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CONVERGENCE OF SCHRÖDINGER OPERATORS

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Abstract: For a large class, containing the Kato class, of real-valued Radon measures m on $\mathbb{R}^d$ the operators $-\Delta + \varepsilon^2\Delta^2 + m$ in $L^2(\mathbb{R}^d, dx)$ tend to the operator $-\Delta + m$ in the norm resolvent sense, as ε tends to zero. If $d \leq 3$ and a sequence (μ_n) of finite real-valued Radon measures on $\mathbb{R}^d$ converges to the finite real-valued Radon measure m weakly and, in addition, $\sup_{n\in\mathbb{N}} \mu_n^{\pm}(\mathbb{R}^d) < \infty$, then the operators $-\Delta + \varepsilon^2\Delta^2 + \mu_n$ converge to $-\Delta + \varepsilon^2\Delta^2 + m$ in the norm resolvent sense. Explicit upper bounds for the rates of convergences are derived. One can choose point measures μ_n with mass at only finitely many points so that a combination of both convergence results leads to an efficient method for the numerical computation of the eigenvalues in the discrete spectrum and corresponding eigenfunctions of Schrödinger operators.

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I Introduction

Weak convergence of potentials implies norm-resolvent convergence of the corresponding one-dimensional Schrödinger operators. This result from [5] may be interesting for several reasons. For instance every finite real-valued Radon measure on $\mathbb{R}$ is the weak limit of a sequence of point measures with mass at only finitely many points. There exist efficient numerical methods for the computation of the eigenvalues and corresponding eigenfunctions of one-dimensional Schrödinger operators with a potential supported by a finite set; actually the effort for the computation of an eigenvalue and corresponding eigenfunction grows at most linearly with the number of points of the support [8]. Since norm resolvent convergence implies convergence of the eigenvalues in the discrete spectra and corresponding eigenspaces, we get an efficient method for the numerical calculation of the points in the discrete spectra and corresponding eigenspaces of one-dimensional Schrödinger operators.

Let us also mention a completely different motivation. In quantum mechanics neutron scattering is often described via so called zero-range Hamiltonians (the monograph [1] is an excellent standard reference to this research area). In a wide variety of models the positions of the neutrons are described via a family $(X_j)_{j=1}^n$ of independent random variables with joint distribution μ. Usually the number n of neutrons is large and one is interested in the limit when n tends to infinity and the strengths of the single size potentials tend to zero. In the one-dimensional case this motivates to investigate the limits of operators of the form

$$H_\omega := -\frac{d^2}{dx^2} + \frac{a}{n}\sum_{j=1}^{n} \delta_{X_j(\omega)}, \quad \omega \in \Omega,$$

$a \neq 0$ being a real constant and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. By the theorem of Glivenko-Cantelli, for $\mathbb{P}$-almost all $\omega \in \Omega$ the sequence $(\frac{a}{n}\sum_{j=1}^{n} \delta_{X_j(\omega)})_{n\in\mathbb{N}}$ converges to the measure $a\mu$ weakly. By the mentioned result from [5], this implies that

$$-\frac{d^2}{dx^2} + a\mu = \lim_{n\longrightarrow\infty} (-\frac{d^2}{dx^2} + \frac{a}{n}\sum_{j=1}^{n} \delta_{X_j(\omega)})$$

in the norm resolvent sense $\mathbb{P}$-a.s.

It is the purpose of the present note to derive analogous results in the two- and three-dimensional case. Here one is not only interested in the absolutely continuous case $dm = V dx$ for some function V where $-\Delta + m$ equals the regular Schrödinger operator $-\Delta + V$, but also in the case that m is absolutely continuous w.r.t. the $(d-1)$-dimensional volume measure of a manifold with codimension one [4], [7].

If $d > 1$, then it seems to be impossible to work directly with operators of the form $-\Delta + \mu$, μ being a point measure. In fact, while the operators $-\frac{d^2}{dx^2} + \sum_{j=1}^{n} a_j \delta_{x_j}$ can be defined via Kato's quadratic form method as the unique lower semibounded self-adjoint operator associated to the energy form

$$\begin{aligned} D(\mathcal{E}) &:= H^1(\mathbb{R}), \\ \mathcal{E}(f,f) &:= \int |f'(x)|^2 dx + \sum_{j=1}^{n} a_j |\tilde{f}(x_j)|^2, \quad f \in D(\mathcal{E}), \end{aligned}$$

$\tilde{f}$ being the unique continuous representative of $f \in H^1(\mathbb{R})$, the quadratic form

$$\begin{aligned} D(\mathcal{E}) &:= \{f \in H^1(\mathbb{R}^d) : f \text{ has a continuous representative } \tilde{f}\}, \\ \mathcal{E}(f,f) &:= \int |\nabla f(x)|^2 dx + \sum_{j=1}^{n} a_j |\tilde{f}(x_j)|^2, \quad f \in D(\mathcal{E}), \end{aligned}$$

is not lower semibounded and closable if $d > 1$ and at least one coefficient a_j is different from zero.

The starting point for the strategy to overcome the mentioned problem in higher dimensions have been the following two simple observations:

1. The lower semibounded self-adjoint operator $\Delta^2 + \mu$ can be defined via Kato's quadratic form method for every real-valued finite Radon measure μ on $\mathbb{R}^d$ (if $d \in \{1, 2, 3\}$).

2. $-\Delta + \varepsilon^2\Delta^2 \longrightarrow -\Delta$ in the norm resolvent sense, as $\varepsilon > 0$ tends to zero.

For a large class of measures m, containing the Kato class, we shall prove that

$$-\Delta + \varepsilon^2\Delta^2 + m \longrightarrow -\Delta + m$$

in the norm resolvent sense, as $\varepsilon > 0$ tends to zero, cf. section III. In section II we shall prove that the sequence $(-\Delta + \varepsilon^2\Delta^2 + \mu_n)_{n\in\mathbb{N}}$ converges to $-\Delta + \varepsilon^2\Delta^2 + m$ in the norm resolvent sense provided $d \leq 3$, $\varepsilon > 0$, the finite real-valued Radon measures μ_n on $\mathbb{R}^d$ converge to the finite real-valued Radon measure m weakly and $\sup_{n\in\mathbb{N}} \mu_n^{\pm}(\mathbb{R}^d) < \infty$, $\mu = \mu^+ - \mu^-$ being the Hahn-Jordan decomposition of μ. Actually we shall not only prove convergence but even give explicit error estimates.

As approximating measures μ_n we can, in particular, choose point measures with mass at only finitely many points. In section IV we shall derive a result which makes it possible to calculate the eigenvalues and corresponding eigenspaces of operators of the form $-\Delta + \varepsilon^2\Delta^2 + \mu$ numerically provided μ is a point measure with mass at n points and $n < \infty$. The effort for these computations grows at most as $O(n^3)$. In particular, we get an efficient method to calculate the eigenvalues in the discrete spectrum and corresponding eigenspaces of Schrödinger operators $-\Delta + m$ numerically. Let us emphasize that our method covers both the absolutely continuous case $dm = V dx$ where $-\Delta + m = -\Delta + V$ is a regular Schrödinger operator and the singular case when m is absolutely continuous w.r.t. the $(d-1)$-dimensional volume measure of a manifold with codimension one. Actually, we will treat a fairly large class of measures m containing the set of all finite real-valued measures belonging to the Kato class.

Notation and auxiliary results: Let μ be a real-valued Radon measure on $\mathbb{R}^d$. By the Hahn-Jordan theorem, there exist unique positive Radon measures $\mu^{\pm}$ on $\mathbb{R}^d$ such that

$$\mu = \mu^+ - \mu^- \text{ and } \mu^+(\mathbb{R}^d \setminus B) = 0 = \mu^-(B)$$

for some suitably chosen Borel set B. We put

$$\| \mu \| := \mu^+(\mathbb{R}^d) + \mu^-(\mathbb{R}^d) \text{ and } |\mu| := \mu^+ + \mu^-.$$

If μ is finite, then we define the Fourier transform $\hat{\mu}$ of μ as

$$\hat{\mu}(p) := (2\pi)^{-d/2} \int e^{ipx} \mu(dx), \quad p \in \mathbb{R}^d.$$

$\hat{f}$ also denotes the Fourier transform of $f \in L^2(dx) := L^2(\mathbb{R}^d, dx)$, dx being the Lebesgue measure.

For $s > 0$ we denote the Sobolev space of order s by $H^s(\mathbb{R}^d)$, i.e.

$$\begin{aligned} H^s(\mathbb{R}^d) &:= \{f \in L^2(dx) : \int (1+p^2)^s |\hat{f}(p)|^2 dp < \infty\}, \\ \| f \|_{H^s} &:= (\int (1+p^2)^s |\hat{f}(p)|^2 dp)^{1/2}, \quad f \in H^s(\mathbb{R}^d). \end{aligned}$$

Occasionally we shall use the abbreviations $L^2(\mu) := L^2(\mathbb{R}^d, \mu)$ and $H^s := H^s(\mathbb{R}^d)$.

$\| T \|_{\mathcal{H}_1, \mathcal{H}_2}$ denotes the operator norm of T as an operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ and $\| T \|_{\mathcal{H}} := \| T \|_{\mathcal{H},\mathcal{H}}$. $\| f \|_{\mathcal{H}}$ and $(f,h)_{\mathcal{H}}$ denotes the norm of f and the scalar product of f and h in the Hilbert $\mathcal{H}$, respectively. If the reference to a measure is missing, then we tacitly refer to the Lebesgue measure dx. For instance "integrable" means "integrable w.r.t. dx" if not stated otherwise, $\| T \|$ denotes the norm of T as an operator in $L^2(dx)$ and (f,h) and $\| f \|$ denote the scalar product of f and h and the norm of f in the Hilbert space $L^2(dx)$, respectively. We denote by $C_0^\infty(\mathbb{R}^d)$ the space of smooth functions with compact support.

For arbitrary $\varepsilon \geq 0$ ($\varepsilon = 0$ will be admitted only in section III) let $\mathcal{E}_\varepsilon$ be the nonnegative closed quadratic form in the Hilbert space $L^2(dx)$ associated to the nonnegative self-adjoint operator $-\Delta + \varepsilon^2 \Delta^2$ in $L^2(dx)$. Obviously we have

$$\begin{aligned} D(\mathcal{E}_\varepsilon) &= H^2(\mathbb{R}^d), \\ \mathcal{E}_\varepsilon(f,f) &= \varepsilon^2 (\Delta f, \Delta f) + (f, \Delta f) \geq \varepsilon^2 (\Delta f, \Delta f), \quad f \in D(\mathcal{E}_\varepsilon), \end{aligned} \tag{1}$$

for every $\varepsilon > 0$.

We put

$$\mathcal{E}_\alpha(f,h) := \mathcal{E}(f,h) + \alpha(f,h), \quad f,h \in D(\mathcal{E}),$$

for every quadratic form $\mathcal{E}$ in $L^2(dx)$ and $\alpha > 0$.

For every $\varepsilon \geq 0$ and $\alpha > 0$ there exists a function $g_{\varepsilon,\alpha}$ with Fourier transform

$$p \mapsto \frac{1}{\varepsilon^2 p^4 + p^2 + \alpha}, \qquad \mathbb{R}^d \longrightarrow \mathbb{R},$$

which is continuous on $\mathbb{R}^d \setminus \{0\}$ (on $\mathbb{R}^d$ if $d = 1$ or if $d \leq 3$ and $\varepsilon > 0$). $g_{\varepsilon,\alpha}(x)$ is unique for every $x \in \mathbb{R}^d \setminus \{0\}$ (even every $x \in \mathbb{R}^d$ if $d = 1$ or if $d \leq 3$ and $\varepsilon > 0$). $g_{\varepsilon,\alpha}$ is radially symmetric and $g_{\varepsilon,\alpha}(x-y)$ is the integral kernel of the operator $G_{\varepsilon,\alpha} := (-\Delta + \varepsilon^2 \Delta^2 + \alpha)^{-1}$ in $L^2(dx)$. $g_{0,\alpha}$ is nonnegative.

II Weak and operator norm convergence

Throughout this section let $d \leq 3$ and μ be a finite real-valued Radon measure on $\mathbb{R}^d$. Then, by Sobolev's embedding theorem, for every $s > 3/2$, and, in particular, for $s = 2$, every $f \in H^s(\mathbb{R}^d)$ has a unique continuous representative $\tilde{f}$ and

$$\| \tilde{f} \|_\infty := \sup\{|\tilde{f}(x)| : x \in \mathbb{R}^d\} \le c_s \| f \|_{H^s}, \quad f \in H^s(\mathbb{R}^d), \tag{2}$$

for some finite constant c_s. Note that $c_s \le 1$ if $s = 2$.

Let $a > 0$ be arbitrary and for $f \in H^2(\mathbb{R}^d)$ put

$$f_a(x) := f(ax), \quad x \in \mathbb{R}^d.$$

Then f_a also belongs to $H^2(\mathbb{R}^d)$ and we get with the aid of (1) and the Sobolev inequality (2) that

$$\begin{aligned} \| \tilde{f} \|_\infty^2 = \| \tilde{f}_a \|_\infty^2 &\le \int \varepsilon^2 |\Delta f_a(x)|^2 dx + \int |f_a(x)|^2 dx \\ &\le a^{4-d} \varepsilon^{-2} \mathcal{E}_\varepsilon(f, f) + a^{-d}(f, f). \end{aligned}$$

Since $d \le 3$ it follows that for every $\varepsilon > 0$ and every $\eta > 0$ there exists an $\alpha = \alpha(\varepsilon, \eta) < \infty$ such that

$$\| \tilde{f} \|_\infty^2 \le \eta \, \mathcal{E}_\varepsilon(f, f) + \alpha(f, f), \quad f \in H^2(\mathbb{R}^d). \tag{3}$$

By (3), for every $\varepsilon > 0$ and every $\eta > 0$ there exists an $\alpha = \alpha(\varepsilon, \eta) < \infty$ such that

$$\left| \int |\tilde{f}|^2 d\mu \right| \le \eta \| \mu \| \, \mathcal{E}_\varepsilon(f, f) + \alpha \| \mu \| \, (f, f), \quad f \in H^2(\mathbb{R}^d). \tag{4}$$

We put

$$\begin{aligned} D(\mathcal{E}_\varepsilon^\mu) &:= H^2(\mathbb{R}^d), \\ \mathcal{E}_\varepsilon^\mu(f, f) &:= \mathcal{E}_\varepsilon(f, f) + \int |\tilde{f}|^2 d\mu, \quad f \in D(\mathcal{E}_\varepsilon^\mu). \end{aligned}$$

By (4) and the KLMN-theorem, $\mathcal{E}_\varepsilon^\mu$ is a lower semibounded closed quadratic form in $L^2(dx)$. We denote the lower semibounded self-adjoint operator in $L^2(dx)$ associated to $\mathcal{E}_\varepsilon^\mu$ by $-\Delta + \varepsilon^2 \Delta^2 + \mu$ and put

$$G_{\varepsilon,\alpha}^\mu := (-\Delta + \varepsilon^2 \Delta^2 + \mu + \alpha)^{-1}$$

provided the operator $-\Delta + \varepsilon^2 \Delta^2 + \mu + \alpha$ is invertible.

Let $\varepsilon, \alpha > 0$, then the function

$$p \mapsto \frac{1}{\varepsilon^2 p^4 + p^2 + \alpha}, \qquad \mathbb{R}^d \longrightarrow \mathbb{R},$$

and all its partial derivatives (of arbitrary order) are integrable with respect to the Lebesgue measure. Thus the inverse Fourier transform $g_{\varepsilon,\alpha}$ of this function is continuous and

$$|x|^j g_{\varepsilon,\alpha}(x) \longrightarrow 0, \text{ as } |x| \longrightarrow \infty, \tag{5}$$

for every $j \in \mathbb{N}$. By the dominated convergence theorem,

$$\| g_{\varepsilon,\alpha} \|_{H^2}^2 = \int \frac{(1+p^2)^2}{|\varepsilon^2 p^4 + p^2 + \alpha|^2} dp \longrightarrow 0, \text{ as } |\alpha| \longrightarrow \infty. \tag{6}$$

By Sobolev's inequality, this implies that

$$\| g_{\varepsilon,\alpha} \|_\infty \longrightarrow 0, \text{ as } |\alpha| \longrightarrow \infty. \tag{7}$$

Since the operator $G_{\varepsilon,\alpha} = (-\Delta + \varepsilon^2\Delta^2 + \alpha)^{-1}$ in $L^2(dx)$ is the integral operator with kernel $g_{\varepsilon,\alpha}(x-y)$, we have

$$\int g_{\varepsilon,\alpha}(x-y)(-\Delta + \varepsilon^2\Delta^2 + \alpha)h(y)dy = h(x) \quad dx\text{-a.e.}, \quad h \in D(-\Delta + \varepsilon^2\Delta^2).$$

The equation above does not only hold almost everywhere w.r.t. the Lebesgue measure dx but even pointwise everywhere, i.e.

$$\int g_{\varepsilon,\alpha}(x-y)(-\Delta + \varepsilon^2\Delta^2 + \alpha)h(y)dy = \tilde{h}(x), \quad x \in \mathbb{R}^d, \quad h \in D(-\Delta + \varepsilon^2\Delta^2). \tag{8}$$

In fact, we have only to show that the the integral on the left hand side is a continuous function of $x \in \mathbb{R}^d$. We choose any sequence $(f_n)_{n\in\mathbb{N}}$ of continuous functions with compact support converging to $(-\Delta+\varepsilon^2\Delta^2+\alpha)h$ in $L^2(dx)$. By (6), $g_{\varepsilon,\alpha} \in H^2(\mathbb{R}^d) \subset L^2(dx)$, therefore we can write

$$\int g_{\varepsilon,\alpha}(x-y)(-\Delta + \varepsilon^2\Delta^2 + \alpha)h(y)dy = \lim_{n\longrightarrow\infty} \int g_{\varepsilon,\alpha}(x-y)f_n(y)dy, \quad x \in \mathbb{R}^d.$$

Obviously the mapping $x \mapsto \int g_{\varepsilon,\alpha}(x-y)f_n(y)dy$, $\mathbb{R}^d \longrightarrow \mathbb{C}$, is the unique continuous representative $\widetilde{G_{\varepsilon,\alpha}f_n}$ of $G_{\varepsilon,\alpha}f_n$ for every $n \in \mathbb{N}$. Since $G_{\varepsilon,\alpha}$ is a bounded operator from $L^2(dx)$ to $H^2(\mathbb{R}^d)$ (even to $H^4(\mathbb{R}^d)$), the sequence $(G_{\varepsilon,\alpha}f_n)_{n\in\mathbb{N}}$ converges in $H^2(\mathbb{R}^d)$ to $G_{\varepsilon,\alpha}(-\Delta + \varepsilon^2\Delta^2 + \alpha)h = h$. By Sobolev's inequality (2), this implies that the sequence $(\widetilde{G_{\varepsilon,\alpha}f_n})_{n\in\mathbb{N}}$ of the unique continuous representatives converges to a continuous function uniformly. By the last equality, $x \mapsto \int g_{\varepsilon,\alpha}(x-y)(-\Delta+\varepsilon^2\Delta^2+\alpha)h(y)dy$, $\mathbb{R}^d \longrightarrow \mathbb{C}$, is this continuous uniform limit and we have proved (8).

By Sobolev's inequality and (5),

$$g_{\varepsilon,\alpha} * \tilde{f}\mu(x) := \int g_{\varepsilon,\alpha}(x-y)\tilde{f}(y)\mu(dy), \quad x \in \mathbb{R}^d,$$

defines a bounded continuous function for every $f \in H^2(\mathbb{R}^d)$. We put

$$G^{\mu}_{\varepsilon,\alpha} f(x) := \int g_{\varepsilon,\alpha}(x-y)\tilde{f}(y)\mu(dy) \qquad dx\text{-a.e., } f \in H^2(\mathbb{R}^d).$$

Using Sobolev's inequality, we arrive at

$$|\widehat{\tilde{f}\mu}(p)|^2 \le (2\pi)^{-d} \parallel \tilde{f} \parallel_\infty^2 \parallel \mu \parallel^2 \le (2\pi)^{-d} \parallel \tilde{f} \parallel_{H^2}^2 \parallel \mu \parallel^2, \quad p \in \mathbb{R}^d, \quad f \in H^2(\mathbb{R}^d).$$

Then the convolution theorem and Sobolev's inequality yield

$$\begin{aligned}
\parallel G^{\mu}_{\varepsilon,\alpha} f \parallel_{H^2}^2 &= \int |(1+p^2)^2| \, |(g_{\varepsilon,\alpha} * \tilde{f}\mu)\hat{}(p)|^2 dp \\
&= (2\pi)^d \int \frac{(1+p^2)^2}{|\varepsilon^2 p^4 + p^2 + \alpha|^2} |\widehat{\tilde{f}\mu}(p)|^2 dp \\
&\le \int \frac{(1+p^2)^2}{|\varepsilon^2 p^4 + p^2 + \alpha|^2} \parallel \tilde{f} \parallel_\infty^2 \parallel \mu \parallel^2 \, dp \\
&\le \int \frac{(1+p^2)^2}{|\varepsilon^2 p^4 + p^2 + \alpha|^2} dp \parallel \tilde{f} \parallel_{H^2}^2 \parallel \mu \parallel^2 < \infty, \quad f \in H^2(\mathbb{R}^d).
\end{aligned}$$

Therefore $G^{\mu}_{\varepsilon,\alpha}$ is an everywhere defined bounded operator on $H^2(\mathbb{R}^d)$ and we get an upper bound for the norm $\parallel G^{\mu}_{\varepsilon,\alpha} \parallel_{H^2,H^2}$ of $G^{\mu}_{\varepsilon,\alpha}$ as an operator on $H^2(\mathbb{R}^d)$ and a uniform upper bound for the norm of $G^{\mu}_{\varepsilon,\alpha} f$ in terms of the supremum norm of the continuous representative $\tilde{f}$ of f:

$$\parallel G^{\mu}_{\varepsilon,\alpha} \parallel_{H^2,H^2} \le \parallel \mu \parallel \left(\int \frac{(1+p^2)^2}{|\varepsilon^2 p^4 + p^2 + \alpha|^2} dp\right)^{1/2}, \tag{9}$$

$$\parallel G^{\mu}_{\varepsilon,\alpha} f \parallel_{H^2} \le \parallel \mu \parallel \left(\int \frac{(1+p^2)^2}{|\varepsilon^2 p^4 + p^2 + \alpha|^2} dp\right)^{1/2} \parallel \tilde{f} \parallel_\infty, \quad f \in H^2(\mathbb{R}^d). \tag{10}$$

Moreover

$$\begin{aligned}
& \int |G^{\mu}_{\varepsilon,\alpha} f(x)|^2 dx \\
= \quad & \int \left| \int g_{\varepsilon,\alpha}(x-y)\tilde{f}(y)\mu^+(dy) - \int g_{\varepsilon,\alpha}(x-y)\tilde{f}(y)\mu^-(dy)\right|^2 dx \\
\le \quad & 2\int \left| \int g_{\varepsilon,\alpha}(x-y)\tilde{f}(y)\mu^+(dy)\right|^2 dx + 2\int \left| \int g_{\varepsilon,\alpha}(x-y)\tilde{f}(y)\mu^-(dy)\right|^2 dx \\
\le \quad & 2\int \int |g_{\varepsilon,\alpha}(x-y)|^2 \mu^+(dy) \int |\tilde{f}(y)|^2 \mu^+(dy) dx \\
+ \quad & 2\int \int |g_{\varepsilon,\alpha}(x-y)|^2 \mu^-(dy) \int |\tilde{f}(y)|^2 \mu^-(dy) dx \\
\le \quad & 2\int |g_{\varepsilon,\alpha}|^2 dx \parallel \mu \parallel \int |\tilde{f}(y)|^2 |\mu|(dy), \quad f \in H^2(\mathbb{R}^d).
\end{aligned} \tag{11}$$

In a similar way we get

$$\int |\widetilde{G^{\mu}_{\varepsilon,\alpha}f}(x)|^2|\mu|(dx) \leq 2 \parallel g_{\varepsilon,\alpha} \parallel_{\infty}^2 \parallel \mu \parallel^2 \int |\tilde{f}(y)|^2|\mu|(dy). \tag{12}$$

General results of [2] (cf. also section III below) provide, in particular, an explicit formula for the resolvent of the operator $-\Delta + \varepsilon^2\Delta^2 + \mu$. In this resolvent formula there occur operators acting in different Hilbert spaces. This is inconvenient when we investigate the convergence of sequences of such operators and we shall derive another resolvent formula. While this could be done with the aid of the mentioned result from [2] it may be convenient for an uninitiated reader to start from the very beginning: by the dominated convergence theorem,

$$\int \frac{1+p^4}{|\varepsilon^2 p^4 + p^2 + \alpha|^2} dp \longrightarrow 0, \text{ as } \alpha \longrightarrow \infty.$$

Hence according to (9), we can choose $\alpha > 0$ such that $\parallel G^{\mu}_{\varepsilon,\alpha} \parallel_{H^2,H^2} < 1$. Then the operator $I + G^{\mu}_{\varepsilon,\alpha}$ is invertible and its inverse is everywhere defined on $H^2(\mathbb{R}^d)$ and bounded; here I denotes the identity on $H^2(\mathbb{R}^d)$. By (4), we can choose $\alpha > 0$ such that, in addition,

$$\mathcal{E}^{\mu}_{\varepsilon,\alpha}(f,f) := \mathcal{E}^{\mu}_{\varepsilon}(f,f) + \alpha(f,f) \geq (f,f), \quad f \in D(\mathcal{E}^{\mu}_{\varepsilon}). \tag{13}$$

Let $f \in L^2(dx)$. Since $\mathcal{E}_\varepsilon$ and $\mathcal{E}^{\mu}_{\varepsilon}$ is associated to $-\Delta + \varepsilon^2\Delta^2$ and $-\Delta + \varepsilon^2\Delta^2 + \mu$, respectively, it follows from Kato's representation theorem that

$$\mathcal{E}_{\varepsilon,\alpha}(G_{\varepsilon,\alpha}f, h) = (f,h) = \mathcal{E}^{\mu}_{\varepsilon,\alpha}(G^{\mu}_{\varepsilon,\alpha}f, h), \quad h \in H^2(\mathbb{R}^d). \tag{14}$$

Moreover we have

$$\begin{aligned}
& \mathcal{E}_{\varepsilon,\alpha}(G^{\mu}_{\varepsilon,\alpha}\psi, h) = (G^{\mu}_{\varepsilon,\alpha}\psi, (-\Delta + \varepsilon^2\Delta^2 + \alpha)h) \\
= & \textstyle\int\int g_{\varepsilon,\alpha}(x-y)\bar{\tilde{\psi}}(y)\mu(dy)(-\Delta + \varepsilon^2\Delta^2 + \alpha)h(x)dx \\
= & \textstyle\int\int g_{\varepsilon,\alpha}(x-y)(-\Delta + \varepsilon^2\Delta^2 + \alpha)h(x)dx\bar{\tilde{\psi}}(y)\mu(dy) \\
= & \textstyle\int \tilde{h}\bar{\tilde{\psi}}\mu(dy), \quad \psi \in H^2(\mathbb{R}^d), \quad h \in D(-\Delta + \varepsilon^2\Delta^2).
\end{aligned} \tag{15}$$

We could change the order of integration in the second step. In fact, as $\mu^{\pm}$ are finite Radon measures and $g_{\varepsilon,\alpha}$ is square integrable w.r.t. the Lebesgue measure dx, the mappings $x \mapsto \int |g_{\varepsilon,\alpha}(x-y)|\mu^{\pm}(dy)$, $\mathbb{R}^d \longrightarrow \mathbb{R}$, are square integrable w.r.t. dx. Since $\tilde{\psi}$ is bounded and $(-\Delta + \varepsilon^2\Delta^2 + \alpha)h \in L^2(dx)$ it follows that

$$\int\int |g_{\varepsilon,\alpha}(x-y)\bar{\tilde{\psi}}(y)|\mu^{\pm}(dy)|(-\Delta + \varepsilon^2\Delta^2 + \alpha)h(x)|dx < \infty$$

and, by Fubini's theorem, we could change the order of integration in the second step. In the last step we have used (8). Employing Sobolev's inequality and the fact that $D(-\Delta+\varepsilon^2\Delta^2)$ is dense in $(D(\mathcal{E}_\varepsilon),\mathcal{E}_{\varepsilon,\alpha})$, we can extend (15) to all functions ψ, $h\in D(\mathcal{E}_\varepsilon)$.

Put

$$\phi := G_{\varepsilon,\alpha}f - G^{\mu}_{\varepsilon,\alpha}(I+G^{\mu}_{\varepsilon,\alpha})^{-1}G_{\varepsilon,\alpha}f.$$

Then $\phi\in H^2(\mathbb{R}^d)=D(\mathcal{E}^{\mu}_{\varepsilon})$ and (14) and extended (15) yield

$$\begin{aligned}
& \mathcal{E}^{\mu}_{\varepsilon,\alpha}(\phi,h) \\
= \quad & \mathcal{E}_{\varepsilon,\alpha}(G_{\varepsilon,\alpha}f,h)-\mathcal{E}_{\varepsilon,\alpha}(G^{\mu}_{\varepsilon,\alpha}(I+G^{\mu}_{\varepsilon,\alpha})^{-1}G_{\varepsilon,\alpha}f,h) \\
+ \quad & \int[G_{\varepsilon,\alpha}f-G^{\mu}_{\varepsilon,\alpha}(I+G^{\mu}_{\varepsilon,\alpha})^{-1}G_{\varepsilon,\alpha}f\tilde{\tilde{]}}\tilde{h}d\mu \\
= \quad & (f,h)-\int[(I+G^{\mu}_{\varepsilon,\alpha})^{-1}G_{\varepsilon,\alpha}f\tilde{\tilde{]}}\tilde{h}d\mu \\
+ \quad & \int[(I+G^{\mu}_{\varepsilon,\alpha})(I+G^{\mu}_{\varepsilon,\alpha})^{-1}G_{\varepsilon,\alpha}f-G^{\mu}_{\varepsilon,\alpha}(I+G^{\mu}_{\varepsilon,\alpha})^{-1}G_{\varepsilon,\alpha}f\tilde{\tilde{]}}\tilde{h}d\mu \\
= \quad & (f,h),\quad h\in H^2(\mathbb{R}^d).
\end{aligned}$$

Due to (13), $\mathcal{E}^{\mu}_{\varepsilon,\alpha}$ is a scalar product on $D(\mathcal{E}^{\mu}_{\varepsilon,\alpha})=H^2(\mathbb{R}^d)$. Thus (14) and the calculation above imply that $\phi=G^{\mu}_{\varepsilon,\alpha}f$ and we have derived the following new resolvent formula:

$$G^{\mu}_{\varepsilon,\alpha}=G_{\varepsilon,\alpha}-G^{\mu}_{\varepsilon,\alpha}(I+G^{\mu}_{\varepsilon,\alpha})^{-1}G_{\varepsilon,\alpha} \tag{16}$$

for every $\alpha>0$ satisfying (13) and

$$\|\,\mu\,\|^2\int\frac{(1+p^2)^2}{|\varepsilon^2p^4+p^2+\alpha|^2}dp<1.$$

We are now well prepared for the proof of the main theorem of this section:

THEOREM 1 *Let m and μ_n, $n\in\mathbb{N}$, be finite real-valued Radon measures on $\mathbb{R}^d$. Suppose that the sequence $(\mu_n)_{n\in\mathbb{N}}$ converges to m weakly and $\sup_{n\in\mathbb{N}}\|\,\mu_n\,\|<\infty$. Let $\varepsilon,\alpha>0$ and $d\in\{1,2,3\}$. Then the operators $-\Delta+\varepsilon^2\Delta^2+\mu_n$ converge to $-\Delta+\varepsilon^2\Delta^2+m$ in the norm resolvent sense.*

Proof: Let $\varepsilon>0$ be arbitrary. We choose $0<c<1$ and $\alpha>0$ such that

$$\|\,\mu_n\,\|^2\int\frac{(1+p^2)^2}{|\varepsilon^2p^4+p^2+\alpha|^2}dp\le c^2,\quad n\in\mathbb{N}, \tag{17}$$

and

$$\|\,m\,\|^2\int\frac{(1+p^2)^2}{|\varepsilon^2p^4+p^2+\alpha|^2}dp\le c^2. \tag{18}$$

By (4), we can choose $\alpha > 0$ such that, in addition,

$$\mathcal{E}^{\mu_n}_{\varepsilon,\alpha}(f,f) \geq (f,f), \quad f \in H^2(\mathbb{R}^d), \quad n \in \mathbb{N}. \tag{19}$$

Since $(\mu_n)_{n\in\mathbb{N}}$ converges to m weakly, (19) also holds when we replace μ_n by m. By (9), (17) and (18),

$$\| G^{\mu_n}_{\varepsilon,\alpha} \|_{H^2,H^2} \leq c, \quad n \in \mathbb{N}, \text{ and } \| G^{m}_{\varepsilon,\alpha} \|_{H^2,H^2} \leq c, \tag{20}$$

and, by (10) and (18),

$$\| G^{m}_{\varepsilon,\alpha} f \|_{H^2} \leq c \| \tilde{f} \|_{\infty}, \quad f \in H^2(\mathbb{R}^d).$$

By (19) and (20), the resolvent formula (16) is valid both for $\mu = m$ and for $\mu = \mu_n$, $n \in \mathbb{N}$. By (6), (7), (11) and (12), we can choose $\alpha > 0$ so large that also

$$\int |G^m_{\varepsilon,\alpha} h(x)|^2 dx \leq c^2 \int |\tilde{h}|^2 d|m| \text{ and } \int |\widetilde{G^m_{\varepsilon,\alpha} h}(x)|^2 |m|(dx) \leq c^2 \int |\tilde{h}|^2 d|m| \tag{21}$$

for every $h \in H^2(\mathbb{R}^d)$.

For notational brevity we put

$$g_0 := g_{0,1}, \quad g := g_{\varepsilon,\alpha}, \quad G := G_{\varepsilon,\alpha}, \quad G^{\mu_n} := G^{\mu_n}_{\varepsilon,\alpha} \text{ and } G^m := G^m_{\varepsilon,\alpha}.$$

With this notation we have

$$\begin{aligned} & (-\Delta + \varepsilon^2\Delta^2 + \mu_n + \alpha)^{-1} - (-\Delta + \varepsilon^2\Delta^2 + m + \alpha)^{-1} \\ = \; & G^m[I + G^m]^{-1}G - G^{\mu_n}[I + G^{\mu_n}]^{-1}G \\ = \; & (G^m - G^{\mu_n})[I + G^m]^{-1}G + (G^{\mu_n} - G^m)[I + G^m]^{-1}(G^{\mu_n} - G^m)[I + G^{\mu_n}]^{-1}G \\ & + G^m[I + G^m]^{-1}(G^{\mu_n} - G^m)[I + G^{\mu_n}]^{-1}G. \end{aligned}$$

Since G is a bounded operator from $L^2(dx)$ to $H^2(\mathbb{R}^d)$ we have only to show that

$$\| G^m - G^{\mu_n} \|_{H^2,L^2(dx)} \longrightarrow 0, \text{ as } n \longrightarrow \infty, \tag{22}$$

$$\| G^m[I + G^m]^{-1}(G^m - G^{\mu_n}) \|_{H^2,L^2(dx)} \longrightarrow 0, \text{ as } n \longrightarrow \infty. \tag{23}$$

We introduce

$$\nu_n := m - \mu_n \text{ and } \nu_{nx}(dy) := g(x-y)\nu_n(dy), \quad x \in \mathbb{R}^d, \quad n \in \mathbb{N}.$$

As $d \leq 3$, the function

$$y \mapsto \int g_0(y-a)f(a)da, \quad \mathbb{R}^d \longrightarrow \mathbb{C},$$

is continuous and bounded for every $f \in L^2(dx)$ (this well known fact can be proved in the same way as (8)).

Since the function g is bounded and g_0 is nonnegative it follows that

$$|\int |g(x-y)| \int |g_0(y-a)||(-\Delta+1)h(a)|da\nu_n^{\pm}(dy)| < \infty,\ x \in \mathbb{R}^d,\ h \in H^2(\mathbb{R}^d).$$

Hence by Fubini's theorem, the function $k_{\nu_{nx}} : \mathbb{R}^d \longrightarrow \mathbb{R}$, defined by

$$k_{\nu_{nx}}(a) := \begin{cases} \int g_0(y-a)g(x-y)\nu_n(dy), & \text{if defined,} \\ 0, & \text{otherwise,} \end{cases}$$

is Borel measurable, the integral on the right hand side is defined and finite for almost all $a \in \mathbb{R}^d$ (almost all w.r.t. the Lebesgue measure) and

$$\begin{aligned} |(G^{\nu_n}\tilde{h})(x)|^2 &= |\int g(x-y)h(y)\nu_n(dy)|^2 \\ &= |\int g(x-y)\int g_0(y-a)(-\Delta+1)h(a)da\nu_n(dy)|^2 \\ &\leq \int |k_{\nu_{nx}}(a)|^2 da \cdot \int |(-\Delta+1)h(a)|^2 da \\ &\leq 2 \parallel h \parallel_{H^2}^2 \int |k_{\nu_{nx}}(a)|^2 da, \quad x \in \mathbb{R}^d,\ h \in H^2(\mathbb{R}^d),\ n \in \mathbb{N}. \end{aligned} \tag{24}$$

Thus in order to prove (22) we have only to show that

$$\int\int |k_{\nu_{nx}}(a)|^2 dadx \longrightarrow 0, \text{ as } n \longrightarrow \infty. \tag{25}$$

We have

$$\begin{aligned} & \int\int |k_{\nu_{nx}}(a)|^2 dadx \\ = & \int\int |\widehat{k_{\nu_{nx}}}(p)|^2 dpdx \\ = & (2\pi)^d \int\int |\widehat{g_0}(p)|^2 |\widehat{\nu_{nx}}(p)|^2 dpdx \\ = & \int\int \tfrac{1}{|1+p^2|^2} \int e^{ipy} g(x-y)\nu_n(dy) \int e^{-ipz} g(x-z)\nu_n(dz)dpdx. \end{aligned} \tag{26}$$

Since $p \mapsto \frac{1}{|1+p^2|^2}$ and g are integrable w.r.t. the Lebesgue measure, g is bounded and the Radon measures ν_n are finite, we can change the order of integration.

As $(2\pi)^{-d/2} \int e^{ipy} e^{-ipz} \frac{1}{|1+p^2|^2} dp$ is the inverse Fourier transform of the integrable function $p \mapsto \frac{1}{|1+p^2|^2}$ at the point $z-y$, the function

$$f(y,z) := \int e^{ipy} e^{-ipz} \frac{1}{|1+p^2|^2} dp, \quad y, z \in \mathbb{R}^d,$$

is bounded and continuous. Let $y \in \mathbb{R}^d$ and K be any compact neighbourhood of y. By (5), there exists a constant $a < \infty$ such that

$$|g(x-y)g(x-z)| \leq a \parallel g \parallel_\infty \mathrm{dist}(x,K)^{-4}, \quad x \in \mathbb{R}^d \setminus K, \quad z \in \mathbb{R}^d, \quad y \in K.$$

Moreover g is continuous. Thus the function

$$h(y,z) := \int g(x-y)g(x-z)dx, \quad y,z \in \mathbb{R}^d,$$

is bounded and continuous. By Stone-Weierstrass theorem, the set of functions of the form $(x,y) \mapsto \sum_{j=1}^N f_j(x)g_j(y)$, $N \in \mathbb{N}$, f_j, g_j are bounded and continuous, is dense in the space of bounded continuous functions w.r.t. the supremum norm. Since the measures ν_n tend to zero weakly and $\sup_{n\in\mathbb{N}} \parallel \nu_n \parallel < \infty$ this implies that the product measures $\nu_n \otimes \nu_n$ tend to zero weakly, too. Hence we get

$$\begin{aligned}\int\int \frac{1}{|1+p^2|^2} \int e^{ipy} g(x-y)\nu_n(dy) \int e^{-ipz} g(x-z)\nu_n(dz)dpdx \\ = \int f(y,z)h(y,z)\nu_n \otimes \nu_n(dydz) \longrightarrow 0, \text{ as } n \longrightarrow \infty.\end{aligned}$$

By (26), it follows that we have proved (25) and therefore also (22).

It only remains to prove (23). For this purpose we first note that

$$c_n := \int\int |k_{\nu_{nx}}(a)|^2 da|m|(dx) \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

This can be shown by mimicking the proof of (25). By (24), it follows that

$$5 \int |(\widetilde{G^{\nu_n} h})(x)|^2 |m|(dx) \leq 2c_n \parallel h \parallel_{H^2}^2, \quad h \in H^2(\mathbb{R}^d).$$

Thus, in order to prove (23), we have only to show that that there exists a finite constant C such that

$$\parallel G^m(I+G^m)^{-1}h \parallel_{L^2(dx)} \leq C\left(\int |\tilde{h}|^2 d|m|\right)^{1/2}, \quad h \in H^2(\mathbb{R}^d). \tag{27}$$

By (20), we have

$$G^m(I+G^m)^{-1} = -\sum_{j=1}^{\infty}(-G^m)^j. \tag{28}$$

According to (21),

$$\parallel (G^m)^{j+1}h \parallel_{L^2(dx)} \leq c\left(\int |\widetilde{(G^m)^j h}|^2 d|m|\right)^{1/2} \leq c \cdot c^j \left(\int |\tilde{h}|^2 d|m|\right)^{1/2},$$

for every $j \in \mathbb{N}$ and hence

$$\| \sum_{j=1}^{\infty}(-G^m)^j h \|_{L^2(dx)} \leq \sum_{j=1}^{\infty} c^j \cdot (\int |\tilde{h}|^2 d|m|)^{1/2} = \frac{c}{1-c} \cdot (\int |\tilde{h}|^2 d|m|)^{1/2}.$$

By (28), this implies (27) and the proof of the theorem is complete. □

REMARK 2 We have shown that

$$\begin{array}{rl} & \| (-\Delta + \varepsilon\Delta^2 + \mu_n + \alpha)^{-1} - (-\Delta + \varepsilon\Delta^2 + m + \alpha)^{-1} \|^2 \\ \leq & c_1 \int \int | \int g_{0,1}(y-a) g_{\varepsilon,\alpha}(x-y)(m-\mu_n)(dy)|^2 da\, dx \\ & +c_2 \int \int | \int g_{0,1}(y-a) g_{\varepsilon,\alpha}(x-y)(m-\mu_n)(dy)|^2 da\, |m|(dx) \end{array}$$

for some finite constants $c_j = c_j(\varepsilon, \alpha)$, $j = 1, 2$, which can be computed with the aid of the proof of the theorem. Thus the proof provides explicit upper bounds for the error one makes when one replaces the operator $-\Delta + \varepsilon\Delta^2 + m$ by $-\Delta + \varepsilon\Delta^2 + \mu_n$.

III Dependence on the coupling constant

In this section we are going to prove that

$$-\Delta + \varepsilon^2\Delta^2 + m \longrightarrow -\Delta + m, \text{ as } \varepsilon \downarrow 0, \tag{29}$$

in the norm resolvent sense. Here m denotes a real-valued Radon measure on $\mathbb{R}^d$ and we assume, in addition, that for every $\eta > 0$ there exists a $\beta_\eta < \infty$ such that

$$\int |f|^2 d|m| \leq \eta(\int |\nabla f|^2 dx + \beta_\eta \int |f|^2 dx), \quad f \in C_0^\infty(\mathbb{R}^d). \tag{30}$$

Note that we neither require that m is finite nor that $d \leq 3$. On the other hand, the condition (30) implies that $m(B) = 0$ for every Borel set B with classical capacity zero and, for instance, it is excluded that m is a point measure if $d > 1$.

The inequality (30) holds, in particular, provided m belongs to the Kato class, i.e.

$$\begin{array}{rcll} \sup_{n\in\mathbb{Z}} |m|([n, n+1]) & < & \infty, & d = 1, \\ \lim_{\varepsilon\to 0} \sup_{x\in\mathbb{R}^2} \int_{B(x,\varepsilon)} |\log(|x-y|)|\, |m|(dy) & = & 0, & d = 2, \\ \lim_{\varepsilon\to 0} \sup_{x\in\mathbb{R}^3} \int_{B(x,\varepsilon)} \frac{1}{|x-y|}\, |m|(dy) & = & 0, & d = 3, \end{array}$$

with $B(x,\varepsilon)$ denoting the ball of radius ε centered at x (cf. [9], Theorem 3.1). We refer to [6], ch. 1.2, for additional examples of measures satisfying (30).

In general, the elements f in the form domain of $-\Delta$ do not possess a continuous representative $\tilde{f}$. Therefore we shall give a definition of $\mathcal{E}_\varepsilon^m$ different from the one in the previous section 2 but equivalent to the definition in section II in the special cases treated there.

Since the space $C_0^\infty(\mathbb{R}^d)$ of smooth functions with compact support is dense in the Sobolev space $H^1(\mathbb{R}^d)$, there exists a unique bounded linear mapping $J_m : H^1(\mathbb{R}^d) \longrightarrow L^2(|m|)$ satisfying

$$J_m f = f, \quad f \in C_0^\infty(\mathbb{R}^d),$$

(strictly speaking J_m maps the dx-equivalence class of the continuous function $\tilde{f} \in C_0^\infty(\mathbb{R}^d)$ to the $|m|$-equivalence class of $\tilde{f}$).

We put

$$\begin{aligned} D(\mathcal{E}_\varepsilon^m) &:= D(\mathcal{E}_\varepsilon), \\ \mathcal{E}_\varepsilon^m(f,f) &:= \mathcal{E}_\varepsilon(f,f) + (A_m J_m f, J_m f)_{L^2(|m|)}, \quad f \in D(\mathcal{E}_\varepsilon^m), \end{aligned}$$

where $D(\mathcal{E}_\varepsilon) = H^1(\mathbb{R}^d)$ for $\varepsilon = 0$, $D(\mathcal{E}_\varepsilon) = H^2(\mathbb{R}^d)$ otherwise and

$$A_m h(x) := \begin{cases} h(x), & x \in B, \\ -h(x), & x \in \mathbb{R}^d \setminus B, \end{cases} \qquad h \in L^2(|m|),$$

with B being any Borel set such that $m^+(\mathbb{R}^d \setminus B) = 0 = m^-(B)$.

By (30) and the KLMN-theorem, the quadratic form $\mathcal{E}_\varepsilon^m$ in $L^2(dx)$ is lower semi-bounded and closed and

$$\mathcal{E}_{\varepsilon,\beta_1}^m(f,f) \geq 0, \quad f \in D(\mathcal{E}_\varepsilon^m).$$

We denote by $-\Delta + \varepsilon^2\Delta^2 + m$ the lower semibounded self-adjoint operator associated to $\mathcal{E}_\varepsilon^m$ and put

$$G_{\varepsilon,\alpha} := (-\Delta + \varepsilon^2\Delta^2 + \alpha)^{-1} \text{ and } G_{\varepsilon,\alpha}^m := (-\Delta + \varepsilon^2\Delta^2 + m + \alpha)^{-1}$$

provided the inverse operators exist.

One key for the proof of the convergence result (29) is the observation that

$$\frac{1}{\varepsilon^2 p^4 + p^2 + \alpha} = \frac{c(\varepsilon)}{p^2 + \alpha(\varepsilon)} - \frac{c(\varepsilon)}{p^2 + \beta(\varepsilon)}, \text{ as } \varepsilon \downarrow 0,$$

with

$$c(\varepsilon) := \frac{1}{\sqrt{1 - 4\varepsilon^2\alpha}} \longrightarrow 1, \text{ as } \varepsilon \downarrow 0, \tag{31}$$

and $-\alpha(\varepsilon)$ and $-\beta(\varepsilon)$ being the roots of the polynomial $\varepsilon^2 x^2 + x + \alpha$, i.e.

$$\alpha(\varepsilon) := \frac{1 - \sqrt{1 - 4\varepsilon^2\alpha}}{2\varepsilon^2} = \frac{2\alpha}{1 + \sqrt{1 - 4\varepsilon^2\alpha}} \longrightarrow \alpha, \text{ as } \varepsilon \downarrow 0, \tag{32}$$

$$\beta(\varepsilon) := \frac{1 + \sqrt{1 - 4\varepsilon^2\alpha}}{2\varepsilon^2} \longrightarrow \infty, \text{ as } \varepsilon \downarrow 0. \tag{33}$$

Using the parameters introduced above, we have

$$g_{\varepsilon,\alpha}(x) = c(\varepsilon) g_{0,\alpha(\varepsilon)}(x) - c(\varepsilon) g_{0,\beta(\varepsilon)}(x), \quad x \in \mathbb{R}^d \setminus \{0\},$$

and hence

$$G_{\varepsilon,\alpha} = c(\varepsilon) G_{0,\alpha(\varepsilon)} - c(\varepsilon) G_{0,\beta(\varepsilon)}. \tag{34}$$

The other key for the proof of the convergence result (29) is a Krein-like resolvent formula from [2], cf. (37) below. First we need some preparation.

Let $\alpha > 0$ and $\varepsilon \geq 0$. We introduce the operator $J_{m,\varepsilon,\alpha}$ from the Hilbert space $(D(\mathcal{E}_\varepsilon), \mathcal{E}_{\varepsilon,\alpha})$ to $L^2(|m|)$ as follows:

$$\begin{aligned} D(J_{m,\varepsilon,\alpha}) &:= D(\mathcal{E}_\varepsilon), \\ J_{m,\varepsilon,\alpha} f &:= J_m f, \quad f \in D(J_{m,\varepsilon,\alpha}). \end{aligned} \tag{35}$$

By (30), the operator norm of $J_{m,\varepsilon,\alpha}$ is less than or equal to η provided $\alpha \geq \beta_\eta$. Thus we can choose $\alpha_0 > 0$ and $c < 1$ such that

$$\| J_{m,\varepsilon,\alpha} \|_{(D(\mathcal{E}_\varepsilon),\mathcal{E}_{\varepsilon,\alpha}),L^2(|m|)} \leq \sqrt{c}, \quad \varepsilon \geq 0, \quad \alpha \geq \alpha_0. \tag{36}$$

By (35) and (36), the hypothesis of Theorem 3 in [2] is satisfied (with $\mathcal{H} = L^2(dx)$, $\mathcal{H}_{aux} = L^2(|m|)$, $\mathcal{E} = \mathcal{E}_\varepsilon$, $J = J_m$, $U_\alpha = J^*_{m,\varepsilon,\alpha}$, $A = A_m$, $H = -\Delta + \varepsilon^2\Delta^2$ and $H^A = -\Delta + \varepsilon^2\Delta^2 + m$) and this theorem implies that $-\alpha$ belongs to the resolvent set of $-\Delta + \varepsilon^2\Delta^2 + m$ and

$$G^m_{\varepsilon,\alpha} = G_{\varepsilon,\alpha} - (J_{m,\varepsilon,\alpha})^* A_m (1 + J_m J^*_{m,\varepsilon,\alpha} A_m)^{-1} J_m G_{\varepsilon,\alpha}, \quad \varepsilon \geq 0, \quad \alpha \geq \alpha_0. \tag{37}$$

In fact, we can write

$$J^*_{m,\varepsilon,\alpha'} = (J_m G_{\varepsilon,\alpha'})^*, \quad \varepsilon \geq 0, \quad \alpha' > 0, \tag{38}$$

since we have

$$(J^*_{m,\varepsilon,\alpha'} f, h) = \mathcal{E}_{\varepsilon,\alpha'}(J^*_{m,\varepsilon,\alpha'} f, G_{\varepsilon,\alpha'} h) = (f, J_{m,\varepsilon,\alpha'} G_{\varepsilon,\alpha'} h)_{L^2(|m|)} = ((J_m G_{\varepsilon,\alpha'})^* f, h)$$

for every $h \in L^2(dx)$, $\varepsilon \geq 0$ and $\alpha' > 0$.

We choose any $\alpha > \alpha_0$, then from (35) and (36) we get

$$\| (1 + J_m J^*_{m,\varepsilon,\alpha} A_m)^{-1} \|_{L^2(|m|)} \leq \frac{1}{1-c}, \quad \varepsilon \geq 0.$$

By the second resolvent identity

$$(1+A)^{-1} - (1+B)^{-1} = (1+A)^{-1}(B-A)(1+B)^{-1},$$

it implies that

$$\| (1 + J_m J^*_{m,\varepsilon,\alpha} A_m)^{-1} - (1 + J_m J^*_{m,0,\alpha} A_m)^{-1} \|_{L^2(|m|)} \longrightarrow 0, \text{ as } \varepsilon \downarrow 0,$$

provided

$$\| J_m J^*_{m,\varepsilon,\alpha} - J_m J^*_{m,0,\alpha} \|_{L^2(|m|)} \longrightarrow 0, \text{ as } \varepsilon \downarrow 0. \tag{39}$$

Employing the resolvent formula (37), this implies that the convergence result (29) is true, provided (39) holds and, in addition,

$$\| G_{\varepsilon,\alpha} - G_{0,\alpha} \|_{L^2(dx)} \longrightarrow 0, \text{ as } \varepsilon \downarrow 0, \tag{40}$$

$$\| J_m G_{\varepsilon,\alpha} - J_m G_{0,\alpha} \|_{L^2(dx), L^2(|m|)} \longrightarrow 0, \text{ as } \varepsilon \downarrow 0. \tag{41}$$

We have

$$\| G_{0,\alpha'} \|^2_{L^2(dx),H^1} \leq k(\alpha'), \quad \alpha' > 0, \tag{42}$$

for some continuous function k vanishing at infinity (actually, $k(x) = 1/x^2$ for $x \leq 2$ and $k(x) = 1/(4(x-1))$ for $x > 2$). By the hypothesis (30), it follows that

$$\| J_m G_{0,\alpha'} \|^2_{L^2(dx),L^2(|m|)} \leq \max(1, \beta_1) k(\alpha'), \quad \alpha' > 0. \tag{43}$$

By the first resolvent formula,

$$G_{0,\alpha(\varepsilon)} - G_{0,\alpha} = (\alpha - \alpha(\varepsilon)) G_{0,\alpha} G_{0,\alpha(\varepsilon)}. \tag{44}$$

Since $\alpha(\varepsilon) \longrightarrow \alpha$, $\beta(\varepsilon) \longrightarrow \infty$, and $c(\varepsilon) \longrightarrow 1$ as $\varepsilon \downarrow 0$, (cf. (31), (32), (33), respectively), the formulae (34), (42) and (44) imply (40). Moreover (34), (43) and (44) imply (41).

From (34) and (38) follows that

$$J_m J^*_{m,\varepsilon,\alpha} = c(\varepsilon) J_m (J_m G_{0,\alpha(\varepsilon)})^* - c(\varepsilon) J_m (J_m G_{0,\beta(\varepsilon)})^*,$$

note that $c(\varepsilon)$ is real for sufficiently small ε. Using this expression and (44), (38), we get

$$\begin{aligned}
& \| J_m J^*_{m,\varepsilon,\alpha} - J_m J^*_{m,0,\alpha} \|_{L^2(|m|)} \\
\le\ & \| (c(\varepsilon)-1) J_m (J_m G_{0,\alpha(\varepsilon)})^* \|_{L^2(|m|)} + \| J_m (J_m G_{0,\alpha(\varepsilon)})^* - J_m (J_m G_{0,\alpha})^* \|_{L^2(|m|)} \\
& + \| c(\varepsilon) J_m (J_m G_{0,\beta(\varepsilon)})^* \|_{L^2(|m|)} \\
=\ & \| (c(\varepsilon)-1) J_{m,0,\alpha(\varepsilon)} J^*_{m,0,\alpha(\varepsilon)} \|_{L^2(|m|)} + \| (\alpha-\alpha(\varepsilon)) J_m G_{0,\alpha} (J_m G_{0,\alpha(\varepsilon)})^* \|_{L^2(|m|)} \\
& + \| c(\varepsilon) J_{m,0,\beta(\varepsilon)} J^*_{m,0,\beta(\varepsilon)} \|_{L^2(|m|)}, \quad \varepsilon > 0. \qquad (45)
\end{aligned}$$

By (30), the mapping $\alpha \mapsto \| J_{m,0,\alpha} J^*_{m,0,\alpha} \|_{L^2(|m|)}$ is locally bounded on $(0,\infty)$ and tends to zero as α tends to infinity. Since $\alpha(\varepsilon) \longrightarrow \alpha$, $c(\varepsilon) \longrightarrow 1$, and $\beta(\varepsilon) \longrightarrow \infty$ as $\varepsilon \downarrow 0$, this implies, in conjunction with (43) and (45), that

$$\| J_m J^*_{m,\varepsilon,\alpha} - J_m J^*_{m,0,\alpha} \|_{L^2(|m|)} \longrightarrow 0, \quad \varepsilon \downarrow 0.$$

This completes the proof of the following theorem.

THEOREM 3 *Let m be a real-valued Radon measure on $\mathbb{R}^d$ satisfying (30). Then the operators $-\Delta + \varepsilon^2\Delta^2 + m$ converge to $-\Delta + m$ in the norm resolvent sense as $\varepsilon \downarrow 0$.*

REMARK 4 By the proof of the theorem, $\| G^m_{\varepsilon,\alpha} - G^m_{0,\alpha} \|$ is upper bounded by an expression of the form $c \cdot (\varepsilon^2 + \eta(m,\varepsilon))$ where the finite constant c can be extracted from the proof and $\eta(m,\varepsilon)$ has to be chosen (and can be chosen) such that (30) holds with η and β replaced by $\eta(m,\varepsilon)$ and $\beta(\varepsilon)$ (cf. (33)), respectively.

IV Eigenvalues and eigenspaces of the approximating operators

Throughout this section let $d \le 3$. First let μ be any finite real-valued Radon measure on $\mathbb{R}^d$. By Sobolev's inequality and [3], Lemma 19, the mapping $f \mapsto \tilde{f}$ from $H^2(\mathbb{R}^d)$ to $L^2(|\mu|)$ is compact. As above let

$$A_\mu h(x) := \begin{cases} h(x), & x \in B, \\ -h(x), & x \in \mathbb{R}^d \setminus B, \end{cases} \qquad h \in L^2(|\mu|),$$

where B is any Borel set such that $\mu^+(\mathbb{R}^d \setminus B) = 0 = \mu^-(B)$. Then we can write

$$G^\mu_{\varepsilon,\alpha} f = \int g_{\varepsilon,\alpha}(\cdot - y) A_\mu \tilde{f}(y) |\mu|(dy) \quad \text{dx- } a.e., \quad f \in H^2(\mathbb{R}^d).$$

By the compactness of the mapping $f \mapsto \tilde{f}$ and (11), $G^{\mu}_{\varepsilon,\alpha}$ is compact if regarded as an operator from $H^2(\mathbb{R}^d)$ to $L^2(dx)$. According to the resolvent formula (16), this implies that the resolvent difference $G^{\mu}_{\varepsilon,\alpha} - G_{\varepsilon,\alpha}$ is compact and hence

$$\sigma_{ess}(-\Delta + \varepsilon^2\Delta^2 + \mu) = \sigma_{ess}(-\Delta + \varepsilon^2\Delta^2) = [0, \infty).$$

Let m be a finite real-valued Radon measure satisfying (30) (e.g., let m be from the Kato class). By the preceding considerations, we can approximate the negative eigenvalues and corresponding eigenspaces of the operator $-\Delta + m$ in $L^2(\mathbb{R}^d, dx)$ by the corresponding eigenvalues and eigenfunctions of operators of the form $-\Delta+\varepsilon^2\Delta^2+\mu$, where $\varepsilon > 0$ and μ is a point measure with mass at only finitely many points. In this section we shall show how to compute the eigenvalues and corresponding eigenspaces of these approximating operators.

We fix $\varepsilon > 0$. In the remaining part of this paper let $\mu = \sum_{j=1}^{N} c_j \delta_{x_j}$, where $N \in \mathbb{N}$, $x_1, \ldots, x_N$ are N distinct points in $\mathbb{R}^d$ and $c_1, \ldots, c_N$ are real numbers different from zero.

Every $f \in D(\mathcal{E}_\varepsilon) = H^2(\mathbb{R}^d)$ has a unique continuous representative $\tilde{f}$ and we define the mapping $J_\mu : D(\mathcal{E}_\varepsilon) \longrightarrow L^2(|\mu|)$ by

$$J_\mu f := \tilde{f} \quad |\mu|\text{-a.e.}, \quad f \in H^2(\mathbb{R}^d).$$

By (8), $\int g_{\varepsilon,\alpha}(\cdot - y) f(y) dy$ is the unique continuous representative of $G_{\varepsilon,\alpha} f$. Hence $J_\mu G_{\varepsilon,\alpha}$ is the integral operator from $L^2(dx)$ to $L^2(|\mu|)$ with kernel $g_{\varepsilon,\alpha}(x - y)$. Thus $(J_\mu G_{\varepsilon,\alpha})^*$ is the integral operator from $L^2(|\mu|)$ to $L^2(dx)$ with the same kernel and we get

$$(J_\mu G_{\varepsilon,\alpha})^* A_\mu h = \sum_{k=1}^{N} c_k g_{\varepsilon,\alpha}(\cdot - x_k) h(x_k)$$

and therefore

$$J_\mu (J_\mu G_{\varepsilon,\alpha})^* A_\mu h(x_j) = \sum_{k=1}^{N} c_k g_{\varepsilon,\alpha}(x_j - x_k) h(x_k), \quad 1 \le j \le N, \tag{46}$$

for every $h \in L^2(|\mu|)$.

By (4), the operator norm of J_μ, regarded as an operator from $(D(\mathcal{E}_\varepsilon), \mathcal{E}_{\varepsilon,\alpha})$ to $L^2(|\mu|)$, tends to zero, as α tends to infinity. Moreover it can be shown as in the proof of (38) that the adjoint of J_μ, with J_μ regarded as an operator from $(D(\mathcal{E}_\varepsilon), \mathcal{E}_{\varepsilon,\alpha})$ to $L^2(|\mu|)$, maps every $h \in L^2(|\mu|)$ to $(J_\mu G_{\varepsilon,\alpha})^* h$. Thus the hypothesis of Theorem 3 in [2] is satisfied (with $\mathcal{H} = L^2(dx)$, $\mathcal{H}_{aux} = L^2(|\mu|)$, $\mathcal{E} = \mathcal{E}_\varepsilon$, $J = J_\mu$, $A = A_\mu$, $H = -\Delta+\varepsilon^2\Delta^2$ and $H^A = -\Delta + \varepsilon^2\Delta^2 + \mu$ (we recall that $\varepsilon > 0$)) and, by this theorem, $-\alpha$ belongs to the resolvent set of $-\Delta + \varepsilon^2\Delta^2 + \mu$ and

$$G^{\mu}_{\varepsilon,\alpha} = G_{\varepsilon,\alpha} - (J_\mu G_{\varepsilon,\alpha})^* A_\mu (1 + J_\mu (J_\mu G_{\varepsilon,\alpha})^* A_\mu)^{-1} J_\mu G_{\varepsilon,\alpha}, \tag{47}$$

provided $\alpha > 0$ and $1 + J_\mu (J_\mu G_{\varepsilon,\alpha})^* A_\mu$ is a bijective mapping on $L^2(|\mu|)$.

Since $L^2(|\mu|)$ is finite dimensional a linear operator in this space is bijective if it is injective. By (46), the operator $1 + J_\mu (J_\mu G_{\varepsilon,\alpha})^* A_\mu$ in $L^2(|\mu|)$ is injective if, and only if

$$\det(\delta_{jk} + c_k g_{\varepsilon,\alpha}(x_j - x_k))_{1\le j,k\le N} \neq 0,$$

with $\delta_{j,k}$ being the Kronecker delta.

As the mapping $\alpha \longrightarrow g_{\varepsilon,\alpha}(x)$ is real analytic on $(0,\infty)$ for every $x \in \mathbb{R}^d$, the function

$$\alpha \mapsto \det(\delta_{jk} + c_k g_{\varepsilon,\alpha}(x_j - x_k))_{1\le j,k\le N}$$

is also real analytic on $(0,\infty)$. By (7), it is different from zero for all sufficiently large α. Thus the set of zeros on $(0,\infty)$ of this function is discrete.

Since $J_\mu G_{\varepsilon,\alpha}$ is surjective and $(J_\mu G_{\varepsilon,\alpha})^* A_\mu$ injective, the resolvent formula (47) implies that

$$\| G^\mu_{\varepsilon,\alpha} \| \longrightarrow \infty, \text{ as } \alpha \longrightarrow \alpha_0,$$

for every $\alpha_0 > 0$ satisfying $\det(\delta_{jk} + c_k g_{\varepsilon,\alpha}(x_j - x_k))_{1\le j,k\le N} = 0$. Thus we have proved that the real number $-\alpha$ is an eigenvalue of $-\Delta + \varepsilon^2\Delta^2 + \mu$ if and only if

$$\det(\delta_{jk} + c_k g_{\varepsilon,\alpha}(x_j - x_k))_{1\le j,k\le N} = 0.$$

Since $\det(\delta_{jk} + c_k g_{\varepsilon,\alpha}(x_j - x_k))_{1\le j,k\le N} = \Pi_{k=1}^N c_k \cdot \det(\delta_{jk}/c_k + g_{\varepsilon,\alpha}(x_j - x_k))_{1\le j,k\le N}$, this implies the assertion a) in the following theorem.

THEOREM 5 *Let $d \le 3$ and $\varepsilon > 0$. Let $\mu = \sum_{j=1}^N c_j \delta_{x_j}$, where $N \in \mathbb{N}$, $x_1, \dots, x_N$ are N distinct points in $\mathbb{R}^d$ and $c_1, \dots, c_N$ are real numbers different from zero. Then the following holds:*

a) The real number $-\alpha < 0$ is an eigenvalue of $-\Delta + \varepsilon^2\Delta^2 + \mu$ if and only if

$$\det(\frac{\delta_{jk}}{c_k} + g_{\varepsilon,\alpha}(x_j - x_k))_{1\le j,k\le N} = 0.$$

b) For every eigenvalue $-\alpha < 0$ the corresponding eigenvalues have the following form

$$\sum_{k=1}^N c_k h_k g_{\varepsilon,\alpha}(\cdot - x_k), \qquad (h_k)^T_{1\le k\le N} \in \ker(\delta_{jk} + c_k g_{\varepsilon,\alpha}(x_j - x_k))_{1\le j,k\le N}$$

Proof: It only remains to prove the assertion b). By the preceding considerations and Lemma 1 in [2],

$$h \mapsto (J_\mu G_{\varepsilon,\alpha})^* A_\mu h$$

is a linear bijective mapping from $\ker(1 + J_\mu (J_\mu G_{\varepsilon,\alpha})^* A_\mu)$ onto $\ker(-\Delta + \varepsilon^2\Delta^2 + \mu + \alpha)$ and b) follows from the preceding considerations. □

REMARK 6 Since the Hilbert space $L^2(|\mu|)$ is N-dimensional with $N < \infty$ and by the resolvent formula (47), the difference $G^{\mu}_{\varepsilon,\alpha} - G_{\varepsilon,\alpha}$ is a finite rank operator with rank less than or equal to N. Thus the number, counting multiplicity, of negative eigenvalues of $-\Delta + \varepsilon^2\Delta^2 + \mu$ is less than or equal to N.

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