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Hopf Structures on Ambiskew Polynomial Rings

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Abstract. We derive necessary and sufficient conditions for an ambiskew polynomial ring to have a Hopf algebra structure of a certain type. This construction generalizes many known Hopf algebras, for example $U(\mathfrak{sl}_2)$, $U_q(\mathfrak{sl}_2)$ and the enveloping algebra of the 3-dimensional Heisenberg Lie algebra. In a torsion-free case we describe the finite-dimensional simple modules, in particular their dimensions and prove a Clebsch-Gordan decomposition theorem for the tensor product of two simple modules. We construct a Casimir type operator and prove that any finite-dimensional weight module is semisimple.

1. Introduction

In [4], the authors define a four parameter deformation of the Heisenberg (oscillator) Lie algebra $W_{\gamma,\alpha,\beta}(q)$ and study its representations. Moreover by requiring this algebra to be invariant under $q \to q^{-1}$, they define a Hopf algebra structure on $W_{\gamma,\alpha,\beta}(q)$ generalizing several previous results.

The quantum group $U_q(\mathfrak{sl}_2(\mathbb{C}))$ has by definition the structure of a Hopf algebra. In [6], an extension of this quantum group to an associative algebra denoted by $U_q(f(H,K))$ (where $f$ is a Laurent polynomial in two variables) is defined and finite-dimensional representations are studied. The authors show that under certain conditions on $f$, a Hopf algebra structure can be introduced. Among these Hopf algebras is for example the Drinfeld double $D(\mathfrak{sl}_2)$.

All of the mentioned algebras fall (after suitable mathematical formalization in the case of $W_{\gamma,\alpha,\beta}(q)$) into the class of so called ambiskew polynomial rings (see Section 2 for the definition). Motivated by these examples of similar classes of algebras, all of which can be equipped with Hopf algebra structures, we consider a certain type of Hopf structures on a class of ambiskew polynomial rings.

In Section 2, we recall some definitions and fix notation. We present the conditions for a certain Hopf structure on an ambiskew polynomial ring in Section 3, while Section 4 is devoted to examples. In Section 5 we introduce some convenient notation and state some useful formulas for viewing $R$ as an algebra of functions on its set of maximal ideals. Finite-dimensional simple modules are studied in Section 6. Those have already been classified in [7], but we focus on describing the dimensions in terms of the highest weights. The main result is stated in Theorem 6.17. The classical Clebsch-Gordan theorem for $U(\mathfrak{sl}_2)$ is generalized in Section 7 to the present more general setting, using the results of the previous section. Finally, in Section 8 we first construct a kind of Casimir operator and prove that it can be used to distinguish non-isomorphic simple modules. This is then used to prove that any weight module is semisimple.

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2. Preliminaries

Throughout, $\mathbb{K}$ will be an algebraically closed field of characteristic zero. All algebras are associative and unital $\mathbb{K}$-algebras.

By a Hopf structure on an algebra $A$ we mean a triple $(\Delta, \varepsilon, S)$ where the coproduct $\Delta : A \to A \otimes A$ is a homomorphism, $(A \otimes A$ is given the tensor product algebra structure) the counit $\varepsilon : A \to \mathbb{K}$ is a homomorphism, and the antipode $S : A \to A$ is an anti-homomorphism such that

\begin{align}
(\text{Coassociativity}) & \\ (\text{Counit axiom}) & \\ (\text{Antipode axiom})
\end{align}

for all $x \in A$. Here $m : A \otimes A \to A$ denotes the multiplication map of $A$. A Hopf algebra is an algebra equipped with a Hopf structure. An element $x \in A$ of a Hopf algebra $A$ is called grouplike if $\Delta(x) = x \otimes x$ and primitive if $\Delta(x) = x \otimes 1 + 1 \otimes x$.

In the former case it follows from the axioms that $\varepsilon(x) = 1$, $x$ is invertible and $S(x) = x^{-1}$ while in the latter $\varepsilon(x) = 0$ and $S(x) = -x$.

If $V_i$ ($i = 1, 2$) are two modules over a Hopf algebra $H$, then $V_1 \otimes V_2$ becomes an $H$-module in the following way

\begin{align}
a(v_1 \otimes v_2) &= \sum_i (a'_i v_1) \otimes (a''_i v_2)
\end{align}

for $v_i \in V_i$ ($i = 1, 2$) if $a \in H$ with $\Delta(a) = \sum_i a'_i \otimes a''_i$. From (2.1) it follows that if $V_i$ ($i = 1, 2, 3$) are modules over $H$ then the natural vector space isomorphism $V_1 \otimes (V_2 \otimes V_3) \simeq (V_1 \otimes V_2) \otimes V_3$ is an isomorphism of $H$-modules. From (2.2) follows that the one-dimensional module $\mathbb{K}_x$ associated to the representation $\varepsilon$ of $H$ is a tensor unit, i.e. $\mathbb{K}_x \otimes V \simeq V \simeq V \otimes \mathbb{K}_x$ as $H$-modules for any $H$-module $V$.

Let $R$ be a finitely generated commutative algebra over $\mathbb{K}$. Let $\sigma$ be an automorphism of $R$, $h \in R$ and $\xi \in \mathbb{K} \setminus \{0\}$. Then we define the algebra $A = A(R, \sigma, h, \xi)$ as the associative $\mathbb{K}$-algebra formed by adjoining to $R$ two symbols $X_+, X_-$ subject to the relations

\begin{align}
X_+ a &= \sigma^{\pm 1}(a)X_+ \quad \text{for } a \in R, \\
X_+ X_- &= h + \xi X_- X_+.
\end{align}

This algebra is called an ambiskew polynomial ring. Its structure and representations were studied by Jordan [8] (see also references therein).

We recall the definition of a generalized Weyl algebra (GWA) (see [1] and references therein). If $B$ is a ring, $\sigma$ an automorphism of $B$, and $t \in B$ a central element, then the generalized Weyl algebra $B(\sigma, t)$ is the ring extension of $B$ generated by two elements $x_+, x_- \subseteq B$ subject to the relations

\begin{align}
x_\pm a &= \sigma^{\pm 1}(a)x_\pm, & a \in B, \\
x_-x_+ &= t, & x_+x_- = \sigma(t).
\end{align}

The relation between these two constructions is the following. Let $A = A(R, \sigma, h, \xi)$ be an ambiskew polynomial ring. Denote by $R[t]$ be the polynomial ring in one variable $t$ with coefficients in $R$ and let us extend the automorphism $\sigma$ of $R$ to a $\mathbb{K}$-algebra automorphism of $R[t]$ satisfying

\begin{align}
\sigma(t) &= h + \xi t.
\end{align}
Then $A$ is isomorphic to the GWA $R[t](\sigma, t)$.

3. The Hopf structure

Let $A = A(R, \sigma, h, \xi)$ be a skew polynomial ring and assume that $R$ has been equipped with a Hopf structure. In this section we will extend the Hopf structure on $R$ to $A$. We make the following ansatzs, guided by [4] and [6]:

(3.1) \[ \Delta(X_\pm) = X_\pm \otimes r_\pm + l_\pm \otimes X_\pm, \]

(3.2) \[ \varepsilon(X_\pm) = 0, \]

(3.3) \[ S(X_\pm) = s_\pm X_\mp. \]

The elements $r_\pm, l_\pm$ and $s_\pm$ will be assumed to belong to $R$.

Theorem 3.1. Formulas (3.1)-(3.3) define a Hopf algebra structure on $A$ which extends that of $R$ iff

(3.4a) \[ (\sigma \otimes \text{Id}) \circ \Delta|_R = \Delta \circ \sigma|_R = (\text{Id} \otimes \sigma) \circ \Delta|_R, \]

(3.4b) \[ S \circ \sigma|_R = \sigma^{-1} \circ S|_R, \]

(3.5a) \[ \Delta(h) = h \otimes r_+ r_- + l_+ l_- \otimes h, \]

(3.5b) \[ \varepsilon(h) = 0, \]

(3.5c) \[ S(h) = -(l_+ l_- r_+ r_-)^{-1} h, \]

(3.6a) \[ r_\pm \text{ and } l_\pm \text{ are grouplike, i.e. } \Delta(x) = x \otimes x \text{ for } x \in \{r_\pm, l_\pm\}, \]

(3.6b) \[ \sigma(l_\pm) \otimes \sigma(r_\mp) = \xi l_\pm \otimes r_\mp, \]

(3.7) \[ (s_\pm)^{-1} = -l_\pm \sigma^{-1}(r_\pm). \]

Proof. From (2.5)-(2.6) we see that $\varepsilon$ extends to a homomorphism $A \to K$ satisfying (3.2) if and only if (3.5b) holds. Assume for a moment that $\Delta$ extends to a homomorphism $A \to A \otimes A$. From (3.1)-(3.2) it follows that $\varepsilon$ is a counit iff

(3.8) \[ \varepsilon(r_+) = \varepsilon(r_-) = \varepsilon(l_+) = \varepsilon(l_-) = 1. \]

$\Delta$ is coassociative iff (dropping the $\pm$)

(3.9) \[ (\text{Id} \otimes \Delta)(\Delta(X)) = (\Delta \otimes \text{Id})(\Delta(X)) \]

which is equivalent to

\[ X \otimes \Delta(r) + l \otimes X \otimes r + l \otimes l \otimes X = X \otimes r \otimes r + l \otimes X \otimes r + \Delta(l) \otimes X, \]

or

(3.9) \[ X \otimes (\Delta(r) - r \otimes r) = (\Delta(l) - l \otimes l) \otimes X. \]

From (2.5)-(2.6) follows that $A$ has a $\mathbb{Z}$-gradation defined by requiring that $\deg r = 0$ for $r \in R$, $\deg X_\pm = \pm 1$. This also induces a $\mathbb{Z}^2$-gradation on $A \otimes A$ in a natural way. The left and right hand sides of equation (3.9) are homogenous of different $\mathbb{Z}^2$-degrees, namely $(\pm 1, 0)$ and $(0, \pm 1)$ respectively. Hence, since homogenous elements of different degrees must be linearly independent, (3.9) is equivalent to both sides being zero which holds iff $r_\pm$ and $l_\pm$ are grouplike.
\( \Delta \) respects (2.5) iff (again dropping \( \pm \))
\[
\Delta(X)\Delta(a) = \Delta(\sigma(a))\Delta(X),
\]
\[
(X \otimes r + l \otimes X)\Delta(a) = \Delta(\sigma(a))(X \otimes r + l \otimes X),
\]
\[
(\sigma \otimes 1)\Delta(a) \cdot (X \otimes r) + (1 \otimes \sigma)\Delta(a) \cdot (l \otimes X) = \Delta(\sigma(a))(X \otimes r + l \otimes X),
\]
\[
((\sigma \otimes 1)\Delta(a) - \Delta(\sigma(a))) \cdot (X \otimes r) + ((1 \otimes \sigma)\Delta(a) - \Delta(\sigma(a))) \cdot (l \otimes X) = 0.
\]

As before the two terms in the last equation have different \( Z^2 \)-degrees and therefore must be zero. So \( \Delta \) respects (2.5) iff (3.4a) holds.

It is straightforward to check that \( \Delta \) respects (2.6) iff
\[
(3.10) \quad h \otimes r_+r_- + l_+l_- \otimes h - \Delta(h) + \frac{1}{2}(l_+ \otimes \sigma(r_-) - \xi \sigma^{-1}(l_+) \otimes r_-)X_- \otimes X_+ + \frac{1}{2}(\sigma(l_-) \otimes -\xi l_- \otimes \sigma^{-1}(r_+))X_+ \otimes X_- = 0.
\]

Again these three terms have different degrees so each of them must be zero. Hence (3.5a) holds. Multiply the second term by \( X_+ \otimes X_- \) from the right:
\[
(l_+ \otimes \sigma(r_-) - \xi \sigma^{-1}(l_+) \otimes r_-)t \otimes \sigma(t) = 0.
\]

Here we use the extension (2.8) of \( \sigma \) to \( R[t] \) where \( t = X_-X_+ \). If we apply \( e_1 \otimes e_1' \) to this equation, where \( e_r (e_r') \) for \( r \in R \) is the evaluation homomorphism \( R[t] \to R \) which maps \( t (\sigma(t)) \) to \( r \), we get:
\[
l_+ \otimes \sigma(r_-) = \xi \sigma^{-1}(l_+) \otimes r_-.
\]

Applying \( \sigma \otimes 1 \) to this we obtain one of the relations in (3.6b). Similarly the vanishing of the third term in (3.10) implies the other.

Assuming that \( S \) is an anti-homomorphism \( A \to A \) satisfying (3.3), we obtain that \( S \) is a an antipode on \( A \) iff
\[
S(X_+)r_+ + S(l_+X_-) = 0 = X_+S(r_+) + l_+S(X_-),
\]
which is equivalent to (3.7), using that \( r_+ \) and \( l_+ \) are grouplike. And \( S \) extends to a well-defined anti-homomorphism \( A \to A \) iff
\[
(3.11) \quad S(a)S(X_{\pm}) = S(X_{\pm})S(\sigma^{\pm 1}(a)), \quad \text{for } a \in R,
\]
\[
(3.12) \quad S(X_-)S(X_+) = S(h) + \xi S(X_+)S(X_-).
\]

Using (3.7) and that \( r_+, l_+ \) are invertible, (3.11) holds iff (3.4b) holds. And (3.12) holds iff
\[
0 = s_-X_-s_+X_+ - S(h) - \xi s_+X_+s_-X_- = s_-\sigma^{-1}(s_+)X_-X_+ - S(h) - s_+\xi \sigma(s_-)X_+X_- = -S(h) - s_+\sigma(s_-)\xi h + (s_-\sigma^{-1}(s_+) - s_+\sigma(s_-))\xi^2 t.
\]

Applying \( e_0 \) and \( e_1 \) we obtain
\[
S(h) = -\xi s_+\sigma(s_-)h,
\]
\[
s_-\sigma^{-1}(s_+) = \xi^2 s_+\sigma(s_-).\]

Substituting (3.7) in these equations and using (3.6b), the first is equivalent to (3.5c), while the other already holds.

□
4. Examples

Many Hopf algebras known in the literature can be viewed as one defined in the previous section.

4.1. Heisenberg algebra. Let $R = \mathbb{C}[c]$ with $c$ primitive, and $\sigma(c) = c$. Choose $h = c$, $\xi = r_+ = r_- = l_+ = l_- = 1$. Then $A$ is the universal enveloping algebra $U(\mathfrak{h}_3)$ of the three-dimensional Heisenberg Lie algebra.

4.2. $U(\mathfrak{s}(1))$ and its quantizations.

4.2.1. $U(\mathfrak{s}(1))$. Let $R = \mathbb{C}[H]$ with Hopf algebra structure $\Delta(H) = H \otimes 1 + 1 \otimes H$, $\varepsilon(H) = 0$, $S(H) = -H$. Define $\sigma(H) = H - 1$. Choose $h = H$, $\xi = r_+ = r_- = l_+ = l_- = 1$. Then $A \simeq U(\mathfrak{s}(1))$ as Hopf algebras.

4.2.2. $U_q(\mathfrak{s}(1))$. Let $R = \mathbb{C}[K, K^{-1}]$ with Hopf structure defined by requiring that $K$ is grouplike. Define $\sigma(K) = q^{-2}K$, where $q \in \mathbb{C}$, $q^2 \neq 1$, and choose $h = \frac{K - K^{-1}}{q - q^{-1}}$, $\xi = r_- = l_+ = 1$ and $r_+ = K$, $l_- = K^{-1}$. Then the equations in Theorem 3.1 are satisfied giving a Hopf algebra $A$ which is isomorphic to $U_q(\mathfrak{s}(1))$.

4.2.3. $\tilde{U}_q(\mathfrak{s}(1))$. For the definition of this algebra, see for example [10]. Let $q \in \mathbb{C}$, $q^4 \neq 1$. Let $R = \mathbb{C}[K, K^{-1}]$ with $K$ grouplike. Define $\sigma(K) = q^{-1}K$, $h = \frac{K^2 - K^{-2}}{q - q^{-1}}$, $\xi = 1$, $r_+ = r_- = K$, $l_+ = l_- = K^{-1}$. Then $A = A(R, \sigma, h, \xi)$ is a Hopf algebra isomorphic to $\tilde{U}_q(\mathfrak{s}(1))$.

4.3. $U_q(f(H, K))$. Let $R = \mathbb{C}[H, H^{-1}, K, K^{-1}]$, $\sigma(H) = q^2H$, $\sigma(K) = q^{-2}K$. Let $\alpha \in \mathbb{K}$ and $M, p, r, s, t, s', t' \in \mathbb{Z}$ such that $M = m - n = m' - n' = p + t - r - s$, $s - t = s' - t'$ and $p - r = p' - r'$. Set $h = \alpha(K^mH^n - K^{-m}H^{-n})$, $\xi = 1$, $r_+ = K^sH^r$, $l_+ = K^sH^r$, $r_- = K^{-s}H^{-r}$, $l_- = K^{-s}H^{-r}$, then $A$ is the Hopf algebra described in [6], Theorem 3.3.

4.4. Down-up algebras. The down-up algebra $A(\alpha, \beta, \gamma)$ where $\alpha, \beta, \gamma \in \mathbb{C}$, was defined in [2] and studied by many authors, see for example [3], [5], [8], [9], and references therein. It is the algebra generated by $u, d$ and relations

$$ddu = \alpha ud + \beta udd + \gamma d,$$
$$duu = \alpha udu + \beta uud + \gamma u.$$

In [8] it is proved that if $\sigma$ is allowed to be any endomorphism, not necessarily invertible, then any down-up algebra is an ambiskew polynomial ring. Here we consider the down-up algebra $B = A(0, 1, 1)$. Thus $B$ is the $\mathbb{C}$-algebra with generators $u, d$ and relations

$$(4.1) \quad d^2u = ud^2 + d, \quad du^2 = u^2d + u.$$ 

Let $R = \mathbb{C}[h]$, $\sigma(h) = h + 1$ and $\xi = -1$. Then $B$ is isomorphic to the ambiskew polynomial ring $A(R, \sigma, h, \xi)$ via $d \mapsto X_+$ and $u \mapsto X_-$.

One can show that $B$ is isomorphic to the enveloping algebra of the Lie super algebra $\mathfrak{osp}(1, 2)$ and hence has a graded Hopf structure. A question was raised in [9] whether there exists a Hopf structure on $B$. We do not answer this question here but we show the existence of a Hopf structure on a larger algebra $B_q$ giving us a formula for the tensor product of weight (in particular finite-dimensional) modules over $B$. 

Let $q \in \mathbb{C}^*$ and fix a value of $\log q$. By $q^a$ we always mean $e^{a \log q}$. Let $B_q$ be the ambiskew polynomial ring $B_q = A(R, \sigma, h, \xi)$ where $R = \mathbb{C}[h, w, w^{-1}]$, $\sigma(h) = h + 1$, $\sigma(w) = qw$, and $\xi = -1$.

**Theorem 4.1.** For any $\rho, \lambda \in \mathbb{Z}$ such that $q^{\rho - \lambda} = -1$ and $q^{2\rho} = 1$, the algebra $B_q$ has a Hopf algebra structure given by

\[
\Delta(X_\pm) = X_\pm \otimes w^{\pm \rho} + w^{\pm \lambda} \otimes X_\pm, \quad \varepsilon(X_\pm) = 0,
\]

\[
S(X_\pm) = -w^{\mp \lambda} X_\pm w^{\mp \rho} = -q^{\rho} w^{\mp (\rho + \lambda)} X_\pm,
\]

and

\[
\Delta(w) = w \otimes w, \quad \varepsilon(w) = 1, \quad S(w) = w^{-1},
\]

\[
\Delta(h) = h \otimes 1 + 1 \otimes h, \quad \varepsilon(h) = 0, \quad S(h) = -h.
\]

**Proof.** The subalgebra $\mathbb{C}[h, w, w^{-1}]$ of $B_q$ has a unique Hopf structure given by the maps above. We must verify (3.4)-(3.7) with $h = v$, $\xi = -1$, $r_\pm = w^{\pm \rho}$, $l_\pm = w^{\pm \lambda}$, and $s_\pm = -q^\rho w^{\mp (\rho + \lambda)}$. This is straightforward. \(\Box\)

This gives us a tensor structure on the category of modules over $B_q$. Next aim is to show how using the Hopf structure on $B_q$ one can define a tensor structure on the category of weight modules over $B$.

In general, if $C$ is a commutative subalgebra of an algebra $A$, we say that an $A$-module $V$ is a weight module with respect to $C$ if

\[V = \bigoplus_{m \in \text{Max}(C)} V_m, \quad V_m = \{v \in V | mv = 0\},\]

where $\text{Max}(C)$ denotes the set of all maximal ideals of $C$. When $C$ is finitely generated this is equivalent to $V$ having a basis in which each $c \in C$ acts diagonally.

By weight modules over $B$ ($B_q$) we mean weight modules with respect to the subalgebra $\mathbb{C}[h]$ ($\mathbb{C}[h, w, w^{-1}]$). We need a simple lemma.

**Lemma 4.2.** Any finite-dimensional module $V$ over $B$ is a weight module.

**Proof.** By Proposition 5.3 in [8], any finite-dimensional $B$-module is semisimple. Since direct sums of weight modules are weight modules we can assume that $V$ is simple. Since $V$ is finite-dimensional, the commutative subalgebra $\mathbb{C}[h]$ has a common eigenvector $v \neq 0$, i.e. $mv = 0$ for some maximal ideal $m$ of $\mathbb{C}[h]$. Acting on this weight vector by $X_\pm$ produces another weight vector: $\sigma^{\pm 1}(m)X_\pm v = X_\pm mv = 0$. Since $B$ is generated by $\mathbb{C}[h]$ and $X_\pm$, any vector in the $B$-submodule of $V$ generated by $v$ is a sum of weight vectors. But $V$ was simple so $V = \bigoplus_m V_m$. \(\Box\)

Let $W(B)$ denote the category of weight $B$-modules and similarly for $B_q$.

**Theorem 4.3.** The category of weight modules over $B$ can be embedded into the category of weight modules over $B_q$, i.e. there exist functors

\[W(B) \xrightarrow{\mathcal{E}} W(B_q) \xrightarrow{\mathcal{R}} W(B)\]

whose composition is the identity functor. In particular, the category of finite-dimensional $B$-modules can be embedded in $W(B_q)$.

**Proof.** $\mathcal{R}$ is given by restriction. It takes weight modules to weight modules. Next we define $\mathcal{E}$. Let $V$ be a weight module over $B$ and define

\[(4.2) \quad vw = q^\alpha v \quad \text{for } v \in V_{(h-\alpha)} \text{ and } \alpha \in \mathbb{C}.\]
It is immediate that $w$ commutes with $h$. Let $v \in V_{(h-\alpha)}$ be arbitrary. Then

$$X_+wv = X_+q^{\alpha}v = q^{\alpha}X_+v.$$ 

On the other hand, since $hX_+v = X_+(h-1)v = (\alpha-1)X_+v$ which shows that $X_+v \in V_{(h-(\alpha-1))}$, we have

$$qwX_+v = qq^{\alpha-1}X_+v = q^{\alpha}X_+v.$$ 

Thus $X_+w = qwX_+$. Similarly $X_-w = q^{-1}wX_-$ on $V$. Thus $V$ becomes a module over $B_q$. That $V$ is a weight module with respect to $\mathbb{C}[h, w, w^{-1}]$ is clear. We define $\mathcal{E}(V)$ to be the same space $V$ with additional action (4.2). If $\varphi : V \rightarrow W$ is a morphism of weight $B$-modules then $\varphi(wv) = w\varphi(v)$ for weight vectors $v$, since $\varphi(V_m) \subseteq W_m$ for any maximal ideal $m$ of $\mathbb{C}[h]$. But then $\varphi(wv) = w\varphi(v)$ for all $v \in V$ since $V$ is a weight module. Thus $\varphi$ is automatically a morphism of $B_q$-modules and we set $\mathcal{E}(\varphi) = \varphi$. It is clear that the composition of the functors is the identity on objects and morphisms. \qed

Note that

$$\mathcal{E}(\mathcal{W}(B)) = \{V \in \mathcal{W}(B_q) \mid \text{Supp}(V) \subseteq \{m = (h - \alpha, w - q^\alpha) \mid \alpha \in \mathbb{C}\}\}.$$ 

It is not difficult to see that

$$\mathcal{E}(V_1) \otimes \mathcal{E}(V_2) \in \mathcal{E}(\mathcal{W}(B))$$ 

and hence there is a unique $V_3 \in \mathcal{W}(B)$ such that

$$\mathcal{E}(V_1) \otimes \mathcal{E}(V_2) = \mathcal{E}(V_3).$$ 

Thus we can define

$$V_1 \otimes V_2 := V_3$$ 

and this will make $\mathcal{W}(B)$ into a tensor category.

4.5. Non Hopf ambiskew polynomial rings. There are many examples of ambiskew polynomial rings which do not have any Hopf structure. One example is the Weyl algebra $W = \langle a, b | ab - ba = 1 \rangle$ which can have no counit $\varepsilon$. Indeed, a counit is in particular a homomorphism $\varepsilon : W \rightarrow \mathbb{C}$ so we would have $1 = \varepsilon(1) = \varepsilon(a)\varepsilon(b) - \varepsilon(b)\varepsilon(a) = 0$. Moreover all down-up algebras are ambiskew polynomial rings (see [8]) and [9] contains necessary conditions for the existence of a Hopf structure on a down-up algebra in terms of the parameters $\alpha, \beta, \gamma$. More precisely, they show that if $A = A(\alpha, \beta, \gamma)$ is a Noetherian down-up algebra that is a Hopf algebra, then $\alpha + \beta = 1$. Moreover if $\gamma = 0$, then $(\alpha, \beta) = (2, -1)$ and as algebras, $A$ is isomorphic to the universal enveloping algebra of the three-dimensional Heisenberg Lie algebra, while if $\gamma \neq 0$, then $-\beta$ is not an $n$th root of unity for $n \geq 3$. It would be of interest to generalize such a result to a more general class of ambiskew polynomial rings and also to other GWAs.

5. $R$ as functions on a group

From now on we assume that $A = A(R, \sigma, h, \xi)$ is an algebra of the form defined in Section 3 and that conditions (3.4)-(3.7) hold so that $A$ becomes a Hopf algebra with $R$ as a Hopf subalgebra. Let $G$ denote the set of all maximal ideals in $R$. Since $\mathbb{K}$ is algebraically closed and $R$ is finitely generated, the inclusion map $i_m : \mathbb{K} \rightarrow R/m$ is onto for any $m \in G$ and we let $\varphi_m : R \rightarrow \mathbb{K}$ denote the composition of the
projection \( R \to R/m \) and \( i_m^{-1} \). Thus \( \varphi_m(a) \) is the unique element of \( \mathbb{K} \) such that 
\[ a - \varphi_m(a) \in m. \]
We define the weight sum of \( m, n \in G \) to be 
\[ m + n := \ker(m \circ (\varphi_m \otimes \varphi_n) \circ \Delta|_R). \]
This is the kernel of a \( \mathbb{K} \)-algebra homomorphisms \( R \to \mathbb{K} \), hence \( m + n \in G \). We will never use the usual addition of ideals so + should not cause any confusion. Using that \( \Delta \) is coassociative, \( \varepsilon \) is a counit and \( S \) is an antipode, one easily deduces that + is associative, that \( \emptyset := \ker \varepsilon \) is a unit element and \( S(m) \) is the inverse of \( m \). Thus \( G \) is a group under +. If \( R \) is cocommutative, \( G \) is abelian.

**Example 5.1.** Let \( R = \mathbb{C}[H] \). Then \( G = \{(H - \alpha) \mid \alpha \in \mathbb{C}\} \). Give \( R \) the Hopf structure \( \Delta(H) = H \otimes 1 + 1 \otimes H \), \( \varepsilon(H) = 0 \) and \( S(H) = -H \). Then the operation + will be 
\[ (H - \alpha) + (H - \beta) = (H - (\alpha + \beta)), \]
i.e. the correspondence \( \mathbb{C} \ni \alpha \mapsto (H - \alpha) \in G \) is an additive group isomorphism.

If \( R = \mathbb{C}[K, K^{-1}] \) then \( G = \{(K - \alpha) \mid \alpha \in \mathbb{C}^*\} \). With the Hopf structure \( \Delta(K) = K \otimes K \), \( \varepsilon(K) = 1 \) and \( S(K) = K^{-1} \), the operation + will be 
\[ (K - \alpha) + (K - \beta) = (K - \alpha\beta) \]
for \( \alpha, \beta \neq 0 \). Thus \( G \simeq \langle \mathbb{C}^*, \cdot \rangle \).

We will often think of elements from \( R \) as \( \mathbb{K} \)-valued functions on \( G \) and for \( x \in R \) and \( m \in G \) we will use the notation \( x(m) \) for \( \varphi_m(x) \). Note however that different elements \( x, y \in R \) can represent the same function. In fact one can check that the map from \( R \) to functions on \( G \) is a homomorphism of \( \mathbb{K} \)-algebras with kernel equal to the radical \( \text{Rad}(R) := \cap_{m \in G} m \).

Define a map 
\[ (5.1) \quad \zeta : \mathbb{Z} \to G, \quad n \mapsto \underline{n} := \sigma^n(\emptyset). \]

**Lemma 5.2.** Let \( m, n \in G \). Then for any \( a \in R \),
\[ (5.2) \quad \sigma(a)(m) = a(\sigma^{-1}(m)), \]
\[ (5.3) \quad a(m + n) = m \circ (\varphi_m \otimes \varphi_n) \circ \Delta(a) = \sum_{(a)} a'(m)a''(n), \]
\[ (5.4) \quad m + 1 = \sigma(m) = 1 + m. \]
Thus \( \zeta \) is a group homomorphism and its image is contained in the center of \( G \).

**Proof.** Since for any \( a \in R \) we have 
\[ \sigma(a)(m) - a = \sigma^{-1}(\sigma(a)(m) - \sigma(a)) \in \sigma^{-1}(m), \]
(5.2) holds. Similarly,
\[ a(m + n) - a \in m + n \]
so applying the map \( m \circ (\varphi_m \otimes \varphi_n) \circ \Delta \) to \( a(m + n) - a \) yields zero. This gives (5.3). Finally we have for any \( a \in m \),
\[ \sigma(a)(m + 1) = m \circ (\varphi_m \otimes \varphi_n) \circ \Delta(\sigma(a)) = m \circ (\varphi_m \otimes \varphi_n) \circ (1 \otimes \sigma) \Delta(a) = m \circ (\varphi_m \otimes \varphi_n) \circ \Delta(a) = a(m + \emptyset) = a(m) = 0. \]
Here we used (5.3) in the first and the fourth equality, (3.4a) in the second and (5.2) in the third. Thus \( \sigma(m) \subseteq m + 1 \) and then equality holds since both sides are maximal ideals. The proof of the other equality in (5.4) is symmetric.
Example 5.3. If $R = \mathbb{C}[K, K^{-1}]$ with $\Delta(K) = K \otimes K, \varepsilon(K) = 1, S(K) = K^{-1}$ and $\sigma(K) = q^{-2}K$, then $\ker \varepsilon = (K - 1)$ so

$$m = \sigma^n(\emptyset) = \sigma^n((K - 1)) = (q^{-2n}K - 1) = (K - q^{2n}).$$

From (5.3) follows that if $x \in R$ is grouplike, then viewed as a function $G \to \mathbb{K}$ it is a multiplicative homomorphism. Using (5.3) and (3.5a)-(3.5c), the following formulas are satisfied by $h$ as a function on $G$.

$$h(m + n) = h(m)r(n) + l(m)h(n),$$

$$h(\emptyset) = 0,$$

$$h(-m) = -r^{-1}l^{-1}h(m),$$

where $r = r_+r_-$ and $l = l_+l_-.

6. Finite-dimensional simple modules

In this section we consider finite-dimensional simple modules over the algebra $A$. The main theorem is Theorem 6.17 where we, under the torsion-free assumption (6.1), characterize the finite-dimensional simple modules of a given dimension in terms of their highest weights. This result will be used in Section 7 to prove a Clebsch-Gordan decomposition theorem.

Throughout the rest of the paper we will assume that

$$\sigma^n(m) \neq m \text{ for any } n \in \mathbb{Z}\setminus\{0\} \text{ and any } m \in G.$$  

By (5.4), this condition holds iff $1$ has infinite order in $G$.

6.1. Weight modules, Verma modules and their finite-dimensional simple quotients. In this section we define weight modules, Verma modules and derive an equation for the dimension of its finite-dimensional simple quotients.

Let $V$ be an $A$-module. We call $m \in G$ a weight of $V$ if $mv = 0$ for some nonzero $v \in V$. The support of $V$, denoted $\text{Supp}(V)$, is the set of weights of $V$. To a weight $m$ we associate its weight space

$$V_m = \{v \in V | mv = 0\}.$$

Elements of $V_m$ are called weight vectors of weight $m$. A module $V$ is a weight module if $V = \oplus_m V_m$. A highest weight vector $v \in V$ of weight $m$ is a weight vector of weight $m$ such that $X_+v = 0$. A module $V$ is called a highest weight module if it is generated by a highest weight vector. From the defining relations of $A$ it follows that

$$X_+V_m \subseteq V_{\sigma^{\pm 1}(m)}.$$

Equation (6.2) implies that a highest weight module is a weight module.

Let $m \in G$. The Verma module $M(m)$ is defined as the left $A$-module $A/I(m)$ where $I(m)$ is the left ideal $AX_+ + Am \subseteq A$. From relations (2.5),(2.6) follows that

$$\{v_n := X^n + I(m) | n \geq 0\}$$

is a basis for $M(m)$. It is clear that $M(m)$ is a highest weight module generated by $v_0$. We also see that the vectors $v_n (n \geq 0)$ are weight vectors of weights $\sigma^n(m)$ respectively. By (6.1) we conclude $\dim M(m)_m = 1$. Therefore the sum of all its proper submodules is proper and equals the unique maximal submodule $N(m)$ of $M(m)$. Thus $M(m)$ has a unique simple quotient $L(m)$. Since it is easy to see that any highest weight module over $A$ of highest weight $m$ is a quotient of $M(m)$ we
deduce that $L(m)$ is the unique irreducible highest weight module over $A$ with given highest weight $m \in G$. We set
\[ G_f := \{ m \in G \mid \dim L(m) < \infty \}. \]

**Proposition 6.1.** Any finite-dimensional simple module over $A$ is isomorphic to $L(m)$ for some $m \in G_f$.

**Proof.** Let $V$ be a finite-dimensional simple $A$-module. Since $K$ is algebraically closed, $R$ has a common eigenvector $v \neq 0$, i.e. there exists $n \in G$ such that $nv = 0$. From (2.5) it follows that $\sigma^n(n)(X_+)^n v = 0$ for any $n \geq 0$. By (6.1), the set $\{X_+^n v \mid n \geq 0\}$ is a set of weight vectors of different weights. Since $V$ is finite-dimensional it follows that $(X_+)^n v = 0$ for some $n > 0$. This proves the existence of a highest weight vector of weight $m$ in $V$ for some weight $m$. Thus $V = L(m)$. \[\Box\]

**Corollary 6.2.** Let $V$ be a finite-dimensional simple module over $A$. Then
\[ \text{Supp}(V) \subseteq G_f + \mathbb{Z} = \{ m + n \mid m \in G_f, n \in \mathbb{Z} \}. \]

**Proof.** Let $m \in \text{Supp}(V)$ and let $0 \neq v \in V_m$. Then $(X_+)^n v = 0$ for some smallest $n > 0$. But then $(X_+)^{n-1} v$ is a highest weight vector so its weight $\sigma^{n-1}(m) = m + n - 1$ must belong to $G_f$. Thus $m = m + n - 1 - n - 1 \in G_f + \mathbb{Z}$. \[\Box\]

The following lemma was essentially proved in [7], Proposition 2.3, and the general result was mentioned in [8]. We give a proof for completeness.

**Proposition 6.3.** The dimension of $L(m)$ is the smallest positive integer $n$ such that
\[ \sum_{k=0}^{n-1} \xi^{n-1-k} h(m-k) = 0. \]

**Proof.** Let $e^m$ be a highest weight vector in $L(m)$. Let $n > 0$ be the smallest positive integer such that $X_+^n e^m = 0$. Then the set spanned by the vectors $X_+^j e^m$, $0 \leq j < n$, is invariant under $X_-$, under $R$ using (2.5), and under $X_+$, using (2.6). Hence it is a nonzero submodule and so coincides with $L(m)$ since the latter is simple. Therefore $n = \dim L(m)$. Let $k > 0$. Then $X_+^k e^m = 0$ implies that $X_+^{k+1} e^m = 0$. Conversely, suppose $X_+^k X_+^l e^m = 0$. Then $X_+^{k+1} X_+^l e^m$ generates a proper submodule and thus is zero. Repeating this argument we obtain $X_+^k e^m = 0$. Hence $\dim L(m)$ is the smallest positive integer $n$ such that $X_+^n X_+^m e^m = 0$. Using induction it is easy to deduce the formulas
\[ X_+ X_+^n = X_+^{n-1} \left( \xi^n X_+ X_+ + \sum_{k=0}^{n-1} \xi^{n-1-k} \sigma^k(h) \right), \]
\[ X_+^n X_+ = \prod_{m=1}^{n} \left( \xi^m X_+ X_+ + \sum_{k=0}^{m-1} \xi^{m-1-k} \sigma^k(h) \right). \]

Applying both sides of this equality to the vector $e^m$ gives
\[ X_+^n X_+^m e^m = \prod_{m=1}^{n} \sum_{k=0}^{m-1} \xi^{m-1-k} \sigma^k(h) e^m. \]
Using that \( e^m \) is a weight vector of weight \( m \) and formula (5.2) we have
\[
\sigma^k(h)e^m = \sigma^k(h)(m)e^m = h(m - k)e^m.
\]
Substituting this into (6.4) we obtain
\[
X^n_+X^n_-e^m = \prod_{m=1}^{n} \sum_{k=0}^{m-1} \xi^{m-1-k}h(m-k)e^m.
\]
The smallest positive \( n \) such that this is zero must be the one such that the last factor is zero. The claim is proved. \( \square \)

**Corollary 6.4.** If \( m, m_0 \in G \) where \( h(m_0) = 0 \), then
\[
\dim L(m_0 + m) = \dim L(m) = \dim L(m + m_0).
\]

**Proof.** Note that (5.5) implies that \( h(n + m_0) = h(n)r(m_0) \) and \( h(m_0 + n) = l(m_0)h(n) \) for any \( n \in G \), recall that \( r \) and \( l \) are invertible and use Proposition 6.3. \( \square \)

### 6.2. Dimension and highest weights

The goal in this subsection is to prove Theorem 6.17 which describes in detail the relationship between the dimension of a finite-dimensional simple module and its highest weight.

We begin with a few useful lemmas. Recall that \( r = r_+r_- \) and \( l = l_+l_- \). For brevity we set \( r_1 = r(\mathbf{1}) \) and \( l_1 = l(\mathbf{1}) \). Since \( r_1, l_1 \) are grouplike so are \( r \) and \( l \) and thus \( r_1, l_1 \) are nonzero scalars.

**Lemma 6.5.** We have
a) \( \xi^2r_1l_1 = 1 \),
b) \( h(-k) = -r_1^{-k}l_1^{-k}h(k) \) for any \( k \in \mathbb{Z} \),
c) for any \( k \in \mathbb{Z} \) and \( m \in G \) we have
\[
(6.5) \quad \xi^kh(m + k) + \xi^{-k}h(m - k) = ((\xi r_1)^k + (\xi r_1)^{-k})h(m).
\]

**Proof.** For a), multiply the two equations in (3.6b) and apply the multiplication map to both sides to obtain
\[
\sigma(l_+l_-r_+r_-) = \xi^2l_+l_-r_+r_-.
\]
Evaluate both sides at \( \mathbf{1} \) to get
\[
1 = lr(\mathbf{1}) = l r(\sigma^{-1}(\mathbf{1})) = \sigma(lr(\mathbf{1})) = \xi^2lr(\mathbf{1}) = \xi^2l_1r_1.
\]
Next (5.5) gives for any \( k \in \mathbb{Z} \),
\[
0 = h(k - k) = h(k)r_1^{-k} + l_1^kh(-k),
\]
hence b) follows. Finally, using (5.5) again, we have
\[
\xi^kh(m + k) + \xi^{-k}h(m - k) = \xi^kh(m)r_1^k + \xi^kl(m)h(k) +
\]
\[
+ \xi^{-k}h(m)r_1^{-k} + \xi^{-k}l(m)h(-k) =
\]
\[
= h(m)((\xi r_1)^k + (\xi r_1)^{-k}) +
\]
\[
+ l(m)h(k)((\xi - \xi^{-k}r_1^{-k}l_1^{-k}).
\]
In the last equality we used part b). Now the second term in the last expression vanishes due to part a). Thus c) follows. \( \square \)
In what follows, we will treat the two cases when \( h(\mathbb{1}) = 0 \) and \( h(\mathbb{1}) \neq 0 \) separately. The algebras satisfying the former condition have a representation theory which reminds of that of the enveloping algebra \( U(h_3) \) of the three-dimensional Heisenberg Lie algebra, while the latter case includes \( U(sl_2) \) and other algebras with similar structure of representations.

### 6.2.1. The case \( h(\mathbb{1}) = 0 \)

**Proposition 6.6.** If \( h(\mathbb{1}) = 0 \), then

\[
\epsilon^2 = r_1^2 = 1, \quad \sigma(h) = r_1 h, \quad \sigma(r) = r_1 r, \quad \text{and} \quad \sigma(l) = r_1 l.
\]

In particular, \( \langle X_+, X_-, h \rangle \) is a subalgebra of \( A \) with relations

\[
\begin{align*}
[X_+, X_] &= h, & [h, X_+] &= 0, & \text{if} \, \xi = 1, \, r_1 = 1, \\
[X_+, X_-] &= h, & [h, X_\pm] &= 0, & \text{if} \, \xi = 1, \, r_1 = -1, \\
\{X_+, X_-\} &= h, & [h, X_\pm] &= 0, & \text{if} \, \xi = -1, \, r_1 = 1, \\
\{X_+, X_-\} &= h, & [h, X_\pm] &= 0, & \text{if} \, \xi = -1, \, r_1 = -1,
\end{align*}
\]

respectively, where \( \{,\} \) denotes anti-commutator.

**Proof.** Suppose \( h(\mathbb{1}) = 0 \). Then, by Lemma 6.5b), \( h(-\mathbb{1}) = 0 \). This means that \( h \in -\mathbb{1} = \sigma^{-1}(\mathbb{1}) = \sigma^{-1}(\ker \epsilon) \). Thus \( \epsilon(\sigma(h)) = 0 \). Using (2.2), (3.4a) and (3.5a) we deduce

\[
\sigma(h) = (\epsilon \otimes 1)(\Delta(\sigma(h))) = (\epsilon \otimes 1)(\sigma \otimes 1)(\Delta(h)) = \epsilon(\sigma(h)) \otimes r + \epsilon(\sigma(l)) \otimes h = \epsilon(\sigma(l)) h.
\]

Analogously one proves \( \sigma(h) = \epsilon(\sigma(r)) h \). Hence \( \epsilon(\sigma(r)) = \epsilon(\sigma(l)) \). But

\[
\epsilon(\sigma(r)) = \sigma(r)(\ker \epsilon) = \sigma(r)(\mathbb{1}) = r^{-1}(\mathbb{1}) = r_1^{-1}
\]

and similarly for \( l \). So \( r_1 = l_1 \). From Lemma 6.5a) we obtain \( \langle \xi r_1 \rangle^2 = 1 \). Now

\[
S(\sigma(h)) = S(r_1^{-1} h) = -r_1^{-1} h, \quad \text{and} \quad \sigma^{-1}(h) = \sigma^{-1}(-h) = -r_1 h,
\]

so (3.4b) implies that \( r_1^2 = 1 \). A similar calculation as above shows that \( \sigma(r) = r_1^{-1} r = r_1 r \) and \( \sigma(l) = l_1^{-1} l = r_1 l \).

We leave it to the reader to prove the following statement.

**Proposition 6.7.** All finite-dimensional simple modules over an algebra \( A(R, \sigma, h, \xi) \) satisfying (6.6) and one of the commutation relations above are either one- or two-dimensional.

**Remark 6.8.** The algebra \( U(h_3) \) is an ambiskew polynomial ring, as shown in Section 4.1. For this algebra we have \( h(\mathbb{1}) = 0 \) and \( \xi = r_1 = 1 \).

### 6.2.2. The case \( h(\mathbb{1}) \neq 0 \)

In this section, we consider the more complicated case when \( h(\mathbb{1}) \neq 0 \). We prove Theorem 6.17 which describes the dimensions of \( L(m) \) in terms of \( m \). The following two subsets of \( G \) will play a vital role:

\[
\begin{align*}
G_0 &= \{ m \in G \mid h(m) = 0 \}, \\
G_{1/2} &= \{ m \in G \mid h(m - \mathbb{1}) + \xi h(m) = 0 \}.
\end{align*}
\]
The reason for this notation is that when $A = U(sl_2)$ as in Section 4.2.1 then we have $G_0 = \{(H - 0)\}$ and $G_{1/2} = \{(H - \frac{1}{2})\}$. From (5.5) it is immediate that $G_0$ is a subgroup of $G$. By Proposition 6.3 we have

$$G_0 = \{ m \in G \mid \dim L(m) = 1 \}. \tag{6.9}$$

The following analogous result holds for $G_{1/2}$.

**Proposition 6.9.**

$$G_{1/2} = \{ m \in G \mid \dim L(m) = 2 \}. \tag{6.10}$$

**Proof.** If $m \in G_{1/2}$, then by Proposition 6.3, $\dim L(m) \leq 2$. But if $\dim L(m) = 1$, then $h(m) = 0$ so using $m \in G_{1/2}$ we get $h(m - 1) = 0$ also. Since $G_0$ is a group we deduce that $\frac{1}{2} \in G_0$, i.e. $h(\frac{1}{2}) = 0$ which is a contradiction. So $\dim L(m) = 2$. The converse inclusion is immediate from Proposition 6.3. □

Set

$$N = \begin{cases} 
\text{order of } \xi r_1 & \text{if } (\xi r_1)^2 \neq 1 \text{ and } \xi r_1 \text{ is a root of unity}, \\
\infty & \text{otherwise}.
\end{cases} \tag{6.11}$$

We also set

$$N' = \begin{cases} 
N, & \text{if } N \text{ is odd}, \\
N/2, & \text{if } N \text{ is even}, \\
\infty, & \text{if } N = \infty.
\end{cases}$$

The next statement describes the intersection of $G_0$ and $G_{1/2}$ with $\mathbb{Z}$.

**Proposition 6.10.** We have

$$G_0 \cap \mathbb{Z} = \begin{cases} 
\{0\}, & \text{if } N = \infty, \\
N'\mathbb{Z}, & \text{otherwise},
\end{cases} \tag{6.12}$$

and

$$G_{1/2} \cap \mathbb{Z}_{>0} = \begin{cases} 
\emptyset, & \text{if } N = \infty, \\
\{ n \in \mathbb{Z}_{>0} : N | 2n - 1 \}, & \text{otherwise}.
\end{cases} \tag{6.13}$$

**Remark 6.11.** The set $G_{1/2} \cap \mathbb{Z}_{\leq 0}$ can be understood using (6.13) and Lemma 6.14a).

**Proof.** We first prove (6.12). Let $n \in \mathbb{Z}$. The right hand side of (6.12) is invariant under $n \mapsto -n$. By Lemma 6.5b) so is the left hand side. Moreover since $h(0) = 0$, the ideal $0$ belongs to both sides of the equality. Thus we can assume $n > 0$.

Using (5.5) and that $r$ and $l$, viewed as functions $G \to \mathbb{K}$, are multiplicative homomorphisms it follows by induction that

$$h(n) = h(1) \sum_{i=0}^{n-1} r_1 l_1^{n-1-i}.$$

By Lemma 6.5a), $r_1/l_1 = (\xi r_1)^2/(\xi^2 r_1 l_1) = (\xi r_1)^2$, so we can rewrite this as

$$h(n) = h(1) l_1^{n-1} \sum_{i=0}^{n-1} (\xi r_1)^{2i}. \tag{6.14}$$
If $N = \infty$ and $(\xi r_1)^2 \neq 1$ then by (6.14) we have $n \in G_0 \cap \mathbb{Z}$ iff $(\xi r_1)^{2n} = 1$, which is false. If $(\xi r_1)^2 = 1$, then (6.14) implies that $n \notin G_0 \cap \mathbb{Z}$. If $N < \infty$, then $(\xi r_1)^2 \neq 1$ so by (6.14), $h(n) = 0$ iff $(\xi r_1)^{2n} = 1$ i.e. if $N \mid 2n$. This is equivalent to $N \mid n$.

Next we prove (6.13). Suppose $n \in \mathbb{Z}_{>0}$. By definition, $n \in G_{1/2}$ iff

$$h\left(\frac{n}{2}\right) + \xi h\left(\frac{n}{2}\right) = 0.$$ 

Using (6.14) on both terms and dividing by $h(1)\xi l_n^{n-1}$, this is equivalent to

$$\xi^{-1}l_n^{-1} \sum_{k=0}^{n-2} (\xi r_1)^{2k} + \sum_{k=0}^{n-1} (\xi r_1)^{2k} = 0.$$

But $\xi^{-1}l_n^{-1} = \xi r_1$ by Lemma 6.5a) so this can be rewritten as

$$(6.15)\quad \sum_{k=0}^{2n-2} (\xi r_1)^k = 0.$$ 

Thus $(\xi r_1)^2 \neq 1$ and multiplying by $\xi r_1 - 1$ we get $(\xi r_1)^{2n-1} = 1$. Therefore $N < \infty$ and $N \mid 2n - 1$. Conversely, if $N < \infty$ and $N \mid 2n - 1$ then $(\xi r_1)^2 \neq 1$ and $(\xi r_1)^{2n-1} = 1$ which implies (6.15). This proves (6.13).

$\square$

**Proposition 6.12.** Suppose $h(1) \neq 0$ and $G_{1/2} \neq \emptyset$. Then

a) $\xi r_1 \neq -1$, and

b) $G_{1/2}$ is a left and right coset of $G_0$ in $G$.

**Proof.** Let $m_{1/2} \in G_{1/2}$. To prove a), suppose that $\xi r_1 = -1$. Then

$$0 = h(m_{1/2} - \mathbf{1}) + \xi h(m_{1/2}) =$$

$$= h(m_{1/2}) r(\mathbf{1}) + l(m_{1/2}) h(\mathbf{1}) + \xi h(m_{1/2}) =$$

$$= h(m_{1/2}) (r_1^{-1} + \xi) + l(m_{1/2}) h(\mathbf{1}) =$$

$$= -l(m_{1/2}) r_1^{-1} l_1^{-1} h(\mathbf{1}),$$

where we used Lemma 6.5b) in the last equality. Since $l$ is invertible we deduce that $h(1) = 0$ which is a contradiction.

To prove part b), we will show that

$$G_{1/2} = G_0 + m_{1/2}.$$ 

One proves $G_{1/2} = m_{1/2} + G_0$ in an analogous way. Let $m \in G_0$ be arbitrary. Then using (5.5) twice,

$$h(m + m_{1/2} - \mathbf{1}) + \xi h(m + m_{1/2}) = l(m)(h(m_{1/2} - \mathbf{1}) + \xi h(m_{1/2})) = 0.$$ 

Since $l$ is invertible we get $m + m_{1/2} \in G_{1/2}$.

Conversely, suppose $m \in G_{1/2}$. Then

$$h(m - \mathbf{1}) + \xi h(m) = 0,$$

$$h(m_{1/2} - \mathbf{1}) + \xi h(m_{1/2}) = 0.$$ 

Multiply the first equation by $r(-m_{1/2})$ and the second by $-r(-m_{1/2}) l(-m_{1/2}) l(m)$ and add them together. Then we get

$$(h(m) r_1^{-1} + l(m) h(-\mathbf{1})) r(-m_{1/2}) -$$

$$= r(-m_{1/2}) l(-m_{1/2}) l(m)(h(m_{1/2} r_1^{-1} + l(m_{1/2}) h(-\mathbf{1})) + \xi h(m - m_{1/2}) = 0,$$
or equivalently,
\[ h(m)\xi_{r}^{-1}r(-m_{1/2}) - r(-m_{1/2})l(-m_{1/2})l(m_{1/2})r_{1}^{-1} + \xi h(m - m_{1/2}) = 0. \]
Using (5.5) this can be written
\[ r_{1}^{-1}(1 + \xi r_{1})h(m - m_{1/2}) = 0. \]
Since \( \xi r_{1} \neq -1 \) by part a), we conclude that \( h(m - m_{1/2}) = 0 \). This shows that \( m \in G_{0} + m_{1/2} \).

The following lemma will be useful.

**Lemma 6.13.** Let \( j \in \mathbb{Z} \). If \( m_{0} \in G_{0} \), then
\[ (6.16) \quad m_{0} + j \in G_{0} \iff j \in G_{0}, \]
and if \( h(1) \neq 0 \) and \( m_{1/2} \in G_{1/2} \), then
\[ (6.17) \quad m_{1/2} + j \in G_{1/2} \iff j \in G_{0}. \]

**Proof.** \((6.16)\) is immediate since \( G_{0} \) is a subgroup of \( G \). If \( j \in G_{0} \), then \( m_{1/2} + j \in G_{1/2} \) by **Proposition 6.12**. Conversely, if \( m_{1/2} + j \in G_{1/2} \) then by **Proposition 6.12**, \( G_{0} \ni m_{1/2} + j - m_{1/2} = j \).

The next statements will be needed in Section 8.

**Lemma 6.14.** Suppose \( h(1) \neq 0 \) and let \( m, n \in G_{1/2} \). Then
\[ a) \quad \bot - m \in G_{1/2}, \text{ and} \]
\[ b) \quad m + n - \bot \in G_{0}. \]

**Proof.** Part a) follows from the calculation
\[
\begin{align*}
\xi(\bot - m) + \xi h(1 - m) &= -l(-m)r(-m)h(m) - \xi l(1 - m)(r(1 - m)h(m - 1)) = \\
&= -l(-m)r(-m)h(m) + \xi r_{1} l_{1} h(m - 1) = \\
&= -l(-m)r(-m)\xi^{-1} (\xi h(m) + h(m - 1)) = 0.
\end{align*}
\]
For part b), use that \( \dim L(\bot - n) = 2 \) by part a), and thus \( m + n - \bot = m - (\bot - n) \in G_{0} \) by **Proposition 6.12b**.

The formulas provided by the following technical lemma are the key to proving our main theorem.

**Lemma 6.15.** Let \( m \in G \) and \( j \in \mathbb{Z}_{\geq 0} \). If \( n = 2j + 1 \) then
\[ (6.18) \quad \sum_{k=0}^{n-1} \xi^{n-1-k} h(m - k) = r_{1}^{-j} \sum_{k=0}^{n-1} (\xi_{r_{1}})^{k} \]
and if \( n = 2j + 2 \) then
\[ (6.19) \quad \sum_{k=0}^{n-1} \xi^{n-1-k} h(m - k) = r_{1}^{-j} (h(m - \bot - 1) + \xi h(m - j)) \sum_{k=0}^{n/2-1} (\xi_{r_{1}})^{2k}. \]

**Proof.** If \( n = 2j + 1 \), we make the change of index \( k \mapsto j - k \), then factor out \( \xi^{j} \) and apply formula (6.5):
\[
\sum_{k=0}^{2j} \xi^{2j-k} h(m - \bot) = \sum_{k=-j}^{j} \xi^{j+k} h(m - \bot + k) = \xi^{j} h(m - j) \sum_{k=-j}^{j} (\xi_{r_{1}})^{k}.
\]
Factoring out \((\xi r_1)^{-j}\) and changing index from \(k\) to \(k - j\) yields (6.18).

For the \(n = 2j + 2\) case we first split the sum in the left hand side of (6.19) into two sums corresponding to odd and even \(k\):

\[
\sum_{k=0}^{j} \xi^{2j-2k}h(m - 2k - 1) + \sum_{k=0}^{j} \xi^{2j+1-2k}h(m - 2k)
\]

Then we make the change of summation index \(k \mapsto -k + j/2\) in both sums

\[
\xi^{j/2} \sum_{k=-j/2}^{j/2} \xi^{2k}h(m - j - 1 + 2k) + \xi^{j+1/2} \sum_{k=-j/2}^{j/2} \xi^{2k}h(m - j + 2k)
\]

and use (6.5) on each of them to get

\[
(h(m - j - 1) + \xi h(m - j))\xi^{j/2} \sum_{k=-j/2}^{j/2} (\xi r_1)^{2k}.
\]

If we factor out \((\xi r_1)^{-j}\) and change summation index from \(k\) to \(k - j/2\) we obtain (6.19).

We now come to the main results in this section.

**Main Lemma 6.16.** Assume that \(b(1) \neq 0\) and let \(m \in G\). Then

a) \(\dim L(m) \leq N\),

b) if \(\dim L(m) = n < N\) then \(m \in G_{i+1}^{h} + j\) where \(n = 2j + i, i \in \{1, 2\}\), \(j \in \mathbb{Z}_{\geq 0}\), and

c) if \(i \in \{1, 2\}\), \(j \in \mathbb{Z}_{\geq 0}\), \(2j + i \leq N\) and \(m \in G_{i+1}^{h}\) then

\[
(6.20) \quad \dim L(m + j) = 2j + i.
\]

d) If \(N' < \infty\) then \(\dim L(m + N'j) = \dim L(m)\) for any \(j \in \mathbb{Z}\).

**Proof.** Part a) is trivial when \(N = \infty\). If \(N\) is finite and odd, Proposition 6.3 and (6.18) imply that \(\dim L(m) \leq N\). If \(N\) is finite and even, then \((\xi r_1)^N = 1\) and \((\xi r_1)^2 \neq 1\) so \(\sum_{k=0}^{N/2} (\xi r_1)^k = 0\). Hence Proposition 6.3 and (6.19) implies \(\dim L(m) \leq N\) in this case as well.

Next we turn to part b). Suppose first that \(\dim L(m) = n = 2j + 1 < N\). Then by Proposition 6.3 and (6.18) the right hand side of (6.18) is zero. The definition of \(N\) implies that \(h(m - j) = 0\), i.e. \(m \in G_0 + j\). If instead \(\dim L(m) = 2j + 2 < N\), Proposition 6.3 and (6.19) similarly implies that \(m \in G_{1/2}^{h} + j\).

To prove (6.20), we proceed by induction on \(j\). For \(j = 0\) it follows from (6.9) and (6.10). Suppose it holds for \(j = 0, 1, \ldots, k - 1\), where \(k > 0\) and \(2k + i \leq N\). We first show that \(\dim L(m + k) \leq 2k + i\). If \(i = 1\) then by (6.18),

\[
\sum_{l=0}^{2k} \xi^{2k-l}h(m + k - l) = r_1^{-k}h(m) \sum_{l=0}^{2k} (\xi r_1)^{l} = 0
\]

since \(m \in G_0\). Similarly, if \(i = 2\), then (6.19) gives

\[
\sum_{l=0}^{2k+1} \xi^{2k+1-l}h(m + k - l) = r_1^{-k}(h(m - 1) + \xi h(m)) \sum_{l=0}^{k} (\xi r_1)^{2l} = 0
\]
since \( m \in G_{1/2} \) in this case. Thus \( \dim L(m + j) \leq 2j + i \) by Proposition 6.3. Write \( \dim L(m + k) = 2k' + i' \) where \( k' \geq 0 \), \( i' \in \{1, 2\} \) and assume that \( 2k' + i' < 2k + i \). By part b) we have \( m + k \in G_{i' - 1} + k' \) which implies that \( \dim L(m + k - k') = i' \) by (6.9) and (6.10). This contradicts the induction hypothesis unless \( k' = 0 \). Assuming \( k' = 0 \) we get \( m + k \in G_{i' - 1} \). If \( i = i' \) then from Lemma 6.13 follows that \( k \in G_{0} \).

Since \( 0 < k < \frac{2k + i}{2} \leq N/2 \leq N' \) this contradicts 6.12. We now show that \( i \neq i' \) is also impossible. If \( i = 1 \) and \( i' = 2 \), then \( m \in G_{0} \) and \( m + k \in G_{1/2} \) so by Proposition 6.12b), \( k \in G_{1/2} \cap Z_{\geq 0} \). By (6.13) we get \( N|2k - 1 \) which is absurd because \( 0 < 2k - 1 < 2k + 1 \leq N \). If \( i = 2 \) and \( i' = 1 \) then \( m \in G_{1/2} \) and \( m + k \in G_{0} \). By Proposition 6.12b) we have \( -k = m - (m + k) \in G_{1/2} \). By Lemma 6.14a), \( 1 + k \in G_{1/2} \) so (6.13) implies that \( N|2(1 + k) - 1 = 2k + 1 \). This is impossible since \( 0 < 2k + 1 < 2k + 2 \leq N \). We have proved that the assumption \( 2k' + i' < 2k + i \) is false and hence that \( \dim L(m + k) = 2k + i \), which proves the induction step.

Finally, part d) follows from Corollary 6.4 and Proposition 6.10.

\[ \square \]

**Theorem 6.17.** Let \( m \in G \).

- If \( N = \infty \), then

\[ \left( 6.21 \right) \quad \dim L(m) < \infty \iff m \in (G_{0} + Z_{\geq 0}) \cup (G_{1/2} + Z_{\geq 0}) \]

and

\[ \left( 6.22 \right) \quad \dim L(m_{0} + j) = 2j + 1, \quad \text{for } m_{0} \in G_{0} \text{ and } j \in Z_{\geq 0}, \]

\[ \left( 6.23 \right) \quad \dim L(m_{1/2} + j) = 2j + 2, \quad \text{for } m_{1/2} \in G_{1/2} \text{ and } j \in Z_{\geq 0}. \]

- If \( N < \infty \) and \( N \) is even, then

\[ \left( 6.24 \right) \quad \dim L(m) < \infty \iff m \in (G_{0} + Z_{1}) \cup (G_{1/2} + Z_{1}) \]

and

\[ \left( 6.25 \right) \quad \dim L(m + (N/2)j) = \dim L(m), \quad \text{for any } m \in G \text{ and } j \in Z, \]

and for \( m_{0} \in G_{0} \) and \( m_{1/2} \in G_{1/2} \) we have

\[ \left( 6.26 \right) \quad \dim L(m_{0} + j) = 2j + 1, \quad \text{if } 0 \leq j < N/2, \]

\[ \left( 6.27 \right) \quad \dim L(m_{1/2} + j) = 2j + 2, \quad \text{if } 0 \leq j < N/2. \]

- If \( N < \infty \) and \( N \) is odd, then

\[ \left( 6.28 \right) \quad \dim L(m) < \infty \iff m \in G_{0} + Z = G_{1/2} + Z \]

and

\[ \left( 6.29 \right) \quad \dim L(m + Nj) = \dim L(m), \quad \text{for any } m \in G \text{ and } j \in Z, \]

and for \( m_{0} \in G_{0} \) and \( m_{1/2} \in G_{1/2} \) we have

\[ \left( 6.30 \right) \quad \dim L(m_{0} + j) = \begin{cases} 2j + 1, & \text{if } 0 \leq j < \frac{N+1}{2}, \\ 2j + 1 - N, & \text{if } \frac{N+1}{2} \leq j < N, \end{cases} \]

\[ \left( 6.31 \right) \quad \dim L(m_{1/2} + j) = \begin{cases} 2j + 2, & \text{if } 0 \leq j < \frac{N-1}{2}, \\ 2j + 2 - N, & \text{if } \frac{N-1}{2} \leq j < N. \end{cases} \]
Proof. When $N = \infty$, relations (6.21)-(6.23) are immediate from Lemma 6.16b) and c).

Suppose $N$ is finite and even. The $\Rightarrow$ implication in (6.24) holds by Lemma 6.16b). And (6.25) follows from (6.12) and Corollary 6.4. Assume that $m \in (G_0 + \mathbb{Z}) \cup (G_{1/2} + \mathbb{Z})$. Using (6.25) we can assume that $m = m' + j$ where $m' \in G_0 \cup G_{1/2}$ and $0 \leq j < N/2$. Then, if $i \in \{1, 2\}$ we have $2j + i \leq N$ and Lemma 6.16c) implies (6.26)-(6.27) and therefore $\dim L(m) < \infty$ so (6.24) is also proved.

Assume that $N$ is finite and odd. By (6.13) we have $(N + 1)/2 \in G_{1/2}$. Therefore $G_0 + \mathbb{Z} = G_0 + (N + 1)/2 + \mathbb{Z} = G_{1/2} + \mathbb{Z}$ since $G_{1/2}$ is a right coset of $G_0$ in $G$ by Proposition 6.12. As before, Lemma 6.16b) implies the $\Rightarrow$ case in (6.28) and (6.29) holds by virtue of (6.12) and Corollary 6.4. If $m \in G_0 + \mathbb{Z}$ we can assume by (6.29) that $m \in G_0 + j$ where $0 \leq j < N$. If $j < N + 2$ so since $N$ is odd we have $2j + 1 \leq N$. By Lemma 6.16c) we deduce that $\dim L(m) = 2j + 1$. If instead $j \geq N + 1$, then $m = (N + 1)/2 + m - (N + 1)/2 \in G_{1/2}$ where $k = j - \frac{N+1}{2} \geq 1$. Therefore $\dim L(m_{1/2} + j) = \dim L(m_0 + j')$, where $j' = j + (N + 1)/2$ and $m_0 = m_{1/2} - (N + 1)/2$. Now $m_0 \in G_0$ since $G_{1/2}$ is a coset of $G_0$ in $G$. If $0 \leq j < \frac{N+1}{2}$, then $\frac{N+1}{2} \leq j' < N$ so by (6.30) we have $\dim L(m_{1/2} + j) = \dim L(m_0 + j') = 2j' + 1 - N = 2j + 2$.

And if $\frac{N+1}{2} \leq j < N$, then $0 \leq j' - N < \frac{N+1}{2}$ and hence $\dim L(m_{1/2} + j) = \dim L(m_0 + j' - N) = 2(j' - N) + 1 = 2j + 1 - N$.

The proof is finished. $\square$

Corollary 6.18. If $N = \infty$ and $m \in G_0 \cup G_{1/2}$, then $L(m + j)$ is infinite-dimensional for any $j \in \mathbb{Z}_{<0}$.

Proof. If the dimension of $L(m + j)$ were finite and odd (even), then $\dim L(m + j - k) = 1 (2)$ for some $k \geq 0$ by Lemma 6.16b). By Lemma 6.16c), $L(m)$ has then dimension $2(j - k) + 1 (2(j - k) + 2)$ and thus $j = k$ which is absurd. $\square$

Corollary 6.19. Suppose $N = \infty$ and let $m \in G_f$. Then $L(m)$ is the unique finite-dimensional quotient of $M(m)$.

Proof. It is enough to prove that the unique maximal proper submodule $N(m)$ of $M(m)$ is simple. By Theorem 6.17 we can write $m = n + j$ where $n \in G_0 \cup G_{1/2}$ and $j \in \mathbb{Z}_{\geq 0}$. From the proof of Proposition 6.3 we have $\text{Supp}(L(m)) = \{n + j, n + j - 1, \ldots, n - j\}$.

Thus $N(m)$ is a highest weight module of highest weight $n - j - 1$. So $N(m)$ is a quotient of $M(n - j - 1)$. But $M(n - j - 1)$ is simple, otherwise it would have a finite-dimensional simple quotient, i.e., $L(n - j - 1)$ would be finite-dimensional, contradicting Corollary 6.18. Thus $N(m)$ is also simple. $\square$
Remark 6.20. We finish this section by remarking that there exist algebras in the class studied in this paper which do not have even-dimensional simple modules as for example the algebra $B_0$ from Section 4.4. Indeed, in this case we have $t_1 = -1$ and so $N = \infty$ by definition. By Proposition 6.12, $G_{1/2} = \emptyset$ so by Theorem 6.17, there can exist no even-dimensional simple modules.

7. TENSOR PRODUCTS AND A CLEBSCH-GORDAN FORMULA

As we have seen in Section 2 the existence of a Hopf structure on an algebra allows one to define tensor product of its representations by (2.4). The aim of this section is to prove a formula which decomposes the tensor product of two simple $A$-modules into a direct sum of simple modules. It generalizes the classical Clebsch-Gordan formula for modules over $U(su_l)$. We will assume that $A = A(R, \sigma, h, \xi)$ is an ambiskew polynomial ring and that it carries a Hopf structure of the type considered in Section 3. We will also assume (6.1) and that $N = \infty$.

Lemma 7.1. Let $V$ and $W$ be two $A$-modules. Then

\begin{equation}
V_m \otimes W_n \subseteq (V \otimes W)_{m+n}
\end{equation}

for any $m, n \in G$. Hence if $V$ and $W$ are weight modules, then so is $V \otimes W$ and

$$
\text{Supp}(V \otimes W) = \{m + n \mid m \in \text{Supp}(V), n \in \text{Supp}(W)\}.
$$

Proof. Let $v \in V_m, w \in W_n$. Then for any $r \in R$,

$$
r(v \otimes w) = \sum_{(r)} r'v \otimes r''w = \sum_{(r)} r'(m)v \otimes r''(n)w = \sum_{(r)} r'(m)r''(n)v \otimes w = r(m + n)v \otimes w
$$

by (5.3), proving (7.1). Thus if $V, W$ are weight modules,

$$
V \otimes W = (\oplus_m V_m) \otimes (\oplus_n W_n) = \oplus_{m,n} V_m \otimes W_n = \oplus_m (\oplus_{m_1 + m_2 = m} V_{m_1} \otimes W_{m_2}).
$$

\Box

Theorem 7.2. Let $m, n \in G_f$. We have the following isomorphism

\begin{equation}
L(m) \otimes L(n) \cong L(m + n) \oplus L(m + n - 1) \oplus \ldots \oplus L(m + n - s + 1)
\end{equation}

where $s = \min\{\dim L(m), \dim L(n)\}$.

Proof. Let $e^m, e^n$ denote highest weight vectors in $L(m), L(n)$ respectively and set $e_j^m := (X_-)^j e^m$ for $j \in \mathbb{Z}_{\geq 0}$ and similarly for $n$. Set $V = L(m) \otimes L(n)$. By Lemma 7.1 we have

$$
V_{m+n-k} = \oplus_{i+j=k} \mathbb{K} e_i^m \otimes e_j^n
$$

for $k \in \mathbb{Z}_{\geq 0}$. Fix $0 \leq k \leq s - 1$. We will prove that

\begin{equation}
\dim \ker X_+|_{V_{m+n-k}} = 1.
\end{equation}

From the calculations in the proof of Proposition 6.3 follows that when $j > 0$, $X_+ e_j^n$ is a nonzero multiple of $e_{j-1}^n$. Let $\nu_j^n$ denote this multiple. Let

$$
u = \sum_{i=0}^k \lambda_i e_i^m \otimes e_{k-i}^n
$$

\Box
be an arbitrary vector in \( V_{m+n-k} \). Then

\[
X_u = \sum_{i=0}^{k} \lambda_i (X_i e_i^m \otimes r_i^{n-1} + l_i e_i^m \otimes X_i e_i^{n-1}) =
\]

\[
= \sum_{i=0}^{k-1} [\lambda_{i+1} \nu_{i+1}^m r_+ (n - \frac{k}{2} + \frac{i}{2} + 1) + \lambda_i l_+ (m - \frac{k}{2}) \nu_{i-1}^n] e_i^m \otimes e_{i-1}^{n-1}.
\]

Setting

\[
c_i = l_+ (m - \frac{k}{2}) \nu_{i-1}^n,
\]

\[
c_i' = \nu_i^m r_+ (n - \frac{k}{2} + \frac{i}{2}),
\]

the condition for \( u \) to be a highest weight vector can hence be written as

\[
(7.4) \quad \begin{bmatrix}
c_0 & c_1' & c_2' & \cdots & c_{k-1}'
c_1 & c_2' & \cdots & \cdots & \cdots
c_2' & \cdots & \cdots & \cdots & \cdots
c_{k-2} & \cdots & \cdots & \cdots & \cdots
c_{k-1} & \cdots & \cdots & \cdots & \cdots
\end{bmatrix} \begin{bmatrix}
\lambda_0 \\
\lambda_1 \\
\vdots \\
\lambda_{k-1}
\end{bmatrix} = 0.
\]

Since \( r_+ \) and \( l_+ \) are grouplike, they are invertible and hence \( c_i \neq 0 \neq c_i' \) for any \( i = 0, 1, \ldots, k - 1 \). Therefore the space of solutions to \((7.4)\) is one-dimensional. Thus \((7.3)\) is proved.

From the definition of Verma modules, it follows that for \( k = 0, 1, \ldots, s - 1 \), there is a nonzero \( A \)-module morphism

\[
M(m + n - k) \to L(m) \otimes L(n)
\]

which maps a highest weight vector in \( M(m + n - k) \) to a highest weight vector in \( L(m) \otimes L(n) \) of weight \( m + n - k \). But \( L(m) \otimes L(n) \) is finite-dimensional so this morphism must factor through \( L(m + n - k) \) by Corollary 6.19. Taking direct sums of these morphisms we obtain an \( A \)-module morphism

\[
\varphi : L(m + n) \oplus L(m + n - 1) \oplus \ldots \oplus L(m + n - s + 1) \to L(m) \otimes L(n).
\]

We claim it is injective. Indeed, the projection of the kernel of \( \varphi \) to any term \( L(m + n - i) \) must be zero, because it is a proper submodule of the simple module \( L(m + n - i) \).

To conclude we now calculate the dimensions of both sides. Write \( \dim L(m) = 2j_1 + i_1 \) and \( \dim L(n) = 2j_2 + i_2 \) where \( j_1, j_2 \in \mathbb{Z}_{\geq 0} \) and \( i_1, i_2 \in \{1, 2\} \). By Lemma 6.16b), \( \dim L(m - j_1) = i_1 \) and \( \dim L(n - j_2) = i_2 \). First note that

\[
\dim L(m - j_1 + n - j_2) = i_1 + i_2 - 1.
\]

When \( i_1 = i_2 = 1 \), this is true because \( G_0 \) is a subgroup of \( G \). When one of \( i_1, i_2 \) is 1 and the other 2, it follows from Proposition 6.12b). And if \( i_1 = i_2 = 2 \), it follows from Lemma 6.14b) and Theorem 6.17.
From Theorem 6.17 also follows that \( \dim L(m+k) = \dim L(m) + 2k \) if \( \dim L(m) < \infty \) and \( k \in \mathbb{Z}_{\geq 0} \). Hence, recalling that \( s = \min\{\dim L(m), \dim L(n)\} \), we have

\[
\sum_{k=0}^{s-1} \dim L(m+n-k) = \sum_{k=0}^{s-1} \dim L(m - j_1 + n - j_2 + j_1 + j_2 - k) = \\
= \sum_{k=0}^{s-1} (i_1 + i_2 - 1 + 2(j_1 + j_2 - k)) = \\
= s(i_1 + i_2 - 1 + 2j_1 + 2j_2) - s(s - 1) = \\
= s(\dim L(m) + \dim L(n) - s) = \\
= \dim L(m) \dim L(n) = \dim (L(m) \otimes L(n)).
\]

This completes the proof of the theorem. \(\square\)

Under some conditions it is possible to introduce a \(*\)-structure on \(A\). In this connection it would be interesting to study Clebsch-Gordan coefficients and the relation with special functions. This will be a subject for future investigation.

8. Casimir Operators and Semisimplicity

Arguing as in the proof of Lemma 4.2, it is easy to see that any finite-dimensional semisimple module over \(A = A(R, \sigma, h, \xi)\) is a weight module. In this section we will prove the converse, that any finite-dimensional weight module over \(A\) is semisimple. Note that in general not all finite-dimensional modules over our algebra \(A\) are semisimple. The corresponding example is constructed in [6] for the algebra from Section 4.3. A necessary and sufficient condition for all finite-dimensional modules over an ambiskew polynomial ring to be semisimple was given in [8], Theorem 5.1.

In this section we assume that \(A = A(R, \sigma, h, \xi)\) is an ambiskew polynomial ring with a Hopf structure of the type introduced in Section 3 such that (6.1) holds. We also assume that \(N = \infty\).

Let \(V\) be a finite-dimensional weight module over \(A\). We will first treat the case when \(\text{Supp}(V) \subseteq m + \mathbb{Z}\) where \(m \in G_0\) is fixed. Define a linear map

\[
C_V : V \to V
\]

by requiring

\[
C_V v = \sigma^j(t)v, \quad \text{for } v \in V_{m+j} \text{ and } j \in \mathbb{Z}.
\]

Here \(\sigma\) denotes the extended automorphism (2.8). More explicitly we have (if \(j \geq 0\))

\[
C_V v = \sigma^j(t)v = \left(\xi^j t + \sum_{k=0}^{j-1} \xi^k \sigma^{j-1-k}(h)\right)v = \xi^j tv + \sum_{k=0}^{j-1} \xi^k h(m+k+1)v
\]

and similarly when \(j < 0\). It is easy to check that \(C_V\) is a morphism of \(A\)-modules. Hence it is constant on each finite-dimensional simple module \(V\) by Schur’s Lemma. Moreover if \(\varphi : V \to W\) is a morphism of weight \(A\)-modules with support in \(m + \mathbb{Z}\), then \(\varphi C_V = C_W \varphi\).

**Proposition 8.1.** Let \(j_1, j_2 \in \mathbb{Z}_{\geq 0}\). If \(C_L(m+j_1) = C_L(m+j_2)\), then \(j_1 = j_2\).
Proof. By applying \( C_{L(m+j)} \) to the highest weight vector of \( L(m+j) \), \( (j \in \mathbb{Z}_{\geq 0}) \) we get

\[
C_{L(m+j)} = \sum_{k=0}^{j-1} \xi^k h(m + k + 1).
\]

We can assume \( j_1 < j_2 \). By assumption we have

\[
0 = \sum_{k=0}^{j_2-1} \xi^k h(m + k + 1) - \sum_{k=0}^{j_1-1} \xi^k h(m + k + 1) = \sum_{k=j_1}^{j_2-1} \xi^k h(m + k + 1) =
\]

\[
= \xi^{j_1} \sum_{k=0}^{j_2-j_1-1} \xi^k h(m + j_2 - (j_2 - j_1) + m + k + 1).
\]

By Proposition 6.3 this means that \( \dim L(m + j_2) \leq j_2 - j_1 \). But this contradicts Theorem 6.17 which says that \( \dim L(m + j_2) = 2j_2 + 1 \). □

Theorem 8.2. Let \( V \) be a finite-dimensional weight module over \( A \) with support in \( G_0 + \mathbb{Z} \). Then \( V \) is semisimple.

Proof. We follow the idea of the proof of Proposition 12 in [10], Chapter 3. Writing

\[
V = \oplus_{m \in C_0} \left( \oplus_{j \in \mathbb{Z}} V_{m+j} \right)
\]

and noting that \( \oplus_{j \in \mathbb{Z}} V_{m+j} \) are submodules, we can reduce to the case when \( \text{Supp}(V) \) is contained in \( m + \mathbb{Z} \) for a fixed \( m \in G_0 \).

Let \( \lambda_1, \ldots, \lambda_k \) be the generalized eigenvalues of the Casimir operator \( C_V \), i.e. the elements of the set

\[
\{ \lambda \in \mathbb{K} \mid \ker(C_V - \lambda \text{Id})^p \neq 0 \text{ for some } p > 0 \}.
\]

Then each generalized eigenspace \( \sum_p \ker(C_V - \lambda_i \text{Id})^p \) is invariant under \( A \), hence they are submodules. It suffices to prove that each such submodule is semisimple. Let \( V \) be one of them. Let \( V_1 = \{ v \in V \mid X_+ v = 0 \} \). Then \( V_1 \) is invariant under \( R \) and since \( V \) is a weight module, \( V_1 = \oplus_{n \in G}(V_1 \cap V_n) \). Now if \( 0 \neq v \in V_1 \cap V_n \), then \( v \) is a highest weight vector of \( V \) and generates a submodule isomorphic to \( L(n) \). Hence if \( V_1 \cap V_n \neq 0 \) for more than one \( n \in G, C_V \) will have two different eigenvalues by Proposition 8.1 which is impossible. Here we used that the restriction of \( C_V \) to a submodule \( W \) coincides with \( C_W \). Hence \( V_1 \) is contained in a single weight space, say \( V_n \). Let \( v_1, \ldots, v_k \) be a basis for \( V_1 \). Then each \( v_i \) generates a simple submodule isomorphic to \( L(n) \). We will show that the sum of these submodules is direct. Vectors of different weights are linearly independent so it suffices to show that if

\[
\sum_{i=1}^{k} \lambda_i (X_-)^m v_k = 0
\]

then all \( \lambda_i = 0 \). Assume the sum was nonzero and act by \( X_+ \) \( m \) times. In each step we get a nonzero result because we have not reached the highest weight \( n \) yet. But then, using (6.3), we have a linear relation among the \( v_k \) — a contradiction. We have shown that \( V \) contains the direct sum \( V' \) of \( k \) copies of \( L(n) \). Now \( X_+ \) acts injectively on \( V/V' \). This is only possible in a torsion-free finite-dimensional weight \( A \)-module if it is 0-dimensional. Thus \( V \) is semisimple. □
We now turn to the general case. Assume now that \( A \) has an even-dimensional irreducible representation. By Lemma 6.16b, \( G_{1/2} \neq \emptyset \). We fix \( m_{1/2} \in G \). Then \( G_{1/2} = G_0 + m_{1/2} \) by Proposition 6.12.

**Theorem 8.3.** Any finite-dimensional weight module \( V \) over \( A \) is semisimple.

**Proof.** By Corollary 6.2 and Theorem 6.17,
\[
\text{Supp}(V) \subseteq (G_0 + \mathbb{Z}) \cup (G_{1/2} + \mathbb{Z})
\]
Thus we have a decomposition
\[
V = \left( \bigoplus_{m \in G_0} V_{m+\mathbb{Z}} \right) \oplus \left( \bigoplus_{m \in G_{1/2}} V_{m+m_{1/2}+\mathbb{Z}} \right)
\]
where \( V_{n+\mathbb{Z}} := \bigoplus_{n \in \mathbb{Z}} V_n \) for \( n \in G \) are submodules. It remains to prove that a weight module \( V \) with support in \( m + m_{1/2} + \mathbb{Z} \) is semisimple. By Lemma 7.1,
\[
\text{Supp} \left( V \otimes L(m_{1/2}) \right) \subseteq m + m_{1/2} + m_{1/2} + \mathbb{Z} = m' + \mathbb{Z}
\]
where \( m' := m + m_{1/2} + m_{3/2} - 1 \in G_0 \) by Lemma 6.14b). Hence \( V \otimes L(m_{1/2}) \) is semisimple by Theorem 8.2. By the Clebsch-Gordan formula (7.2), the tensor product of two semisimple modules is semisimple again. Therefore \( V \otimes L(m_{1/2}) \otimes L(1 - m_{1/2}) \) is semisimple, where \( \dim L(1 - m_{1/2}) = 2 \) by Lemma 6.14a). On the other hand, by (7.2) again we have
\[
V \otimes L(m_{1/2}) \otimes L(1 - m_{1/2}) \cong V \otimes (L(0) \oplus L(m)) \cong (V \otimes L(0)) \oplus (V \otimes L(m)).
\]
Finally, it is easy to verify the isomorphism \( V \cong V \otimes L(0), v \mapsto v \otimes e \) where \( 0 \neq e \in L(0) \) is fixed. Thus \( V \) is isomorphic to a submodule of the semisimple module \( V \otimes L(m_{1/2}) \otimes L(1 - m_{1/2}) \) and is therefore itself semisimple. \( \square \)

**References**


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