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INTEGRAL REPRESENTATION WITH WEIGHTS II, DIVISION AND INTERPOLATION

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ABSTRACT. Let f be a $r \times m$ -matrix of holomorphic functions that is generically surjective. We provide explicit integral representation of holomorphic ψ such that $\phi = f\psi$, provided that ϕ is holomorphic and annihilates a certain residue current with support on the set where f is not surjective. We also consider formulas for interpolation. As applications we obtain generalizations of various results previously known for the case $r = 1$.

1. INTRODUCTION

This paper is a continuation of [1] where we introduced a new way to generate weighted representation formulas for holomorphic functions, generalizing [11]. In this paper we focus on division and interpolation and we introduce new formulas for matrices of holomorphic functions. As applications we obtain generalizations of various results previously known for a row matrix.

Let $f = (f_1, \dots, f_m)$ be a tuple of holomorphic functions defined in, say, a neighborhood of the closure of the unit ball D in \mathbb{C}^n with common zero set Z , and assume that $df_1 \wedge \dots \wedge df_n \neq 0$ on Z . In [12] was constructed a representation formula

$$(1.1) \quad \phi(z) = f(z) \cdot \int_{\zeta} T(\zeta, z)\phi(\zeta) + \int_{\zeta} S(\zeta, z)\phi(\zeta), \quad z \in D,$$

for holomorphic functions ϕ , where both T and S are holomorphic in z , $T(\cdot, z)$ is integrable, and $S(\cdot, z)$ is a current of order zero (i.e., with measure coefficients) with support on Z . If ϕ vanishes on Z , thus (1.1) provides an explicit representation of ϕ as an element of the ideal generated by f . Moreover, if ϕ is just defined on Z , then

$$\int_{\zeta} S(\zeta, z)\phi(\zeta)$$

is a holomorphic extension, i.e., a holomorphic function in D that interpolates ϕ on Z . The formula (1.1) was extended to the case where f

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defines a complete intersection, i.e., $\text{codim } Z = m$, in [28]. In this case, as expected, $S(\cdot, z)$ is closely related to the Coleff-Herrera current

$$\bar{\partial} \frac{1}{f_1} \wedge \dots \wedge \bar{\partial} \frac{1}{f_m};$$

however also the singularities of $T(\cdot, z)$ are rather complicated, and contains terms that are concentrated on Z , see also Remark 1 below. Formulas of this kind have been used for various purposes by several authors; notably for instance the explicit proof of the duality theorem for a complete intersection in [28], explicit versions of the fundamental principle, [13], sharp approximation by polynomials [34], and estimates of solutions to the Bezout equation, [8]; for further examples see [10] and the references given there. More recent applications can be found in [20] and [9]. One can also use such formulas to obtain sharp estimates at the boundary, such as H^p -estimates, of explicit solutions to division problems, [6].

In [27] and [1] independently, was constructed a similar formula where $T(\cdot, z)$ has quite simple, principal value, singularities but instead spread out over the larger set $Y = \{f_1 f_2 \cdots f_m = 0\}$. In [2] we introduced a new formula like (1.1) for an arbitrary f , where the singularity of $T(\cdot, z)$ is a principal value at Z , and $S(\cdot, z)$ is a smooth form times a Bochner-Martiny type residue current $R^f(\zeta)$ with support on Z .

The purpose of this paper is to extend this kind of formulas to the case when f is a generically surjective $r \times m$ matrix of holomorphic functions. Given such a matrix it was defined in [4] an associated (matrix-valued) current R^f with support on the analytic set

$$Z = \{z; f(z) \text{ is not surjective}\},$$

with the property that if ϕ is an r -column of holomorphic functions such that $R^f \phi = 0$, then $f\psi = \phi$ has holomorphic solutions ψ locally. In the generic case, i.e., $\text{codim } Z = m + r - 1$, the converse also holds. When $r = 1$, and $\text{codim } Z = m$ this is precisely the duality theorem for a complete intersection ([19] and [28]). In particular we get an explicit proof of the following statement from [4], generalizing the Briançon-Skoda theorem, [15] (for an explicit proof in the case $r = 1$, see [10] and [22]): *Suppose that f is an $r \times m$ matrix of holomorphic functions that is generically surjective, and that ϕ is an r -column of holomorphic functions. If $\|\phi\|^2 \leq C(\det f f^*)^{\min(n, m-r+1)}$, then $R\phi = 0$ and hence $\phi = f\psi$ locally. Here $\|\phi\|$ is the norm $\|\phi\|^2 = \langle \widehat{f f^*} \phi, \phi \rangle$, where $\widehat{f f^*}$ is the transpose of the co-matrix of $f f^*$.*

We also obtain the following result, which for the case $r = 1$ appeared in [3]. Let $\bar{\partial}^\alpha = \partial^\alpha / \partial \bar{z}^\alpha$ for multiindices α .

Theorem 1.1. *Suppose that f is an $r \times m$ -matrix of holomorphic functions that is generically surjective. Let ϕ be an r -column of smooth*

functions such that

$$(1.2) \quad R^f(\bar{\partial}^\alpha \phi) = 0$$

for all α . Then $\phi = f\psi$ has a smooth solution ψ . In case $\text{codim } Z = m - r + 1$ the condition (1.2) is also necessary.

We also present variants of this result for lower regularity. The division formulas admit sharp estimates at the boundary, and as an example we indicate how one can obtain an explicit solution of the matrix H^p -corona problem. Finally we present formulas for division problems for $\bar{\partial}$ -closed forms.

2. REPRESENTATION OF HOLOMORPHIC FUNCTIONS

For a fixed point z in the open set X in \mathbb{C}^n , we let $\delta_{\zeta-z}$ denote interior multiplication with the vector field

$$2\pi i \sum (\zeta_j - z_j) \frac{\partial}{\partial \zeta_j},$$

and let $\nabla_{\zeta-z} = \delta_{\zeta-z} - \bar{\partial}$. We begin with a slight generalization of the main result in [1] (lower indices denote bidegree).

Proposition 2.1. *Assume that z is a fixed point in X and $g = g_{0,0} + \dots + g_{n,n}$ is a current in X with compact support such that $\nabla_{\zeta-z}g = 0$. Moreover, assume that g is smooth in a neighborhood of z and $g_{0,0}(z) = 1$. Then*

$$(2.1) \quad \phi(z) = \int g\phi = \int g_{n,n}\phi$$

for each holomorphic function ϕ in X .

For the reader's convenience we supply the simple proof.

Proof. Let $u = u_{1,0} + \dots + u_{n,n-1}$ be a current that is smooth outside the point z and such that $\nabla_{\zeta-z}u = 1 - [z]$, where $[z]$ denotes the (n, n) -current point evaluation at z . For instance one can take

$$u = \frac{b}{\nabla_{\zeta-z}b} = b + b \wedge \bar{\partial}b + \dots + b \wedge (\bar{\partial}b)^{n-1},$$

where $b = \partial|\zeta - z|^2/|\zeta - z|^2 2\pi i$, see [1]. Then $u \wedge g$ is a well-defined current with compact support, and

$$\nabla_{\zeta-z}(u \wedge g) = g - [z] \wedge g = g - [z]$$

since $g_{0,0}(z) = 1$. Therefore, $\bar{\partial}(u \wedge g)_{n,n-1}\phi = \phi(z)[z] - g_{n,n}\phi$, which implies (2.1) by Stokes' theorem. \square

A form $g = g_{0,0} + g_{1,1} + \dots + g_{n,n}$ which is smooth in a neighborhood of our fixed point z , and such that $g_{0,0}(z) = 1$, will be called a weight (with respect to z). Notice that if g^1 and g^2 are weights with disjoint singular supports, then again the product $g = g^1 \wedge g^2$ is a weight. Moreover, if

ϕ takes values in the vector bundle $Q \rightarrow X$, and g takes values in $\text{Hom}(Q, Q)$, $\nabla_{\zeta-z} = 0$ and $g_{0,0}(z) = I_Q$, I_Q denoting the identity morphism $Q \rightarrow Q$, then (2.1) still holds; this follows by the same proof as for the scalar-valued case.

Example 1. Assume that D is a smoothly bounded domain in X that admits a smooth family of holomorphic support functions, i.e., $\Gamma(\zeta, z) \in C^\infty(\partial D \times U)$, where $U \supset \bar{D}$, depending holomorphically on z , such that $\Gamma(\zeta, z)$ is non-vanishing for $z \in \bar{D} \setminus \{\zeta\}$ and $\Gamma(z, z) = 0$ for $z \in \partial D$. Then D is necessarily pseudoconvex and we may assume that $\Gamma(\zeta, z) = \delta_{\zeta-z}\gamma(\zeta, z)$, where γ is a smooth $(1, 0)$ -form that is holomorphic for $z \in D$. If $s = \gamma(\zeta, z)/\delta_{\zeta-z}\gamma(\zeta, z)$, then for each $z \in D$,

$$g = \chi_D - \bar{\partial}\chi_D \wedge \frac{s}{\nabla_{\zeta} s} = 1 - \bar{\partial}\chi_D \wedge [s + s \wedge \bar{\partial}s + s \wedge (\bar{\partial}s)^2 + \cdots + s \wedge (\bar{\partial}s)^{n-1}]$$

is a weight (with respect to z) with support on \bar{D} , smooth outside ∂D , and depending holomorphically on z . Let g' be any weight (with respect to z) that is smooth in a neighborhood of ∂D . Since $\bar{\partial}\chi_D = -[\partial D]_{0,1}$ we get from (2.1) the formula

$$(2.2) \quad \phi(z) = \int_D g' \phi + \int_{\partial D} g' \phi \wedge [s + s \wedge \bar{\partial}s + s \wedge (\bar{\partial}s)^2 + \cdots + s \wedge (\bar{\partial}s)^{n-1}].$$

The existence of such families of holomorphic support functions for strictly pseudoconvex domains is due to Henkin and Ramirez, see, e.g., [24]. In [17] and [18] are constructed families of holomorphic support functions, admitting sharp estimates, for (linearly) convex domains of finite type. If D is the unit ball in \mathbb{C}^n we can take $s(\zeta, z) = \partial|\zeta|^2/2\pi i(1 - \bar{\zeta} \cdot z)$; we then get (2.2) with

$$s \wedge (\bar{\partial}s)^{k-1} = \frac{1}{(2\pi i)^k} \frac{\partial|\zeta|^2 \wedge (\bar{\partial}\partial|\zeta|^2)^{k-1}}{(1 - \bar{\zeta} \cdot z)^k}.$$

□

For our purposes it is convenient to have a formula like this for a weight g' that is not necessarily smooth on ∂D .

Example 2. Assume that X is pseudoconvex and let $K \subset X$ be a holomorphically convex compact subset. Moreover let χ be a cutoff function that is identically 1 in a neighborhood of K . It is easy to find a $(1, 0)$ -form $s(\zeta, z)$ on the support of $\bar{\partial}\chi$, depending holomorphically on z in a neighborhood of K , such that $\delta_{\zeta-z}s = 1$. Then for each $z \in K$,

$$g = \chi - \bar{\partial}\chi \wedge \frac{s}{\nabla_{\zeta-z} s} = \chi - \bar{\partial}\chi \wedge [s + s \wedge \bar{\partial}s + s \wedge (\bar{\partial}s)^2 + \cdots + s \wedge (\bar{\partial}s)^{n-1}]$$

is a compactly supported weight that depends holomorphically on z . If K is the closure of the unit ball D we can take

$$s(\zeta, z) = \frac{\partial|\zeta|^2}{2\pi i(|\zeta|^2 - \bar{\zeta} \cdot z)}.$$

If g' is any weight in X (with respect to $z \in K$), then we get the representation formula

$$(2.3) \quad \phi(z) = \int \chi g' \phi - \int \bar{\partial} \chi \wedge [s + s \wedge \bar{\partial} s + s \wedge (\bar{\partial} s)^2 + \cdots + s \wedge (\bar{\partial} s)^{n-1}] \wedge g' \phi, \quad z \in K.$$

□

3. DIVISION FORMULAS IN THE CASE $r = 1$

To begin with, let f be a row matrix of holomorphic functions in a pseudoconvex domain $X \subset \mathbb{C}^n$. In [2] were introduced formulas for division and interpolation, and more generally, homotopy formulas for the Koszul complex induced by f . In this section we derive these formulas in a new way that will model the construction when $r > 1$. It is convenient to introduce a trivial rank m bundle E over X and think of f as a section of the dual bundle E^* . If we let $\delta_f: \Lambda^{k+1} E \rightarrow \Lambda^k E$ denote interior multiplication with f , we have the Koszul complex

$$(3.1) \quad 0 \rightarrow \Lambda^m E \xrightarrow{\delta_f} \cdots \xrightarrow{\delta_f} \Lambda^2 E \xrightarrow{\delta_f} E \xrightarrow{\delta_f} \mathbb{C} \rightarrow 0,$$

where \mathbb{C} is the trivial line bundle. We consider currents with values in ΛE as sections of the bundle $\Lambda(E \oplus T^*(X))$, so that, e.g., differentials and sections of E anti-commute, and δ_f and $\bar{\partial}$ anti-commute; for more details, see [2]. Assume that we have $(0, k-1)$ -currents U_k , smooth outside some analytic variety, and $(0, k)$ -currents R_k with values in $\Lambda^k E$, R_k having support on $Z = \{f = 0\}$, such that

$$(3.2) \quad (\delta_f - \bar{\partial})U = 1 - R,$$

where $U = U_1 + \cdots + U_m$ and $R = R_1 + \cdots + R_m$. Specific choices will be discussed below. Assume that ϕ is a holomorphic section of $\Lambda^\ell E$ such that $\delta_f \phi = 0$ and $R \wedge \phi = 0$. Then it follows from (3.2) that $(\delta_f - \bar{\partial})(R \wedge \phi) = 0$, and by solving a sequence of $\bar{\partial}$ -equations one finds (locally) a holomorphic section ψ of $\Lambda^{\ell+1} E$ such that $\delta_f \psi = \phi$, see [2]. We will now provide an explicit formula for such a solution ψ .

Let e_j be a global frame for E with dual frame e_j^* for E^* so that $f = \sum_1^m f_j e_j^*$. One can find holomorphic $(1, 0)$ -forms h_j such that $\delta_{\zeta-z} h_j = f_j(\zeta) - f_j(z)$, so called Hefer forms. Now $h = \sum h_j \wedge e_j^*$ induces a mapping δ_h , taking a (p, q) -current-valued section of $\Lambda^{k+1} E$ to a $(p+1, q)$ -current-valued section of $\Lambda^k E$. Since h has total degree 2, δ_h commutes with δ_f and $\delta_{\zeta-z}$. If $(\delta_h)_k = \delta_h^k/k!$ and $\delta_{f(z)}$ is interior

multiplication with the section $f(z) = \sum f_j(z)e_j^*$ of E^* , then for a $(0, q)$ -current ξ with values in $\Lambda^k E$ we have

$$(3.3) \quad \delta_{\zeta-z}(\delta_h)_k \xi = (\delta_h)_{k-1}(\delta_f - \delta_{f(z)})\xi$$

for all integers k , if $(\delta_h)_k$ is interpreted as 1 for $k = 0$ and zero for $k < 0$. Assume that ϕ is holomorphic and takes values in $\Lambda^\ell E$. Using (3.2) and (3.3), a straight forward computation shows that

$$g' = \delta_{f(z)} \sum_k (\delta_h)_{k-1} (U_k \wedge \phi) + \sum_k (\delta_h)_{k-1} (U_k \wedge \delta_f \phi) + \sum_k (\delta_h)_k (R_k \wedge \phi)$$

is $\nabla_{\zeta-z}$ -closed for each fixed z . Moreover, by (3.2), $\delta_f U_1 \phi = \phi$, so

$$g'_{0,0}(z) = \delta_{f(z)}(U_1 \wedge \phi)|_{\zeta=z} = \phi(z).$$

For each z outside Z and the set where U is not smooth, by (an immediate consequence of) Proposition 2.1 we get the representation

$$(3.4) \quad \phi(z) = \delta_{f(z)} T\phi(z) + T(\delta_f \phi)(z) + S\phi(z),$$

where

$$T\phi(z) = \int \sum_k (\delta_h)_{k-1} (U_k \wedge \phi) \wedge g,$$

and

$$(3.5) \quad S\phi(z) = \int \sum_k (\delta_h)_k (R_k \wedge \phi) \wedge g,$$

if g is a smooth weight with compact support. Since each term in (3.4) is holomorphic in z , the equality must hold everywhere.

Example 3. With the following choice of currents U and R , the representation (3.4) is precisely the formula in Theorem 9.3 in [2] expressed in a new way. Assume that E is equipped with a Hermitian metric and let σ be the section of E over $X \setminus Z$ with pointwise minimal norm such that $f\sigma = 1$. If E has the trivial metric with respect to the global holomorphic frame e_j , then $\sigma = \sum_j \bar{f}_j e_j / |f|^2$. Let

$$u^f = \frac{\sigma}{\nabla_f \sigma} = \sigma + \sigma \wedge \bar{\partial} \sigma + \sigma \wedge (\bar{\partial} \sigma)^2 + \dots$$

Then the principal value current $U^f = \lim_{\epsilon \rightarrow 0} \chi_{|f| > \epsilon} u^f$, exists and is a current extension of u^f across Z . Moreover $R^f = \lim_{\epsilon \rightarrow 0} \bar{\partial} \chi_{|f| > \epsilon} \wedge u^f$, exists and (3.2) holds, see [2]. Alternatively, U^f and R^f can be defined as the value at $\lambda = 0$ of the analytic continuation of $|f|^{2\lambda} u^f$ and $\bar{\partial} |f|^{2\lambda} \wedge u^f$, respectively. Moreover, these currents can be obtained as limits of smooth forms. Notice that the section $s = |f|^2 \sigma$ is smooth. Now, see [32] and [33],

$$U^f = \lim_{\epsilon \rightarrow 0} \sum_{\ell} \frac{s \wedge (\bar{\partial} s)^{\ell-1}}{(|f|^2 + \epsilon)^\ell},$$

and

$$R^f = \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{|f|^2 + \epsilon} \sum_{\ell} \frac{(\bar{\partial}s)^\ell}{(|f|^2 + \epsilon)^\ell}.$$

For degree reasons $R_k^f = 0$ if $k > \min(m, n)$ and $U_k^f = 0$ if $k > \min(m, n + 1)$. It turns out also that $R_k^f = 0$ if $k < p = \text{codim } Z$, and therefore $R^f = R_p^f + \dots + R_{\min(m, n)}^f$. Thus the sum in (3.5) only runs from $k = p$ to $k = \min(n, m - \ell)$. Therefore, if ϕ is a holomorphic section of $\Lambda^\ell E$ such that $\delta_f \phi = 0$ and, in addition, $R^f \wedge \phi = 0$, then

$$\psi(z) = \int_{\zeta} \sum_{k=1}^{\min(n+1, m-\ell)} (\delta_h)_{k-1} (U_k^f \wedge \phi) \wedge g$$

is a holomorphic solution to $\delta_f \psi = \phi$. \square

Example 4. Assume that f defines a complete intersection, and let

$$U_1 = \frac{e_1}{f_1}, \quad U_{k+1} = \frac{e_{k+1}}{f_{k+1}} \wedge \bar{\partial} U_k, \quad R = \bar{\partial} \frac{1}{f_1} \wedge \dots \wedge \bar{\partial} \frac{1}{f_m} \wedge e_m \wedge \dots \wedge e_1,$$

where the currents are of Coleff-Herrera type. Then, using the calculus from [29], cf., also Section 5 in [1], one can check that (3.2) holds. Thus (3.4) gives a division formula in (3.4), which is singular over the set $\{f_1 \cdots f_m = 0\}$. This formula is similar to, but even simpler than, the formula in Section 5 of [1]. \square

Remark 1 (Berndtsson's division formula). In our notation, Berndtsson's classical division formula, [12], can be described in the following way. For $\epsilon > 0$, let $\sigma^\epsilon = s/(|f|^2 + \epsilon)$ (here s is as in Example 3) and let h be a Hefer form as above. Then

$$g' = 1 - \nabla_\eta h \cdot \sigma^\epsilon = \frac{\epsilon}{|f|^2 + \epsilon} + f(z) \cdot \sigma^\epsilon + h \cdot \bar{\partial} \sigma^\epsilon$$

is a weight, so by Proposition 2.1 we have the representation

$$\phi(z) = \int \left(\frac{\epsilon}{|f|^2 + \epsilon} + f(z) \cdot \sigma^\epsilon + h \cdot \bar{\partial} \sigma^\epsilon \right)^{\min(n+1, m)} \wedge g \phi.$$

if ϕ is a holomorphic function (i.e., $\ell = 0$). Possibly besides the choice of form g , this is precisely the formula introduced in [12]. One can prove that it converges to a decomposition like (1.1), for a quite arbitrary tuple f , when $\epsilon \rightarrow 0$; the non-complete intersection case is studied in [10] and [9] (but using analytic continuation) and in [22]. Making the most natural decomposition, letting all terms without a factor $f(z)$ together constitute $S\phi$, the resulting current in $T\phi$ is not of simple principal value type but will involve terms concentrated on Z . However, in the case when f defines a complete intersection it seems that these bad terms disappear. Moreover, in the the general case, and under the hypothesis that $|\phi| \leq C|f|^{\min(m, n)}$ that is considered in [10] and [22] to get an explicit proof of the Briançon-Skoda theorem (in [22] even a

more general form of this theorem is considered), these terms vanish. Therefore it seems that with no essential loss, one could incorporate these bad terms in $S\phi$ from the beginning. \square

4. GENERALIZED HEFER FORMS IN THE UNIT BALL

We shall now make an explicit computation of the formula (3.4) in the case when f is the tuple

$$f(\zeta) = \sum_1^n (\zeta_j - w_j) e_j^* = (\zeta - w) \cdot e^*$$

in the unit ball D for a fixed $w \in D$; a similar computation works in any domain that admits a smooth family of holomorphic support functions, cf., Example 1 above. In order to get holomorphic dependence of w we define $u = \sigma + \sigma \wedge (\bar{\partial}\sigma) + \sigma \wedge (\bar{\partial}\sigma)^2 + \dots$ with

$$\sigma = \frac{\sum \bar{\zeta}_j e_j}{|\zeta|^2 - \bar{\zeta} \cdot w} = \frac{\bar{\zeta} \cdot e}{|\zeta|^2 - \bar{\zeta} \cdot w}$$

outside the singularity w , and since u so defined is integrable we can let U be the trivial extension across w . Since $f(\zeta) - f(z) = (\zeta - z) \cdot e^*$ we can take

$$h = \frac{1}{2\pi i} \sum_1^n d\zeta_j \wedge e_j^*$$

as our Hefer form. Then, see, e.g., [1], Proposition 2.2 (just replace e_j by $d\zeta_j$), we have that $(\delta_f - \bar{\partial})U = 1 - R$, where

$$(4.1) \quad (\delta_h)_k R_k = 0, \quad k < n, \quad (\delta_h)_n R_n = [w].$$

Since U and R have no singularities at the boundary we can use the weight g in (2.2).

We first consider (3.4) when ϕ is a function. Since $\sum_1^n (\delta_h)_{k-1} U_k$ has no component of bidegree (n, n) , formula (3.4) becomes

$$\phi(z) - \phi(w) = \delta_{f(z)} \int_{\partial D} \sum_{k=1}^n (\delta_h)_{k-1} (\sigma \wedge (\bar{\partial}\sigma)^{k-1} \phi) \wedge s \wedge (\bar{\partial}s)^{n-k}.$$

Noting that $\delta_h \sigma \wedge s = 0$, more explicitly we have

$$\phi(z) - \phi(w) = \delta_{f(z)} \int_{\partial D} \sum_{k=1}^n \frac{\bar{\zeta} \cdot e \wedge \partial|\zeta|^2 \wedge (\frac{i}{2\pi} \partial \bar{\partial} |\zeta|^2)^{n-1}}{(1 - \bar{\zeta} \cdot w)^k (1 - \bar{\zeta} \cdot z)^{n-k+1}}.$$

Since ϕ clearly depends holomorphically on w , this is an explicit Hefer decomposition of the function ϕ in D . In fact this is precisely what we get if we express $\phi(z) - \phi(w)$ by means of the Szegő integral as, e.g., in [6]. The new interesting case here is when ϕ takes values in $\Lambda^\ell E$, $\ell > 0$. In view of (4.1), then $R \wedge \phi = 0$, so we get instead

$$\phi = \delta_f T\phi + T(\delta_f \phi),$$

where

$$T\phi(z) = \int_{\partial D} \sum_{k=1}^{n-\ell} \frac{\bar{\zeta} \cdot e \wedge (\delta_h)_{k-1} [(d\bar{\zeta} \cdot e)^k \wedge \phi] \wedge \partial|\zeta|^2 \wedge (\frac{i}{2\pi} \partial\bar{\partial}|\zeta|^2)^{n-k}}{(1 - \bar{\zeta} \cdot w)^k (1 - \bar{\zeta} \cdot z)^{n-k+1}}.$$

In particular, if $\delta_f \phi = 0$ we have that $\delta_f T\phi = \phi$. It is clear from this formula that $T\phi$ depends holomorphically on w . Moreover, it is proved in [6] that mappings like $\phi \mapsto T\phi$ admit certain sharp estimates at the boundary.

5. DIVISION FORMULAS IN THE CASE $r > 1$

In order to generalize the formulas in Section 3 to the case with matrices f we first consider a quite abstract setting. Assume that we have a finite complex of Hermitian holomorphic vector bundles over X

$$(5.1) \quad 0 \rightarrow E_N \xrightarrow{f_N} \dots \xrightarrow{f_3} E_2 \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0 \rightarrow 0.$$

We will consider currents with values in $E = \bigoplus E_k$ and in $\text{Hom}(E_0, E)$, i.e., sections of the bundles $\mathcal{D}'_\bullet(X, E) = \mathcal{D}'_\bullet(X) \otimes_{\mathcal{E}(X)} \mathcal{E}(X, E)$ and $\mathcal{D}'_\bullet(X, \text{End}E) = \mathcal{D}'_\bullet(X) \otimes_{\mathcal{E}(X)} \mathcal{E}(X, \text{Hom}(E_0, E))$. Clearly, $f = \sum f_k$ and $\bar{\partial}$ act on these spaces and we will arrange so that $f\bar{\partial} = -\bar{\partial}f$. To obtain this, it is natural to consider E as a superbundle, $E = E^+ \oplus E^-$, with $E^+ = \bigoplus E_{2k}$ and $E^- = \bigoplus E_{2k+1}$, so that sections of E^+ have even degree and sections of E^- have odd degree. The space $\mathcal{D}'_\bullet(X, E)$ has a natural structure as a left $\mathcal{E}_\bullet(X)$ -module, and it gets a natural Z_2 -grading by combining that gradings of $\mathcal{D}_\bullet(X)$ and $\mathcal{E}(X, E)$. We make $\mathcal{D}'_\bullet(X, E)$ into a right $\mathcal{E}_\bullet(X)$ -module by letting $\xi\phi = (-1)^{\deg \xi \deg \phi} \phi\xi$ for sections ξ of $\mathcal{E}(X, E)$ and smooth forms ϕ . The superstructure on E induces a superstructure $\text{End}E = \text{End}(E)^+ \oplus \text{End}(E)^-$ so that a mapping is odd if, like f , it maps $E^+ \rightarrow E^-$ and $E^- \rightarrow E^+$. In the same way we get a Z_2 -grading of $\mathcal{D}'_\bullet(X, \text{End}E)$. For instance, $\bar{\partial}$ extends to an odd mapping on $\mathcal{D}'_\bullet(X, E)$, as well as on $\mathcal{D}'_\bullet(X, \text{End}E)$. Since f is holomorphic and of odd degree, we have that $\bar{\partial} \circ f = -f\bar{\partial}$.

Let us now assume that we have $(0, k-1)$ -currents U_k and $(0, k)$ -currents R_k with values in $\text{Hom}(E_0, E_k)$ such that

$$(5.2) \quad f_1 U_1 = I_{E_0}, \quad f_{k+1} U_{k+1} - \bar{\partial} U_k = R_k.$$

Moreover, we assume that $U = \sum U_k$ is smooth outside some analytic variety Z and that $R = \sum R_k$ has its support on Z . A possible choice of such currents will be discussed in the next section. We will also use the short-hand notation $(f - \bar{\partial})U = I_{E_0} - R$ for (5.2). Notice that $f - \bar{\partial}$ is (minus) the $(0, 1)$ -part of the super connection $D - f$ introduced by Quillen, [31], where D is the Chern connection on E .

Proposition 5.1. *Assume that ϕ is a holomorphic section of E_0 such that $R\phi = 0$. Then, locally, $f_1\psi = \phi$ has holomorphic solutions ψ .*

Proof. In fact, by assumption $(f - \bar{\partial})(U\phi) = \phi$, and hence by successively solving the equations $\bar{\partial}w_k = U_k\phi + f_{k+1}w_{k+1}$, we get the holomorphic solution $\psi = U_1\phi + f_2w_2$. \square

Our aim now is to construct an explicit formula that provides the desired solution ψ . If we restrict our attention to some open domain X where we can choose global frames for all the bundles, then, for each fixed $z \in \Omega$, $f_k(z): E_k \rightarrow E_{k-1}$ is a well-defined morphism, which coincides with f_k on the fiber over z . For fixed $z \in X$, as before, let $\delta_{\zeta-z}$ and $\nabla_{\zeta-z}$ be as in Section 2.

Lemma 5.2. (i) *Assume that X is pseudoconvex. For any holomorphic function ϕ we can find a holomorphic $(1,0)$ -form h , depending holomorphically on z , such that $\delta_{\zeta-z}h = \phi(\zeta) - \phi(z)$.*

(ii) *If ξ is a holomorphic $(k,0)$ -form, $k \geq 1$, depending holomorphically on the parameter z , such that $\delta_{\zeta-z}\xi = 0$, then we can find a holomorphic $(k+1,0)$ -form ξ' depending holomorphically on z such that $\delta_{\zeta-z}\xi' = \xi$.*

These facts are well-known and follow from Cartan's theorem. For an explicit construction in the unit ball, see Section 4 above.

Proposition 5.3 (Existence of Hefer forms). *There are $(k-\ell,0)$ -form-valued holomorphic morphisms $H_k^\ell: E_k \rightarrow E_\ell$, depending holomorphically on z , such that $H_k^\ell = 0$ for $k < \ell$, $H_\ell^\ell = I_{E_\ell}$, and in general,*

$$(5.3) \quad \delta_{\zeta-z}H_k^\ell = H_{k-1}^\ell f_k(\zeta) - f_{\ell+1}(z)H_k^{\ell+1}.$$

Proof. In fact, for $k = \ell + 1$ the right hand side of (5.3) is just $f_{\ell+1}(\zeta) - f_{\ell+1}(z)$ so the existence of $H_{\ell+1}^\ell$ is ensured by the first part of the lemma applied to the entries in the matrix representation of $f_{\ell+1}$. If $k > \ell + 1$, then the right hand side of (5.3) is $\delta_{\zeta-z}$ -closed, so in view of the second part of the lemma, the proposition follows by induction over ℓ downwards, starting with $\ell = N - 1$, and over k up-wards. \square

Assuming U is smooth outside Z and R supported on Z , for fixed $z \notin Z$, we can define the current

$$g' = f_1(z) \sum_1^\mu H_k^1 U_k + \sum_1^\mu H_k^0 R_k,$$

and it is easily checked that

$$(5.4) \quad \nabla_{\zeta-z}g' = 0, \quad g'_{0,0}(z) = I_{E_0}.$$

In fact, noticing that (5.3) holds for all k and ℓ , we can write $g' = f(z)H^1U + H^0R$ and use (5.3) to get

$$\begin{aligned} \nabla_{\zeta-z}g' &= (\delta_{\zeta-z} - \bar{\partial})g' = \\ &= -f(z) \left((H^1\delta - f(z)H^2)U - H\bar{\partial}U \right) + (H^0f(\zeta) - f(z)H^1)R - H^0\bar{\partial}R = \\ &= -f(z)H^1I_{E_0} = 0, \end{aligned}$$

where we have used that $f \circ f = 0$ (this holds since (5.1) is a complex) and that $f = 0$ on E_0 .

Proposition 5.4. *Let ϕ be a holomorphic section of E_0 in X , let g be a scalar weight with compact support in X for each z in $\mathcal{U} \subset\subset X$, and assume that g depends holomorphically on z . Then we have the holomorphic decomposition*

$$(5.5) \quad \phi(z) = f_1(z) \int_{\zeta} H^1 U \phi \wedge g + \int_{\zeta} H^0 R \phi \wedge g, \quad z \in \mathcal{U}.$$

Proof. In view of (5.4) the case when $z \notin Z$ follows from Proposition 2.1. Since each term in (5.5) is holomorphic for z in \mathcal{U} , the equality must hold also across Z . \square

In particular we see directly that

$$(5.6) \quad T\phi = \int H^1 U \phi \wedge g$$

is a holomorphic solution to $f_1\psi = \phi$ in \mathcal{U} if $R\phi = 0$. We have thus obtained an explicit representation of the solution in Proposition 5.1.

One can also use (5.5) for interpolation.

Proposition 5.5. *Assume that ϕ is a holomorphic section of E_0 in a neighborhood of Z in X . Then*

$$S\phi = \int_{\zeta} H^0 R \phi \wedge g$$

is a holomorphic section E_0 in \mathcal{U} such that $\phi - S\phi$ belongs to the image of f_1 locally at $Z \cap \mathcal{U}$.

Proof. Recall that $S\phi$ only depends on $R\phi$ and thus only depends on the values of ϕ on Z up to the order of the current R . In virtue of Cartan's theorem there is some holomorphic section Φ in X that coincides with ϕ up to the prescribed order on Z . From (5.5) it then follows that $S\phi = S\Phi = \Phi - f_1 T\Phi$ from which the proposition follows. \square

It is possible to give a direct argument of this proposition, with no reference to Cartan's theorem, cf., Remark 3 in [2]. In fact, suppose that ϕ is holomorphic in the open set $U \supset Z$, and take a cutoff function χ with support in U and equal to 1 in a small neighborhood of Z . For a fixed z^0 on Z we can find a $(1, 0)$ -form $s(\zeta)$ such that $\delta_{\zeta-z^0} s(\zeta) \neq 0$ for ζ on $\text{supp } \bar{\partial}\chi$. By continuity this will hold also for z in a small neighborhood V of z^0 . Therefore, $g' = \chi - \bar{\partial}\chi \wedge s / \nabla_{\zeta-z} s$ is a weight for each $z \in V$ and hence

$$(5.7) \quad \phi(z) = f_1(z) \int H^1 U \phi \wedge g \wedge g' + \int H^0 R \phi \wedge g \wedge g', \quad z \in V.$$

However, since $g' \equiv 1$ in a neighborhood of Z , the last term coincides with $S\phi(z)$ for $z \in V$, and hence (5.7) shows that $\phi - S\phi$ is in the image of f_1 there.

6. GENERICALLY SURJECTIVE MORPHISMS

Our main application in this paper of the abstract procedure developed in the preceding section is when E and Q are given (trivial) Hermitian holomorphic vector bundles and $f: E \rightarrow Q$ is a pointwise surjective, or possibly generically surjective, holomorphic morphism; we look for an explicit formula for a holomorphic solution to $f\psi = \phi$, where ϕ is a section of Q . Let us assume that E and Q are bundles over a neighborhood of the closed unit ball and let Z be the analytic variety where f is not surjective. Following [4] we take $E_0 = Q$, $E_1 = E$, and

$$E_k = \Lambda^{k+r-1} E \otimes S^{k-2} Q^* \otimes \det Q^*, \quad k \geq 2.$$

Let e_j and ϵ_k be global holomorphic frames for E and Q respectively, and let e_j^* and ϵ_k^* be their dual frames. Then $f = \sum f^k \otimes \epsilon_k$, where f^k are sections of E^* and $\det f = f^1 \wedge \dots \wedge f^r \otimes \epsilon_r \wedge \dots \wedge \epsilon_1$. We obtain a complex (5.1) by taking $f_1 = f$, $f_2 = \det f$ and f_k as interior multiplication δ_f with f for $k > 2$, see [4] for details, which is known as the Eagon-Northcott complex. In the case $r = 1$ this is just the Koszul complex. If r is odd, the natural grading of $\Lambda(T_{0,1}^*(X) \oplus E)$ gives rise to the desired Z_2 -grading of $\oplus E_j$; if r is even, then $\bar{\partial}$ and $\det f$ do not anti-commute and therefore one has to compensate with a factor $(-1)^{r+1}$ at some places, see [4]; in what follows it is therefore tacitly understood that r is odd, and we leave it to the interested reader to find out where to put necessary minus signs in the case when r is even.

Outside Z let σ_k be the sections of E with minimal norms such that $f^j \sigma_k = \delta_{jk}$. Then $\sigma = \sigma_1 \otimes \epsilon_1^* + \dots + \sigma_r \otimes \epsilon_r^*$ is the minimal section of $\text{Hom}(E_0, E_1)$ such that $f\sigma = I_{E_0}$. Moreover, the section $\det f$ is nonvanishing, and if $\mathcal{O} = \sigma_1 \wedge \dots \wedge \sigma_r \otimes \epsilon_r^* \wedge \dots \wedge \epsilon_1^*$, for ξ with values in E_1 , then $\mathcal{O}\xi$ is the minimal inverse. Now we define

$$(6.1) \quad u_1 = \sigma, \quad u_k = (\bar{\partial}\sigma)^{\otimes(k-2)} \otimes \mathcal{O} \otimes \bar{\partial}\sigma, \quad k \geq 2,$$

where \otimes shall be interpreted as \wedge on the factors in $\Lambda(T_{0,1}^*(X) \oplus E)$ and \otimes on the Q^* -factors, and where the rightmost factor $\bar{\partial}\sigma$ is supposed to act on E_0 . If we extend u across Z as $U = |\det f|^{2\lambda} u|_{\lambda=0}$, cf., Example 3 and see [4] for details, and let $R = \bar{\partial}|\det f|^{2\lambda} \wedge u|_{\lambda=0}$, then we get currents satisfying (5.2). Moreover, U is smooth outside Z and R has support on Z .

To construct the division formula we also need suitable Hefer forms, whose existence are ensured by Proposition 5.3. However, we can be somewhat more explicit. To begin with let $h(\zeta, z)$ be a $(1,0)$ -form with values in $\text{Hom}(E_1, E_0)$ such that $\delta_{\zeta-z} h = f(\zeta) - f(z)$, and let $(\delta_h)_\ell = (\delta_h)^\ell / \ell!$. Notice that since δ_f is an odd mapping, δ_h is even. For $\ell \geq 2$ we can now take $H_k^\ell = (\delta_h)_{k-\ell}$. In fact,

$$\delta_{\zeta-z} (\delta_h)_{k-\ell} = (\delta_h)_{k-\ell-1} (\delta_{f(\zeta)} - \delta_{f(z)}) = (\delta_h)_{k-\ell-1} \delta_{f(\zeta)} - \delta_{f(z)} (\delta_h)_{k-\ell-1},$$

which shows that (5.3) is fulfilled for $\ell \geq 2$.

Theorem 6.1. *Let $f: E \rightarrow Q$ be a generically surjective morphism in a neighborhood of the closure of the unit ball D , and let the currents R, U and the Hefer forms H be defined as above. If g for instance is the smooth weight from Example 2, then for any holomorphic section ϕ of Q (r -tuple of holomorphic functions) we have the explicit holomorphic decomposition*

$$(6.2) \quad \phi(z) = f(z) \int H^1 U \phi \wedge g + \int H^0 R \phi \wedge g, \quad z \in D.$$

Example 5. This division formula is non-trivial even when Z is empty so that $R = 0$, and as an example let us compute it explicitly in the ball in \mathbb{C}^2 . Since Z is empty, U is smooth, and therefore we can use the weight g from (2.2), and so the formula is $\phi(z) = f(z)T\phi(z)$, where

$$T\phi(z) = \frac{1}{(2\pi i)^2} \int_{|\zeta|=1} H_1^1 U_1 \phi \wedge \frac{\partial|\zeta|^2 \wedge (\bar{\partial}\partial|\zeta|^2)^2}{(1 - \bar{\zeta} \cdot z)^2} + \frac{1}{2\pi i} \int_{|\zeta|=1} H_2^1 U_2 \phi \wedge \frac{\partial|\zeta|^2 \wedge \bar{\partial}\partial|\zeta|^2}{1 - \bar{\zeta} \cdot z} + \int_{|\zeta|<1} H_3^1 U_3 \phi.$$

If $f = f^1 \otimes \epsilon_1 + \dots + f^r \otimes \epsilon_r$ and $\sigma = \sigma_1 \otimes \epsilon_1^* + \dots + \sigma_r \otimes \epsilon_r^*$ as above, and $\phi = \phi_1 \epsilon_1 + \dots + \phi_r \epsilon_r$, then (suppressing the basis elements $\epsilon_1 \wedge \dots \wedge \epsilon_r$ and its dual $\epsilon_1^* \wedge \dots \wedge \epsilon_r^*$) we have that

$$U_1 \phi = \sigma \phi = \sum_1^r \phi_k \sigma_k, \quad U_2 \phi = \sigma \wedge \bar{\partial} \sigma \phi = \sigma_1 \wedge \dots \wedge \sigma_r \wedge \sum_1^r \bar{\partial} \phi_k \sigma_k,$$

and

$$U_3 \phi = \sum_{j=1}^r \bar{\partial} \sigma_j \otimes \epsilon_j \wedge \sigma_1 \wedge \dots \wedge \sigma_r \wedge \sum_1^r \bar{\partial} \phi_k \sigma_k.$$

Next we have to compute H_k^1 for $k = 1, 2, 3$. To begin with, $H_1^1 = I_{E_0}$ whereas H_2^1 has to be a holomorphic solution to

$$\delta_{\zeta-z} H_2^1 = \delta_{f^1(\zeta)} \cdots \delta_{f^r(\zeta)} - \delta_{f^1(z)} \cdots \delta_{f^r(z)}.$$

If $r = 2$ one can take, e.g., $H_2^1 = \delta_{f^1} \delta_{h_2} - \delta_{h_1} \delta_{f^2(z)}$, where $h = h_1 \otimes \epsilon_1 + \dots + h_r \otimes \epsilon_r$ and h_j are $(1, 0)$ -forms such that $\delta_{\zeta-z} h_j = f^j(\zeta) - f^j(z)$. Finally, H_3^1 has to solve (recall that r is assumed to be odd; in case r is even, the first term on the right should have a minus sign)

$$\delta_{\zeta-z} H_3^1 = H_2^1 \delta_{f(\zeta)} - \delta_{f^1(z)} \cdots \delta_{f^r(z)} \delta_h,$$

i.e., $H_3^1 = (H_3^1)_1 \otimes \delta_{\epsilon_1} + \dots + (H_3^1)_r \otimes \delta_{\epsilon_r}$, where

$$\delta_{\zeta-z} (H_3^1)_k = H_2^1 \delta_{f^k(\zeta)} - \delta_{f^1(z)} \cdots \delta_{f^r(z)} \delta_{h_k}.$$

□

7. VARIOUS APPLICATIONS

In this section we illustrate the utility of our new formula by presenting some matrix variants of previously known results. In most cases, the proofs are very similar the case $r = 1$, so we only indicate them.

7.1. A cohomological duality result. Let $f: E \rightarrow Q$ be generically surjective, assume that $\text{codim } Z = m - r + 1$, and let U and R be the currents from the preceding section. We then know from [4] that $f\psi = \phi$ has a holomorphic solution locally if and only if $R\phi = R_{m-r+1}\phi = 0$. Moreover, a solution ψ is given by (5.6).

For degree reasons, $R_{m-r+1} = \bar{\partial}U_{m-r+1}$, and we have a mapping

$$G\phi: \xi \mapsto \int \bar{\partial}\xi \wedge u_{m-r+1}\phi$$

for test forms ξ such that $\bar{\partial}\xi = 0$ in a neighborhood of Z .

Proposition 7.1. *If $\text{codim } Z = m - r + 1$, then $G\phi = 0$ if and only if $f\psi = \phi$ locally has holomorphic solutions.*

In the case $r = 1$ this duality result is proved in [19] and [28]. In the case that $m = n$, then $G\phi$ is the classical Grothendieck residue. One can prove (see [5]) that G as well as R are independent of the Hermitian metrics on E and Q , and essentially only depends on the sheaf $J = \text{Im}(\mathcal{O}(E) \rightarrow \mathcal{O}(Q))$.

Proof. By Stokes' theorem it follows that $G\phi = 0$ if $R_{m-r+1}\phi = 0$, i.e., if $f\psi = \phi$ is locally solvable. To prove the converse, we mimick the argument given in [28] (the proof of Theorem 6.3.1). Clearly the statement is local, so let us fix a point on Z that we may assume is the origin. After a suitable linear change of coordinates we may assume that if $W = \{|z'|, |z''| < 1\}$, where $z = (z', z'') \in \mathbb{C}^{n-(m-r+1)} \times \mathbb{C}^{m-r+1}$, then $Z \cap W$ is contained in $\{|z''| < \delta\}$. Take $\chi = \chi'\chi''$, where $\chi' = \chi'(z')$ has support in $|z'| < \delta$ and is identically 1 for small z' , and χ'' is a cutoff function that is 1 in a neighborhood W . Moreover, take

$$s = \frac{\partial|\zeta'|^2}{2\pi i(|\zeta'|^2 - \bar{\zeta}' \cdot z')}$$

in W and extend it outside W so that $\delta_{\zeta-z}s \neq 0$ for z close to the origin, and depends holomorphically of z there. Let $g = \chi - \bar{\partial}\chi \wedge s / \nabla_{\zeta-z}s$ as before. In W then g only depends on z' so for degree reasons $g_{\mu,\mu}$, $\mu = n - (m - r + 1)$, is $\bar{\partial}$ -closed there. The obstruction for $T\phi$ being a solution to $f\psi = \phi$ is the residue term, cf., (6.2)

$$\int H^0 R_{m-r+1}\phi \wedge g_{\mu,\mu}.$$

However, $R_{m-r+1} = \bar{\partial}U_{m-r+1}$ so an integration by part gives

$$\int H^0 u_{m-r+1}\phi \wedge \bar{\partial}g_{\mu,\mu},$$

which by assumption vanishes for z close to 0. This proves the statement. \square

7.2. A division problem for smooth sections. Let $f: E \rightarrow Q$ be a generically surjective morphism as before, and assume to begin with that $\text{codim } Z = m - r + 1$. Let ϕ be a smooth section of Q and assume that there is a smooth section of E such that $f\psi = \phi$. Arguing as in the proof of Theorem 1.2 in [4] it follows that $R\phi = R(f\psi) = (\bar{\partial}R')\psi = 0$. Let $\bar{\partial}^\alpha = \partial^\alpha / \partial \bar{z}^\alpha$. Then $\bar{\partial}^\alpha \phi = f(\bar{\partial}^\alpha \psi)$, and it therefore follows that

$$(7.1) \quad R(\bar{\partial}^\alpha \phi) = 0$$

for all α . For a general f , i.e., not necessarily such that $\text{codim } Z = m - r + 1$, we have the converse statement.

Theorem 7.2. *Suppose that ϕ smooth and assume that (7.1) holds for all α . Then $f\psi = \phi$ has a smooth solution.*

This was first proved for $r = 1$ in [3]. We do not know any argument based on the Koszul complex and successively solving of $\bar{\partial}$ -equations as in the proof of Proposition 5.1 above. However, in [14] is recently given a quite simple proof based on a deep criterion for closedness of ideals of smooth functions in terms of formal power series due to Malgrange, [26].

If we replace C^∞ with real-analytic functions C^ω , then the corresponding statement follows directly from the holomorphic case, by embedding X in the anti-diagonal $\{(z, \bar{z}) \in \mathbb{C}^{2n}; z \in X\}$.

Corollary 7.3. *Suppose that ϕ smooth and $\|\bar{\partial}^\alpha \phi\| \lesssim \det(ff^*)^{\min(n, m-r+1)}$ for all α . Then $f\psi = \phi$ has a smooth solution.*

Remark 2. The corollary can be seen as an extension of the Briançon-Skoda theorem and follows by a standard estimate from the theorem.

In the real-analytic case it is easy to see that the size condition in the corollary is fulfilled if $r = 1$ and ϕ belongs to the integral closure of the ideal $J = \mathcal{E}(f)$, i.e., if there are functions $a_k \in J^k$ such that $\phi^N + a_1 \phi^{N-1} + \dots + a_N = 0$. We do not know if the same is true in the smooth case. \square

We also have an analogous result for lower regularity.

Theorem 7.4. *Assume that M is the order of the current R . There is a number c_n , only depending on n , such that if $\phi \in C^{c_n+2M+k}$ and (7.1) holds for all $|\alpha| \leq c_n + M + k$, then there is a section ψ of E of class C^k such that $f\psi = \phi$.*

Once we have the appropriate division formula these theorems follows in the same way as for the case $r = 1$ in [3], and we therefore omit the proofs.

7.3. Matrix H^p -corona theorems. Suppose that $F(z)$ is a pointwise surjective $r \times m$ -matrix of bounded holomorphic functions in a strictly pseudoconvex domain D and assume furthermore that $|\det f(z)| \geq \delta > 0$. Then for each r -tuple ϕ in $H^p(D)$, $p < \infty$, one can find an m -tuple ψ in H^p such that $F\psi = \phi$. This was proved in [25] (and with a sharper estimate in [4], see [4] also for a further discussion), by reducing it to the case $r = 1$ via the Fuhrmann trick, [21], and the case $r = 1$ is known since long ago, see [7] and the references given there. An explicit solution formula in case $r = 1$ is given in [6], and copying the arguments there, and using the special choice of Hefer forms defined in Section 4, (most likely)

$$T\phi = \int_D H^1 U \phi \wedge g$$

is such a solution in H^p in the unit ball provided that

$$g = \left(\frac{1 - \bar{\zeta} \cdot z}{1 - |\zeta|^2} - \omega \right)^{-\alpha},$$

and α is large enough.

7.4. Division formulas for $\bar{\partial}$ -closed forms. In [4] we proved Briançon-Skoda type results also for $\bar{\partial}$ -closed smooth (p, q) -forms, and even in this case we can provide explicit representations of the solutions. Again let $f: E \rightarrow Q$ be a generically surjective holomorphic morphism. We want explicit expression for a $\bar{\partial}$ -closed solution to $f\psi = \phi$ provided that $R\phi = 0$. Following [1] we now consider forms in $X_\zeta \times X_z$ with values in the exterior algebra spanned by $T_{0,1}^*(X \times X)$ and the $(1, 0)$ -forms $d\eta_1, \dots, d\eta_n$, where $\eta_k = \zeta_k - z_k$. Then interior multiplication δ_η with $\eta = \sum_1^n \eta_j (\partial/\partial \eta_j)$ has a meaning and we can build up formulas pretty much as when z is just considered as a parameter. If we let $v = b/\nabla_\eta b$, where $b = (2\pi i)^{-1} \sum \partial|\eta|^2/|\eta|^2$, and $\nabla_\eta = \delta_\eta - \bar{\partial}$, then $\nabla_\eta v = 1 - [\Delta]$, where $[\Delta]$ is the (n, n) -current of integration over the diagonal in $X \times X$. Let $g' = f(z)H^1 U + H^0 R$ as before and let g be the form from Example 2 but with all $d\zeta_k$ replaced by $d\eta_k$. Then $\nabla_\eta(g' \wedge g) = 0$ as before and therefore at least formally we have

$$\nabla_\eta(v \wedge g' \wedge g) = g' \wedge g - [\Delta]I_Q,$$

and for degree reasons thus

$$\bar{\partial}(v \wedge g' \wedge g)_{n,n-1} = [\Delta] - (g' \wedge g)_{n,n}.$$

Therefore,

$$\phi(z) = \pm \int_\zeta v \wedge g \wedge g' \wedge \bar{\partial} \phi \pm \bar{\partial}_z \int_\zeta v \wedge g \wedge g' \wedge \phi + \int g \wedge g' \wedge \phi, \quad z \in \mathcal{U}.$$

Assuming that $\bar{\partial}\phi = 0$, we get

$$(7.2) \quad \phi(z) = \pm f(z)\bar{\partial}_z \int_{\zeta} H^1 U(\zeta) \wedge v \wedge g \wedge \phi + f(z) \int_{\zeta} g \wedge H^1 U \wedge \phi + \\ \pm \bar{\partial}_z \int_{\zeta} v \wedge g \wedge H^0 R \wedge \phi + \int_{\zeta} g \wedge H^0 R \wedge \phi.$$

Since the integrals with v are essentially convolutions of currents and the locally integrable functions $\zeta \mapsto \zeta_j/|\zeta|^{2k}$, $k \leq n$, they have meaning. One can prove (7.2) strictly by a suitable approximation argument that we omit.

If ϕ is holomorphic, then we get back formula (6.2). Notice that g and g' are holomorphic in z and therefore cannot contain any component of positive degree in $d\bar{z}$. Therefore we have

Proposition 7.5. *If ϕ is a $\bar{\partial}$ -closed (p, q) -form with values in Q , $q > 0$, such that $R\phi = 0$, then a $\bar{\partial}$ -closed solution to $f\psi = \phi$ is provided by the formula*

$$\psi(z) = \bar{\partial}_z \int_{\zeta} H^1 U \wedge v \wedge g \wedge \phi.$$

Notice that in general ψ is not, and cannot be, smooth.

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