a) = \Delta(\sigma(a))(X\otimes r + l\otimes X),\\
(\sigma\otimes 1)(\Delta(a))\cdot(X\otimes r) + (1\otimes\sigma)(\Delta(a))\cdot(l\otimes X) = \Delta(\sigma(a))(X\otimes r + l\otimes X),\\
\big((\sigma\otimes 1)(\Delta(a)) - \Delta(\sigma(a))\big)\cdot(X\otimes r) + \big((1\otimes\sigma)(\Delta(a)) - \Delta(\sigma(a))\big)\cdot(l\otimes X) = 0.
\end{gathered}$$

As before the two terms in the last equation have different $\mathbb{Z}^2$-degrees and therefore must be zero. So Δ respects (2.5) iff (3.4a) holds.

It is straightforward to check that Δ respects (2.6) iff

$$\begin{aligned}
(3.10)\quad & \mathsf{h}\otimes r_+r_- + l_+l_-\otimes\mathsf{h} - \Delta(\mathsf{h}) +\\
&\quad + \Big(l_+\otimes\sigma(r_-) - \xi\sigma^{-1}(l_+)\otimes r_-\Big)X_-\otimes X_+ +\\
&\qquad + \Big(\sigma(l_-)\otimes -\xi l_-\otimes\sigma^{-1}(r_+)\Big)X_+\otimes X_- = 0.
\end{aligned}$$

Again these three terms have different degrees so each of them must be zero. Hence (3.5a) holds. Multiply the second term by $X_+\otimes X_-$ from the right:

$$\big(l_+\otimes\sigma(r_-) - \xi\sigma^{-1}(l_+)\otimes r_-\big)t\otimes\sigma(t) = 0.$$

Here we use the extension (2.8) of σ to $R[t]$ where $t = X_-X_+$. If we apply $e_1\otimes e_1'$ to this equation, where e_r (e_r') for $r\in R$ is the evaluation homomorphism $R[t]\to R$ which maps t ($\sigma(t)$) to r, we get

$$l_+\otimes\sigma(r_-) = \xi\sigma^{-1}(l_+)\otimes r_-.$$

Applying $\sigma\otimes 1$ to this we obtain one of the relations in (3.6b). Similarly the vanishing of the third term in (3.10) implies the other.

Assuming that S is an anti-homomorphism $A\to A$ satisfying (3.3), we obtain that S is a an antipode on A iff

$$S(X_\pm)r_\pm + S(l_\pm)X_\pm = 0 = X_\pm S(r_\pm) + l_\pm S(X_\pm),$$

which is equivalent to (3.7), using that $r_\pm$ and $l_\pm$ are grouplike. And S extends to a well-defined anti-homomorphism $A\to A$ iff

$$(3.11)\qquad S(a)S(X_\pm) = S(X_\pm)S(\sigma^{\pm 1}(a)),\quad \text{for } a\in R,$$

$$(3.12)\qquad S(X_-)S(X_+) = S(\mathsf{h}) + \xi S(X_+)S(X_-).$$

Using (3.7) and that $r_\pm, l_\pm$ are invertible, (3.11) holds iff (3.4b) holds. And (3.12) holds iff

$$\begin{aligned}
0 &= s_-X_-s_+X_+ - S(\mathsf{h}) - \xi s_+X_+s_-X_- =\\
&= s_-\sigma^{-1}(s_+)X_-X_+ - S(\mathsf{h}) - s_+\xi\sigma(s_-)X_+X_- =\\
&= -S(\mathsf{h}) - s_+\sigma(s_-)\xi\mathsf{h} +\\
&\quad + \big(s_-\sigma^{-1}(s_+) - s_+\sigma(s_-)\xi^2\big)t.
\end{aligned}$$

Applying e_0 and e_1 we obtain

$$\begin{aligned}
S(\mathsf{h}) &= -\xi s_+\sigma(s_-)\mathsf{h},\\
s_-\sigma^{-1}(s_+) &= \xi^2 s_+\sigma(s_-).
\end{aligned}$$

Substituting (3.7) in these equations and using (3.6b), the first is equivalent to (3.5c), while the other already holds. □

4. Examples

Many Hopf algebras known in the literature can be viewed as one defined in the previous section.

4.1. **Heisenberg algebra.** Let $R = \mathbb{C}[c]$ with c primitive, and $\sigma(c) = c$. Choose $\mathsf{h} = c$, $\xi = r_+ = r_- = l_+ = l_- = 1$. Then A is the universal enveloping algebra $U(\mathfrak{h}_3)$ of the three-dimensional Heisenberg Lie algebra.

4.2. $U(\mathfrak{sl}_2)$ **and its quantizations.**

4.2.1. $U(\mathfrak{sl}_2)$. Let $R = \mathbb{C}[H]$ with Hopf algebra structure $\Delta(H) = H \otimes 1 + 1 \otimes H$, $\varepsilon(H) = 0$, $S(H) = -H$. Define $\sigma(H) = H - 1$. Choose $\mathsf{h} = H$, $\xi = r_+ = r_- = l_+ = l_- = 1$. Then $A \simeq U(\mathfrak{sl}_2)$ as Hopf algebras.

4.2.2. $U_q(\mathfrak{sl}_2)$. Let $R = \mathbb{C}[K, K^{-1}]$ with Hopf structure defined by requiring that K is grouplike. Define $\sigma(K) = q^{-2}K$, where $q \in \mathbb{C}, q^2 \neq 1$, and choose $\mathsf{h} = \frac{K-K^{-1}}{q-q^{-1}}$, $\xi = r_- = l_+ = 1$ and $r_+ = K$, $l_- = K^{-1}$. Then the equations in Theorem 3.1 are satisfied giving a Hopf algebra A which is isomorphic to $U_q(\mathfrak{sl}_2)$.

4.2.3. $\breve{U}_q(\mathfrak{sl}_2)$. For the definition of this algebra, see for example [10]. Let $q \in \mathbb{C}, q^4 \neq 1$. Let $R = \mathbb{C}[K, K^{-1}]$ with K grouplike. Define $\sigma(K) = q^{-1}K$, $\mathsf{h} = \frac{K^2-K^{-2}}{q-q^{-1}}$, $\xi = 1$, $r_+ = r_- = K$, $l_+ = l_- = K^{-1}$. Then $A = A(R, \sigma, \mathsf{h}, \xi)$ is a Hopf algebra isomorphic to $\breve{U}_q(\mathfrak{sl}_2)$.

4.3. $U_q(f(H, K))$**.** Let $R = \mathbb{C}[H, H^{-1}, K, K^{-1}]$, $\sigma(H) = q^2 H$, $\sigma(K) = q^{-2}K$. Let $\alpha \in \mathbb{K}$ and $M, p, r, s, t, p', r', s', t' \in \mathbb{Z}$ such that $M = m - n = m' - n' = p + t - r - s$, $s - t = s' - t'$ and $p - r = p' - r'$. Set $\mathsf{h} = \alpha(K^m H^n - K^{-m'} H^{-n'})$, $\xi = 1$, $r_+ = K^p H^r$, $l_+ = K^s H^t$, $r_- = K^{-s'} H^{-t'}$, $l_- = K^{-p'} H^{-r'}$. Then A is the Hopf algebra described in [6], Theorem 3.3.

4.4. **Down-up algebras.** The down-up algebra $A(\alpha, \beta, \gamma)$ where $\alpha, \beta, \gamma \in \mathbb{C}$, was defined in [2] and studied by many authors, see for example [3], [5], [8], [9], and references therein. It is the algebra generated by u, d and relations

$$ddu = \alpha dud + \beta udd + \gamma d,$$
$$duu = \alpha udu + \beta uud + \gamma u.$$

In [8] it is proved that if σ is allowed to be any endomorphism, not necessarily invertible, then any down-up algebra is an ambiskew polynomial ring. Here we consider the down-up algebra $B = A(0, 1, 1)$. Thus B is the $\mathbb{C}$-algebra with generators u, d and relations

$$d^2u = ud^2 + d, \quad du^2 = u^2 d + u. \tag{4.1}$$

Let $R = \mathbb{C}[\mathsf{h}]$, $\sigma(\mathsf{h}) = \mathsf{h} + 1$ and $\xi = -1$. Then B is isomorphic to the ambiskew polynomial ring $A(R, \sigma, \mathsf{h}, \xi)$ via $d \mapsto X_+$ and $u \mapsto X_-$.

One can show that B is isomorphic to the enveloping algebra of the Lie super algebra $\mathfrak{osp}(1, 2)$ and hence has a *graded* Hopf structure. A question was raised in [9] whether there exists a Hopf structure on B. We do not answer this question here but we show the existence of a Hopf structure on a larger algebra B_q giving us a formula for the tensor product of weight (in particular finite-dimensional) modules over B.

Let $q \in \mathbb{C}^*$ and fix a value of $\log q$. By q^a we always mean $e^{a \log q}$. Let B_q be the ambiskew polynomial ring $B_q = A(R, \sigma, \mathsf{h}, \xi)$ where $R = \mathbb{C}[\mathsf{h}, w, w^{-1}]$, $\sigma(\mathsf{h}) = \mathsf{h} + 1$, $\sigma(w) = qw$, and $\xi = -1$.

Theorem 4.1. *For any $\rho, \lambda \in \mathbb{Z}$ such that $q^{\rho-\lambda} = -1$ and $q^{2\rho} = 1$, the algebra B_q has a Hopf algebra structure given by*

$$\Delta(X_\pm) = X_\pm \otimes w^{\pm\rho} + w^{\pm\lambda} \otimes X_\pm, \quad \varepsilon(X_\pm) = 0,$$
$$S(X_\pm) = -w^{\mp\lambda} X_\pm w^{\mp\rho} = -q^\rho w^{\mp(\rho+\lambda)} X_\pm,$$

and

$$\Delta(w) = w \otimes w, \quad \varepsilon(w) = 1, \quad S(w) = w^{-1},$$
$$\Delta(\mathsf{h}) = \mathsf{h} \otimes 1 + 1 \otimes \mathsf{h}, \quad \varepsilon(\mathsf{h}) = 0, \quad S(\mathsf{h}) = -\mathsf{h}.$$

Proof. The subalgebra $\mathbb{C}[\mathsf{h}, w, w^{-1}]$ of B_q has a unique Hopf structure given by the maps above. We must verify (3.4)-(3.7) with $\mathsf{h} = v$, $\xi = -1$, $r_\pm = w^{\pm\rho}$, $l_\pm = w^{\pm\lambda}$, and $s_\pm = -q^\rho w^{\mp(\rho+\lambda)}$. This is straightforward. □

This gives us a tensor structure on the category of modules over B_q. Next aim is to show how using the Hopf structure on B_q one can define a tensor structure on the category of weight modules over B.

In general, if C is a commutative subalgebra of an algebra A, we say that an A-module V is a *weight module* with respect to C if

$$V = \oplus_{\mathfrak{m} \in \mathrm{Max}(C)} V_\mathfrak{m}, \qquad V_\mathfrak{m} = \{v \in V | \mathfrak{m}v = 0\},$$

where $\mathrm{Max}(C)$ denotes the set of all maximal ideals of C. When C is finitely generated this is equivalent to V having a basis in which each $c \in C$ acts diagonally.

By weight modules over B (B_q) we mean weight modules with respect to the subalgebra $\mathbb{C}[\mathsf{h}]$ ($\mathbb{C}[\mathsf{h}, w, w^{-1}]$). We need a simple lemma.

Lemma 4.2. *Any finite-dimensional module V over B is a weight module.*

Proof. By Proposition 5.3 in [8], any finite-dimensional B-module is semisimple. Since direct sums of weight modules are weight modules we can assume that V is simple. Since V is finite-dimensional, the commutative subalgebra $\mathbb{C}[\mathsf{h}]$ has a common eigenvector $v \neq 0$, i.e. $\mathfrak{m}v = 0$ for some maximal ideal $\mathfrak{m}$ of $\mathbb{C}[\mathsf{h}]$. Acting on this weight vector by $X_\pm$ produces another weight vector: $\sigma^{\pm 1}(\mathfrak{m}) X_\pm v = X_\pm \mathfrak{m} v = 0$. Since B is generated by $\mathbb{C}[\mathsf{h}]$ and $X_\pm$, any vector in the B-submodule of V generated by v is a sum of weight vectors. But V was simple so $V = \oplus_\mathfrak{m} V_\mathfrak{m}$. □

Let $\mathcal{W}(B)$ denote the category of weight B-modules and similarly for B_q.

Theorem 4.3. *The category of weight modules over B can be embedded into the category of weight modules over B_q, i.e. there exist functors*

$$\mathcal{W}(B) \xrightarrow{\mathcal{E}} \mathcal{W}(B_q) \xrightarrow{\mathcal{R}} \mathcal{W}(B)$$

whose composition is the identity functor. In particular, the category of finite-dimensional B-modules can be embedded in $\mathcal{W}(B_q)$.

Proof. $\mathcal{R}$ is given by restriction. It takes weight modules to weight modules. Next we define $\mathcal{E}$. Let V be a weight module over B and define

$$wv = q^\alpha v \qquad \text{for } v \in V_{(\mathsf{h}-\alpha)} \text{ and } \alpha \in \mathbb{C}. \tag{4.2}$$

It is immediate that w commutes with h. Let $v \in V_{(\mathsf{h}-\alpha)}$ be arbitrary. Then

$$X_+ wv = X_+ q^\alpha v = q^\alpha X_+ v.$$

On the other hand, since $\mathsf{h}X_+v = X_+(\mathsf{h}-1)v = (\alpha-1)X_+v$ which shows that $X_+v \in V_{(\mathsf{h}-(\alpha-1))}$, we have

$$qwX_+v = qq^{\alpha-1}X_+v = q^\alpha X_+ v.$$

Thus $X_+w = qwX_+$. Similarly $X_-w = q^{-1}wX_-$ on V. Thus V becomes a module over B_q. That V is a weight module with respect to $\mathbb{C}[\mathsf{h}, w, w^{-1}]$ is clear. We define $\mathcal{E}(V)$ to be the same space V with additional action (4.2). If $\varphi : V \to W$ is a morphism of weight B-modules then $\varphi(wv) = w\varphi(v)$ for weight vectors v, since $\varphi(V_\mathfrak{m}) \subseteq W_\mathfrak{m}$ for any maximal ideal $\mathfrak{m}$ of $\mathbb{C}[\mathsf{h}]$. But then $\varphi(wv) = w\varphi(v)$ for all $v \in V$ since V is a weight module. Thus φ is automatically a morphism of B_q-modules and we set $\mathcal{E}(\varphi) = \varphi$. It is clear that the composition of the functors is the identity on objects and morphisms. □

Note that

$$\mathcal{E}\big(\mathcal{W}(B)\big) = \big\{V \in \mathcal{W}(B_q) \mid \operatorname{Supp}(V) \subseteq \{\mathfrak{m} = (\mathsf{h}-\alpha, w - q^\alpha) \mid \alpha \in \mathbb{C}\}\big\}. \tag{4.3}$$

It is not difficult to see that

$$\mathcal{E}(V_1) \otimes \mathcal{E}(V_2) \in \mathcal{E}\big(\mathcal{W}(B)\big)$$

and hence there is a unique $V_3 \in \mathcal{W}(B)$ such that

$$\mathcal{E}(V_1) \otimes \mathcal{E}(V_2) = \mathcal{E}(V_3).$$

Thus we can define

$$V_1 \otimes V_2 := V_3$$

and this will make $\mathcal{W}(B)$ into a tensor category.

4.5. **Non Hopf ambiskew polynomial rings.** There are many examples of ambiskew polynomial rings which do not have any Hopf structure. One example is the Weyl algebra $W = \langle a, b | ab - ba = 1\rangle$ which can have no counit ε. Indeed, a counit is in particular a homomorphism $\varepsilon : W \to \mathbb{C}$ so we would have $1 = \varepsilon(1) = \varepsilon(a)\varepsilon(b) - \varepsilon(b)\varepsilon(a) = 0$. Moreover all down-up algebras are ambiskew polynomial rings (see [8]) and [9] contains necessary conditions for the existence of a Hopf structure on a down-up algebra in terms of the parameters α, β, γ. More precisely, they show that if $A = A(\alpha, \beta, \gamma)$ is a Noetherian down-up algebra that is a Hopf algebra, then $\alpha + \beta = 1$. Moreover if $\gamma = 0$, then $(\alpha, \beta) = (2, -1)$ and as algebras, A is isomorphic to the universal enveloping algebra of the three-dimensional Heisenberg Lie algebra, while if $\gamma \neq 0$, then $-\beta$ is not an nth root of unity for $n \geq 3$. It would be of interest to generalize such a result to a more general class of ambiskew polynomial rings and also to other GWAs.

5. R as functions on a group

From now on we assume that $A = A(R, \sigma, \mathsf{h}, \xi)$ is an algebra of the form defined in Section 3 and that conditions (3.4)-(3.7) hold so that A becomes a Hopf algebra with R as a Hopf subalgebra. Let G denote the set of all maximal ideals in R. Since $\mathbb{K}$ is algebraically closed and R is finitely generated, the inclusion map $i_\mathfrak{m} : \mathbb{K} \to R/\mathfrak{m}$ is onto for any $\mathfrak{m} \in G$ and we let $\varphi_\mathfrak{m} : R \to \mathbb{K}$ denote the composition of the projection $R \to R/\mathfrak{m}$ and $i_{\mathfrak{m}}^{-1}$. Thus $\varphi_{\mathfrak{m}}(a)$ is the unique element of $\mathbb{K}$ such that $a - \varphi_{\mathfrak{m}}(a) \in \mathfrak{m}$. We define the *weight sum* of $\mathfrak{m}, \mathfrak{n} \in G$ to be

$$\mathfrak{m} + \mathfrak{n} := \ker(m \circ (\varphi_{\mathfrak{m}} \otimes \varphi_{\mathfrak{n}}) \circ \Delta|_R).$$

This is the kernel of a $\mathbb{K}$-algebra homomorphisms $R \to \mathbb{K}$, hence $\mathfrak{m} + \mathfrak{n} \in G$. We will never use the usual addition of ideals so $+$ should not cause any confusion. Using that Δ is coassociative, ε is a counit and S is an antipode, one easily deduces that $+$ is associative, that $\underline{0} := \ker \varepsilon$ is a unit element and $S(\mathfrak{m})$ is the inverse of $\mathfrak{m}$. Thus G is a group under $+$. If R is cocommutative, G is abelian.

Example 5.1. Let $R = \mathbb{C}[H]$. Then $G = \{(H - \alpha) \mid \alpha \in \mathbb{C}\}$. Give R the Hopf structure $\Delta(H) = H \otimes 1 + 1 \otimes H$, $\varepsilon(H) = 0$ and $S(H) = -H$. Then the operation $+$ will be

$$(H - \alpha) + (H - \beta) = (H - (\alpha + \beta)),$$

i.e. the correspondence $\mathbb{C} \ni \alpha \mapsto (H - \alpha) \in G$ is an additive group isomorphism.

If $R = \mathbb{C}[K, K^{-1}]$ then $G = \{(K - \alpha) \mid \alpha \in \mathbb{C}^*\}$. With the Hopf structure $\Delta(K) = K \otimes K$, $\varepsilon(K) = 1$ and $S(K) = K^{-1}$, the operation $+$ will be

$$(K - \alpha) + (K - \beta) = (K - \alpha\beta)$$

for $\alpha, \beta \neq 0$. Thus $G \simeq \langle \mathbb{C}^*, \cdot \rangle$.

We will often think of elements from R as $\mathbb{K}$-valued functions on G and for $x \in R$ and $\mathfrak{m} \in G$ we will use the notation $x(\mathfrak{m})$ for $\varphi_{\mathfrak{m}}(x)$. Note however that different elements $x, y \in R$ can represent the same function. In fact one can check that the map from R to functions on G is a homomorphism of $\mathbb{K}$-algebras with kernel equal to the radical $\mathrm{Rad}(R) := \cap_{\mathfrak{m} \in G} \mathfrak{m}$.

Define a map

$$\zeta : \mathbb{Z} \to G,\ n \mapsto \underline{n} := \sigma^n(\underline{0}). \tag{5.1}$$

Lemma 5.2. *Let $\mathfrak{m}, \mathfrak{n} \in G$. Then for any $a \in R$,*

$$\sigma(a)(\mathfrak{m}) = a\big(\sigma^{-1}(\mathfrak{m})\big), \tag{5.2}$$

$$a(\mathfrak{m} + \mathfrak{n}) = m \circ (\varphi_{\mathfrak{m}} \otimes \varphi_{\mathfrak{n}}) \circ \Delta(a) = \sum_{(a)} a'(\mathfrak{m}) a''(\mathfrak{n}), \tag{5.3}$$

$$\mathfrak{m} + \underline{1} = \sigma(\mathfrak{m}) = \underline{1} + \mathfrak{m}. \tag{5.4}$$

Thus ζ is a group homomorphism and its image is contained in the center of G.

Proof. Since for any $a \in R$ we have

$$\sigma(a)(\mathfrak{m}) - a = \sigma^{-1}\big(\sigma(a)(\mathfrak{m}) - \sigma(a)\big) \in \sigma^{-1}(\mathfrak{m}),$$

(5.2) holds. Similarly,

$$a(\mathfrak{m} + \mathfrak{n}) - a \in \mathfrak{m} + \mathfrak{n}$$

so applying the map $m \circ (\varphi_{\mathfrak{m}} \otimes \varphi_{\mathfrak{n}}) \circ \Delta$ to $a(\mathfrak{m} + \mathfrak{n}) - a$ yields zero. This gives (5.3). Finally we have for any $a \in \mathfrak{m}$,

$$\begin{aligned}\sigma(a)(\mathfrak{m} + \underline{1}) = m \circ (\varphi_{\mathfrak{m}} \otimes \varphi_{\underline{1}}) \circ \Delta(\sigma(a)) = m \circ (\varphi_{\mathfrak{m}} \otimes \varphi_{\underline{1}}) \circ (1 \otimes \sigma)\Delta(a) = \\ = m \circ (\varphi_{\mathfrak{m}} \otimes \varphi_{\underline{0}}) \circ \Delta(a) = a(\mathfrak{m} + \underline{0}) = a(\mathfrak{m}) = 0.\end{aligned}$$

Here we used (5.3) in the first and the fourth equality, (3.4a) in the second and (5.2) in the third. Thus $\sigma(\mathfrak{m}) \subseteq \mathfrak{m} + \underline{1}$ and then equality holds since both sides are maximal ideals. The proof of the other equality in (5.4) is symmetric. □

Example 5.3. If $R=\mathbb{C}[K,K^{-1}]$ with $\Delta(K)=K\otimes K, \varepsilon(K)=1, S(K)=K^{-1}$ and $\sigma(K)=q^{-2}K$, then $\ker\varepsilon=(K-1)$ so

$$\underline{n}=\sigma^n(\underline{0})=\sigma^n((K-1))=(q^{-2n}K-1)=(K-q^{2n}).$$

From (5.3) follows that if $x\in R$ is grouplike, then viewed as a function $G\to\mathbb{K}$ it is a multiplicative homomorphism. Using (5.3) and (3.5a)-(3.5c), the following formulas are satisfied by h as a function on G.

$$\begin{aligned} \mathsf{h}(\mathfrak{m}+\mathfrak{n}) &= \mathsf{h}(\mathfrak{m})r(\mathfrak{n})+l(\mathfrak{m})\mathsf{h}(\mathfrak{n}),\\ \mathsf{h}(\underline{0}) &= 0,\\ \mathsf{h}(-\mathfrak{m}) &= -r^{-1}l^{-1}\mathsf{h}(\mathfrak{m}), \end{aligned} \tag{5.5}$$

where $r=r_+r_-$ and $l=l_+l_-$.

6. Finite-dimensional simple modules

In this section we consider finite-dimensional simple modules over the algebra A. The main theorem is Theorem 6.17 where we, under the torsion-free assumption (6.1), characterize the finite-dimensional simple modules of a given dimension in terms of their highest weights. This result will be used in Section 7 to prove a Clebsch-Gordan decomposition theorem.

Throughout the rest of the paper we will assume that

$$\sigma^n(\mathfrak{m})\neq\mathfrak{m} \text{ for any } n\in\mathbb{Z}\backslash\{0\} \text{ and any } \mathfrak{m}\in G. \tag{6.1}$$

By (5.4), this condition holds iff $\underline{1}$ has infinite order in G.

6.1. Weight modules, Verma modules and their finite-dimensional simple quotients.

In this section we define weight modules, Verma modules and derive an equation for the dimension of its finite-dimensional simple quotients.

Let V be an A-module. We call $\mathfrak{m}\in G$ a *weight* of V if $\mathfrak{m}v=0$ for some nonzero $v\in V$. The *support* of V, denoted $\mathrm{Supp}(V)$, is the set of weights of V. To a weight $\mathfrak{m}$ we associate its *weight space*

$$V_\mathfrak{m}=\{v\in V\mid \mathfrak{m}v=0\}.$$

Elements of $V_\mathfrak{m}$ are called *weight vectors of weight* $\mathfrak{m}$. A module V is a *weight module* if $V=\oplus_\mathfrak{m}V_\mathfrak{m}$. A *highest weight vector* $v\in V$ of weight $\mathfrak{m}$ is a weight vector of weight $\mathfrak{m}$ such that $X_+v=0$. A module V is called a *highest weight module* if it is generated by a highest weight vector. From the defining relations of A it follows that

$$X_\pm V_\mathfrak{m}\subseteq V_{\sigma^{\pm1}(\mathfrak{m})}. \tag{6.2}$$

Equation (6.2) implies that a highest weight module is a weight module.

Let $\mathfrak{m}\in G$. The *Verma module* $M(\mathfrak{m})$ is defined as the left A-module $A/I(\mathfrak{m})$ where $I(\mathfrak{m})$ is the left ideal $AX_++A\mathfrak{m}\subseteq A$. From relations (2.5),(2.6) follows that

$$\{v_n:=X_-^n+I(\mathfrak{m})\mid n\geq 0\}$$

is a basis for $M(\mathfrak{m})$. It is clear that $M(\mathfrak{m})$ is a highest weight module generated by v_0. We also see that the vectors v_n $(n\geq 0)$ are weight vectors of weights $\sigma^n(\mathfrak{m})$ respectively. By (6.1) we conclude $\dim M(\mathfrak{m})_\mathfrak{m}=1$. Therefore the sum of all its proper submodules is proper and equals the unique maximal submodule $N(\mathfrak{m})$ of $M(\mathfrak{m})$. Thus $M(\mathfrak{m})$ has a unique simple quotient $L(\mathfrak{m})$. Since it is easy to see that any highest weight module over A of highest weight $\mathfrak{m}$ is a quotient of $M(\mathfrak{m})$ we deduce that $L(\mathfrak{m})$ is the unique irreducible highest weight module over A with given highest weight $\mathfrak{m} \in G$. We set

$$G_f := \{\mathfrak{m} \in G \mid \dim L(\mathfrak{m}) < \infty\}.$$

Proposition 6.1. *Any finite-dimensional simple module over A is isomorphic to $L(\mathfrak{m})$ for some $\mathfrak{m} \in G_f$.*

Proof. Let V be a finite-dimensional simple A-module. Since $\mathbb{K}$ is algebraically closed, R has a common eigenvector $v \neq 0$, i.e. there exists $\mathfrak{n} \in G$ such that $\mathfrak{n}v = 0$. From (2.5) it follows that $\sigma^n(\mathfrak{n})(X_+)^n v = 0$ for any $n \geq 0$. By (6.1), the set $\{X_+^n v \mid n \geq 0\}$ is a set of weight vectors of different weights. Since V is finite-dimensional it follows that $(X_+)^n v = 0$ for some $n > 0$. This proves the existence of a highest weight vector of weight $\mathfrak{m}$ in V for some weight $\mathfrak{m}$. Thus $V = L(\mathfrak{m})$. □

Corollary 6.2. *Let V be a finite-dimensional weight module over A. Then*

$$\operatorname{Supp}(V) \subseteq G_f + \underline{\mathbb{Z}} = \{\mathfrak{m} + \underline{n} \mid \mathfrak{m} \in G_f, n \in \mathbb{Z}\}.$$

Proof. Let $\mathfrak{m} \in \operatorname{Supp}(V)$ and let $0 \neq v \in V_{\mathfrak{m}}$. Then $(X_+)^n v = 0$ for some smallest $n > 0$. But then $(X_+)^{n-1} v$ is a highest weight vector so its weight $\sigma^{n-1}(\mathfrak{m}) = \mathfrak{m} + \underline{n-1}$ must belong to G_f. Thus $\mathfrak{m} = \mathfrak{m} + \underline{n-1} - \underline{n-1} \in G_f + \underline{\mathbb{Z}}$. □

The following lemma was essentially proved in [7], Proposition 2.3, and the general result was mentioned in [8]. We give a proof for completeness.

Proposition 6.3. *The dimension of $L(\mathfrak{m})$ is the smallest positive integer n such that*

$$\sum_{k=0}^{n-1} \xi^{n-1-k} \mathsf{h}(\mathfrak{m} - \underline{k}) = 0.$$

Proof. Let $e^{\mathfrak{m}}$ be a highest weight vector in $L(\mathfrak{m})$. Let $n > 0$ be the smallest positive integer such that $X_-^n e^{\mathfrak{m}} = 0$. Then the set spanned by the vectors $X_-^j e^{\mathfrak{m}}$, $0 \leq j < n$, is invariant under X_-, under R using (2.5), and under X_+, using (2.6). Hence it is a nonzero submodule and so coincides with $L(\mathfrak{m})$ since the latter is simple. Therefore $n = \dim L(\mathfrak{m})$. Let $k > 0$. Then $X_-^k e^{\mathfrak{m}} = 0$ implies that $X_+^k X_-^k e^{\mathfrak{m}} = 0$. Conversely, suppose $X_+^k X_-^k e^{\mathfrak{m}} = 0$. Then $X_+^{k-1} X_-^k e^{\mathfrak{m}}$ generates a proper submodule and thus is zero. Repeating this argument we obtain $X_-^k e^{\mathfrak{m}} = 0$. Hence $\dim L(\mathfrak{m})$ is the smallest positive integer n such that $X_+^n X_-^n e^{\mathfrak{m}} = 0$. Using induction it is easy to deduce the formulas

$$X_+ X_-^n = X_-^{n-1}\Big(\xi^n X_- X_+ + \sum_{k=0}^{n-1} \xi^{n-1-k} \sigma^k(\mathsf{h})\Big),$$

$$X_+^n X_-^n = \prod_{m=1}^{n} \Big(\xi^m X_- X_+ + \sum_{k=0}^{m-1} \xi^{m-1-k} \sigma^k(\mathsf{h})\Big). \tag{6.3}$$

Applying both sides of this equality to the vector $e^{\mathfrak{m}}$ gives

$$X_+^n X_-^n e^{\mathfrak{m}} = \prod_{m=1}^{n} \sum_{k=0}^{m-1} \xi^{m-1-k} \sigma^k(h) e^{\mathfrak{m}}. \tag{6.4}$$

Using that $e^{\mathfrak{m}}$ is a weight vector of weight $\mathfrak{m}$ and formula (5.2) we have

$$\sigma^k(\mathsf{h})e^{\mathfrak{m}} = \sigma^k(\mathsf{h})(\mathfrak{m})e^{\mathfrak{m}} = \mathsf{h}(\mathfrak{m} - \underline{k})e^{\mathfrak{m}}.$$

Substituting this into (6.4) we obtain

$$X_+^n X_-^n e^{\mathfrak{m}} = \prod_{m=1}^{n} \sum_{k=0}^{m-1} \xi^{m-1-k}\mathsf{h}(\mathfrak{m} - \underline{k})e^{\mathfrak{m}}.$$

The smallest positive n such that this is zero must be the one such that the last factor is zero. The claim is proved. □

Corollary 6.4. *If* $\mathfrak{m}, \mathfrak{m}_0 \in G$ *where* $\mathsf{h}(\mathfrak{m}_0) = 0$*, then*

$$\dim L(\mathfrak{m}_0 + \mathfrak{m}) = \dim L(\mathfrak{m}) = \dim L(\mathfrak{m} + \mathfrak{m}_0).$$

Proof. Note that (5.5) implies that $\mathsf{h}(\mathfrak{n}+\mathfrak{m}_0) = \mathsf{h}(\mathfrak{n})r(\mathfrak{m}_0)$ and $\mathsf{h}(\mathfrak{m}_0+\mathfrak{n}) = l(\mathfrak{m}_0)\mathsf{h}(\mathfrak{n})$ for any $\mathfrak{n} \in G$, recall that r and l are invertible and use Proposition 6.3. □

6.2. **Dimension and highest weights.** The goal in this subsection is to prove Theorem 6.17 which describes in detail the relationship between the dimension of a finite-dimensional simple module and its highest weight.

We begin with a few useful lemmas. Recall that $r = r_+r_-$ and $l = l_+l_-$. For brevity we set $r_1 = r(\underline{1})$ and $l_1 = l(\underline{1})$. Since $r_\pm, l_\pm$ are grouplike so are r and l and thus r_1, l_1 are nonzero scalars.

Lemma 6.5. *We have*
a) $\xi^2 r_1 l_1 = 1$,
b) $\mathsf{h}(-\underline{k}) = -r_1^{-k}l_1^{-k}\mathsf{h}(\underline{k})$ *for any* $k \in \mathbb{Z}$,
c) for any $k \in \mathbb{Z}$ *and* $\mathfrak{m} \in G$ *we have*

$$\xi^k\mathsf{h}(\mathfrak{m} + \underline{k}) + \xi^{-k}\mathsf{h}(\mathfrak{m} - \underline{k}) = \big((\xi r_1)^k + (\xi r_1)^{-k}\big)\mathsf{h}(\mathfrak{m}). \tag{6.5}$$

Proof. For a), multiply the two equations in (3.6b) and apply the multiplication map to both sides to obtain

$$\sigma(l_+l_-r_+r_-) = \xi^2 l_+l_-r_+r_-.$$

Evaluate both sides at $\underline{1}$ to get

$$1 = lr(\underline{0}) = lr(\sigma^{-1}(\underline{1})) = \sigma(lr)(\underline{1}) = \xi^2 lr(\underline{1}) = \xi^2 l_1 r_1.$$

Next (5.5) gives for any $k \in \mathbb{Z}$,

$$0 = \mathsf{h}(\underline{k} - \underline{k}) = \mathsf{h}(\underline{k})r_1^{-k} + l_1^k\mathsf{h}(-\underline{k}),$$

hence b) follows. Finally, using (5.5) again, we have

$$\begin{aligned}\xi^k\mathsf{h}(\mathfrak{m} + \underline{k}) + \xi^{-k}\mathsf{h}(\mathfrak{m} - \underline{k}) &= \xi^k\mathsf{h}(\mathfrak{m})r_1^k + \xi^k l(\mathfrak{m})\mathsf{h}(\underline{k}) + \\ &\quad + \xi^{-k}\mathsf{h}(\mathfrak{m})r_1^{-k} + \xi^{-k}l(\mathfrak{m})\mathsf{h}(-\underline{k}) = \\ &= \mathsf{h}(\mathfrak{m})\big((\xi r_1)^k + (\xi r_1)^{-k}\big) + \\ &\quad + l(\mathfrak{m})\mathsf{h}(\underline{k})(\xi^k - \xi^{-k}r_1^{-k}l_1^{-k}).\end{aligned}$$

In the last equality we used part b). Now the second term in the last expression vanishes due to part a). Thus c) follows. □

In what follows, we will treat the two cases when $\mathsf{h}(\underline{1}) = 0$ and $\mathsf{h}(\underline{1}) \neq 0$ separately. The algebras satisfying the former condition have a representation theory which reminds of that of the enveloping algebra $U(\mathfrak{h}_3)$ of the three-dimensional Heisenberg Lie algebra, while the latter case includes $U(\mathfrak{sl}_2)$ and other algebras with similar structure of representations.

6.2.1. *The case* $\mathsf{h}(\underline{1}) = 0$.

Proposition 6.6. *If* $\mathsf{h}(\underline{1}) = 0$*, then*

$$\xi^2 = r_1^2 = 1,\ \sigma(\mathsf{h}) = r_1\mathsf{h},\ \sigma(r) = r_1 r,\ \text{and } \sigma(l) = r_1 l. \tag{6.6}$$

In particular, $\langle X_+, X_-, \mathsf{h}\rangle$ *is a subalgebra of* A *with relations*

$$\begin{aligned}
&[X_+, X_-] = h, && [h, X_\pm] = 0, && \text{if } \xi = 1,\ r_1 = 1,\\
&[X_+, X_-] = h, && \{h, X_\pm\} = 0, && \text{if } \xi = 1,\ r_1 = -1,\\
&\{X_+, X_-\} = h, && [h, X_\pm] = 0, && \text{if } \xi = -1,\ r_1 = 1,\\
&\{X_+, X_-\} = h, && \{h, X_\pm\} = 0, && \text{if } \xi = -1,\ r_1 = -1,
\end{aligned}$$

respectively, where $\{\cdot,\cdot\}$ *denotes anti-commutator.*

Proof. Suppose $\mathsf{h}(\underline{1}) = 0$. Then, by Lemma 6.5b), $\mathsf{h}(-\underline{1}) = 0$. This means that $\mathsf{h} \in \underline{-1} = \sigma^{-1}(\underline{0}) = \sigma^{-1}(\ker\varepsilon)$. Thus $\varepsilon(\sigma(\mathsf{h})) = 0$. Using (2.2), (3.4a) and (3.5a) we deduce

$$\begin{aligned}
\sigma(\mathsf{h}) &= (\varepsilon\otimes 1)(\Delta(\sigma(\mathsf{h}))) = (\varepsilon\otimes 1)(\sigma\otimes 1)(\Delta(\mathsf{h})) =\\
&= \varepsilon(\sigma(\mathsf{h}))\otimes r + \varepsilon(\sigma(l))\otimes \mathsf{h} = \varepsilon(\sigma(l))\mathsf{h}.
\end{aligned}$$

Analogously one proves $\sigma(\mathsf{h}) = \varepsilon(\sigma(r))\mathsf{h}$. Hence $\varepsilon(\sigma(r)) = \varepsilon(\sigma(l))$. But

$$\varepsilon(\sigma(r)) = \sigma(r)(\ker\varepsilon) = \sigma(r)(\underline{0}) = r(-\underline{1}) = r_1^{-1}$$

and similarly for l. So $r_1 = l_1$. From Lemma 6.5a) we obtain $(\xi r_1)^2 = 1$. Now

$$S(\sigma(\mathsf{h})) = S(r_1^{-1}\mathsf{h}) = -r_1^{-1}\mathsf{h}, \quad \text{and} \quad \sigma^{-1}(S(\mathsf{h})) = \sigma^{-1}(-h) = -r_1\mathsf{h},$$

so (3.4b) implies that $r_1^2 = 1$. A similar calculation as above shows that $\sigma(r) = r_1^{-1}r = r_1 r$ and $\sigma(l) = l_1^{-1}l = r_1 l$. □

We leave it to the reader to prove the following statement.

Proposition 6.7. *All finite-dimensional simple modules over an algebra* $A(R,\sigma,\mathsf{h},\xi)$ *satisfying* (6.6) *and one of the commutation relations above are either one- or two-dimensional.*

Remark 6.8. The algebra $U(\mathfrak{h}_3)$ is an ambiskew polynomial ring, as shown in Section 4.1. For this algebra we have $\mathsf{h}(\underline{1}) = 0$ and $\xi = r_1 = 1$.

6.2.2. *The case* $\mathsf{h}(\underline{1}) \neq 0$. In this section, we consider the more complicated case when $\mathsf{h}(\underline{1}) \neq 0$. We prove Theorem 6.17 which describes the dimensions of $L(\mathfrak{m})$ in terms of $\mathfrak{m}$. The following two subsets of G will play a vital role:

$$G_0 = \{\mathfrak{m}\in G \mid \mathsf{h}(\mathfrak{m}) = 0\}, \tag{6.7}$$

$$G_{1/2} = \{\mathfrak{m}\in G \mid \mathsf{h}(\mathfrak{m}-\underline{1}) + \xi\mathsf{h}(\mathfrak{m}) = 0\}. \tag{6.8}$$

The reason for this notation is that when $A = U(\mathfrak{sl}_2)$ as in Section 4.2.1 then we have $G_0 = \{(H-0)\}$ and $G_{1/2} = \{(H-\frac{1}{2})\}$. From (5.5) it is immediate that G_0 is a subgroup of G. By Proposition 6.3 we have

$$G_0 = \{\mathfrak{m} \in G \mid \dim L(\mathfrak{m}) = 1\}. \tag{6.9}$$

The following analogous result holds for $G_{1/2}$.

Proposition 6.9.

$$G_{1/2} = \{\mathfrak{m} \in G \mid \dim L(\mathfrak{m}) = 2\}. \tag{6.10}$$

Proof. If $\mathfrak{m} \in G_{1/2}$, then by Proposition 6.3, $\dim L(\mathfrak{m}) \leq 2$. But if $\dim L(\mathfrak{m}) = 1$, then $\mathsf{h}(\mathfrak{m}) = 0$ so using $\mathfrak{m} \in G_{1/2}$ we get $\mathsf{h}(\mathfrak{m} - \underline{1}) = 0$ also. Since G_0 is a group we deduce that $\underline{1} \in G_0$, i.e. $\mathsf{h}(\underline{1}) = 0$ which is a contradiction. So $\dim L(\mathfrak{m}) = 2$. The converse inclusion is immediate from Proposition 6.3. □

Set

$$N = \begin{cases} \text{order of } \xi r_1 & \text{if } (\xi r_1)^2 \neq 1 \text{ and } \xi r_1 \text{ is a root of unity,} \\ \infty & \text{otherwise.} \end{cases} \tag{6.11}$$

We also set

$$N' = \begin{cases} N, & \text{if } N \text{ is odd,} \\ N/2, & \text{if } N \text{ is even,} \\ \infty, & \text{if } N = \infty. \end{cases}$$

The next statement describes the intersection of G_0 and $G_{1/2}$ with $\underline{\mathbb{Z}}$.

Proposition 6.10. *We have*

$$G_0 \cap \underline{\mathbb{Z}} = \begin{cases} \{\underline{0}\}, & \text{if } N = \infty, \\ \underline{N'\mathbb{Z}}, & \text{otherwise,} \end{cases} \tag{6.12}$$

and

$$G_{1/2} \cap \underline{\mathbb{Z}_{>0}} = \begin{cases} \emptyset, & \text{if } N = \infty, \\ \{\underline{n} \in \underline{\mathbb{Z}_{>0}} : N \big| 2n-1\}, & \text{otherwise.} \end{cases} \tag{6.13}$$

Remark 6.11. The set $G_{1/2} \cap \underline{\mathbb{Z}_{\leq 0}}$ can be understood using (6.13) and Lemma 6.14a).

Proof. We first prove (6.12). Let $n \in \mathbb{Z}$. The right hand side of (6.12) is invariant under $n \mapsto -n$. By Lemma 6.5b) so is the left hand side. Moreover since $\mathsf{h}(\underline{0}) = 0$, the ideal $\underline{0}$ belongs to both sides of the equality. Thus we can assume $n > 0$.

Using (5.5) and that r and l, viewed as functions $G \to \mathbb{K}$, are multiplicative homomorphisms it follows by induction that

$$\mathsf{h}(\underline{n}) = \mathsf{h}(\underline{1}) \sum_{i=0}^{n-1} r_1^i l_1^{n-1-i}.$$

By Lemma 6.5a), $r_1/l_1 = (\xi r_1)^2/(\xi^2 r_1 l_1) = (\xi r_1)^2$, so we can rewrite this as

$$\mathsf{h}(\underline{n}) = \mathsf{h}(\underline{1}) l_1^{n-1} \sum_{i=0}^{n-1} (\xi r_1)^{2i}. \tag{6.14}$$

If $N=\infty$ and $(\xi r_1)^2\neq 1$ then by (6.14) we have $\underline{n}\in G_0\cap\underline{\mathbb{Z}}$ iff $(\xi r_1)^{2n}=1$, which is false. If $(\xi r_1)^2=1$, then (6.14) implies that $\underline{n}\notin G_0\cap\underline{\mathbb{Z}}$. If $N<\infty$, then $(\xi r_1)^2\neq 1$ so by (6.14), $\mathsf{h}(\underline{n})=0$ iff $(\xi r_1)^{2n}=1$ i.e. iff $N\big|2n$. This is equivalent to $N'\big|n$.

Next we prove (6.13). Suppose $n\in\mathbb{Z}_{>0}$. By definition, $n\in G_{1/2}$ iff

$$\mathsf{h}(\underline{n}-\underline{1})+\xi\mathsf{h}(\underline{n})=0.$$

Using (6.14) on both terms and dividing by $\mathsf{h}(\underline{1})\xi l_1^{n-1}$, this is equivalent to

$$\xi^{-1}l_1^{-1}\sum_{k=0}^{n-2}(\xi r_1)^{2k}+\sum_{k=0}^{n-1}(\xi r_1)^{2k}=0.$$

But $\xi^{-1}l_1^{-1}=\xi r_1$ by Lemma 6.5a) so this can be rewritten as

$$\sum_{k=0}^{2n-2}(\xi r_1)^k=0. \tag{6.15}$$

Thus $(\xi r_1)^2\neq 1$ and multiplying by ξr_1-1 we get $(\xi r_1)^{2n-1}=1$. Therefore $N<\infty$ and $N\big|2n-1$. Conversely, if $N<\infty$ and $N\big|2n-1$ then $(\xi r_1)^2\neq 1$ and $(\xi r_1)^{2n-1}=1$ which implies (6.15). This proves (6.13). □

Proposition 6.12. *Suppose* $\mathsf{h}(\underline{1})\neq 0$ *and* $G_{1/2}\neq\emptyset$. *Then*
a) $\xi r_1\neq -1$, *and*
b) $G_{1/2}$ *is a left and right coset of* G_0 *in* G.

Proof. Let $\mathfrak{m}_{1/2}\in G_{1/2}$. To prove a), suppose that $\xi r_1=-1$. Then

$$\begin{aligned}0&=\mathsf{h}(\mathfrak{m}_{1/2}-\underline{1})+\xi\mathsf{h}(\mathfrak{m}_{1/2})=\\&=\mathsf{h}(\mathfrak{m}_{1/2})r(-\underline{1})+l(\mathfrak{m}_{1/2})\mathsf{h}(-\underline{1})+\xi\mathsf{h}(\mathfrak{m}_{1/2})=\\&=\mathsf{h}(\mathfrak{m}_{1/2})(r_1^{-1}+\xi)+l(\mathfrak{m}_{1/2})\mathsf{h}(-\underline{1})=\\&=-l(\mathfrak{m}_{1/2})r_1^{-1}l_1^{-1}\mathsf{h}(\underline{1}),\end{aligned}$$

where we used Lemma 6.5b) in the last equality. Since l is invertible we deduce that $\mathsf{h}(\underline{1})=0$ which is a contradiction.

To prove part b), we will show that

$$G_{1/2}=G_0+\mathfrak{m}_{1/2}.$$

One proves $G_{1/2}=\mathfrak{m}_{1/2}+G_0$ in an analogous way. Let $\mathfrak{m}\in G_0$ be arbitrary. Then using (5.5) twice,

$$\mathsf{h}(\mathfrak{m}+\mathfrak{m}_{1/2}-\underline{1})+\xi\mathsf{h}(\mathfrak{m}+\mathfrak{m}_{1/2})=l(\mathfrak{m})\big(\mathsf{h}(\mathfrak{m}_{1/2}-\underline{1})+\xi\mathsf{h}(\mathfrak{m}_{1/2})\big)=0.$$

Since l is invertible we get $\mathfrak{m}+\mathfrak{m}_{1/2}\in G_{1/2}$.

Conversely, suppose $\mathfrak{m}\in G_{1/2}$. Then

$$\mathsf{h}(\mathfrak{m}-\underline{1})+\xi\mathsf{h}(\mathfrak{m})=0,$$
$$\mathsf{h}(\mathfrak{m}_{1/2}-\underline{1})+\xi\mathsf{h}(\mathfrak{m}_{1/2})=0.$$

Multiply the first equation by $r(-\mathfrak{m}_{1/2})$ and the second by $-r(-\mathfrak{m}_{1/2})l(-\mathfrak{m}_{1/2})l(\mathfrak{m})$ and add them together. Then we get

$$\begin{aligned}&\big((\mathsf{h}(\mathfrak{m})r_1^{-1}+l(\mathfrak{m})\mathsf{h}(-\underline{1})\big)r(-\mathfrak{m}_{1/2})-\\&\qquad r(-\mathfrak{m}_{1/2})l(-\mathfrak{m}_{1/2})l(\mathfrak{m})\big(\mathsf{h}(\mathfrak{m}_{1/2}r_1^{-1}+l(\mathfrak{m}_{1/2})\mathsf{h}(-\underline{1})\big)+\xi\mathsf{h}(\mathfrak{m}-\mathfrak{m}_{1/2})=0,\end{aligned}$$

or equivalently,

$$\mathsf{h}(\mathfrak{m})r_1^{-1}r(-\mathfrak{m}_{1/2}) - r(-\mathfrak{m}_{1/2})l(-\mathfrak{m}_{1/2})l(\mathfrak{m})\mathsf{h}(\mathfrak{m}_{1/2})r_1^{-1} + \xi\mathsf{h}(\mathfrak{m}-\mathfrak{m}_{1/2}) = 0.$$

Using (5.5) this can be written

$$r_1^{-1}(1+\xi r_1)\mathsf{h}(\mathfrak{m}-\mathfrak{m}_{1/2}) = 0.$$

Since $\xi r_1 \neq -1$ by part a), we conclude that $\mathsf{h}(\mathfrak{m}-\mathfrak{m}_{1/2}) = 0$. This shows that $\mathfrak{m} \in G_0 + \mathfrak{m}_{1/2}$. ☐

The following lemma will be useful.

Lemma 6.13. *Let $j \in \mathbb{Z}$. If $\mathfrak{m}_0 \in G_0$, then*

$$\mathfrak{m}_0 + \underline{j} \in G_0 \iff \underline{j} \in G_0, \tag{6.16}$$

and if $\mathsf{h}(\underline{1}) \neq 0$ and $\mathfrak{m}_{1/2} \in G_{1/2}$, then

$$\mathfrak{m}_{1/2} + \underline{j} \in G_{1/2} \iff \underline{j} \in G_0. \tag{6.17}$$

Proof. (6.16) is immediate since G_0 is a subgroup of G. If $\underline{j} \in G_0$, then $\mathfrak{m}_{1/2} + \underline{j} \in G_{1/2}$ by Proposition 6.12. Conversely, if $\mathfrak{m}_{1/2} + \underline{j} \in G_{1/2}$ then by Proposition 6.12, $G_0 \ni \mathfrak{m}_{1/2} + \underline{j} - \mathfrak{m}_{1/2} = \underline{j}$. ☐

The next statements will be needed in Section 8.

Lemma 6.14. *Suppose $\mathsf{h}(\underline{1}) \neq 0$ and let $\mathfrak{m}, \mathfrak{n} \in G_{1/2}$. Then*

a) $\underline{1} - \mathfrak{m} \in G_{1/2}$, *and*
b) $\mathfrak{m} + \mathfrak{n} - \underline{1} \in G_0$.

Proof. Part a) follows from the calculation

$$\begin{aligned}\mathsf{h}(\underline{1}-\mathfrak{m}-\underline{1}) + \xi\mathsf{h}(\underline{1}-\mathfrak{m}) &= -l(-\mathfrak{m})r(-\mathfrak{m})\mathsf{h}(\mathfrak{m}) - \xi l(\underline{1}-\mathfrak{m})(r(\underline{1}-\mathfrak{m})\mathsf{h}(\mathfrak{m}-\underline{1}) = \\ &= -l(-\mathfrak{m})r(-\mathfrak{m})\big(\mathsf{h}(\mathfrak{m}) + \xi r_1 l_1 \mathsf{h}(\mathfrak{m}-\underline{1})\big) = \\ &= -l(-\mathfrak{m})r(-\mathfrak{m})\xi^{-1}\big(\xi\mathsf{h}(\mathfrak{m}) + \mathsf{h}(\mathfrak{m}-\underline{1})\big) = 0.\end{aligned}$$

For part b), use that $\dim L(\underline{1}-\mathfrak{n}) = 2$ by part a), and thus $\mathfrak{m}+\mathfrak{n}-\underline{1} = \mathfrak{m}-(\underline{1}-\mathfrak{n}) \in G_0$ by Proposition 6.12b). ☐

The formulas provided by the following technical lemma are the key to proving our main theorem.

Lemma 6.15. *Let $\mathfrak{m} \in G$ and $j \in \mathbb{Z}_{\geq 0}$. If $n = 2j+1$ then*

$$\sum_{k=0}^{n-1} \xi^{n-1-k}\mathsf{h}(\mathfrak{m}-\underline{k}) = r_1^{-j}\mathsf{h}(\mathfrak{m}-\underline{j}) \sum_{k=0}^{n-1} (\xi r_1)^k \tag{6.18}$$

and if $n = 2j+2$ then

$$\sum_{k=0}^{n-1} \xi^{n-1-k}\mathsf{h}(\mathfrak{m}-\underline{k}) = r_1^{-j}\big(\mathsf{h}(\mathfrak{m}-\underline{j}-\underline{1}) + \xi\mathsf{h}(\mathfrak{m}-\underline{j})\big) \sum_{k=0}^{n/2-1} (\xi r_1)^{2k}. \tag{6.19}$$

Proof. If $n = 2j+1$, we make the change of index $k \mapsto j-k$, then factor out ξ^j and apply formula (6.5):

$$\sum_{k=0}^{2j} \xi^{2j-k}\mathsf{h}(\mathfrak{m}-\underline{k}) = \sum_{k=-j}^{j} \xi^{j+k}\mathsf{h}(\mathfrak{m}-\underline{j}+\underline{k}) = \xi^j\mathsf{h}(\mathfrak{m}-\underline{j}) \sum_{k=-j}^{j} (\xi r_1)^k.$$

Factoring out $(\xi r_1)^{-j}$ and changing index from k to $k-j$ yields (6.18).

For the $n = 2j+2$ case we first split the sum in the left hand side of (6.19) into two sums corresponding to odd and even k:

$$\sum_{k=0}^{j} \xi^{2j-2k} \mathsf{h}(\mathfrak{m} - \underline{2k} - \underline{1}) + \sum_{k=0}^{j} \xi^{2j+1-2k} \mathsf{h}(\mathfrak{m} - \underline{2k})$$

Then we make the change of summation index $k \mapsto -k + j/2$ in both sums

$$\xi^j \sum_{k=-j/2}^{j/2} \xi^{2k} \mathsf{h}(\mathfrak{m} - \underline{j} - \underline{1} + \underline{2k}) + \xi^{j+1} \sum_{k=-j/2}^{j/2} \xi^{2k} \mathsf{h}(\mathfrak{m} - \underline{j} + \underline{2k})$$

and use (6.5) on each of them to get

$$\big(\mathsf{h}(\mathfrak{m} - \underline{j} - \underline{1}) + \xi \mathsf{h}(\mathfrak{m} - \underline{j})\big) \xi^j \sum_{k=-j/2}^{j/2} (\xi r_1)^{2k}.$$

If we factor out $(\xi r_1)^{-j}$ and change summation index from k to $k - j/2$ we obtain (6.19). □

We now come to the main results in this section.

Main Lemma 6.16. *Assume that* $\mathsf{h}(\underline{1}) \neq 0$ *and let* $\mathfrak{m} \in G$. *Then*

a) $\dim L(\mathfrak{m}) \leq N$,

b) *if* $\dim L(\mathfrak{m}) = n < N$ *then* $\mathfrak{m} \in G_{\frac{i-1}{2}} + \underline{j}$ *where* $n = 2j + i$, $i \in \{1, 2\}$, $j \in \mathbb{Z}_{\geq 0}$, *and*

c) *if* $i \in \{1, 2\}$, $j \in \mathbb{Z}_{\geq 0}$, $2j + i \leq N$ *and* $\mathfrak{m} \in G_{\frac{i-1}{2}}$ *then*

$$\dim L(\mathfrak{m} + \underline{j}) = 2j + i. \tag{6.20}$$

d) *If* $N' < \infty$ *then* $\dim L(\mathfrak{m} + \underline{N'j}) = \dim L(\mathfrak{m})$ *for any* $j \in \mathbb{Z}$.

Proof. Part a) is trivial when $N = \infty$. If N is finite and odd, Proposition 6.3 and (6.18) imply that $\dim L(\mathfrak{m}) \leq N$. If N is finite and even, then $(\xi r_1)^N = 1$ and $(\xi r_1)^2 \neq 1$ so $\sum_{k=0}^{N/2-1} (\xi r_1)^{2k} = 0$. Hence Proposition 6.3 and (6.19) implies $\dim L(\mathfrak{m}) \leq N$ in this case as well.

Next we turn to part b). Suppose first that $\dim L(\mathfrak{m}) = n = 2j + 1 < N$. Then by Proposition 6.3 and (6.18) the right hand side of (6.18) is zero. The definition of N implies that $\mathsf{h}(\mathfrak{m} - \underline{j}) = 0$, i.e. $\mathfrak{m} \in G_0 + \underline{j}$. If instead $\dim L(\mathfrak{m}) = 2j + 2 < N$, Proposition 6.3 and (6.19) similarly implies that $\mathfrak{m} \in G_{1/2} + \underline{j}$.

To prove (6.20), we proceed by induction on j. For $j = 0$ it follows from (6.9) and (6.10). Suppose it holds for $j = 0, 1, \ldots, k-1$, where $k > 0$ and $2k + i \leq N$. We first show that $\dim L(\mathfrak{m} + \underline{k}) \leq 2k + i$. If $i = 1$ then by (6.18),

$$\sum_{l=0}^{2k} \xi^{2k-l} \mathsf{h}(\mathfrak{m} + \underline{k} - \underline{l}) = r_1^{-k} \mathsf{h}(\mathfrak{m}) \sum_{l=0}^{2k} (\xi r_1)^l = 0$$

since $\mathfrak{m} \in G_0$. Similarly, if $i = 2$, then (6.19) gives

$$\sum_{l=0}^{2k+1} \xi^{2k+1-l} \mathsf{h}(\mathfrak{m} + \underline{k} - \underline{l}) = r_1^{-k} \big(\mathsf{h}(\mathfrak{m} - \underline{1}) + \xi \mathsf{h}(\mathfrak{m})\big) \sum_{l=0}^{k} (\xi r_1)^{2l} = 0$$

since $\mathfrak{m} \in G_{1/2}$ in this case. Thus $\dim L(\mathfrak{m} + \underline{j}) \leq 2j + i$ by Proposition 6.3. Write $\dim L(\mathfrak{m} + \underline{k}) = 2k' + i'$ where $k' \geq 0$, $i' \in \{1, 2\}$ and assume that $2k' + i' < 2k + i$. By part b) we have $\mathfrak{m} + \underline{k} \in G_{\frac{i'-1}{2}} + \underline{k'}$ which implies that $\dim L(\mathfrak{m} + \underline{k} - \underline{k'}) = i'$ by (6.9) and (6.10). This contradicts the induction hypothesis unless $k' = 0$. Assuming $k' = 0$ we get $\mathfrak{m} + \underline{k} \in G_{\frac{i'-1}{2}}$. If $i = i'$ then from Lemma 6.13 follows that $\underline{k} \in G_0$. Since $0 < k < \frac{2k+i}{2} \leq N/2 \leq N'$ this contradicts 6.12. We now show that $i \neq i'$ is also impossible. If $i = 1$ and $i' = 2$, then $\mathfrak{m} \in G_0$ and $\mathfrak{m} + \underline{k} \in G_{1/2}$ so by Proposition 6.12b), $\underline{k} \in G_{1/2} \cap \underline{\mathbb{Z}_{>0}}$. By (6.13) we get $N \big| 2k - 1$ which is absurd because $0 < 2k - 1 < 2k + 1 \leq N$. If $i = 2$ and $i' = 1$ then $\mathfrak{m} \in G_{1/2}$ and $\mathfrak{m} + \underline{k} \in G_0$. By Proposition 6.12b) we have $\underline{-k} = \mathfrak{m} - (\mathfrak{m} + \underline{k}) \in G_{1/2}$. By Lemma 6.14a), $\underline{1+k} \in G_{1/2}$ so (6.13) implies that $N \big| 2(1 + k) - 1 = 2k + 1$. This is impossible since $0 < 2k + 1 < 2k + 2 \leq N$. We have proved that the assumption $2k' + i' < 2k + i$ is false and hence that $\dim L(\mathfrak{m} + \underline{k}) = 2k + i$, which proves the induction step.

Finally, part d) follows from Corollary 6.4 and Proposition 6.10. □

Theorem 6.17. *Let* $\mathfrak{m} \in G$.

- *If* $N = \infty$, *then*

$$\dim L(\mathfrak{m}) < \infty \iff \mathfrak{m} \in (G_0 + \underline{\mathbb{Z}_{\geq 0}}) \cup (G_{1/2} + \underline{\mathbb{Z}_{\geq 0}}) \tag{6.21}$$

and

$$\dim L(\mathfrak{m}_0 + \underline{j}) = 2j + 1, \quad \textit{for } \mathfrak{m}_0 \in G_0 \textit{ and } j \in \mathbb{Z}_{\geq 0}, \tag{6.22}$$

$$\dim L(\mathfrak{m}_{1/2} + \underline{j}) = 2j + 2, \quad \textit{for } \mathfrak{m}_{1/2} \in G_{1/2} \textit{ and } j \in \mathbb{Z}_{\geq 0}. \tag{6.23}$$

- *If* $N < \infty$ *and* N *is even, then*

$$\dim L(\mathfrak{m}) < \infty \iff \mathfrak{m} \in (G_0 + \underline{\mathbb{Z}}) \cup (G_{1/2} + \underline{\mathbb{Z}}) \tag{6.24}$$

and

$$\dim L(\mathfrak{m} + \underline{(N/2)j}) = \dim L(\mathfrak{m}), \quad \textit{for any } \mathfrak{m} \in G \textit{ and } j \in \mathbb{Z}, \tag{6.25}$$

and for $\mathfrak{m}_0 \in G_0$ *and* $\mathfrak{m}_{1/2} \in G_{1/2}$ *we have*

$$\dim L(\mathfrak{m}_0 + \underline{j}) = 2j + 1, \quad \textit{if } 0 \leq j < N/2, \tag{6.26}$$

$$\dim L(\mathfrak{m}_{1/2} + \underline{j}) = 2j + 2, \quad \textit{if } 0 \leq j < N/2. \tag{6.27}$$

- *If* $N < \infty$ *and* N *is odd, then*

$$\dim L(\mathfrak{m}) < \infty \iff \mathfrak{m} \in G_0 + \underline{\mathbb{Z}} = G_{1/2} + \underline{\mathbb{Z}} \tag{6.28}$$

and

$$\dim L(\mathfrak{m} + \underline{Nj}) = \dim L(\mathfrak{m}), \quad \textit{for any } \mathfrak{m} \in G \textit{ and } j \in \mathbb{Z}, \tag{6.29}$$

and for $\mathfrak{m}_0 \in G_0$ *and* $\mathfrak{m}_{1/2} \in G_{1/2}$ *we have*

$$\dim L(\mathfrak{m}_0 + \underline{j}) = \begin{cases} 2j + 1, & \textit{if } 0 \leq j < \frac{N+1}{2}, \\ 2j + 1 - N, & \textit{if } \frac{N+1}{2} \leq j < N, \end{cases} \tag{6.30}$$

$$\dim L(\mathfrak{m}_{1/2} + \underline{j}) = \begin{cases} 2j + 2, & \textit{if } 0 \leq j < \frac{N-1}{2}, \\ 2j + 2 - N, & \textit{if } \frac{N-1}{2} \leq j < N. \end{cases} \tag{6.31}$$

Proof. When $N = \infty$, relations (6.21)-(6.23) are immediate from Lemma 6.16b) and c).

Suppose N is finite and even. The $\Rightarrow$ implication in (6.24) holds by Lemma 6.16b). And (6.25) follows from (6.12) and Corollary 6.4. Assume that $\mathfrak{m} \in (G_0 + \underline{\mathbb{Z}}) \cup (G_{1/2} + \underline{\mathbb{Z}})$. Using (6.25) we can assume that $\mathfrak{m} = \mathfrak{m}' + \underline{j}$ where $\mathfrak{m}' \in G_0 \cup G_{1/2}$ and $0 \le j < N/2$. Then, if $i \in \{1, 2\}$ we have $2j + i \le N$ and Lemma 6.16c) implies (6.26)-(6.27) and therefore $\dim L(\mathfrak{m}) < \infty$ so (6.24) is also proved.

Assume that N is finite and odd. By (6.13) we have $\underline{(N+1)/2} \in G_{1/2}$. Therefore $G_0 + \underline{\mathbb{Z}} = G_0 + \underline{(N+1)/2} + \underline{\mathbb{Z}} = G_{1/2} + \underline{\mathbb{Z}}$ since $G_{1/2}$ is a right coset of G_0 in G by Proposition 6.12. As before, Lemma 6.16b) implies the $\Rightarrow$ case in (6.28) and (6.29) holds by virtue of (6.12) and Corollary 6.4. If $\mathfrak{m} \in G_0 + \underline{\mathbb{Z}}$ we can assume by (6.29) that $\mathfrak{m} \in G_0 + \underline{j}$ where $0 \le j < N$. If $j < \frac{N+1}{2}$, then $2j + 1 < N + 2$ so since N is odd we have $2j + 1 \le N$. By Lemma 6.16c) we deduce that $\dim L(\mathfrak{m}) = 2j + 1$. If instead $j \ge \frac{N+1}{2}$, then $\mathfrak{m} = \underline{(N+1)/2} + \mathfrak{m} - \underline{(N+1)/2} \in G_{1/2} + \underline{k}$ where $k = j - \frac{N+1}{2}$ so $0 \le k < \frac{N-1}{2}$. Thus $2k + 2 \le N$ so Lemma 6.16c) implies that $\dim L(\mathfrak{m}) = 2k + 2 = 2j + 1 - N$. This proves (6.30) and the $\Leftarrow$ implication in (6.28). Finally (6.31) is equivalent to (6.30) in the following sense. Let $0 \le j < N$ and $\mathfrak{m}_{1/2} \in G_{1/2}$. Then

$$\dim L(\mathfrak{m}_{1/2} + \underline{j}) = \dim L(\mathfrak{m}_0 + \underline{j'}),$$

where $j' = j + (N+1)/2$ and $\mathfrak{m}_0 = \mathfrak{m}_{1/2} - \underline{(N+1)/2}$. Now $\mathfrak{m}_0 \in G_0$ since $G_{1/2}$ is a coset of G_0 in G. If $0 \le j < \frac{N-1}{2}$, then $\frac{N+1}{2} \le j' < N$ so by (6.30) we have

$$\dim L(\mathfrak{m}_{1/2} + \underline{j}) = \dim L(\mathfrak{m}_0 + \underline{j'}) = 2j' + 1 - N = 2j + 2.$$

And if $\frac{N-1}{2} \le j < N$, then $0 \le j' - N < \frac{N+1}{2}$ and hence

$$\dim L(\mathfrak{m}_{1/2} + \underline{j}) = \dim L(\mathfrak{m}_0 + \underline{j' - N}) = 2(j' - N) + 1 = 2j + 1 - N.$$

The proof is finished. □

Corollary 6.18. *If $N = \infty$ and $\mathfrak{m} \in G_0 \cup G_{1/2}$, then $L(\mathfrak{m} + \underline{j})$ is infinite-dimensional for any $j \in \mathbb{Z}_{<0}$.*

Proof. If the dimension of $L(\mathfrak{m} + \underline{j})$ were finite and odd (even), then $\dim L(\mathfrak{m} + \underline{j - k}) = 1$ (2) for some $k \ge 0$ by Lemma 6.16b). By Lemma 6.16c), $L(\mathfrak{m})$ has then dimension $2(j-k)+1$ $(2(j-k)+2)$ and thus $j = k$ which is absurd. □

Corollary 6.19. *Suppose $N = \infty$ and let $\mathfrak{m} \in G_f$. Then $L(\mathfrak{m})$ is the unique finite-dimensional quotient of $M(\mathfrak{m})$.*

Proof. It is enough to prove that the unique maximal proper submodule $N(\mathfrak{m})$ of $M(\mathfrak{m})$ is simple. By Theorem 6.17 we can write $\mathfrak{m} = \mathfrak{n} + \underline{j}$ where $\mathfrak{n} \in G_0 \cup G_{1/2}$ and $j \in \mathbb{Z}_{\ge 0}$. From the proof of Proposition 6.3 we have

$$\mathrm{Supp}(L(\mathfrak{m})) = \{\mathfrak{n} + \underline{j}, \mathfrak{n} + \underline{j} - \underline{1}, \ldots, \mathfrak{n} - \underline{j}\}.$$

Thus $N(\mathfrak{m})$ is a highest weight module of highest weight $\mathfrak{n} - \underline{j} - \underline{1}$. So $N(\mathfrak{m})$ is a quotient of $M(\mathfrak{n} - \underline{j} - \underline{1})$. But $M(\mathfrak{n} - \underline{j} - \underline{1})$ is simple, otherwise it would have a finite-dimensional simple quotient, i.e. $L(\mathfrak{n} - \underline{j} - \underline{1})$ would be finite-dimensional, contradicting Corollary 6.18. Thus $N(\mathfrak{m})$ is also simple. □

Remark 6.20. We finish this section by remarking that there exist algebras in the class studied in this paper which do not have even-dimensional simple modules as for example the algebra B_q from Section 4.4. Indeed, in this case we have $\xi r_1 = -1$ and so $N = \infty$ by definition. By Proposition 6.12, $G_{1/2} = \emptyset$ so by Theorem 6.17, there can exist no even-dimensional simple modules.

7. Tensor products and a Clebsch-Gordan formula

As we have seen in Section 2 the existence of a Hopf structure on an algebra allows one to define tensor product of its representations by (2.4). The aim of this section is to prove a formula which decomposes the tensor product of two simple A-modules into a direct sum of simple modules. It generalizes the classical Clebsch-Gordan formula for modules over $U(\mathfrak{sl}_2)$. We will assume that $A = A(R, \sigma, \mathsf{h}, \xi)$ is an ambiskew polynomial ring and that it carries a Hopf structure of the type considered in Section 3. We will also assume (6.1) and that $N = \infty$.

Lemma 7.1. *Let V and W be two A-modules. Then*

$$V_{\mathfrak{m}} \otimes W_{\mathfrak{n}} \subseteq (V \otimes W)_{\mathfrak{m}+\mathfrak{n}} \tag{7.1}$$

for any $\mathfrak{m}, \mathfrak{n} \in G$. Hence if V and W are weight modules, then so is $V \otimes W$ and

$$\mathrm{Supp}(V \otimes W) = \{\mathfrak{m} + \mathfrak{n} \mid \mathfrak{m} \in \mathrm{Supp}(V), \mathfrak{n} \in \mathrm{Supp}(W)\}.$$

Proof. Let $v \in V_{\mathfrak{m}}, w \in W_{\mathfrak{n}}$. Then for any $r \in R$,

$$r(v \otimes w) = \sum_{(r)} r'v \otimes r''w = \sum_{(r)} r'(\mathfrak{m})v \otimes r''(\mathfrak{n})w = \\ = \sum_{(r)} r'(\mathfrak{m})r''(\mathfrak{n})v \otimes w = r(\mathfrak{m}+\mathfrak{n})v \otimes w$$

by (5.3), proving (7.1). Thus if V, W are weight modules,

$$V \otimes W = (\oplus_{\mathfrak{m}} V_{\mathfrak{m}}) \otimes (\oplus_{\mathfrak{n}} W_{\mathfrak{n}}) = \oplus_{\mathfrak{m},\mathfrak{n}} V_{\mathfrak{m}} \otimes W_{\mathfrak{n}} = \oplus_{\mathfrak{m}} \big(\oplus_{\mathfrak{m}_1+\mathfrak{m}_2=\mathfrak{m}} V_{\mathfrak{m}_1} \otimes W_{\mathfrak{m}_2} \big).$$

□

Theorem 7.2. *Let $\mathfrak{m}, \mathfrak{n} \in G_f$. We have the following isomorphism*

$$L(\mathfrak{m}) \otimes L(\mathfrak{n}) \simeq L(\mathfrak{m}+\mathfrak{n}) \oplus L(\mathfrak{m}+\mathfrak{n}-\underline{1}) \oplus \ldots \oplus L(\mathfrak{m}+\mathfrak{n}-\underline{s}+\underline{1}) \tag{7.2}$$

where $s = \min\{\dim L(\mathfrak{m}), \dim L(\mathfrak{n})\}$.

Proof. Let $e^{\mathfrak{m}}, e^{\mathfrak{n}}$ denote highest weight vectors in $L(\mathfrak{m})$, $L(\mathfrak{n})$ respectively and set $e^{\mathfrak{m}}_j := (X_-)^j e^{\mathfrak{m}}$ for $j \in \mathbb{Z}_{\geq 0}$ and similarly for $\mathfrak{n}$. Set $V = L(\mathfrak{m}) \otimes L(\mathfrak{n})$. By Lemma 7.1 we have

$$V_{\mathfrak{m}+\mathfrak{n}-\underline{k}} = \oplus_{i+j=k} \mathbb{K} e^{\mathfrak{m}}_i \otimes e^{\mathfrak{n}}_j$$

for $k \in \mathbb{Z}_{\geq 0}$. Fix $0 \leq k \leq s-1$. We will prove that

$$\dim \ker X_+|_{V_{\mathfrak{m}+\mathfrak{n}-\underline{k}}} = 1. \tag{7.3}$$

From the calculations in the proof of Proposition 6.3 follows that when $j > 0$, $X_+ e^{\mathfrak{m}}_j$ is a nonzero multiple of $e^{\mathfrak{m}}_{j-1}$. Let $\nu^{\mathfrak{m}}_j$ denote this multiple. Let

$$u = \sum_{i=0}^{k} \lambda_i e^{\mathfrak{m}}_i \otimes e^{\mathfrak{n}}_{k-i}$$

be an arbitrary vector in $V_{\mathfrak{m}+\mathfrak{n}-\underline{k}}$. Then

$$\begin{aligned} X_+u &= \sum_{i=0}^{k} \lambda_i (X_+ e_i^{\mathfrak{m}} \otimes r_+ e_{k-i}^{\mathfrak{n}} + l_+ e_i^{\mathfrak{m}} \otimes X_+ e_{k-i}^{\mathfrak{n}}) = \\ &= \sum_{i=0}^{k-1} \big[\lambda_{i+1} \nu_{i+1}^{\mathfrak{m}} r_+(\mathfrak{n} - \underline{k} + \underline{i} + \underline{1}) + \lambda_i l_+(\mathfrak{m} - \underline{i}) \nu_{k-i}^{\mathfrak{n}}\big] e_i^{\mathfrak{m}} \otimes e_{k-1-i}^{\mathfrak{n}}. \end{aligned}$$

Setting

$$\begin{aligned} c_i &= l_+(\mathfrak{m} - \underline{i}) \nu_{k-i}^{\mathfrak{n}}, \\ c_i' &= \nu_i^{\mathfrak{m}} r_+(\mathfrak{n} - \underline{k} + \underline{i}), \end{aligned}$$

the condition for u to be a highest weight vector can hence be written as

$$\begin{bmatrix} c_0 & c_1' & & & \\ & c_1 & c_2' & & \\ & & \ddots & \ddots & \\ & & & c_{k-1} & c_k' \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_k \end{bmatrix} = 0. \tag{7.4}$$

Since r_+ and l_+ are grouplike, they are invertible and hence $c_i \neq 0 \neq c_{i+1}'$ for any $i = 0, 1, \ldots, k-1$. Therefore the space of solutions to (7.4) is one-dimensional. Thus (7.3) is proved.

From the definition of Verma modules, it follows that for $k = 0, 1, \ldots, s-1$, there is a nonzero A-module morphism

$$M(\mathfrak{m} + \mathfrak{n} - \underline{k}) \to L(\mathfrak{m}) \otimes L(\mathfrak{n})$$

which maps a highest weight vector in $M(\mathfrak{m} + \mathfrak{n} - \underline{k})$ to a highest weight vector in $L(\mathfrak{m}) \otimes L(\mathfrak{n})$ of weight $\mathfrak{m} + \mathfrak{n} - \underline{k}$. But $L(\mathfrak{m}) \otimes L(\mathfrak{n})$ is finite-dimensional so this morphism must factor through $L(\mathfrak{m} + \mathfrak{n} - \underline{k})$ by Corollary 6.19. Taking direct sums of these morphisms we obtain an A-module morphism

$$\varphi : L(\mathfrak{m} + \mathfrak{n}) \oplus L(\mathfrak{m} + \mathfrak{n} - \underline{1}) \oplus \ldots \oplus L(\mathfrak{m} + \mathfrak{n} - \underline{s} + \underline{1}) \to L(\mathfrak{m}) \otimes L(\mathfrak{n}).$$

We claim it is injective. Indeed, the projection of the kernel of φ to any term $L(\mathfrak{m} + \mathfrak{n} - \underline{i})$ must be zero, because it is a proper submodule of the simple module $L(\mathfrak{m} + \mathfrak{n} - \underline{i})$.

To conclude we now calculate the dimensions of both sides. Write $\dim L(\mathfrak{m}) = 2j_1 + i_1$ and $\dim L(\mathfrak{n}) = 2j_2 + i_2$ where $j_1, j_2 \in \mathbb{Z}_{\geq 0}$ and $i_1, i_2 \in \{1, 2\}$. By Lemma 6.16b), $\dim L(\mathfrak{m} - \underline{j_1}) = i_1$ and $\dim L(\mathfrak{n} - \underline{j_2}) = i_2$. First note that

$$\dim L(\mathfrak{m} - \underline{j_1} + \mathfrak{n} - \underline{j_2}) = i_1 + i_2 - 1.$$

When $i_1 = i_2 = 1$, this is true because G_0 is a subgroup of G. When one of i_1, i_2 is 1 and the other 2, it follows from Proposition 6.12b). And if $i_1 = i_2 = 2$, it follows from Lemma 6.14b) and Theorem 6.17.

From Theorem 6.17 also follows that $\dim L(\mathfrak{m}+\underline{k}) = \dim L(\mathfrak{m})+2k$ if $\dim L(\mathfrak{m}) < \infty$ and $k \in \mathbb{Z}_{\geq 0}$. Hence, recalling that $s = \min\{\dim L(\mathfrak{m}), \dim L(\mathfrak{n})\}$, we have

$$\begin{aligned}\sum_{k=0}^{s-1} \dim L(\mathfrak{m}+\mathfrak{n}-\underline{k}) &= \sum_{k=0}^{s-1} \dim L(\mathfrak{m}-\underline{j_1}+\mathfrak{n}-\underline{j_2}+\underline{j_1}+\underline{j_2}-\underline{k}) = \\ &= \sum_{k=0}^{s-1} \big(i_1+i_2-1+2(j_1+j_2-k)\big) = \\ &= s(i_1+i_2-1+2j_1+2j_2) - s(s-1) = \\ &= s(\dim L(\mathfrak{m})+\dim L(\mathfrak{n})-s) = \\ &= \dim L(\mathfrak{m}) \dim L(\mathfrak{n}) = \dim \big(L(\mathfrak{m})\otimes L(\mathfrak{n})\big).\end{aligned}$$

This completes the proof of the theorem. □

Under some conditions it is possible to introduce a $*$-structure on A. In this connection it would be interesting to study Clebsch-Gordan coefficients and the relation with special functions. This will be a subject for future investigation.

8. Casimir operators and semisimplicity

Arguing as in the proof of Lemma 4.2, it is easy to see that any finite-dimensional semisimple module over $A = A(R, \sigma, \mathsf{h}, \xi)$ is a weight module. In this section we will prove the converse, that any finite-dimensional weight module over A is semisimple. Note that in general not all finite-dimensional modules over our algebra A are semisimple. The corresponding example is constructed in [6] for the algebra from Section 4.3. A necessary and sufficient condition for all finite-dimensional modules over an ambiskew polynomial ring to be semisimple was given in [8], Theorem 5.1.

In this section we assume that $A = A(R, \sigma, \mathsf{h}, \xi)$ is an ambiskew polynomial ring with a Hopf structure of the type introduced in Section 3 such that (6.1) holds. We also assume that $N = \infty$.

Let V be a finite-dimensional weight module over A. We will first treat the case when $\mathrm{Supp}(V) \subseteq \mathfrak{m}+\underline{\mathbb{Z}}$ where $\mathfrak{m} \in G_0$ is fixed. Define a linear map

$$C_V : V \to V$$

by requiring

$$C_V v = \sigma^j(t)v, \quad \text{for } v \in V_{\mathfrak{m}+\underline{j}} \text{ and } j \in \mathbb{Z}.$$

Here σ denotes the extended automorphism (2.8). More explicitly we have (if $j \geq 0$)

$$C_V v = \sigma^j(t)v = \Big(\xi^j t + \sum_{k=0}^{j-1} \xi^k \sigma^{j-1-k}(\mathsf{h})\Big)v = \xi^j t v + \sum_{k=0}^{j-1} \xi^k \mathsf{h}(\mathfrak{m}+\underline{k+1})v$$

and similarly when $j < 0$. It is easy to check that C_V is a morphism of A-modules. Hence it is constant on each finite-dimensional simple module V by Schur's Lemma. Moreover if $\varphi : V \to W$ is a morphism of weight A-modules with support in $\mathfrak{m}+\underline{\mathbb{Z}}$, then $\varphi C_V = C_W \varphi$.

Proposition 8.1. *Let $j_1, j_2 \in \mathbb{Z}_{\geq 0}$. If $C_{L(\mathfrak{m}+\underline{j_1})} = C_{L(\mathfrak{m}+\underline{j_2})}$, then $j_1 = j_2$.*

Proof. By applying $C_{L(\mathfrak{m}+\underline{j})}$ to the highest weight vector of $L(\mathfrak{m}+\underline{j})$, $(j \in \mathbb{Z}_{\geq 0})$ we get

$$C_{L(\mathfrak{m}+\underline{j})} = \sum_{k=0}^{j-1} \xi^k \mathsf{h}(\mathfrak{m} + \underline{k+1}). \tag{8.1}$$

We can assume $j_1 < j_2$. By assumption we have

$$0 = \sum_{k=0}^{j_2-1} \xi^k \mathsf{h}(\mathfrak{m} + \underline{k+1}) - \sum_{k=0}^{j_1-1} \xi^k \mathsf{h}(\mathfrak{m} + \underline{k+1}) = \sum_{k=j_1}^{j_2-1} \xi^k \mathsf{h}(\mathfrak{m} + \underline{k+1}) =$$
$$= \xi^{j_1} \sum_{k=0}^{j_2-j_1-1} \xi^k \mathsf{h}(\mathfrak{m} + \underline{j_2} - \underline{(j_2-j_1)} + \underline{k+1}).$$

By Proposition 6.3 this means that $\dim L(\mathfrak{m}+\underline{j_2}) \leq j_2 - j_1$. But this contradicts Theorem 6.17 which says that $\dim L(\mathfrak{m}+\underline{j_2}) = 2j_2 + 1$. □

Theorem 8.2. *Let V be a finite-dimensional weight module over A with support in $G_0 + \underline{\mathbb{Z}}$. Then V is semisimple.*

Proof. We follow the idea of the proof of Proposition 12 in [10], Chapter 3. Writing

$$V = \oplus_{\mathfrak{m}\in G_0} \big(\oplus_{j\in\mathbb{Z}} V_{\mathfrak{m}+\underline{j}} \big)$$

and noting that $\oplus_{j\in\mathbb{Z}} V_{\mathfrak{m}+\underline{j}}$ are submodules, we can reduce to the case when $\mathrm{Supp}(V)$ is contained in $\mathfrak{m}+\mathbb{Z}$ for a fixed $\mathfrak{m} \in G_0$.

Let $\lambda_1, \ldots, \lambda_k$ be the generalized eigenvalues of the Casimir operator C_V, i.e. the elements of the set

$$\{\lambda \in \mathbb{K} \mid \ker(C_V - \lambda\,\mathrm{Id})^p \neq 0 \text{ for some } p > 0\}.$$

Then each generalized eigenspace $\sum_p \ker(C_V - \lambda_i\,\mathrm{Id})^p$ is invariant under A, hence they are submodules. It suffices to prove that each such submodule is semisimple. Let V be one of them. Let $V_1 = \{v \in V \mid X_+ v = 0\}$. Then V_1 is invariant under R and since V is a weight module, $V_1 = \oplus_{\mathfrak{n}\in G}(V_1 \cap V_{\mathfrak{n}})$. Now if $0 \neq v \in V_1 \cap V_{\mathfrak{n}}$, then v is a highest weight vector of V and generates a submodule isomorphic to $L(\mathfrak{n})$. Hence if $V_1 \cap V_{\mathfrak{n}} \neq 0$ for more than one $\mathfrak{n} \in G$, C_V will have two different eigenvalues by Proposition 8.1 which is impossible. Here we used that the restriction of C_V to a submodule W coincides with C_W. Hence V_1 is contained in a single weight space, say $V_{\mathfrak{n}}$. Let $v_1, \ldots, v_k$ be a basis for V_1. Then each v_i generates a simple submodule isomorphic to $L(\mathfrak{n})$. We will show that the sum of these submodules is direct. Vectors of different weights are linearly independent so it suffices to show that if

$$\sum_{i=1}^{k} \lambda_i (X_-)^m v_k = 0$$

then all $\lambda_i = 0$. Assume the sum was nonzero and act by X_+ m times. In each step we get a nonzero result because we have not reached the highest weight $\mathfrak{n}$ yet. But then, using (6.3), we have a linear relation among the v_k – a contradiction. We have shown that V contains the direct sum V' of k copies of $L(\mathfrak{n})$. Now X_+ acts injectively on V/V'. This is only possible in a torsion-free finite-dimensional weight A-module if it is 0-dimensional. Thus V is semisimple. □

We now turn to the general case. Assume now that A has an even-dimensional irreducible representation. By Lemma 6.16b), $G_{1/2} \neq \emptyset$. We fix $\mathfrak{m}_{1/2} \in G$. Then $G_{1/2} = G_0 + \mathfrak{m}_{1/2}$ by Proposition 6.12.

Theorem 8.3. *Any finite-dimensional weight module V over A is semisimple.*

Proof. By Corollary 6.2 and Theorem 6.17,

$$\operatorname{Supp}(V) \subseteq (G_0 + \underline{\mathbb{Z}}) \cup (G_{1/2} + \underline{\mathbb{Z}})$$

Thus we have a decomposition

$$V = \Big(\bigoplus_{\mathfrak{m} \in G_0} V_{\mathfrak{m}+\underline{\mathbb{Z}}} \Big) \oplus \Big(\bigoplus_{\mathfrak{m} \in G_0} V_{\mathfrak{m}+\mathfrak{m}_{1/2}+\underline{\mathbb{Z}}} \Big)$$

where $V_{\mathfrak{n}+\mathbb{Z}} := \oplus_{j\in\mathbb{Z}} V_{\mathfrak{n}+\underline{j}}$ for $\mathfrak{n} \in G$ are submodules. It remains to prove that a weight module V with support in $\mathfrak{m} + \mathfrak{m}_{1/2} + \mathbb{Z}$ is semisimple. By Lemma 7.1,

$$\operatorname{Supp}\big(V \otimes L(\mathfrak{m}_{1/2})\big) \subseteq \mathfrak{m} + \mathfrak{m}_{1/2} + \mathfrak{m}_{1/2} + \mathbb{Z} = \mathfrak{m}' + \mathbb{Z}$$

where $\mathfrak{m}' := \mathfrak{m} + \mathfrak{m}_{1/2} + \mathfrak{m}_{1/2} - 1 \in G_0$ by Lemma 6.14b). Hence $V \otimes L(\mathfrak{m}_{1/2})$ is semisimple by Theorem 8.2. By the Clebsch-Gordan formula (7.2), the tensor product of two semisimple modules is semisimple again. Therefore $V \otimes L(\mathfrak{m}_{1/2}) \otimes L(\underline{1} - \mathfrak{m}_{1/2})$ is semisimple, where $\dim L(\underline{1} - \mathfrak{m}_{1/2}) = 2$ by Lemma 6.14a). On the other hand, by (7.2) again we have

$$V \otimes L(\mathfrak{m}_{1/2}) \otimes L(\underline{1} - \mathfrak{m}_{1/2}) \simeq V \otimes \big(L(0) \oplus L(\mathfrak{m})\big) \simeq \big(V \otimes L(0)\big) \oplus \big(V \otimes L(\mathfrak{m})\big).$$

Finally, it is easy to verify the isomorphism $V \simeq V \otimes L(0)$, $v \mapsto v \otimes e$ where $0 \neq e \in L(0)$ is fixed. Thus V is isomorphic to a submodule of the semisimple module $V \otimes L(\mathfrak{m}_{1/2}) \otimes L(\underline{1} - \mathfrak{m}_{1/2})$ and is therefore itself semisimple. □

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