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Refinements of Stochastic Domination

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Abstract

In a recent paper by two of the authors, the concepts of upwards and downwards ϵ-movability were introduced, mainly as a technical tool for studying dynamical percolation of interacting particle systems. In this paper, we further explore these concepts which can be seen as refinements or quantifications of stochastic domination, and we relate them to previously studied concepts such as uniform insertion tolerance and extractability.

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1 Introduction

In [3], certain refinements of stochastic domination were introduced, which we call upwards and downwards ϵ-movability; see Definition 1.1 below. These concepts were introduced mainly as a technical tool in the analysis of dynamical percolation for interacting particle systems, but they turn out to be interesting in their own right.

The purpose of the present paper is to relate them to other concepts that have arisen in a number of problems and that we feel belong to the same circle of ideas. These include finite energy [16] and insertion and deletion tolerance [13]; see Definition 1.5. Later, we also define the term *extractability*; see Definition 1.6. Although the term is our own, this concept does have a

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history; in particular, there has been interest in finding lower bounds on ϵ for which ϵ-extractability holds. The question of uniform extractability has been studied for the Ising model as well as other Markov random fields in [1, 8, 15]. Earlier, in [5, 6, 7], a similar question was studied for Markov chains and autoregressive processes. Of related interest is the result in [8] that for Markov random fields, uniform finite energy implies uniform extractability.

Let S be a countable set. For $p \in [0,1]$, let $\pi_p = \prod_{s \in S} p\delta_1 + (1-p)\delta_0$ be the standard product measure with density p. When talking about product measures on $\{0,1\}^S$, we will always mean these uniform ones (with the same p for every $s \in S$).

Let μ be an arbitrary probability measure on $\{0,1\}^S$. For $\epsilon \in (0,1)$, we will let $\mu^{(+,\epsilon)}$ denote the distribution of the process obtained by first choosing an element of $\{0,1\}^S$ according to μ and then independently changing each 0 to a 1 with probability ϵ. Similarly, we will let $\mu^{(-,\epsilon)}$ denote the distribution of the process obtained by first choosing an element of $\{0,1\}^S$ according to μ and then independently changing each 1 to a 0 with probability ϵ. Finally, for $\delta \in (0,1)$, we let $\mu^{(-,\epsilon,+,\delta)}$ denote the distribution of the process obtained by first choosing an element of $\{0,1\}^S$ according to μ and then independently changing each 0 to a 1 with probability δ and each 1 to a 0 with probability ϵ.

It is elementary to check that for any $\epsilon \in [0,1)$, $\mu_1^{(+,\epsilon)} = \mu_2^{(+,\epsilon)}$ or $\mu_1^{(-,\epsilon)} = \mu_2^{(-,\epsilon)}$ implies that $\mu_1 = \mu_2$.

For $\sigma, \sigma' \in \{0,1\}^S$ we write $\sigma \preceq \sigma'$ if $\sigma(s) \le \sigma'(s)$ for every $s \in S$. A function $f : \{0,1\}^S \to \mathbb{R}$ is increasing if $f(\sigma) \le f(\sigma')$ whenever $\sigma \preceq \sigma'$. For two probability measures μ, μ' on $\{0,1\}^S$, we say that μ **is stochastically dominated by** μ', and write $\mu \preceq \mu'$, if for every continuous increasing function f we have that $\mu(f) \le \mu'(f)$. ($\mu(f)$ is shorthand for $\int f d\mu$.) By Strassens theorem (see [9, p. 72]), this is equivalent to the existence of random variables $X, X' \in \{0,1\}^S$ such that $X \sim \mu$, $X' \sim \mu'$, and $X \preceq X'$ a.s.; here and throughout, "$\sim$" means "has distribution".

For a probability measure μ on $\{0,1\}^S$, define $p_{\max,\mu}$ by

$$p_{\max,\mu} := \sup\{p \in [0,1] : \pi_p \preceq \mu\};$$

the supremum is easily seen to be achieved.

Definition 1.1 *Let (μ_1, μ_2) be a pair of probability measures on $\{0,1\}^S$, where S is a countable set. Assume that $\mu_1 \preceq \mu_2$. If, given $\epsilon > 0$, we have*

$$\mu_1 \preceq \mu_2^{(-,\epsilon)},$$

then we say that the pair (μ_1, μ_2) is **downwards ϵ-movable**. *(μ_1, μ_2) is said to be* **downwards movable** *if it is downwards ϵ-movable for some $\epsilon > 0$. Analogously, if, given $\epsilon > 0$, we have*

$$\mu_1^{(+,\epsilon)} \preceq \mu_2,$$

then we say that the pair (μ_1, μ_2) is **upwards ϵ-movable**, *and we say that (μ_1, μ_2) is* **upwards movable** *if the pair is upwards ϵ-movable for some $\epsilon > 0$.*

Note that if we restrict to the case where both μ_1 and μ_2 are product measures, then these concepts become trivial.

Following is a natural example where a stochastically ordered pair of probability measures is neither downwards nor upwards movable. We assume that the reader is familiar with the Ising model; for a definition and survey, see, e.g., [4] or [9].

Example 1.2 Let $\mu^{+,\beta}$ and $\mu^{-,\beta}$ be the plus and minus states for the Ising model on $\mathbb{Z}^d$ with zero external field at inverse temperature $\beta > 0$. It is well known that $\mu^{-,\beta} \preceq \mu^{+,\beta}$, and it is known (see [12]) that $p_{\max,\mu^{+,\beta}} = p_{\max,\mu^{-,\beta}}$. Assume that the pair $(\mu^{-,\beta}, \mu^{+,\beta})$ is upwards ϵ-movable for some $\epsilon > 0$. It then follows that

$$(\pi_{p_{\max,\mu^{+,\beta}}})^{(+,\epsilon)} = (\pi_{p_{\max,\mu^{-,\beta}}})^{(+,\epsilon)} \preceq (\mu^{-,\beta})^{(+,\epsilon)} \preceq \mu^{+,\beta},$$

which of course contradicts the definition of $p_{\max,\mu^{+,\beta}}$. Therefore the pair is not upwards movable. By symmetry of the model, it is not downwards movable either. $\square$

Next, we provide an easy example of a pair of measures which is downwards but not upwards movable.

Example 1.3 Let $\nu = \frac{1}{2}\pi_q + \frac{1}{2}\delta_0$ and $\mu = \frac{1}{2}\pi_p + \frac{1}{2}\delta_0$ where $q < p$ and where δ_0 is the measure which puts probability 1 on the configuration of all zeros. It is trivial to check that when $|S| = \infty$, (ν, μ) is downwards but not upwards movable. $\square$

In [3] a considerable amount of effort was spent on trying to show downwards movability when the pair considered was two stationary distributions, corresponding to two different parameter values, for some specific interacting particle system. In particular, the so called contact process (see [10] for definitions and a survey) was investigated. Considering (μ_1, μ_2), where μ_i is the upper invariant measure for the contact process with infection rate λ_i, it was shown in [3] that if $\lambda_1 < \lambda_2$, then the pair is downwards movable.

Another result from [3] is that if $\mu_1 \preceq \mu_2$, μ_2 satisfies the FKG lattice condition (see [9, p. 78]) and

$$\inf_{\substack{\tilde{S} \subset S \\ |\tilde{S}| < \infty}} \inf_{\substack{s \in \tilde{S} \\ \xi \in \{0,1\}^{\tilde{S} \setminus s}}} [\mu_2(\sigma(s) = 1 | \sigma(\tilde{S} \setminus s) \equiv \xi) - \mu_1(\sigma(s) = 1 | \sigma(\tilde{S} \setminus s) \equiv \xi)] > 0$$

then (μ_1, μ_2) is downwards movable. This however is not sufficient to get the result for the contact process mentioned above since by [11], the upper invariant measure for the contact process on $\mathbb{Z}$ does not satisfy the FKG lattice condition when $\lambda < 2$.

In the present paper, we will concentrate on the case where μ_1 is a product measure but μ_2 is not. We now proceed with some further explanations and definitions needed to state our main results, Theorems 1.7 and 1.10 below.

If $p_{\max,\mu} = 0$, then trivially $(\pi_{p_{\max,\mu}}, \mu)$ is downwards movable but not upwards movable. Assume next that μ is a probability measure with $p_{\max,\mu} > 0$. If $p \in [0, p_{\max,\mu})$, then the pair (π_p, μ) is trivially upwards movable. It is also easy to see that it is downwards movable by arguing as follows. By Strassen's theorem, we may choose $X \sim \mu$ and $Y \sim \pi_{p_{\max,\mu}}$ such that $X \geq Y$ a.s. Then choose $\epsilon > 0$ such that $(1-\epsilon)p_{\max,\mu} > p$, and let $Z \sim \pi_{1-\epsilon}$ be independent of both X and Y. We obtain $\min(X,Z) \geq \min(Y,Z)$ a.s., and since $\min(Y,Z) \sim \pi_{p_{\max,\mu}(1-\epsilon)}$ we conclude that $\pi_p \preceq \mu^{(-,\epsilon)}$, as desired.

The final case we are left with (when one of the measures is a uniform product measure) is $(\pi_{p_{\max,\mu}}, \mu)$ with $p_{\max,\mu} > 0$. This pair is by definition not upwards movable, but we believe it is an interesting question to ask if it is downwards movable and this question motivates the following definition.

Definition 1.4 *We say that μ is* **nonrigid** *if the pair $(\pi_{p_{\max,\mu}}, \mu)$ is downwards movable and otherwise we will say that μ is* **rigid**.

All uniform product measures other than δ_0 are trivially rigid while all μ such that $p_{\max,\mu} = 0$ are trivially nonrigid. Heuristically, it is natural to expect that as long as $p_{\max,\mu} > 0$, then typically μ should be rigid. This issue turns out to be quite intricate, however; see Proposition 3.1 and Theorem 3.2 below.

Definition 1.5 *We say that μ is* **ϵ-insertion tolerant** *if for any $s \in S$, we have that*

$$\mu(\sigma(s) = 1 | \sigma(S \setminus s)) \geq \epsilon \text{ a.s.} \tag{1}$$

We say that μ is **uniformly insertion tolerant** *if it is ϵ-insertion tolerant for some $\epsilon > 0$. The analogous notions of* **ϵ-deletion tolerant** *and* **uniformly deletion tolerant** *are defined similarly (the "1" is replaced by "0"). Finally, we say μ has* **finite ϵ-energy** *if it is both ϵ-insertion tolerant and ϵ-deletion tolerant, and that it has* **uniform finite energy** *if it has finite ϵ-energy for some $\epsilon > 0$.*

Closely related are the following notions of extractability.

Definition 1.6 *We call μ* **ϵ-upwards extractable** *if there exists a probability measure ν such that $\mu = \nu^{(+,\epsilon)}$. We call μ* **uniformly upwards extractable** *if it is ϵ-upwards extractable for some $\epsilon > 0$. The notions of* **ϵ-downwards extractable** *and* **uniformly downwards extractable** *are defined analogously (the "+" is replaced by "−"). Finally, μ is called* **ϵ-extractable** *if there exists a probability measure ν such that $\mu = \nu^{(-,\epsilon,+,\epsilon)}$, and it is called* **uniformly extractable** *if it is ϵ-extractable for some $\epsilon > 0$.*

We are now equipped with all the definitions needed to state our main theorem. We refer to Figure 1 for a comprehensive diagram over the implications and non-implications that the theorem asserts.

Theorem 1.7 *Let S be a countable set and consider the following properties of a probability measure μ on $\{0,1\}^S$:*

(I) *μ is uniformly upwards extractable.*

(II) *μ is uniformly insertion tolerant.*

(III) *μ is rigid.*

(IV) *There exists a $p > 0$ such that $\pi_p \preceq \mu$.*

We then have that (I) $\Rightarrow$ (II) $\Rightarrow$ (IV) *and that* (I) $\Rightarrow$ (III) $\Rightarrow$ (IV) *while none of the four corresponding reverse implications hold. Also,* (III) *does not imply* (II). *Moreover, with $S = \mathbb{Z}$, there exist translation invariant examples for all of the asserted nonimplications.*

In addition, it turns out that (IV) does not even imply "(II) or (III)"; see Remark 3.4. Note that we have not managed to work out whether or not (II) implies (III).

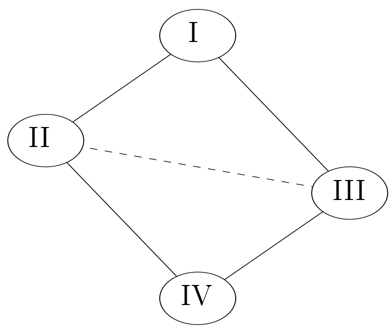

Figure 1. Hasse diagram of the implications between properties (I), (II), (III) and (IV) in Theorem 1.7: we have proved that one property implies another iff there is a downwards path in the diagram from the former to the latter. We do not know whether the dashed line between (II) and (III) should be there or not, i.e., whether or not uniform insertion tolerance implies rigidity. As will be seen in Theorem 1.9, the desired implication (II) $\Rightarrow$ (III) holds under an additional FKG-like assumption. If we restrict to finite S, then some of the implications will turn into equivalences; see Theorem 1.10.

Some of the asserted implications are easy: (I) trivially implies (II). The implication (III) $\Rightarrow$ (IV) is also trivial as we saw. It is a direct application of Holley's inequality (see, e.g., [4, Theorem 4.8]) to see that ϵ-insertion tolerance implies that $\pi_\epsilon \preceq \mu$, whence (II) implies (IV). Thus, apart from the implication (I) $\Rightarrow$ (III) (which is in fact not so hard either), we see all the implications claimed in the theorem. Therefore our interest in Theorem 1.7 is more in the counterexamples showing the distinction between some of these properties rather than in the implications.

As mentioned above, we do not know in general whether (II) implies (III). However Theorem 1.9 provides us with a partial answer, telling us that this is true under the extra assumption of μ being downwards FKG, a property weaker than satisfying the FKG lattice condition and defined as follows.

Definition 1.8 *A measure μ on $\{0,1\}^S$ is downwards FKG if for any finite $S' \subset S$ and any increasing subsets A, B*

$$\mu(A \cap B | \sigma(S') \equiv 0) \geq \mu(A | \sigma(S') \equiv 0) \mu(B | \sigma(S') \equiv 0).$$

The concept of downwards FKG was made explicit in [12], and was further studied in [2], where it was proved that the upper invariant measure for the contact process is downwards FKG.

Theorem 1.9 *Let μ be a translation invariant downwards FKG measure on $\{0,1\}^{\mathbb{Z}^d}$. Then* (II) *implies* (III)*.*

Of course, some of the nonimplications in Theorem 1.7 can become implications under appropriate auxiliary assumptions. For instance, for probability measures μ on $\{0,1\}^{\mathbb{Z}}$ satisfying translation invariance and conditional negative association (see [14] for a definition of the latter), (IV) implies (III). This follows readily from results in [14]; we omit the proof.

Another situation in which further implications between the various properties arise, is when S is taken to be finite. By the support of a measure μ on $\{0,1\}^S$, denoted $\text{supp}(\mu)$, we mean $\{\xi \in \{0,1\}^S : \mu(\sigma(S) \equiv \xi) > 0\}$.

Theorem 1.10 *Let S be finite, and consider properties* (I)–(IV) *of probability measures on $\{0,1\}^S$. We then have*

$$\text{(I)} \Leftrightarrow \text{(II)} \Leftrightarrow \text{supp}(\mu) \textit{ is an up-set,} \tag{2}$$

and

$$\text{(III)} \Leftrightarrow \text{(IV)} \Leftrightarrow \mu(\sigma(S) \equiv 1) > 0. \tag{3}$$

Consequently, the properties in (2) *imply those in* (3) *but not vice versa. Note in particular that if we are in the full support case, then* (I)–(IV) *all hold.*

The rest of this paper is organized as follows. In Sections 2–3, we will establish a number of auxiliary results, in Section 4, we prove Theorem 1.9 and in Section 5, we tie things together giving proofs of Theorems 1.7 and 1.10. Finally, in Section 6, we list some open problems.

2 Uniform insertion tolerance and upwards extractability

In this section we focus on uniform upwards extractability (property (I)) and uniform insertion tolerance (property (II)). Proposition 2.1 provides an equivalence between these properties when S is finite, while Theorem 2.2 exhibits a contrasting scenario for S countable.

Proposition 2.1 *If S is finite and μ is a probability measure on $\{0,1\}^S$, then the following are equivalent:*

(i) *uniform insertion tolerance,*

(ii) *uniform upwards extractability, and*

(iii) $\mathrm{supp}(\mu)$ *is an up-set.*

Theorem 2.2 *For S countably infinite, there exists a probability measure μ on $\{0,1\}^S$ that is uniformly insertion tolerant but not uniformly upwards extractable. Moreover, we can take μ to be a translation invariant measure on $\{0,1\}^{\mathbb{Z}}$.*

Proof of Proposition 2.1. (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (i) are immediate, and so it only remains to show (iii) $\Rightarrow$ (ii).

In what follows, given a configuration $\sigma \in \{0,1\}^S$, $|\sigma|$ will be the number of 1's in σ. If there is to exist a ν such that $\mu = \nu^{(+,\epsilon)}$ with $\epsilon \in [0,1)$, it is not hard to see that we must have

$$\nu(\sigma) = \sum_{\tilde{\sigma} \preceq \sigma} (-\epsilon)^{|\sigma|-|\tilde{\sigma}|}(1-\epsilon)^{|\tilde{\sigma}|-|S|}\mu(\tilde{\sigma}) \ \forall \sigma \in \{0,1\}^S. \tag{4}$$

This can be verified through a direct calculation, but it is easier to calculate $\nu^{(+,\epsilon)}(\sigma)$ and check that it is indeed equal to $\mu(\sigma)$, as follows.

$$\begin{aligned}
\nu^{(+,\epsilon)}(\sigma) &= \sum_{\sigma_1 \preceq \sigma} \epsilon^{|\sigma|-|\sigma_1|}(1-\epsilon)^{|S|-|\sigma|}\nu(\sigma_1) \\
&= \sum_{\sigma_1 \preceq \sigma} \epsilon^{|\sigma|-|\sigma_1|}(1-\epsilon)^{|S|-|\sigma|} \sum_{\sigma_2 \preceq \sigma_1} (-\epsilon)^{|\sigma_1|-|\sigma_2|}(1-\epsilon)^{|\sigma_2|-|S|}\mu(\sigma_2) \\
&= \sum_{\sigma_1 \preceq \sigma} \sum_{\sigma_2 \preceq \sigma_1} \epsilon^{|\sigma|-|\sigma_1|}(-\epsilon)^{|\sigma_1|-|\sigma_2|}(1-\epsilon)^{|\sigma_2|-|\sigma|}\mu(\sigma_2) \\
&= \sum_{\sigma_2} (1-\epsilon)^{|\sigma_2|-|\sigma|}\mu(\sigma_2) \sum_{\sigma_1 : \sigma_2 \preceq \sigma_1 \preceq \sigma} \epsilon^{|\sigma|-|\sigma_1|}(-\epsilon)^{|\sigma_1|-|\sigma_2|}.
\end{aligned}$$

If we fix σ_2, then the binomial theorem gives that the last summation is equal to 0 unless $\sigma_2 = \sigma$ in which case it is equal to 1. We therefore easily obtain that $\nu^{(+,\epsilon)}(\sigma) = \mu(\sigma)$ for every σ.

What remains is to check that $\nu(\sigma) \geq 0$ for all σ. From (4) it is immediate that $\nu(\sigma) = 0$ for every $\sigma \notin \mathrm{supp}(\mu)$ since $\mathrm{supp}(\mu)$ is an up-set. For $\sigma \in \mathrm{supp}(\mu)$ on the other hand, it is easy to see that if we do this construction for different ϵ's, then we get

$$\lim_{\epsilon \to 0} \nu(\sigma) = \mu(\sigma).$$

Since $\mu(\sigma) > 0$ for all $\sigma \in \mathrm{supp}(\mu)$ and $|S| < \infty$, for $\epsilon > 0$ small enough, we get that $\nu(\sigma) > 0$ for all $\sigma \in \mathrm{supp}(\mu)$. This shows that μ is ϵ-upwards extractable for all such ϵ. $\square$

Proof of Theorem 2.2. Let $S = \cup_{k=2}^{\infty} S_k$, where

$$S_k = ((k,1),(k,2),\ldots,(k,k)).$$

We will take the probability measure μ on $\{0,1\}^S$ to be the product measure

$$\mu = \mu_2 \times \mu_3 \times \cdots \tag{5}$$

where each μ_k is a probability measure on $\{0,1\}^{S_k}$. The μ_k's are constructed as follows, drawing heavily on an example in [8]. For $\sigma \in \{0,1\}^{S_k}$, let

$$\mu_k(\sigma) = \begin{cases} \frac{4}{3}2^{-k} & \text{if the number of 1's in } \sigma \text{ is even} \\ \frac{2}{3}2^{-k} & \text{if the number of 1's in } \sigma \text{ is odd.} \end{cases} \tag{6}$$

We may think of μ_k as the distribution of a $\{0,1\}^{S_k}$-valued random variable X_k obtained by first tossing a biased coin with heads-probability $\frac{2}{3}$, and if heads pick the components of X_k i.i.d. $(\frac{1}{2},\frac{1}{2})$ conditioned on an even number of 1's, while if tails pick the components i.i.d. $(\frac{1}{2},\frac{1}{2})$ conditioned on an odd number of 1's. One can also check that this distribution is the same as choosing all but (an arbitrary) one of the variables according to $\pi_{1/2}$ and then taking the last variable to be 1 with probability 1/3 (2/3) if there are an even (odd) number of 1's in the other bits. This last description immediately implies that μ_k is $\frac{1}{3}$-insertion tolerant. Because of the product structure in (5), this property is inherited by μ, which therefore is uniformly insertion tolerant.

It remains to show that μ is not uniformly upwards extractable. To this end, let X be a $\{0,1\}^S$-valued random variable with distribution μ, and for $k = 2,3,\ldots$ let Y_k denote the number of 1's in $X(S_k)$. It is immediate from (6) that

$$\mathbb{P}(Y_k \text{ is even}) = \tfrac{2}{3} \tag{7}$$

for each k. Using our last description of μ_k, the weak law of large numbers implies that

$$\frac{Y_k}{k} \to \tfrac{1}{2} \text{ in probability as } k \to \infty.$$

Hence, in particular,

$$\lim_{k\to\infty} \mathbb{P}(Y_k \le k - m) \;=\; 1 \tag{8}$$

for any fixed m.

Now assume (for contradiction) that $\mu = \nu^{(+,\epsilon)}$ for some fixed $\epsilon > 0$; since μ being ϵ_2-upwards extractable implies it is ϵ_1-upwards extractable for $\epsilon_1 < \epsilon_2$, we may without loss of generality assume that $\epsilon \le 1/3$. Pick X' according to ν; we may then suppose that X has been obtained from X' by randomly switching 0's to 1's independently with probability ϵ. The intuition behind the argument leading up to a contradiction is that the process of independently flipping 0's to 1's will cancel all preferences of ending up with an even number of 1's.

If $X'(S_k)$ contains precisely l 0's, then the conditional probability (given X') that an even number of these switch to 1's when going from X' to X is easily seen to equal

$$\tfrac{1}{2} + \tfrac{1}{2}(1-2\epsilon)^l \,.$$

The easiest way to see this is using an equivalent random mechanism where each 0 independently "updates" with probability 2ϵ and then all the sites which have updated then independently actually switch to a 1 with probability $1/2$. It follows that the conditional probability (again given X') that Y_k is odd is at least

$$\min\{\tfrac{1}{2} + \tfrac{1}{2}(1-2\epsilon)^l, \tfrac{1}{2} - \tfrac{1}{2}(1-2\epsilon)^l\} \;=\; \tfrac{1}{2} - \tfrac{1}{2}(1-2\epsilon)^l.$$

Now pick m large enough so that $\frac{1}{2} - \frac{1}{2}(1-2\epsilon)^m > \frac{5}{12}$. Since $X' \preceq X$ a.s., we get from (8) that

$$\lim_{k\to\infty} \mathbb{P}(A_k) \;=\; 1$$

where A_k is the event that there are at least m 0's in $X'(S_k)$. This gives

$$\begin{aligned}\lim_{k\to\infty} \mathbb{P}(Y_k \text{ is odd}) \;&\ge\; \lim_{k\to\infty} \mathbb{P}(Y_k \text{ is odd} \,|\, A_k)\mathbb{P}(A_k) \\ &\ge\; (\tfrac{1}{2} - \tfrac{1}{2}(1-2\epsilon)^m) \lim_{k\to\infty} \mathbb{P}(A_k) > \tfrac{5}{12}\,.\end{aligned}$$

This clearly contradicts (7).

We now translate this example into the setting of translation invariant distributions on $\{0,1\}^{\mathbb{Z}}$.

Begin with randomly designating either all even integers or all odd integers (each with probability $\frac{1}{2}$) in the index set $\mathbb{Z}$ to represent copies of S_2. Assume that we happened to choose the even integers (the other case is handled analogously). Then we toss another fair coin to decide whether to put i.i.d. copies of $X(S_2)$ on the pairs $\{\ldots,(-4,-2),(0,2),(4,6),\ldots\}$ in $\mathbb{Z}$, or on $\{\ldots(-2,0),(2,4),(6,8),\ldots\}$. Then use one more fair coin to decide whether $\{\ldots-3,1,5,9,\ldots\}$ or $\{\ldots,-1,3,7,11,\ldots\}$ should be designated for i.i.d. copies of $X(S_3)$, and once this is decided toss a fair three-sided coin to choose one of the three possible placements of the length-3 blocks in this subsequence to put these copies. And so on.

This makes the resulting process X^* translation invariant. Also, since the property of ϵ-insertion tolerance is obviously closed under convex combinations, we easily obtain that X^* is $\frac{1}{3}$-insertion tolerant and therefore uniformly insertion tolerant.

Furthermore, for any $k \geq 2$, we may apply (7) to the i.i.d. copies of $X(S_k)$ to deduce that with probability 1 there will exist $i \in \{0, 1, \ldots, k2^{k-1} - 1\}$ such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \mathbf{1}_{B_{i,j,k}} = \tfrac{2}{3} \tag{9}$$

where $B_{i,j,k}$ denotes the event that the number of 1's in

$$\{i+jk2^{k-1}, i+jk2^{k-1}+2^{k-1}, i+jk2^{k-1}+2\cdot 2^{k-1}, \ldots, i+jk2^{k-1}+(k-1)2^{k-1}\}$$

is even. The right way to think of i is that it is the first place to the right of the origin where a copy of $X(S_k)$ starts. The summation variable j on the other hand, makes us jump to the starting points of all the other copies of $X(S_k)$ to the right of the origin. Furthermore, by arguing as in for the non-translation invariant construction, we have that if X^* is uniformly upwards extractable, then for large k the limit in (9) will be less than $1 - \frac{5}{12} = \frac{7}{12}$ for all $i \in \{0, 1, \ldots, k2^{k-1} - 1\}$. But this contradicts (9), so we can conclude that X^* is not uniformly upwards extractable. $\square$

Note, finally, that the examples in the above proof also show that uniform finite energy does not imply uniform extractability.

3 Rigidity

We now proceed to discuss the issue of when a measure is rigid. As mentioned in the introduction, any measure which does not dominate a nontrivial product measure is trivially nonrigid and so it would be more interesting to have a nonrigid measure which dominates a nontrivial product measure; such a measure is provided in Theorem 3.2 below.

Proposition 3.1 *If S is finite and μ is a probability measure on $\{0,1\}^S$, then the following are equivalent.*

(i) *μ dominates π_p for some $p > 0$,*

(ii) *μ is rigid, and*

(iii) *$\mu(\sigma(S) \equiv 1) > 0$.*

This does not extend to infinite S, as shown in the following result.

Theorem 3.2 *For S countably infinite, there exists a μ which dominates a nontrivial product measure π_p but is nevertheless nonrigid. Moreover, we can take μ to be a translation invariant measure on $\{0,1\}^{\mathbb{Z}}$.*

Proof of Proposition 3.1. It is easy to see that the condition that μ dominates π_p for some $p > 0$ is equivalent to the condition that $\mu(\sigma(S) \equiv 1) > 0$. Also, recall that if μ is rigid it must dominate a non-trivial product measure.

To make the proof complete, it only remains to show that (i) and (iii) of the statement imply that μ is rigid. We have $\pi_{p_{\max,\mu}} \preceq \mu$, so that

$$\pi_{p_{\max,\mu}}(A) \;\le\; \mu(A) \tag{10}$$

for all increasing events $A \subseteq \{0,1\}^S$. We next claim that

$$\exists A \neq \emptyset, \{0,1\}^S \text{ such that } A \text{ is increasing and } \pi_{p_{\max,\mu}}(A) = \mu(A). \tag{11}$$

To see this, note that if we had strict inequality in (10) for all such nontrivial increasing events A, then we could find a sufficiently small $\delta > 0$ so that

$$\pi_{p_{\max,\mu}+\delta}(A) \;<\; \mu(A)$$

for all such A (this uses the finiteness of S), contradicting the definition of $p_{\max,\mu}$. Now, for such an A we have that $\mu(A) \ge \mu(\sigma(S) \equiv 1) > 0$ and hence for any $\epsilon > 0$

$$\mu^{(-,\epsilon)}(A) \;<\; \mu(A)$$

(again because S is finite), which in combination with (11) yields

$$\pi_{p_{\max,\mu}} \;\not\preceq\; \mu^{(-,\epsilon)} .$$

Since $\epsilon > 0$ was arbitrary, μ is rigid. □

It will be convenient for the proof of Theorem 3.2 to have the following lemma, whose elementary proof we omit.

Lemma 3.3 *For $k \ge 1$, $p \in (0,1)$ and $m \in \{0,1,\ldots,k\}$, write $\rho_{k,p,m}$ for the distribution of a Binomial(k,p) random variable conditioned on taking value at least m. For $p_1 \le p_2$, we have*

$$\rho_{k,p_1,m} \;\preceq\; \rho_{k,p_2,m} \,.$$

Proof of Theorem 3.2. As in the proof of Theorem 2.2, we take $S = \cup_{k=2}^{\infty} S_k$ where $S_k = ((k,1),(k,2),\ldots,(k,k))$, and the probability measure μ on $\{0,1\}^S$ to be the product measure

$$\mu \;=\; \mu_2 \times \mu_3 \times \cdots$$

where each μ_k is a probability measure on $\{0,1\}^{S_k}$. This time, we take the μ_k's to be as follows. For $\sigma \in \{0,1\}^{S_k}$, set

$$\mu_k(\sigma) = \begin{cases} k^{-1}2^{-k} & \text{if the number of 1's in } \sigma \text{ is exactly 1,} \\ 1-2^{-k} & \text{if } \sigma = (1,1,1,\ldots,1), \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

We now make three claims about the μ_k measures:

CLAIM 1. $p_{\max,\mu_k} \geq \frac{1}{2}$ for all k.

CLAIM 2. $\lim_{k\to\infty} p_{\max,\mu_k} = \frac{1}{2}$.

CLAIM 3. For any fixed $\epsilon < \frac{1}{2}$, we have for all k sufficiently large that

$$\mu_k^{(-,\epsilon)} \succeq \pi_{\frac{1}{2}}$$

where $\pi_{\frac{1}{2}}$ is product measure with $p = \frac{1}{2}$ on $\{0,1\}^{S_k}$.

We slightly postpone proving the claims, and first show how they imply the existence of a nonrigid measure that dominates $\pi_{\frac{1}{2}}$.

Let us modify S and μ slightly by setting, for $m \geq 2$,

$$\tilde{S}_m \ = \ \cup_{k=m}^{\infty} S_k$$

and

$$\tilde{\mu}_m \ = \ \mu_m \times \mu_{m+1} \times \cdots \tag{13}$$

so that in other words $\tilde{\mu}_m$ is the probability measure on $\{0,1\}^{\tilde{S}_m}$ which arises by projecting μ on $\{0,1\}^{\tilde{S}_m}$.

Using the product structure (13), we get from CLAIM 1 that $p_{\max,\tilde{\mu}_m} \geq \frac{1}{2}$ (for any m), and from CLAIM 2 that $p_{\max,\tilde{\mu}_m} \leq \frac{1}{2}$ (for any m). Hence

$$p_{\max,\tilde{\mu}_m} \ = \ \tfrac{1}{2}$$

for any m. Fixing $\epsilon \in (0,1/2)$, we can also deduce from (13) and CLAIM 3 that

$$\tilde{\mu}_m^{(-,\epsilon)} \ \succeq \ \pi_{\frac{1}{2}} \ = \ \pi_{p_{\max,\tilde{\mu}_m}} \tag{14}$$

for m sufficiently large. For such m we thus have that $\tilde{\mu}_m$ is nonrigid.

It remains to prove CLAIM 1, CLAIM 2 and CLAIM 3.

CLAIM 1 is the same as saying that $\mu_k \succeq \pi_{\frac{1}{2}}$. This is immediate to verify, but the best way to think about it is as follows. Suppose that we pick $X_k \in \{0,1\}^{S_k}$ according to $\pi_{\frac{1}{2}}$, and if $X_k = (0,0,\ldots,0)$ then we switch one of the 0's (chosen uniformly at random) to a 1, while otherwise we switch *all* 0's to 1's. The resulting random element of $\{0,1\}^{S_k}$ then has distribution μ_k.

To prove CLAIM 2, it suffices (in view of CLAIM 1) to prove that

$$\limsup_{k\to\infty} p_{\max,\mu_k} \leq \frac{1}{2}$$

and to this end it is enough to show for any $\delta > 0$ that

$$\mu_k \not\preceq \pi_{\frac{1}{2}+\delta} \tag{15}$$

for all sufficiently large k. Let A_k denote the event of seeing at most one 1 in $\{0,1\}^{S_k}$; then A_k is a decreasing event and its complement $\neg A_k$ is increasing. Now simply note that

$$\frac{\mu_k(A_k)}{\pi_{\frac{1}{2}+\delta}(A_k)} = \frac{(\frac{1}{2})^k}{(\frac{1}{2}-\delta)^k + k(\frac{1}{2}+\delta)(\frac{1}{2}-\delta)^{k-1}} \tag{16}$$

which tends to ∞ as $k \to \infty$. Hence, taking k large enough gives $\mu_k(A_k) > \pi_{\frac{1}{2}+\delta}(A_k)$, so that $\mu_k(\neg A_k) < \pi_{\frac{1}{2}+\delta}(\neg A_k)$ and (15) is established, proving CLAIM 2.

To prove CLAIM 3, note first that both $\pi_{\frac{1}{2}}$ and $\mu_k^{(-,\epsilon)}$ are invariant under permutations of S_k, so that it suffices to show for k large that

$$\mu_k^{(-,\epsilon)}(B_n) \leq \pi_{\frac{1}{2}}(B_n) \tag{17}$$

for $n = 0, 1, \ldots, k-1$, where B_n is the event of seeing at most n 1's in S_k. For $n = 0$ we get

$$\frac{\mu_k^{(-,\epsilon)}(B_0)}{\pi_{\frac{1}{2}}(B_0)} = \frac{(\frac{1}{2})^k\epsilon + (1-(\frac{1}{2})^k)\epsilon^k}{(\frac{1}{2})^k} \tag{18}$$

while for $n = 1$

$$\frac{\mu_k^{(-,\epsilon)}(B_1)}{\pi_{\frac{1}{2}}(B_1)} = \frac{(\frac{1}{2})^k + (1-(\frac{1}{2})^k)(\epsilon^k + k\epsilon^{k-1}(1-\epsilon))}{(k+1)(\frac{1}{2})^k} . \tag{19}$$

The right-hand sides of (18) and (19) tend to ϵ and 0, respectively, as $k \to \infty$, so (17) is verified for $n = 0$ and 1 (and k large enough). To verify (17) for $n \geq 2$ (and all such k), define two random variables Y and Y' as the number of 1's in two random elements of $\{0,1\}^{S_k}$ with respective distributions $\mu_k^{(-,\epsilon)}$ and $\pi_{\frac{1}{2}}$. Note that Y conditioned on taking value at least 2 has the same distribution as a Bin $(k, 1-\epsilon)$ random variable conditional on taking value at least 2, while the conditional distribution of Y' given that it is at least 2, is that of a Bin $(k, \frac{1}{2})$ variable conditioned on being at least 2. Defining $\rho_{k,(1-\epsilon),2}$ and $\rho_{k,\frac{1}{2},2}$ as in Lemma 3.3, we thus have for $n \in \{2, \ldots, k-1\}$ that

$$\mu_k^{(-,\epsilon)}(B_n) = 1 - (1-\mu_k^{(-,\epsilon)}(B_1))(1-\rho_{k,(1-\epsilon),2}(B_n)) \tag{20}$$

and

$$\pi_{\frac{1}{2}}(B_n) \;=\; 1-(1-\pi_{\frac{1}{2}}(B_1))(1-\rho_{k,\frac{1}{2},2}(B_n))\,. \qquad (21)$$

But we have already seen that $\mu_k^{(-,\epsilon)}(B_1) \le \pi_{\frac{1}{2}}(B_1)$, and Lemma 3.3 tells us that $\rho_{k,(1-\epsilon),2}(B_n) \le \rho_{k,\frac{1}{2},2}(B_n)$, so (20) and (21) yield

$$\mu_k^{(-,\epsilon)}(B_n) \;\le\; \pi_{\frac{1}{2}}(B_n)\,,$$

and CLAIM 3 is established.

Finally, we translate this example into the setting of translation invariant distributions on $\{0,1\}^{\mathbb{Z}}$. The measure $\tilde{\mu}_m$ can be turned into a translation invariant measure $\tilde{\mu}_m^*$ on $\{0,1\}^{\mathbb{Z}}$ by the same independent-copy-and-paste procedure as in Theorem 2.2. The property

$$\pi_{\frac{1}{2}} \;\preceq\; (\tilde{\mu}_m^*)^{(-,\epsilon)}$$

is obviously inherited from (14). Thus, in order to show that $\tilde{\mu}_m^*$ is nonrigid, it only remains to show that it does not stochastically dominate $\pi_{\frac{1}{2}+\delta}$ for any $\delta > 0$. This follows using (16) by an argument analogous to (9) in Theorem 2.2: If we pick k depending on δ as in the justification of CLAIM 2, then, under $\tilde{\mu}_m^*$, certain infinite arithmetic progressions will have subsequences of length k which contain at most one 1 often enough (under spatial averaging) that the corresponding event has $\pi_{\frac{1}{2}+\delta}$-measure 0. We omit the details. □

Remark 3.4 The measure $\tilde{\mu}_m$ is obviously not uniformly insertion tolerant, and we have thus demonstrated the existence of a measure for which property (IV) holds while neither (II) nor (III) does. □

Remark 3.5 For any $p \in (0,1)$, the construction above can be modified by replacing 2^{-k} by p^k in (12). Proceeding as in the rest of the proof yields the result that for any $p, \epsilon \in (0,1)$ such that $p+\epsilon < 1$, there exists a measure μ on $\{0,1\}^S$ where S is countably infinite, with the property that $p_{\max,\mu} = p$ and

$$\pi_{p_{\max,\mu}} \preceq \mu^{(-,\epsilon)}\,.$$

This is obviously sharp. □

4 Further results on rigidity

In this section, we continue the study of rigidity, and prove Theorem 1.9.

The proof of Theorem 1.9 will make use of the following technical lemma.

Lemma 4.1 *Let μ be a measure on $\{0,1\}^{\mathbb{Z}^d}$. Assume that it is δ-insertion tolerant for some $\delta > 0$. If for some $p \in (0,1)$ and $\epsilon > 0$*

$$\mu^{(-,\epsilon)}(\sigma(\{1,\ldots,n\}^d) \equiv 0) \leq (1-p)^{n^d} \text{ for all } n \geq 0, \tag{22}$$

then there exists $p' > p$ such that

$$\mu(\sigma(\{1,\ldots,n\}^d) \equiv 0) \leq (1-p')^{n^d} \text{ for all } n \geq 0.$$

Proof. Let $X \sim \mu$ and $Z \sim \pi_{1-\epsilon}$ be independent and let $X^{(-,\epsilon)} = \min(X,Z)$. It is easy to see using the δ-insertion tolerance that for any $s \in \{1,\ldots,n\}^d$, and any $\zeta \in \{0,1\}^{\{1,\ldots,n\}^d \setminus s}$

$$\begin{aligned}&\mathbb{P}(X(s) = 1 \cap X(\{1,\ldots,n\}^d \setminus s) \equiv \zeta)\\ &\quad\geq \frac{\delta}{1-\delta}\mathbb{P}(X(s) = 0 \cap X(\{1,\ldots,n\}^d \setminus s) \equiv \zeta).\end{aligned}$$

Iterating this, we get that for any $\xi \in \{0,1\}^{\{1,\ldots,n\}^d}$

$$\mathbb{P}(X(\{1,\ldots,n\}^d) \equiv \xi) \geq \left(\frac{\delta}{1-\delta}\right)^{|\xi|} \mathbb{P}(X(\{1,\ldots,n\}^d) \equiv 0).$$

Here $|\xi|$ denotes the cardinality of the set $\{s \in \{1,\ldots,n\}^d : \xi(s) = 1\}$. An elementary and straightforward calculation will now yield that

$$\mathbb{P}(X^{(-,\epsilon)}(\{1,\ldots,n\}^d) \equiv 0) \geq \left(1 + \frac{\epsilon\delta}{1-\delta}\right)^{n^d} \mathbb{P}(X(\{1,\ldots,n\}^d) \equiv 0).$$

Using this in combination with (22) proves the lemma. □

Proof of Theorem 1.9. The case $p_{\max,\mu} = 1$ is trivial and we therefore assume that $p_{\max,\mu} \in (0,1)$. In [12], it is shown that if μ is downwards FKG and if

$$\mu(\sigma(\{1,\ldots,n\}^d) \equiv 0) \leq (1-p)^{n^d} \text{ for all } n \geq 0, \tag{23}$$

then $\pi_p \preceq \mu$. Therefore if $\pi_{p_{\max,\mu}} \preceq \mu^{(-,\epsilon)}$ for some $\epsilon > 0$, then (22) trivially holds (with $p = p_{\max,\mu}$) and so we can conclude from Lemma 4.1 and the above result in [12] that $\pi_{p'} \preceq \mu$ for some $p' > p_{\max,\mu}$, a contradiction. □

5 Proof of main result

Lemma 5.1 *If μ is uniform upwards extractable, then for any $\epsilon > 0$ there exists a $\delta > 0$ such that $(\mu^{(-,\epsilon)})^{(+,\delta)} \preceq \mu$.*

Proof. Let ν and $\alpha > 0$ be such that $\mu = \nu^{(+,\alpha)}$. One can easily compute that for any α, ϵ, and δ, we have that

$$((\mu^{(+,\alpha)})^{(-,\epsilon)})^{(+,\delta)} = \mu^{(-,\epsilon(1-\delta),+,\alpha(1-\epsilon)+\alpha\epsilon\delta+(1-\alpha)\delta)}.$$

Now, given $\epsilon > 0$, choose $\delta > 0$ such that $\alpha(1-\epsilon)+\alpha\epsilon\delta+(1-\alpha)\delta < \alpha$. We therefore get that

$$(\mu^{(-,\epsilon)})^{(+,\delta)} = ((\nu^{(+,\alpha)})^{(-,\epsilon)})^{(+,\delta)} \preceq \nu^{(-,\epsilon(1-\delta),+,\alpha)} \preceq \nu^{(+,\alpha)} = \mu.$$

□

Lemma 5.2 *Given a probability measure μ on $\{0,1\}^S$, assume that for every $\epsilon > 0$, there exists a $\delta > 0$ such that $(\mu^{(-,\epsilon)})^{(+,\delta)} \preceq \mu$. Then μ is rigid.*

Proof. The case $p_{\max,\mu} = 1$ is trivial, and we will therefore assume that $p_{\max,\mu} \in (0,1)$. Assume for contradiction that μ is nonrigid. Then there exists an $\epsilon > 0$ such that $\pi_{p_{\max,\mu}} \preceq \mu^{(-,\epsilon)}$. By assumption there exists a $\delta > 0$ such that $(\mu^{(-,\epsilon)})^{(+,\delta)} \preceq \mu$. Hence $(\pi_{p_{\max,\mu}})^{(+,\delta)} \preceq (\mu^{(-,\epsilon)})^{(+,\delta)} \preceq \mu$. Since $p_{\max,\mu} < 1$, $(\pi_{p_{\max,\mu}})^{(+,\delta)}$ is a product measure with density strictly larger than $p_{\max,\mu}$. This is a contradiction. □

Our next example provides us with an example of a μ which is on one hand rigid but on the other hand not uniformly insertion tolerant. It is a variant of [14, Remark 6.4] and shows that the reverse statement of Lemma 5.1 is false.

Example 5.3 Let $\{X_i\}_{i\in\mathbb{N}}$ be defined in the following way. For every even $i \geq 0$, let independently (X_i, X_{i+1}) be $(1,1)$ or $(0,0)$ with probability $1/2$ each. Let μ_e denote the distribution of this process. For $\epsilon,\delta > 0$ let $\{X_i^{(-,\epsilon(1-\delta),+,\delta)}\}_{i\in\mathbb{N}}$ be a sequence of random variables with distribution $\mu_e^{(-,\epsilon(1-\delta),+,\delta)} = (\mu_e^{(-,\epsilon)})^{(+,\delta)}$. By noting that for any $\epsilon > 0$ there exists a $\delta > 0$ such that for even i

$$\mathbb{P}(\max(X_i^{(-,\epsilon(1-\delta),+,\delta)}, X_{i+1}^{(-,\epsilon(1-\delta),+,\delta)}) = 1) < \tfrac{1}{2},$$

we see that for the same choice of ϵ,δ we get that $(\mu_e^{(-,\epsilon)})^{(+,\delta)} \preceq \mu_e$. Lemma 5.2 gives us that μ_e is rigid. However, it is easy to see that μ_e is not uniform insertion tolerant. Furthermore it is possible to make this example translation invariant by some easy manipulations. □

Proof of Theorem 1.7. Lemma 5.1 together with Lemma 5.2 shows that property (I) implies property (III) and all the other implications were indicated in the introduction. As far as all of the reversed implications claimed not to hold, we continue as follows. Example 5.3 together with Lemma 5.2 shows that (III) does not imply (II) (and hence that (III) does not imply (I) and that (IV) does not imply (II)). Theorem 3.2 implies that (IV) does not imply (III). Finally, Theorem 2.2 shows that (II) does not imply (I). Also, all of these examples were translation invariant measures on $\{0,1\}^{\mathbb{Z}}$. □

Proof of Theorem 1.10. This follows immediately from Propositions 2.1 and 3.1. □

Of some interest in this context is the following result, which is an easy consequence of Lemma 5.1.

Corollary 5.4 *Assume that* (μ_1, μ_2) *is downwards movable and that* μ_2 *is uniformly upwards extractable. Then* (μ_1, μ_2) *is also upwards movable.*

Although it would take us too far afield to discuss details, let us mention that the processes studied in [14] provide a nice source of examples illustrating the various concepts in this paper. For example, one can find there examples of processes which are rigid but are not uniformly insertion tolerant.

6 Open problems

We end the paper with a list of open problems.

1. Does property (II) imply (III) in Theorem 1.7?

2. Let S be countable and μ be a uniformly insertion tolerant probability measure on $\{0,1\}^S$. Is it the case that for all $\epsilon > 0$, there exists a $\delta > 0$ such that $(\mu^{(-,\epsilon)})^{(+,\delta)} \preceq \mu$? A positive answer to this question would of course yield a positive answer to question 1.

3. Is the reverse statement of Lemma 5.2 true?

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