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NOETHERIAN RESIDUE CURRENTS

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Abstract. Given an ideal sheaf (or a finitely generated subsheaf of a free analytic sheaf) we construct a vectorvalued residue current whose annihilator is precisely the given sheaf. Using explicit integral formulas in $\mathbb{C}^n$ we obtain a residue version of the Ehrenpreis-Palamodov fundamental principle. Also other results, previously known for a complete intersection, such as characterization of ideals of smooth functions extend to general ideals.

1. Introduction

Let $f = f_1, \ldots, f_m$ be a tuple of holomorphic functions in some domain X in $\mathbb{C}^n$ and assume that their common zero set Z has codimension m, i.e., f define a complete intersection. The duality theorem, due to Dickenstein-Sessa and Passare, [15] and [25], asserts that the annihilator of the Coleff-Herrera current

$$R_{ch} = \bar\partial \frac{1}{f_1} \wedge \ldots \wedge \bar\partial \frac{1}{f_m} \tag{1.1}$$

is equal to the ideal sheaf generated by f, i.e., a holomorphic function ϕ is locally in the ideal $(f_1, \ldots, f_m)$ if and only of $\phi R_{ch} = 0$. This fact, combined by integral formulas for division and interpolation from [10], made it possible, [11], to obtain an explicit proof of the fundamental principle in the case where the symbols of the differential operators define a complete intersection in $\mathbb{C}^n$.

Inspired by [26], the first author introduced in [1] a vector-valued residue current R for an arbitrary tuple f, based on the Koszul complex, with the property that the annihilator $\text{ann}R$ of R is contained in the ideal sheaf; this, e.g., led to a simple proof of the Briançon-Skoda theorem. Moreover, the construction is global, and if f is a section of a Hermitian vector bundle E^* over a complex manifold X, then R is a global current on X taking values in ΛE. In case when f defines a complete intersection, this current coincides with the Coleff-Herrera current and thus $\text{ann}R = (f)$. However, the inclusion $\text{ann}R \subset (f)$ may be strict; recently the second author has proved, [28], that in case

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of monomial ideals of dimension zero, the inequality is always strict unless (f) is defined by a complete intersection.

Using the Eagon-Northcott complex the construction in [1] was extended, [4] and [5], to a generically surjective morphism $f\colon E \to Q$, as well as to the corresponding determinant ideal, i.e., the ideal generated by $\det f$. Again, in the generic case, i.e., when $\operatorname{codim} Z = m - r + 1$, the annihilator of the resulting residue currents are equal to the module sheaf $\operatorname{Im} f$ and ideal sheaf $(\det f)$, respectively, if Z is the set where f is not surjective, and m, r are the ranks of E and Q, respectively.

In this paper we consider general generically exact complexes

$$0 \to E_N \xrightarrow{f_N} \dots \xrightarrow{f_3} E_2 \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0 \to 0 \tag{1.2}$$

of Hermitian holomorphic vector bundles over a complex manifold X. We introduce currents R^ℓ with support on the variety Z where (1.2) is not exact, taking values in $\operatorname{Hom}(E_\ell, E_\bullet)$, and with the property that if ϕ is a holomorphic section of E_ℓ such that $f_\ell \phi = 0$ and morever $R^\ell \phi = 0$, then locally $\phi = f_{\ell+1}\psi$ for some holomorhic ψ. To each such complex there is a corresponding complex of sheaves of locally free $\mathcal{O}$-modules,

$$0 \to \mathcal{O}(E_N) \to \cdots \to \mathcal{O}(E_1) \to \mathcal{O}(E_0) \to 0. \tag{1.3}$$

that is exact outside Z. Conversely, given such a complex of sheaves of locally free $\mathcal{O}$-modules that is exaxt outside some analytic set, there is a generically exact complex of vector bundles which we can equip with a Hermitian structure. It turns out that (1.3) is exact at $\mathcal{O}(E_\ell)$ for all $\ell \geq 1$, if and only if $R^\ell = 0$ for all $\ell \geq 1$. Moreover, we have that if $R^{\ell+1} = 0$, then $R^\ell \phi = 0$ and $f_\ell \phi = 0$ if and only if $\phi = f_{\ell+1}\psi$ is solvable locally. In particular, if (1.3) is exact, then $\operatorname{ann} R^0 = J$, where J is the sheaf $\operatorname{Im} f_1$. Since any finitely generated subsheaf J of $\mathcal{O}^r$ (locally) admits a resolution, this is Hilbert's syzygy theorem, we thus obtain a current R such that $\operatorname{ann} R = J$; we will call R a Noetherian residue current for J.

In the case when J is a Cohen-Macaulay sheaf, by choosing a minimal resolution, one gets a current R which is independent of the choice of Hermitian metric, and in fact essentially independent of the choice of minimal resolution as well; this generalizes the fact that in the complete intersection case, the resulting current is just the Coleff-Herrera current (times a non-vanishing holomorphic function).

Let $F(z)$ be an $r \times m$-matrix of polynomials in $\mathbb{C}^n$ of generic rank r. The fundamental principle of Ehrenpreis and Palamodov, [17] and [24], states that every homogeneous solution to the system of equations $F^*(i\partial/\partial t)\xi(t) = 0$ on a convex compact set in $\mathbb{R}^n$ is a superposition of exponential solutions to this equation, with frequencies in the algebraic set $Z = \{z;\ \operatorname{rank} F(z) < r\}$. After a primary decomposition $J = \cap J_k$ of the module $J = \operatorname{Im}(\mathcal{O}^m \to \mathcal{O}^r)$, a principal step is to prove the existence of Noetherian operators. This is a finite set of holomorphic differential operators $\mathcal{L}_{jk}$ such that $\phi \in J_k$ if and only if $\mathcal{L}_{jk}\phi = 0$ on Z_k for all j, where Z_k is the irreducible algebraic variety associated to the primary module J_k. The next principal step is to solve a certain interpolation problem with precise bounds. For an accessible account of these matters, see [22].

The Noetherian currents defines in this paper fit perfectly into the framework of integral formulas developped in [6], and we obtain a current version of the fundamental principle for a matrix of constant coefficient differential operators, generalizing [11]. Indeed, if $\xi(t)$ is a smooth homogeneous solution of $F^*(i\partial/\partial t)\xi = 0$ on K we have a representation

$$\xi(t) = \int_{\mathbb{C}^n} F^*(\zeta)R^*(\zeta)A(\zeta)e^{-i\langle t,\zeta\rangle}$$

for an appropriate (explicitly given matrix of functions) A. Conversely, any ξ given in this way is a homogenous solution; in fact,

$$F^*(i\partial/\partial t)\xi(t) = \int_{\mathbb{C}^n} F^*(\zeta)R^*(\zeta)A(\zeta)e^{-i\langle t,\zeta\rangle} = 0$$

since $RF = 0$. The principal ingredients in this proof of the fundamental principle is the existence of a graded resolution in $\mathbb{C}^{n+1}$ of the homogenized module induced by F, Hironaka's theorem and toric resolutions of singularities, which are needed to define the residue currents.

We also obtain a residue characterization of the sheaf $\mathcal{E}J$: If R is a Noetherian residue current for J, then a smooth tuple of functions belongs to $\mathcal{E}J$ if and only of $R(\bar\partial^\alpha\phi) = 0$ for all multi-indices α, generalizing the result in [2] for a complete intersection.

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2. Some preliminaries

Assume that E and Q are holomorphic Hermitian vector bundles over an n-dimensional complex manifold X, and let $f\colon E \to Q$ be a holomorphic vector bundle morphism. If we consider f as a section of $E^* \otimes Q \simeq E^*\wedge Q$, then for any positive integer q, f^q is a well-defined section of $\Lambda^q E^* \otimes \Lambda^q Q \simeq \Lambda^q E^*\wedge\Lambda^q Q$, and it is easily seen that f^q is nonvanishing at a point z if and only if $\operatorname{rank} f(z) = \dim \operatorname{Im} f(z) \geq q$. In fact, if e_j is a local frame for E, with dual frame e_j^* for E^*, and ϵ_k is a frame for Q, then $f = \sum_{jk} f_{jk}e_j^*\wedge\epsilon_k$, and

$$f^q = q! \sum_{|I|=q}{}' \sum_{|K|=q}{}' f_{I,K}e_I^*\wedge\epsilon_K,$$

where $f_{I,K}$ is $\pm$ the determinant of the q-minor of the matrix $(f_{jk}(z))_{jk}$ determined by the multiindices I and K. Hence $f^q(z)$ is nonvanishing at z if and only if there is some invertible $q \times q$-minor of the matrix

$(f_{jk}(z))_{jk}$, and this in turn holds if and only if the rank if the mapping $f(z)$ is at least q.

Assume now that $\operatorname{rank} f(z) = \dim \operatorname{Im} f(z) \le q$ for all $z \in X$. If $F = f^q/q!$ it follows that $Z = \{z;\ \operatorname{rank} f(z) < q\}$ is equal to the analytic variety $\{F = 0\}$. For any section ξ of a Hermitian bundle we let ξ^* be its dual section, i.e., the section of the dual bundle with minimal norm such that $\xi^*\xi = |\xi|^2$. Let S be the section of $\Lambda^q E \otimes \Lambda^q Q^*$ that is dual to F and let f^* be the section of $E \otimes Q^*$ that is dual to f. Notice that f induces a natural mapping

$$\delta_f \colon \Lambda^{\ell+1} E \otimes \Lambda^{\ell+1} Q^* \to \Lambda^{\ell} E \otimes \Lambda^{\ell} Q^*$$

and let $(\delta_f)_\ell = \delta_f^\ell/\ell!$. Moreover, in $X \setminus Z$, let $\sigma \colon Q \to E$ be the minimal inverse of f, i.e., such that $f\sigma = \Pi_{\operatorname{Im} f}$ and $\Pi_{\operatorname{Ker} f}\sigma = 0$.

Lemma 2.1. *In $X \setminus Z$ we have that*

$$S = (f^*)^q/q! \tag{2.1}$$

and

$$\sigma = (\delta_f)_{q-1} S/|F|^2. \tag{2.2}$$

Proof. Since the statements are pointwise we may assume that $f \colon E \to Q$ is just a linear mapping between finite-dimensional Hermitian vector spaces. Let ϵ_k be an ON-basis for Q such that $\operatorname{Im} f$ is spanned by $\epsilon_1, \ldots, \epsilon_q$. Then $f = \sum_1^q f_k \otimes \epsilon_k$ with $f_k \in E^*$, and it is easy to see that

$$f^* = \sum_1^q f_k^* \otimes \epsilon_k^*,$$

where ϵ^* is the dual basis. Now $F = f^q/q! = f_1 \wedge \ldots \wedge f_q \otimes \epsilon_1 \wedge \ldots \wedge \epsilon_q$, and since $f_1^* \wedge \ldots \wedge f_q^*$ is the dual of $f_1 \wedge \ldots \wedge f_q$ it follows that $(f^*)^q/q! = f_1^* \wedge \ldots \wedge f_q^* \otimes \epsilon_1^* \wedge \ldots \wedge \epsilon_q^*$ is the dual of F, and thus (2.1) is shown. In particular,

$$|F|^2 = \delta_{f_1} \cdots \delta_{f_q}(f_1^* \wedge \ldots f_q^*), \tag{2.3}$$

where δ_{f_j} is interior multiplication with f_j. To see (2.2), notice that

$$(\delta_f)_{q-1} S = \sum_{j=1}^q (-1)^j \delta_{f_1} \cdots \delta_{f_{j-1}} \delta_{f_{j+1}} \cdots \delta_{f_q}(f_1^* \wedge \ldots \wedge f_q^*) \otimes \epsilon_j^*.$$

If we consider $\alpha = (\delta_f)_{q-1} S$ as an element in $\operatorname{Hom}(Q, E) \simeq E \otimes Q^*$, and compose with f we get, cf., (2.3),

$$f\alpha = \sum_1^q \delta_{f_1} \cdots \delta_{f_q}(f_1^* \wedge \ldots f_q^*) \epsilon_j \otimes \epsilon_j^* = |F|^2 \Pi_{\operatorname{Im} f}.$$

Thus $f\sigma = \Pi_{\operatorname{Im} f}$. Moreover, if $v \in \operatorname{Ker} f$, then

$$\langle \delta_{f_1} \cdots \delta_{f_{j-1}} \delta_{f_{j+1}} \cdots \delta_{f_q}(f_1^* \wedge \ldots \wedge f_q^*), v \rangle_E = \\ \delta_v \delta_{f_1^*} \cdots \delta_{f_{j-1}^*} \delta_{f_{j+1}^*} \cdots \delta_{f_q^*}(f_1 \wedge \ldots \wedge f_q) = 0$$

since $\delta_v f_j = 0$ for $1 \leq j \leq q$. Thus $\operatorname{Im}\sigma$ is orthogonal to $\operatorname{Ker} f$. $\square$

Clearly σ is smooth outside Z. We also have

Proposition 2.2. *If $F = F^0F'$ in X, where F^0 is a holomorphic function and F' is non-vanishing, then $F^0\sigma$ is smooth across Z.*

Proof. Since $F = F^0F'$ we have that $S = \overline{F^0}S'$, where S' is the dual of F', and $|F|^2 = |F^0|^2|F'|^2$, where $|F'|^2$ is smooth and non-vanishing. Thus by Lemma 2.1,

$$F^0\sigma = F^0(\delta_f)_{q-1}S/|F|^2 = (\delta_f)_{q-1}S'/|F'|^2,$$

which is smooth across Z. $\square$

3. Residue currents of generically exact complexes

Let

$$0 \to E_N \xrightarrow{f_N} \dots \xrightarrow{f_3} E_2 \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0 \to 0 \tag{3.1}$$

a complex of Hermitian vector bundles over the n-dimensional complex manifold X, and assume that it is pointwise exact outside an analytic set Z of positive codimension. Then clearly for each k, the rank of f_k, $\operatorname{rank} f_k = \dim \operatorname{Im} f_k$, is constant over $X \setminus Z$, and equal to

$$\rho_k = \dim E_{k-1} - \dim E_{k-2} + \cdots + (-1)^{k+1} \dim E_0.$$

Since $z \mapsto \operatorname{rank} f_k(z)$ is lower semicontinuous it follows that $\operatorname{rank} f_k(z) \leq \rho_k$ everywhere in X.

The bundle $E = \oplus E_k$ has a natural superbundle structure, i.e., $\mathbb{Z}_2$-grading, $E = E^+ \oplus E^-$, E^+ and E^- being the subspaces of even and odd elements, respectively, by letting $E^+ = \oplus_{2k} E_k$ and $E^- = \oplus_{2k+1} E_k$. The space of E-valued currents

$$\mathcal{D}'_\bullet(X, E) = \mathcal{D}'_\bullet(X) \otimes_{\mathcal{E}(X)} \mathcal{E}(X, E)$$

has a natural structure as a left $\mathcal{E}_\bullet(X)$-module, and it gets a natural grading by combining that gradings of $\mathcal{D}_\bullet(X)$ and $\mathcal{E}(X, E)$. We make $\mathcal{D}'_\bullet(X, E)$ into a right $\mathcal{E}_\bullet(X)$-module, by letting $\xi\phi = (-1)^{\deg\xi \deg\phi}\phi\xi$ for sections ξ of $\mathcal{E}_\bullet(X, E)$ and smooth forms ϕ. The superstructure on E induces a superstructure $\operatorname{End}E = \operatorname{End}(E)^+ \oplus \operatorname{End}(E)^-$, such that a mapping is odd if, like $f = f_1 + \ldots + f_N$, maps $E^+ \to E^-$ and $E^- \to E^+$. In the same way we get a $\mathbb{Z}_2$-grading of $\mathcal{D}'_\bullet(X, \operatorname{End}E)$. For instance, $\bar\partial$ extends to an odd mapping on $\mathcal{D}'_\bullet(X, E)$, as well as on $\mathcal{D}'_\bullet(X, \operatorname{End}E)$; if A is a section of $\mathcal{D}'_\bullet(X, \operatorname{End}E)$, then $\bar\partial A = \bar\partial \circ A - (-1)^{\deg A} A \circ \bar\partial$. Here $\bar\partial \circ A$ means A composed with $\bar\partial$ so that for a section ξ of E we have $(\bar\partial \circ A)\xi = \bar\partial(A\xi)$, whereas $(\bar\partial A)\xi = \bar\partial(A\xi) - (-1)^{\deg A} A(\bar\partial\xi)$. Recall that two mappings A and B supercommutes if the supercommutator $[A, B] = AB - (-1)^{\deg A \deg B} BA$ vanishes. Since f is holomorphic and of odd degree, we have that $\bar\partial \circ f = -f \circ \bar\partial$, i.e., $\bar\partial$ and f supercommutes. Thus $\nabla = f - \bar\partial$ is an odd mapping, and it extends to a mapping on an endomorphism A by the formula $\nabla_{\mathrm{End}} A = \nabla \circ A - (-1)^{\deg A} A \circ \nabla$. In fact, ∇ is (minus) the $(0,1)$-part of the super connection $D - f$ introduced by Quillen, [27], where D is the Chern connection on E.

In $X \setminus Z$ we have the minimal inverses $\sigma_k \colon E_{k-1} \to E_k$ of f_k, and we let $\sigma = \sigma_1 + \cdots + \sigma_N \colon E \to E$. Then

$$f\sigma = I - \sigma f. \tag{3.2}$$

We claim that

$$f(\bar\partial\sigma) = (\bar\partial\sigma)f \quad \text{and} \quad \sigma(\bar\partial\sigma) = (\bar\partial\sigma)\sigma. \tag{3.3}$$

In fact, by (3.2), $f\bar\partial\sigma = -\bar\partial(f\sigma) = -\bar\partial(I - \sigma f) = (\bar\partial\sigma)f$; the second assertion is verified in a similar way, using that $\sigma\sigma = 0$. It is also easily checked that

$$\nabla_{\mathrm{End}}\sigma = I - \bar\partial\sigma. \tag{3.4}$$

In $X \setminus Z$ we now define the $\mathrm{End}E$-valued form

$$u = \sigma(\nabla_{\mathrm{End}}\sigma)^{-1} = \sigma(I - \bar\partial\sigma)^{-1} = \sigma + \sigma(\bar\partial\sigma) + \sigma(\bar\partial\sigma)^2 + \ldots. \tag{3.5}$$

Since $\nabla_{\mathrm{End}}^2 = 0$ and u is odd, (3.4) immediately implies

Proposition 3.1. *If $\nabla = f - \bar\partial$, then $\nabla \circ u = I - u \circ \nabla$, i.e., $\nabla_{\mathrm{End}} u = I$.*

Notice that

$$u = \sum_{\ell \ge 0} \sum_{k \ge \ell+1} u_k^\ell$$

where

$$u_{\ell+k}^\ell = \sigma_{\ell+k}(\bar\partial\sigma_{\ell+k-2}) \cdots (\bar\partial\sigma_{\ell+1}) \in \mathcal{E}_{0,k-1}(X \setminus Z, \mathrm{Hom}\,(E_\ell, E_{\ell+k})).$$

In view of (3.3) we also have

$$u_{\ell+k}^\ell = (\bar\partial\sigma_{\ell+k-1})(\bar\partial\sigma_{\ell+k-2}) \cdots (\bar\partial\sigma_{\ell+1})\sigma_\ell.$$

Let

$$u^\ell = u\Pi_{E_\ell} = \sum_{k \ge \ell+1} u_k^\ell.$$

In particular we have $\nabla \circ u^0 = I_{E_0}$ and $\nabla \circ u^1 = I_{E_1} - u^0 \circ \nabla$. Following [26] and [1] we are now going to make a current extension of u across Z.

Proposition 3.2. *Let F be any holomorphic function (or tuple of holomorphic functions) that vanishes on Z. Then $\lambda \mapsto |F|^{2\lambda}u$, a priori defined for $\mathrm{Re}\,\lambda >> 0$, has a continuation as a current-valued analytic function to $\mathrm{Re}\,\lambda > -\epsilon$. Moreover,*

$$U = |F|^{2\lambda}u|_{\lambda=0}$$

is a current extension of u across Z that is independent of the choice of F.

Proof. The proof is very similar to the proof of Theorem 1.1 in [1] so we only provide an outline. For each σ_k, following Section 2, we have a section F_k of $\Lambda^{\rho_k} E_k^* \otimes \Lambda^{\rho_k} E_{k-1}$, and its dual S_k such that $\sigma_k = (\delta_{f_k})_{\rho_k - 1} S_k / |F_k|^2$. After a sequence of suitable resolutions of singularities we may assume that, for all k, $F_k = F_k^0 F_k'$, where F_k^0 is a monomial and F_k' is nonvanishing, and that also F is a monomial times a nonvanishing factor. By Proposition 2.2 therefore $\sigma_k = \alpha_k / F_k^0$, where α_k is smooth across Z. Since $\alpha_{j+1}\alpha_j = 0$ outside the set $\{F_{j+1}^0 F_j^* = 0\}$ thus $\alpha_{j+1}\alpha_j = 0$ everywhere. Therefore, it is easy to see that

$$u_{\ell+k}^{\ell} = \frac{(\bar\partial \alpha_{\ell+k-1})(\bar\partial \alpha_{\ell+k-2}) \cdots (\bar\partial \alpha_{\ell+1})\alpha_\ell}{F_{\ell+k-1}^0 \cdots F_\ell^0}.$$

Since the monomials F_k only vanish on Z and F vanishes there, F must contain each coordinate factor that occurs in any F_k^0. Therefore the proposed analytic continuation exists and the value at $\lambda = 0$ is the natural principal value current extension. □

In the same way we can define the residue current

$$R = \bar\partial |F|^{2\lambda} \wedge u|_{\lambda=0}$$

which has its support on Z. Our main result is

Theorem 3.3. *Let* (3.1) *be a generically exact complex of Hermitian holomorphic vector bundles and let U and R be the currents defines above. Then*

$$\nabla_{\mathrm{End}} U = I - R, \qquad \nabla_{\mathrm{End}} R = 0. \tag{3.6}$$

Moreover, if $\operatorname{codim} Z = p$, *then $R_{\ell+k}^{\ell}$ vanishes if $k < p$.*

We can also write (3.6) as

$$\nabla \circ U = I - U \circ \nabla - R, \qquad \nabla \circ R = R \circ \nabla.$$

Proof. In fact,

$$\nabla_{\mathrm{End}}\big(|F|^{2\lambda} u\big) = |F|^{2\lambda} \nabla_{\mathrm{End}} u - \bar\partial |F|^{2\lambda} \wedge u = |F|^{2\lambda} I - \bar\partial |F|^{2\lambda} \wedge u.$$

The first statement in (3.6) now follows by taking $\lambda = 0$. The second statement follows immediately since $\nabla_{\mathrm{End}}^2 = 0$. The vanishing of $R_{\ell+k}^{\ell}$ for $k < p$ follows from the basic principle that a residue current of bidegree $(0, k)$ cannot have support on a variety with higher codimension than k. For a precise argument see [26] or [1]. □

Corollary 3.4. *Assume that ϕ is a holomorphic section of E_ℓ such that $f_\ell \phi = 0$.*
(i) If $R^\ell \phi = 0$, then locally there is a holomorphic section ψ of $E_{\ell+1}$ such that $f_{\ell+1}\psi = \phi$.
(ii) If moreover $R^{\ell+1} = 0$, then the existence of such a local solution ψ implies that $R^\ell \phi = 0$.

Proof. By (3.6) we we have that $\nabla(U^\ell\phi) = \phi - U_\ell^{\ell-1}(\nabla\phi) - R^\ell\phi$ and by the assumptions of ϕ therefore $\nabla(U^\ell\phi) = 0$. Thus we have a current solution v to $f_{\ell+1}v_{\ell+1} = \phi, \quad f_{\ell+k+1}v_{\ell+k+1} = \bar\partial v_{\ell+k}$. By solving a sequence of $\bar\partial$-equations, we end up with the desired holomorphic solution, cf., [1]. For the second part, assume that $f_{\ell+1}\psi = \phi$. Then by (3.6), $R^\ell\phi = R\phi = R(\nabla\psi) = \nabla(R\psi) = \nabla(R^{\ell+1}\psi) = 0$. □

4. Definition of Noetherian residue currents

We will now discuss how one can find a current whose annihilator coincides with a given ideal sheaf (or subsheaf of $\mathcal{O}^r$). Notice that the complex (3.1) corresponds to a complex of locally free analytic sheaves

$$0 \to \mathcal{O}(E_N) \to \cdots \to \mathcal{O}(E_1) \to \mathcal{O}(E_0) \to 0, \tag{4.1}$$

that is exact outside Z, and conversely, any such sequence of locally free sheaves that is exact outside some analytic set Z gives rise to a generically exact complex (3.1) of vector bundles. From Corollary 3.4 above we get one of the implications in the following basic result.

Theorem 4.1. *Assume that* (3.1) *is generically exact, let R be the associated residue current, and let* (4.1) *be the associated complex of sheaves. Then $R^\ell = 0$ for all $\ell \geq 1$ if and only if* (4.1) *is exact at $\mathcal{O}(E_\ell)$ for all $\ell \geq 1$.*

Thus, if J is the subsheaf $\mathrm{Im}\,(\mathcal{O}(E_1) \to \mathcal{O}(E_0)$ of $\mathcal{O}(E_0)$, then

$$0 \to \mathcal{O}(E_N) \to \cdots \to \mathcal{O}(E_1) \to J \to 0 \tag{4.2}$$

is a resolution of J if and only if $R^\ell = 0$ for all $\ell \geq 1$.

Proof. Since one direction is already settled, let us assume that (4.1) is exact, and let

$$Z_j = \{z;\ \mathrm{rank}\, f_j < \rho_j\}.$$

According to a theorem of Buchsbaum-Eisenbud, see [18] Theorem 20.9,

$$\mathrm{codim}\, Z_j \geq \rho_j. \tag{4.3}$$

The intuitive idea in the proof is based on the (somewhat vague) principle that a residue current of degree $(0,q)$ cannot be supported on a variety of codimension $q+1$. To begin with, $R_2^1 = \bar\partial|F|^{2\lambda}\wedge\sigma_2|_{\lambda=0}$ is a $(0,1)$-current and has its support on Z_2 which has codimension 2 and hence it must vanish. Now, σ_3 is smooth outside Z_3, and hence $R_3^1 = \bar\partial\sigma_3\wedge R_2^1 = 0$ outside Z_3; thus R_3^1 is supported on Z_3 and again by the same principle R_3^1 must vanish etc. To make this into a strict argument we will need the following simple lemma.

Lemma 4.2. *Suppose $\gamma(s,\tau)$ is smooth and moreover that $\omega(s,\tau) = \gamma(s,\tau)/\bar s$ is smooth where $\tau_1\cdots\tau_k \neq 0$. Then $\gamma(s,\tau)/\bar s$ is smooth everywhere.*

Proof. Assume that $\gamma(s,\tau)=\bar{s}\omega(s,\tau)$ where $\tau_1\cdots\tau_k\neq 0$. It follows that $(\partial^k/\partial s^k)\gamma(0,\tau)=0$ when $\tau_1\cdots\tau_k\neq 0$, and hence by continuity it holds also when $\tau_1\cdots\tau_k=0$. It now follows from a Taylor expansion in s that $\gamma(s,\tau)/\bar{s}$ is smooth. $\square$

We have to show that for each k,

$$\int\bar\partial|F|^{2\lambda}\wedge\frac{\bar\partial\alpha_k}{F_k^0}\wedge\frac{\bar\partial\alpha_{k-1}}{F_{k-1}^0}\wedge\ldots\wedge\frac{\bar\partial\alpha_3}{F_3^0}\wedge\frac{\alpha_2}{F_2^0}\wedge\tilde\xi\Big|_{\lambda=0}=0,$$

where $\tilde\xi$ is the pullback of a test form ξ. To be precise, there are also cutoff functions involved that we suppress for simplicity. Observe that $\bar\partial|F|^{2\lambda}$ is a sum of terms like $a\lambda|F|^{2\lambda}d\bar{s}/\bar{s}$. We have to show that all the corresponding integrals vanish. First suppose that s is a factor in F_k. Since ξ has degree $n-k+1$ in $d\bar{z}$ it must vanish on Z_k and hence by standard argument, see, e.g., [26] or [1], $(d\bar{s}/\bar{s})\wedge\tilde\xi$ is smooth (i.e., each term of $\tilde\xi$ contains either a factor $\bar{s}$ or $d\bar{s}$. If s is not a factor in $F_k^0,\ldots,F_{\ell+1}^0$ but in F_ℓ, then where $F_k^0\cdots F_{\ell+1}^0\neq 0$ we have that

$$\frac{d\bar{s}}{\bar{s}}\wedge\frac{\bar\partial\alpha_k}{F_k^0}\wedge\ldots\wedge\frac{\bar\partial\alpha_{\ell+1}}{F_{\ell+1}^0}\wedge\tilde\xi$$

is smooth, since outside where $F_k\cdots F_{\ell+1}=0$, the form $\bar\partial\sigma_k\wedge\ldots\bar\partial\sigma_{\ell+1}\wedge\xi$ must vanish on Z_ℓ for degree reasons. From the lemma it follows now that

$$\frac{d\bar{s}}{\bar{s}}\wedge\bar\partial\alpha_k\wedge\ldots\wedge\bar\partial\alpha_{\ell+1}\wedge\tilde\xi$$

is smooth, and therefore the corresponding integral vanishes at $\lambda=0$. $\square$

Definition 1. A current R satisfying one of the equivalent conditions in Theorem 4.1 will be called a Noetherian residue current for the sheaf $J=\mathrm{Im}\,(\mathcal{O}(E_1)\to\mathcal{O}(E_0))$.

Corollary 4.3. *Assume that R is a Noetherian residue current for the sheaf J. Then R has support on the support Z of $\mathcal{O}(E_0)/J$, $R=R^0$, and* $\mathrm{ann}R=J$.

For a Noetherian current, with no ambiguity, we will write R_k instead of R_k^0.

Proof. If (4.2) is exact, then (4.1) is exact outside the support Z of $\mathcal{O}(E_0)/J$, and therefore (3.1) is pointwise exact outside Z, and hence the corresponding residue current R is supported on Z. From Theorem 4.1 it follows that $R^\ell=0$ for $\ell\geq 1$, and from Corollary 3.4 it follows that $\mathrm{ann}R=J$. $\square$

The degree of explicitness is directly depending on the degree of explicitness of a resolution of J; notice that there are no assumption here of minimality of the resolution.

5. Examples

Given a finitely generated subsheaf J of $\mathcal{O}^{r_0}$ in, e.g., a polydisk X we can always find a resolution of J in any slightly smaller polydisk $X' \subset\subset X$, see, [21], and hence, if $\mathcal{O}^{r_0}/J$ has support on a variety of positive codimension, we get a Noetherian residue current R in X' for J. We will now consider some more explicit examples.

Example 1 (The Koszul complex). Let E_1 be a Hermitian bundle over X of rank m, let $E_0 \simeq \mathbb{C}$ be the trivial line bundle, and let f be a nontrivial section of E_1^*. If δ is interior multiplication with f, we have the Koszul complex

$$0 \to \Lambda^m E_1 \xrightarrow{\delta} \cdots \xrightarrow{\delta} \Lambda^2 E_1 \xrightarrow{\delta} E_1 \xrightarrow{\delta} E_0 \to 0$$

which is exact precisely where f is non-vanishing. Notice that in this case the total bundle $E = \oplus E_k$ is just ΛE_1, and the superbundle structure is obtained from the grading in ΛE. Moreover, the desired $\mathcal{E}_\bullet(X)$-module structure of $\mathcal{D}'_\bullet(X, E)$ is obtained from the wedge product in $\Lambda(E \oplus T^*(X))$. The induced complex of sheaves is exact for $\ell \geq 1$ if and only if $\operatorname{codim} Z = m$, see, e.g., [18]. In that case R just consists of the single term R_m. If $f = f_1 e_1^* + \cdots + f_m e_m^*$ in some local holomorphic frame e_j^* for E^*, then R_m is just the Coleff-Herrera current (1.1) times $e_1 \wedge \ldots \wedge e_m$, where e_j is the dual frame, see [1]. $\square$

Example 2 (The Eagon-Northcott complex). Suppose that E and Q are Hermitian bundles of ranks m and r, and $\Phi \colon E \to Q$ is a generically surjective morphism. Let $f_1 = \det \Phi \colon \Lambda^r E \otimes \det Q^* \to \mathbb{C}$. The Eagon-Northcott complex is obtained by letting $E_0 = \mathbb{C}$ and $E_k = \Lambda^{r+k-1} E \otimes S^{r+k-1} Q^*$ for $k \geq 1$, where f_k for $k \geq 2$ is the natural mappings induced by Φ. The corresponding complex of sheaves is exact for $\ell \geq 1$ in the generic case when $\operatorname{codim} Z = m - r + 1$, see, e.g., [18]. This also follows from Theorem 4.1 since the corresponding residues R^ℓ must vanish for $\ell \geq 1$ for codimension reasons, see also [5] for details. Thus $R = R^0$ is a Noetherian residue current for the ideal sheaf $J = (\det \Phi)$. This was already proved in [5].

Now let instead $E_1 = E$ and $E_0 = Q$. There is a closely related complex, with

$$E_k = \Lambda^{r+k-1} E \otimes S^{k-2} Q^* \otimes \det Q^*, \quad k \geq 2,$$

where f_2 is $\det \Phi$ and f_k is the natural mapping induced by Φ for $k \geq 3$, see [4]. Again, if $\operatorname{codim} Z = m - r + 1$, the induced complex of sheaves is exact for $\ell \geq 1$ and hence $R = R^0$ is a Noetherian residue current for the sheaf $J = \operatorname{Im} \Phi$. This was already proved in [4]. $\square$

There are simple algorithms that produce resolutions of monomial ideals, see, e.g., [19]. We conclude this section by computing a couple of Noetherian residue currents in two variables. We begin with the possibly simplest example of a non-complete intersection ideal.

Example 3. Consider the ideal $J = (z_1^2, z_1z_2)$, with zero variety $\{0\}$. It is easy to see that

$$0 \to \mathcal{O} \xrightarrow{f_2} \mathcal{O}^2 \xrightarrow{f_1} J \to 0, \tag{5.1}$$

where

$$f_1 = \begin{bmatrix} z_1^2 & z_1z_2 \end{bmatrix} \quad \text{and} \quad f_2 = \begin{bmatrix} z_2 \\ -z_1 \end{bmatrix} \tag{5.2}$$

is a (minimal) resolution of J. We assume that the corresponding vector bundles are equipped with the trivial Hermitian metrics. Observe that Z is of dimension 1, so R consists of the two parts $R_2 = \bar\partial|F|^{2\lambda}\wedge u_2|_{\lambda=0}$ and $R_1 = \bar\partial|F|^{2\lambda}\wedge u_1|_{\lambda=0}$, where $u_2 = \sigma_2\bar\partial\sigma_1$ and $u_1 = \sigma_1$ respectively. Notice that $\sigma_1 = f_1^*(f_1f_1^*)^{-1}$ and $\sigma_2 = (f_2^*f_2)^{-1}f_2^*$. To compute R we consider the proper mapping $\Pi : \widetilde{\mathcal{U}} \to \mathcal{U}$, where $\mathcal{U}$ is a neighborhood of the origin and $\widetilde{\mathcal{U}}$ is the blow up at the origin of $\mathcal{U}$. We cover $\widetilde{\mathcal{U}}$ by the two coordinate neighborhoods

$$\Omega_1 = \{t;\ (t_1t_2, t_1) = z \in \mathcal{U}\} \quad \text{and} \quad \Omega_2 = \{s;\ (s_1, s_1s_2) = z \in \mathcal{U}\}.$$

In Ω_1 we get

$$\Pi^*f_1 = t_1^2t_2 \begin{bmatrix} t_2 & 1 \end{bmatrix} \quad \text{so} \quad \Pi^*\sigma_1 = \frac{1}{t_1^2t_2(1+|t_2|^2)}\begin{bmatrix} \bar t_2 \\ 1 \end{bmatrix}. \tag{5.3}$$

Moreover

$$\Pi^*f_2 = \begin{bmatrix} t_2 \\ 1 \end{bmatrix} \quad \text{which gives} \quad \Pi^*\sigma_2 = \frac{1}{t_1(1+|t_2|^2)}\begin{bmatrix} 1 & -\bar t_2 \end{bmatrix}. \tag{5.4}$$

It follows that

$$u_2^0 = \frac{d\bar t_2}{t_1^3t_2(1+|t_2|^2)^2}$$

To compute R_2 take a test form $\phi = \varphi(z)dz_1\wedge dz_2$; in Ω_1, $\Pi^*dz_1\wedge dz_2 = -t_1dt_1 \wedge dt_2$ and thus

$$R_2^0.\phi = -\int \bar\partial\left[\frac{1}{t_1^2}\right]\wedge\left[\frac{1}{t_2}\right]\frac{d\bar t_2}{(1+|t_2|^2)^2}\varphi(t_1t_2, t_1)\ dt_1 \wedge dt_2, \tag{5.5}$$

where the brackets denote one-variable principal value currents. In view of the one-variable formula

$$\bar\partial\left[\frac{1}{s}\right]\wedge\frac{ds}{s} = 2\pi i[s=0]$$

($[V]$ means the current of integration over V), a Taylor expansion of φ and symmetri considerations reveals that (5.5) is equal to

$$2\pi i\int_{t_2}\frac{d\bar t_2 \wedge dt_2}{(1+|t_2|^2)^2}\varphi_{1,0}(0,0) = (2\pi i)^2\varphi_{1,0}(0,0),$$

where $\varphi_{1,0} = \partial\varphi/\partial z_1$. One can check that there is no extra contribution from the other coordinate chart, and hence

$$R_2 = \bar\partial\left[\frac{1}{z_1^2}\right] \wedge \bar\partial\left[\frac{1}{z_2}\right].$$

Notice that R_1, taking values in $\mathrm{Hom}\,(\mathbb{C},\mathbb{C}^2)$, is a column matrix. A similar computation yields that

$$R_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left[\frac{1}{z_2}\right]\bar\partial\left[\frac{1}{z_1}\right].$$

We see that $\mathrm{ann}R_2 = (z_1^2, z_2)$ and $\mathrm{ann}R_1 = (z_1)$, and hence $\mathrm{ann}R = (z_1^2, z_2) \cap (z_1) = J$ as expected. $\square$

We now consider a nontrivial zero-dimensional example.

Example 4. Consider the ideal $J = (z_1^5, z_1^3 z_2, z_2^4)$ in $\mathcal{O}_0$ with variety $Z = \{0\} \subset \mathcal{U}$, where $\mathcal{U}$ is a neighborhood of the origin in $\mathbb{C}^2$. Notice that J is Cohen-Macaulay, since Z is zero-dimensional, and therefore R is essentially canonical, see Section 7. We have a minimal resolution

$$0 \to \mathcal{O}^2 \xrightarrow{f_2} \mathcal{O}^3 \xrightarrow{f_1} J \to 0, \tag{5.6}$$

where

$$f_1 = \begin{bmatrix} z_1^5 & z_1^3 z_2 & z_2^4 \end{bmatrix} \text{ and } f_2 = \begin{bmatrix} 0 & z_2 \\ -z_2^3 & -z_1^2 \\ -z_1^3 & 0 \end{bmatrix}. \tag{5.7}$$

Since Z is of dimension 0, $R = R_2 = \bar\partial|F|^{2\lambda}\wedge u_2|_{\lambda=0}$. To compute R we consider the proper mapping $\Pi : \widetilde{\mathcal{U}} \to \mathcal{U}$, where $\widetilde{\mathcal{U}}$ is a toric variety that can be covered by the three coordinate neihgborhoods

$$\Omega_1 = \{t;\ (t_1t_2, t_2) = z \in \mathcal{U}\}\ ,\ \Omega_2 = \{s;\ (s_1s_2, s_1s_2^2) = z \in \mathcal{U}\} \text{ and }$$
$$\Omega_3 = \{r;\ (r_1, r_1^2 r_2) = z \in \mathcal{U}\}.$$

By considerations inspired by [28] it is enough to make the computation in Ω_2. We get

$$\Pi^* f_1 = s_1^4 s_2^5 \begin{bmatrix} s_1 & 1 & s_2^3 \end{bmatrix} \text{ and } \Pi^* f_2 = s_1 s_2^2 \begin{bmatrix} 0 & 1 \\ s_1^2 s_2^4 & -s_1 \\ -s_1^2 s_2 & 0 \end{bmatrix}. \tag{5.8}$$

It follows that

$$\Pi^*\sigma_1 = \frac{1}{s_1^4 s_2^5 \nu(s)} \begin{bmatrix} \bar s_1 \\ 1 \\ \bar s_2 \end{bmatrix}, \tag{5.9}$$

where $\nu(s) = (1 + |s_1|^2 + |s_2^3|^2)$. A simple computation yields

$$\Pi^*\sigma_2 = \frac{1}{s_1^3 s_2^3 \nu(s)} \begin{bmatrix} s_1 \bar s_2^3 & \bar s_2^3 & -(1+|s_1|^2) \\ s_1^2 s_2(1+|s_2^3|^2) & -s_1^2 \bar s_1 s_2 & -s_1^2 \bar s_1 s_2^4 \end{bmatrix}, \tag{5.10}$$

and thus

$$(5.11)\qquad u_2 = \frac{1}{s_1^7 s_2^8 \nu(s)^2}\begin{bmatrix} s_1\bar{s}_2^3 d\bar{s}_1 - 3\bar{s}_2(1-|s_1|^2)d\bar{s}_2 \\ s_1^2 s_2(1+|s_2^3|^2)d\bar{s}_1 - 3s_1^2\bar{s}_1 s_2^4\bar{s}_2^2 d\bar{s}_2 \end{bmatrix}.$$

Let us compute the action of R_2^0 on a test form $\phi = \varphi dz_1 \wedge dz_2$. In Ω_2, $\Pi^* dz_1 \wedge dz_2 = s_1 s_2^2 ds_1 \wedge ds_2$, and so

$$(5.12)\quad R_2.\phi = \int \bar\partial\left[\frac{1}{s_1^6}\right]\left[\frac{1}{s_2^6}\right]\frac{1}{\nu^2}\begin{bmatrix} 3\bar{s}_2 d\bar{s}_2 \\ 0 \end{bmatrix}\varphi(s_1 s_2, s_1 s_2^2)\, ds_1\wedge ds_2 +$$
$$\int \bar\partial\left[\frac{1}{s_2^5}\right]\left[\frac{1}{s_1^4}\right]\frac{1}{\nu^2}\begin{bmatrix} 0 \\ d\bar{s}_1 \end{bmatrix}\varphi(s_1 s_2, s_1 s_2^2)\, ds_1\wedge ds_2.$$

Let us start by considering the first term. Evaluating the s_1-integral the “upper” integral becomes

$$(5.13)\quad 2\pi i \int \frac{3|s_2|^4}{(1+|s_2|^6)^2}\,\varphi_{2,3}(0,0)\, d\bar{s}_2\wedge ds_2 = 12\,\bar\partial\left[\frac{1}{z_1^3}\right]\wedge\bar\partial\left[\frac{1}{z_2^4}\right].\phi;$$

indeed, for symmetry reasons everything else vanishes as in Example 3. Continuing with the second term, the “lower” integral is

$$(5.14)\quad 2\pi i \int \frac{1}{(1+|s_1|^2)^2}\,\varphi_{4,0}(0,0)\, d\bar{s}_1\wedge ds_1 = 24\,\bar\partial\left[\frac{1}{z_1^5}\right]\wedge\bar\partial\left[\frac{1}{z_2}\right].\phi$$

Thus $\mathrm{ann} R = (z_1^3, z_2^4)\cap(z_1^5, z_2) = J$ as expected. □

6. Division and interpolation formulas

The currents U and R constructed in Section 3 fits perfectly into a general scheme for constructing division and interpolation formulas in pseudoconvex domains in $\mathbb{C}^n$, developed in [6]. For simplicity we restrict here to the unit ball $D = \{z;\ |z| < 1\}$; for more general cases see [6]. Let (3.1) be a complex of (trivial) bundles over a neighborhood of the closed unit ball in $\mathbb{C}^n$, and let $J = \mathrm{Im}\, f_1$.

Let $\delta_{\zeta - z}$ denote interior multiplication by the vector field

$$2\pi i \sum_1^n (\zeta_j - z_j)(\partial/\partial\zeta_j)$$

and let $\nabla_{\zeta-z} = \delta_{\zeta-z} - \bar\partial$. Moreover, let

$$s = \frac{\partial|\zeta|^2}{2\pi i(|\zeta|^2 - \bar\zeta\cdot z)}$$

and let χ be a cutoff function that is 1 in a neighborhood of $\overline{D}$. For each fixed $z \in D$ we define the form

$$g = \chi - \bar\partial\chi\wedge\frac{s}{\nabla_{\zeta-z}s} = \chi - \bar\partial\chi\wedge[s + s\wedge\bar\partial s + s\wedge(\bar\partial s)^2 + \cdots + s\wedge(\bar\partial s)^{n-1}].$$

In the terminology of [6] it is a compactly supported weight that depends holomorphically on $z \in D$, i.e., $\nabla_{\zeta-z} g = 0$ and $g_{0,0}(z) = 1$, where lower indices denote bidegree.

Let us fix global frames for the bundles E_k. The morphisms f_k are then just matrices of holomorphic functions, and one can find (see [6] for explicit choices) $(k-\ell, 0)$-form-valued holomorphic morphisms $H_k^\ell\colon E_k \to E_\ell$, depending holomorphically on z, such that $H_k^\ell = 0$ for $k < \ell$, $H_\ell^\ell = I_{E_\ell}$, and in general,

$$\delta_{\zeta - z} H_k^\ell = H_{k-1}^\ell f_k - f_{\ell+1}(z) H_k^{\ell+1}, \quad k \geq \ell; \tag{6.1}$$

here f stands for $f(\zeta)$. Let

$$H^{\ell+1}U = \sum_k H_k^{\ell+1} U_k^\ell, \quad H^\ell R = \sum_k H_k^\ell R_k^\ell;$$

thus $H^{\ell+1}U$ takes a section of E_ℓ depending on ζ into a (current-valued) section of $E_{\ell+1}$ depending on both ζ and z, and similarily, $H^\ell R$ takes a section of E_ℓ into section of E_ℓ. We let $HU = \sum_\ell H^\ell U$ and $HR = \sum_\ell H^\ell R$. Then, precisely as in [6], a straight-forward computation, using (6.1), yields that

$$g' = f(z)HU + HUf + HR$$

is an E-valued weight, i.e., $\nabla_{\zeta - z} g' = 0$ and $g'_{0,0} = I_E$. Therefore, see [6], we get the representation

$$\phi(z) = \int g' \phi \wedge g,$$

or expressed in another way,

$$\phi(z) = f(z)(T\phi)(z) + T(f\phi)(z) + S\phi(z), \tag{6.2}$$

where

$$T\phi(z) = \int_\zeta HU(\zeta, z)\phi \wedge g, \qquad S\phi(z) = \int_\zeta HR(\zeta, z)\phi(\zeta) \wedge g.$$

In particular, we get an explicit (in terms of U and R) realization of a solution $\psi = T\phi$ of $f\psi = \phi$, if $f\phi = 0$ and $R\phi = 0$, thus providing an explicit proof of Corollary 3.4 (i).

If now R is a Noetherian residue current we see that $S\phi = 0$ as soon as ϕ belongs to J or ϕ is a section of E_ℓ for $\ell \geq 1$.

In the same way as in [2] one can extend these formulas slightly, and get a characterization of the module $\mathcal{E}J$ of smooth tuples of functions generated by J, i.e., the set of all $\phi = f_1\psi$ for smooth ψ. First notice that if $\phi = f_1\psi$, then $R\phi = R^0\phi = R^0 f_1 \psi - R^1 \bar\partial\psi = R\nabla\psi = \nabla R^1 \psi = 0$, so that $R^0\phi = 0$. Since each partial derivative $\partial/\partial\bar z_j$ commutes with ∇, we get that

$$R(\partial^\alpha \phi / \partial \bar z^\alpha) = 0 \tag{6.3}$$

for all multiindices α. The converse is obtained by integral formulas precisely as in [2], and hence we have

Theorem 6.1. *Assume that $J \subset \mathcal{O}^{r_0}$ is an analytic sheaf such that the support of $\mathcal{O}^{r_0}/J$ has positive codimension, and let R be a Noetherian residue current for J. Then an r_0-tuple $\phi \in (\mathcal{E})^{r_0}$ of smooth functions is in $\mathcal{E}J$ if and only if* (6.3) *holds for all α.*

One can also obtain analogous results with lower regularity, see [2] and [6].

7. Cohen-Macaulay ideals and modules

Let J_x be an ideal in the local ring $\mathcal{O}_x$ at $x \in X$. The length ν_x of a minimal resolution of $\mathcal{O}_x/J_x$ is precisely $n - \operatorname{depth}(\mathcal{O}_x/J_x)$. We always have that $\operatorname{depth}(\mathcal{O}_x/J_x) \le n - \operatorname{codim} J_x$ and it may happen that the inequality is strict; e.g., if J_x has embedded primary components. In particular, the minimal length can vary along Z. However, if J is Cohen-Macauley, i.e., $\operatorname{depth}(\mathcal{O}_x/J_x) = \operatorname{codim} J_x$ for each x, thus ν is equal to the codimension everywhere.

More generally, if $J \subset \mathcal{O}^r$ is finitely generated and $\mathcal{O}^r/J$ is a sheaf of Cohen-Macaulay modules, then, see [18], (locally) each primary factor has the same codimension p, and any minimal resolution ends up at position p. Special cases are the sheaves in Examples 1 and 2 above, i.e., (f) if f is a complete intersection, $J = (\det \Phi)$ or $J = \operatorname{Im} \Phi$ if $\Phi\colon E \to Q$ and $\operatorname{codim} Z = m - r + 1$. We have the following generalization of the corresponding known result for a complete intersection.

Theorem 7.1. *Suppose that J is a finitely generated subsheaf of a locally free sheaf of $\mathcal{O}$-modules $\mathcal{O}(E_0)$, and suppose that $\mathcal{O}(E_0)/J$ is Cohen-Macaulay. If*

$$0 \to \mathcal{O}(E_p) \to \cdots \to \mathcal{O}(E_1) \to J \to 0$$

is a minimal resolution, then the corresponding Noetherian residue current $R = R_p^0$ is independent of the choice of Hermitian metric. Moreover, if we choose another minimal resolution

$$0 \to \mathcal{O}(E'_p) \to \cdots \to \mathcal{O}(E'_1) \to J \to 0$$

and R' is the corresponding residue current, then there is a holomorphic isomorphism $g_p\colon E_p \simeq E'_p$ such that $R' = g_p R$.

Proof. Assume that u and v are the forms in $X \setminus Z$ constructed by means of two different choices of metrics on E. Then $\nabla_{\mathrm{End}} u = I$ and $\nabla_{\mathrm{End}} v = I$ in $X \setminus Z$, and hence if $w = uv$ we have

$$\nabla_{\mathrm{End}} w = \nabla_{\mathrm{End}}(uv) = (\nabla_{\mathrm{End}} u)v - u\nabla_{\mathrm{End}} v = v - u,$$

where the minus sign occurs since u has odd order. Thus

$$\nabla_{\mathrm{End}}\big(|F|^{2\lambda} w\big) = |F|^{2\lambda} v - |F|^{2\lambda} u - \bar\partial |F|^{2\lambda} \wedge w,$$

and evaluating at $\lambda = 0$ we get

$$\nabla_{\mathrm{End}} W = V - U - M,$$

where M is the residue current $M = \bar\partial|F|^{2\lambda}\wedge w|_{\lambda=0}$. However, since the complex ends up at p, w has at most bidegree $(0, p-2)$ and hence the current M has at most bidegree $(0, p-1)$. Therefore W must vanish since it is supported on Z which has codimension p. Thus we have

$$0 = \nabla^2_{\mathrm{End}} W = I - R^v - I + R^u = R^u - R^v.$$

For the second statement, first recall that for two minimal resolutions there are isomorphisms $g_k\colon E_k \to E'_k$ such that the corresponding diagram commutes, and such that g_0 is the identity on E_0. Given any metric in E, we equip E' with the induced metric $|\xi| = |g^{-1}\xi|$. Then $\sigma' = g\sigma g^{-1}$ in $X \setminus Z$ and therefore $u' = \sigma' + \sigma'(\bar\partial\sigma') + \cdots = g(\sigma + \sigma(\bar\partial\sigma) + \cdots)g^{-1} = gug^{-1}$. Therefore, $(u')^0_p = g_p u^0_p$, and from this the statement follows. $\square$

Notice that in $X \setminus Z$ the form u_p is a $\bar\partial$-closed $\mathrm{Hom}\,(E_0, E_p)$-valued form and thus defines a Dolbeault cohomology class, and in view of the proof of Theorem 7.1 this class is independent of the choice of Hermitian metric. For a holomorphic section ϕ of E_0 we therefore have a well-defined map

$$G\phi\colon \xi \mapsto \int u_p\phi\wedge\bar\partial\xi$$

for test-forms ξ of bidegree $(n, n-p)$ that are $\bar\partial$-closed in some neighborhood of Z. Precisely as for a complete intersection, [15] and [25], we have a cohomological version of the duality principle.

Theorem 7.2. *Suppose that J is a finitely generated subsheaf of a locally free sheaf of $\mathcal{O}$-modules $\mathcal{O}(E_0)$, and suppose that $\mathcal{O}(E_0)/J$ is Cohen-Macaulay. Then a section of $\mathcal{O}(E_0)$ is in J if and only if $G\phi = 0$.*

The "only if" direction follows from Stokes' theorem. The converse can be proved, using the decomposition formula (6.2), and mimicking the proof of the corresponding statement for a complete intersection in [25], see also Proposition 7.1 in [6].

Example 5. Let J be an ideal in $\mathcal{O}_0$ of dimension zero. Then it is Cohen-Macaulay and for each germ ϕ in $\mathcal{O}_0$, $G\phi$ is a functional on $\mathcal{O}_0^{r_n}$, where $r_n = \dim E_n$. Moreover, $G\phi = 0$ if and only if $\phi \in J$. If J is generated by n functions, then a minimal resolution is given by the Koszul complex, so $r_n = 1$, and the resulting mapping G is precisely the classical Grothendieck residue. $\square$

8. Noetherian residue currents of homogeneous ideals

Let S be the graded ring of polynomials in $\mathbb{C}^{n+1}$, and let

$$0 \to M_N \to \cdots \to M_1 \to M_0 \to 0 \tag{8.1}$$

be a graded complex of free S-modules, i.e.,

$$M_k = S(-d_1^k) \oplus \cdots \oplus S(-d_{r^k}^k), \tag{8.2}$$

and the mappings are given by matrices of elements in S, see [19] for a background. We can associate to (8.1) a complex of vector bundles over $\mathbb{P}^n$,

$$0 \to E_N \xrightarrow{f_N} \dots \xrightarrow{f_3} E_2 \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0 \to 0, \tag{8.3}$$

in the following way. Let $\mathcal{O}(\ell)$ be the holomorphic line bundle over $\mathbb{P}^n$ whose sections are naturally identified with ℓ-homogeneous functions in $\mathbb{C}^{n+1}$. Moreover, let E^i_j be disjoint trivial line bundles over $\mathbb{P}^n$ and let

$$E_k = \big(E_1^k \otimes \mathcal{O}(-d_1^k)\big) \oplus \cdots \oplus \big(E_{r_k}^k \otimes \mathcal{O}(-d_{r_k}^k)\big).$$

The mappings in (8.1) induce vector bundle morphisms $f_k\colon E_k \to E_{k-1}$. If we equip E_k with the natural Hermitian metric

$$|\xi(z)|^2 = \sum_{j=1}^{r_k} |\xi_j(z)|^2/|z|^{2d_j^k}$$

we can then define the associated currents U and R as before, following the general scheme, provided that (8.3) is generically exact.

Let ϵ_j^k be a global frame element for the bundle E_j^k. In the affine part $\mathcal{U}_0 = \{[z] \in \mathbb{P}^n;\ z_0 \neq 0\}$ we then have a local holomorphic frame

$$e_j^k = z_0^{-d_j^k}\epsilon_j^k, \quad j = 1, \dots, r_k,$$

for the bundle E_k. In these local frames

$$R_k^\ell = \sum_{i=1}^{r_\ell}\sum_{j=1}^{r_k} (R_k^\ell)_{ij} \otimes e_i^k \otimes (e_j^\ell)^*,$$

where $(R_k^\ell)_{ij}$ are (scalar-valued) currents in $\mathcal{U}_0 \simeq \mathbb{C}^n$. For later reference we notice that these currents have natural extensions as currents on $\mathbb{P}^n$.

Recall that, see, e.g., [14], $H^{0,q}(\mathbb{P}^n, \mathcal{O}(\nu)) = 0$ for all ν if $0 < q < n$, whereas $H^{0,n}(\mathbb{P}^n, \mathcal{O}(\nu)) = 0$ if $\nu \geq -n$.

We have the following analogue of Theorem 4.1.

Theorem 8.1. *Let* (8.1) *be a graded complex of free S-modules, $N \leq n+1$, and let* (8.3) *be the corresponding complex of vector bundles over $\mathbb{P}^n$ equipped with the natural Hermitian metric. Then R^ℓ vanish for all $\ell \geq 1$ if and only if* (8.1) *is exact at M_ℓ for $\ell \geq 1$.*

Whatever set of generators $M_1 \to M_0$ for $J = \mathrm{Im}\,(M_1 \to M_0)$ we start with, we can always extend to a resolution of J such that $N \leq n+1$.

We notice that a graded resolution of S^{r_0}/J gives rise to a Noetherian residue current R for the corresponding analytic sheaf (generated by) J.

Proof. First assume that (8.1) is exact for $\ell \geq 1$. According to the Baumgarten-Eisenbud theorem in the homogeneous case, see [19], the set in $\mathbb{C}^{n+1}$ (or equivalently in $\mathbb{P}^n$) where the rank of f_k is strictly less than ρ_k has at least codimension k. Precisely as in the proof of Theorem 4.1 it follows that $R^\ell = 0$ for $\ell \geq 1$.

Conversely, assume that $R^\ell = 0$ for all $\ell \geq 1$. Let ϕ be an element in M_ℓ, $\ell \geq 1$, of pure degree that is mapped onto zero in $M_{\ell-1}$. It corresponds to a global section of $E_\ell \otimes \mathcal{O}(r)$ for a certain r, and $f_\ell \phi = 0$. Since $R^\ell = 0$ we therefore have that $\nabla(U^\ell \phi) = \phi$. The first $\bar\partial$-equation to be solved is then $\bar\partial w = U^\ell_N \phi$ and since $N \leq n+1$ and $\ell \geq 1$ the right hand side is a $(0,q)$-current with $q \leq n-1$. Thus there is no cohomologous obstruction, and so we obtain a holomorphic section ψ of $E_{\ell+1} \otimes \mathcal{O}(r)$ such that $f_\ell \psi = \phi$, and thus ψ corresponds to the desired element in $M_{\ell+1}$. $\square$

In view of the preceding proof we see that if ϕ is a section of $E_0 \otimes \mathcal{O}(r)$ such that $R^0 \xi = 0$, then we can find a holomorphic solution to $f_1 \psi = \phi$ if either the complex terminates at (at most) level n, or if the occurring $\bar\partial$-equation of top degree is solvable, which it indeed is if $r - d_j^{n+1} \geq -n$ for all j.

Given a S-module $J \subset S^{r_0}$, there always exists a resolution (8.1), and the length of a minimal resolution is equal to $n+1-\operatorname{depth}(S^{r_0}/J)$, so we can avoid the $\bar\partial$-equation of top degree if (and only if) S^{r_0}/J contains a (non-trivial) nonzerodivisor. We sum up this as

Theorem 8.2. *Let $J \subset S^{r_0}$ be an S-module and let R be the residue current associated with a minimal resolution.*

(i) Suppose that S^{r_0}/J contains a nonzerodivisor. Then a section ϕ of $E_0 \otimes \mathcal{O}(r)$ lies in the image of f_1 if and only if $R\phi = 0$.

(ii) Assume that $r \geq \max_j(r_j^{n+1}) - n$. Then a section ϕ of $E_0 \otimes L^r$ lies in the image of f_1 if and only if $R\phi = 0$.

Some remarks. If J is defined by a complete intersection then clearly the case (i) holds. Also if Z is discrete and all the zeros are of first order, then $\operatorname{depth} S/J = 1$, see [19], so that case (i) holds.

In case (ii) an estimate of $\max_j(r_j^{n+1}) - n$ follows from the degree of regularity of J, see, e.g., [19]. $\square$

We conclude this section by relating to modules of polynomials in $\mathbb{C}^n$. Let $z' = (z'_1, \ldots, z'_n)$ be the standard coordinates in $\mathbb{C}^n$ that we identify with $\mathcal{U}_0 = \{[z] \in \mathbb{P}^n;\ z_0 \neq 0\}$. Let F be a $\operatorname{Hom}(\mathbb{C}^{r_1}, \mathbb{C}^{r_0})$-valued polynomial in $\mathbb{C}^n$, whose columns $F_1, \ldots, F_{r_1}$ have at most degrees $d^1_1, \ldots, d^1_{r_1}$. After the homogenizations $f_k(z) = z_0^{d_k} F(z'/z_0)$, we get an

$r_1 \times r_0$-matrix f whose columns are d_k^1-homogeneous forms in $\mathbb{C}^{n+1}$; thus a graded mapping

$$f_1 \colon S(-d_1^1) \oplus \cdots \oplus S(-d_{r_1}^1) \to S^{r_0}.$$

Extending to a graded resolution we thus obtain a Noetherian residue current on $\mathbb{P}^n$ for the sheaf generated by f_1, and taking a local trivialization in $\mathbb{C}^n \simeq \mathcal{U}_0$, we get a Noetherian residue current R for F in $\mathbb{C}^n$.

Proposition 8.3. *Given an r_0-tuple of polynomials Φ in $\mathbb{C}^n$, there are polynomials Ψ such that $\Phi = F\Psi$ in $\mathbb{C}^n$ if and only if $R\Phi = 0$.*

Proof. Take a homogenization $\phi(z) = z_0^r \Phi(z'/z_0)$. The condition $R\Phi = 0$ in $\mathbb{C}^n$ means that $R\phi = 0$ outside the hyperplane at infinity, so for a large enough r, $R\phi = 0$ on $\mathbb{P}^n$. Now (for a large enough r) part (ii) of Theorem 8.2 applies and provides a solution ψ. After dehomogenization we get the desired solution Ψ. $\square$

Clearly the final degree of Ψ in the preceding proof depends on the choice of r. We conclude with an example where we have optimal control of the degree of the solution; it is a generalization of Max Noether's classical theorem.

Proposition 8.4. *Let $F_1, \ldots, F_N$ be polynomials in $\mathbb{C}^n$ such that the homogenized forms $f_1, \ldots, f_N$ define a Cohen-Macaulay ideal J in S and assume that no irreducible component of Z is contained in the hyperplane at infinity. If Φ belongs to the ideal (F) in $\mathbb{C}^n$, then there are polynomials Ψ_j with $\deg(F_j\Psi_j) \leq \deg\Phi$ such that $F_1\Psi_1 + \cdots + F_N\Psi_N = \Phi$.*

To see this one just has to imitate the proof of Theorem 1.2 in [3]. As in [4] one can just as well assume that J is a submodule of S^{r_0} for some $r_0 > 1$.

9. The fundamental principle

Let F be a $\mathrm{Hom}\,(\mathbb{C}^{r_1}, \mathbb{C}^{r_0})$-valued polynomial of generic rank r_0 and let K be the closure of an open convex bounded set in $\mathbb{R}^n$. We want to find a description of all homogeneous solutions $\xi = (\xi_1, \ldots, \xi_{r_0})$ in $\mathcal{E}(K)$ to $F^*(D)\xi = 0$, where F^* is the transposed matrix and $D = i\partial/\partial t$.

We let R be the Noetherian residue current in $\mathbb{C}^n$ obtained from F by the procedure in the preceding section. Notice that this current has a current extension to $\mathbb{P}^n$.

Let ρ be the support function for K but smoothen out in a neighborhood of the origin in $\mathbb{R}^n$. If ν is a distribution of order at most M with support in K, $\nu \in \mathcal{E}^{\prime,M}(K)$, then

$$|\hat{\nu}(z)| \leq C(1 + |z|)^M e^{\rho(\operatorname{Im} z)}, \tag{9.1}$$

and conversely if ν satisfies such an estimate it is at least in $\mathcal{E}'(K)$. Let $\rho(\zeta)=\rho(\operatorname{Im}\zeta)$. Recall that $\nabla_{\zeta-z}=\delta_{\zeta-z}-\bar\partial$. Notice that

$$\nabla_{\zeta-z}\partial\rho/2\pi=i\langle\zeta-z,\rho'(\zeta)\rangle+\partial\bar\partial\rho(\zeta),$$

so by the convexity and homogeneity of ρ we have that

$$|e^{\nabla_{\zeta-z}\partial\rho/2\pi}|\le Ce^{\rho(z)}. \tag{9.2}$$

It is easy to see that we can choose Hefer forms, as in Section 6, that are polynomials in both ζ and z. In view of (9.2), therefore H^1U and H^0R will be currents in $\mathbb{C}^n$ such that $\psi\wedge H^1U$ and $\psi\wedge H^0R$ have current extensions to $\mathbb{P}^n$ if ψ is smooth in $\mathbb{C}^n$ and vanishes to high enough order at the hyperplane at infinity (depending on the order of the current R at infinity as well as the degrees of the Hefer forms). Now

$$g=e^{\nabla_{\zeta-z}\partial\rho/2\pi}\wedge\Big(1+\nabla_{\zeta-z}\frac{\partial|\zeta|^2}{1+|\zeta|^2}\Big)^{M'}$$

is a weight in $\mathbb{C}^n$ (i.e., $\nabla_{\zeta-z}g=0$ and $g_{0,0}(z)=1$) which vanishes to high order at the hyperplane at infinity if M' is large. Given M we can therefore choose M' so that we get, for $\nu\in\mathcal{E}'^{,M}(K,E_0)$, the decomposition

$$\hat\nu(z)=F(z)(T\hat\nu)(z)+S\hat\nu(z);$$

and $S\hat\nu(z)$ vanishes if $\nu=F(-D)\mu$ for $\mu\in\mathcal{E}'^{,M}(\operatorname{int}(K),E_1)$. Since $T\hat\nu$ and $S\hat\nu$ satisfies (9.1) for some power, we get mappings

$$\mathcal{T}\colon\mathcal{E}'^{,M}(K,E_0)\to\mathcal{E}'(K,E_1),\qquad \mathcal{S}\colon\mathcal{E}'^{,M}(K,E_0)\to\mathcal{E}'(K,E_0),$$

such that

$$\nu=F(-D)\mathcal{T}\nu+\mathcal{S}\nu,$$

and $\mathcal{S}\nu=0$ if $\nu=F(-D)\mu$ for some $\mu\in\mathcal{E}'^{,M}(\operatorname{int}(K),E_1)$.

Theorem 9.1. *If $\xi\in\mathcal{E}(K,E_0^*)$, then $\mathcal{S}^*\xi\in C^M(K,E_0^*)$ satisfies $F^*(D)\mathcal{S}^*\xi=0$. If in addition $F^*(D)\xi=0$, then $S^*\xi=\xi$.*

Thus $\mathcal{S}^*$ is a projection onto the space of homogeneous solutions.

Proof. For a μ with values in E_1, and support in $\operatorname{int}(K)$, we have that $\mu.F^*(D)\mathcal{S}^*\xi=F(-D)\mu.\mathcal{S}^*\xi=\mathcal{S}(F(-D)\mu).\xi=0$, and hence first part OK. On the other hand, if $F^*(D)\xi=0$, then τ of order M and with values in E_0 we have

$$\tau.\mathcal{S}^*\xi=\mathcal{S}\tau.\xi=(\tau-F(-D)\tau).\xi=\tau.\xi-\tau.F^*(D)\xi=\tau.\xi,$$

which shows the second assertion. □

We can write (recall that $(R=R^0)$)

$$S\hat\nu(\zeta)=\int_\zeta\alpha(\zeta,z)R(\zeta)\hat\nu(\zeta)e^{i\langle\zeta-z,\rho'(\zeta)\rangle}$$

where $\alpha(\cdot, z)$ is a polynomial i z, and precisely as in [11] we then get the formula

$$\mathcal{S}^*\phi(t) = \int_\zeta R^*(\zeta)\alpha^*(\zeta, D)\phi(\rho')e^{-i\langle\zeta,t-\rho'\rangle},$$

where $\alpha^*(\zeta, z)\phi(\rho'(\zeta))$ is the result when replacing each occurrence of z in $\alpha^*(\zeta, z)$ by D, letting it act on $\phi(t)$ and evaluate at the point $\rho'(\zeta)$. Thus we have

Theorem 9.2. *For any solution $\phi \in \mathcal{E}(K)$ of $F^*(D)\phi = 0$, there is a smooth $A(\zeta)$ such that*

$$\phi(t) = \int_\zeta R^*(\zeta)A(\zeta)e^{-i\langle\zeta,t-\rho'(\zeta)\rangle}.$$

Conversely, for any smooth $A(\zeta)$ with not too high polynomial growth (depending on the choice of M'), the residue integral defines a homogeneous solution.

References

[1] M. Andersson: *Residue currents and ideals of holomorphic functions*, Bull. Sci. Math., **128**, (2004), 481–512.

[2] M. Andersson: *Ideals of smooth functions and residue currents*, J. Functional Anal. **212** (2004), no. 1, 76–88.

[3] M. Andersson: *The membership problem for polynomial ideals in terms of residue currents*, Ann. Inst. Fourier (to appear).

[4] M. Andersson: *Residue currents of holomorphic morphisms*, J. Reine Angew. Math. (to appear).

[5] M. Andersson: *Explicit versions of the Briançon-Skoda theorem with variations*, Michigan Math. J. (to appear).

[6] M. Andersson: *Integral representation with weights II, division and interpolation formulas*, Preprint Göteborg (2005).

[7] C. Berenstein & A. Yger: *Effective Bezout identities in $Q[z_1, \cdots, z_n]$*, Acta Math. **166** (1991), 69–120.

[8] C. Berenstein & A. Yger: *Analytic residue theory in the non-complete intersection case*, J. Reine Angew. Math. **527** (2000), 203–235.

[9] C. Berenstein & R. Gay & A. Vidras & A. Yger: *Residue Currents and Bézout Identities*, Birkhäuser (1993).

[10] B. Berndtsson: *A formula for division and interpolation*, Math Ann. **263** (1983).

[11] B. Berndtsson & M. Passare: *Integral formulas and an explicit version of the fundamental principle*, J. Func. Analysis **84** (1989).

[12] J-E Björk: *Annihilating ideals of residue currents*, Preprint Stockholm (2005).

[13] J. Briançon & H. Skoda: *Sur la clôture intégrale d'un idéal de germes de fonctions holomorphes en un point de $\mathbb{C}^n$.*, C. R. Acad. Sci. Paris Sér. A 278 (1974), 949–951.

[14] J-P Demailly: *Complex Analytic and Differential Geometry*, Monograph Grenoble (1997).

[15] A. Dickenstein & C. Sessa: *Canonical representatives in moderate cohomology*, Invent. Math. **80** (1985), 417–434..

[16] A. Dickenstein & R. Gay & C. Sessa & A. Yger: *Analytic functionals annihilated by ideals*, Manuscripta Math. **90** (1996), 175–223.

[17] L. Ehrenpreis: *Fourier analysis in several complex variables*, Pure and Applied Mathematics, Vol. XVII Wiley-Interscience Publishers A Division of John Wiley & Sons, New York-London-Sydney 1970.

[18] D. Eisenbud: *Commutative algebra. With a view toward algebraic geometry*, Graduate Texts in Mathematics, 150. Springer-Verlag, New York, 1995.

[19] D. Eisenbud: *The geometry of syzygies. A second course in commutative algebra and algebraic geometry*, Graduate Texts in Mathematics, 229. Springer-Verlag, New York, 2005.

[20] Ph. Griffiths & J. Harris: *Principles of Algebraic Geometry*, John Wiley and Sons, (1978).

[21] R. Gunning & H. Rossi: *Analytic functions of several complex variables*, Prentice-Hall, Inc., Englewood Cliffs, N.J. 1965.

[22] L. Hörmander: *An introduction to complex analysis in several variables*, Second revised edition. North-Holland Mathematical Library, Vol. 7. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973.

[23] B. Malgrange: *Une remarque sur les idéaux de fonctions différentiables*, Invent. Math. **9** (1969/1970), 279–283.

[24] V.P. Palamodov: *Linear differential operators with constant coefficients*, Translated from the Russian by A. A. Brown. Die Grundlehren der mathematischen Wissenschaften, Band 168 Springer-Verlag, New York-Berlin 1970.

[25] M. Passare: *Residues, currents, and their relation to ideals of holomorphic functions*, Math. Scand. **62** (1988), 75–152.

[26] M. Passare & A. Tsikh & A. Yger: *Residue currents of the Bochner-Martinelli type*, Publ. Mat. **44** (2000), 85-117.

[27] D. Quillen: *Superconnections and the Chern character*, Topology **24** (1985), 89–95.

[28] E. Wulcan: *Residue currents of monomial ideals*, Preprint Göteborg (2005).

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