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ABSTRACT. Let L be a (semi)-positive line bundle over a Kähler manifold, X, fibered over a complex manifold Y. Assuming the fibers are compact and non-singular we prove that the hermitian vector bundle E whose fibers are the space of global sections to $L\otimes K_{X/Y}$ endowed with the L^2-metric is (semi)-positive in the sense of Nakano. As an application we prove a partial result on a conjecture of Griffiths on the positivity of ample bundles. This is a revised and much expanded version of a previous preprint with the title " Bergman kernels and the curvature of vector bundles".

1. INTRODUCTION

Let us first consider a domain $D = U \times \Omega$ in $\mathbb{C}^m \times \mathbb{C}^n$ and a function ϕ, plurisubharmonic in D. We also assume for simplicity that ϕ is smooth up to the boundary and strictly plurisubharmonic in D. Then, for each t in U, $\phi^t(\cdot) := \phi(t, \cdot)$ is plurisubharmonic in Ω and we denote by A^2_t the Bergman spaces of holomorphic functions in Ω with norm

$$\|h\|^2 = \|h\|^2_t = \int_\Omega |h|^2 e^{-\phi^t}.$$

The spaces A^2_t are then all equal as vector spaces but have norms that vary with t. The - infinite rank - vector bundle E over U with fiber $E_t = A^2_t$ is therefore trivial as a bundle but is equipped with a nontrivial metric. The first result of this paper is the following theorem.

Theorem 1.1. *The hermitian bundle $(E, \|\cdot\|_t)$ is strictly positive in the sense of Nakano.*

Of the two main differential geometric notions of positivity (see section 2, where these matters will be reviewed in the slightly non standard setting of bundles of infinite rank), positivity in the sense of Nakano is the stronger one and implies the weaker property of positivity in the sense of Griffiths. On the other hand the Griffiths notion of positivity has nicer functorial properties and implies in particular that the dual bundle is negative (in the sense of Griffiths). This latter property is in turn equivalent to the condition that if ξ is any nonvanishing local holomorphic section to the dual bundle, then the function

$$\log \|\xi\|^2$$

is strictly plurisubharmonic. We can obtain such holomorphic sections to the dual bundle from point evaluations. More precisely, let f be a holomorphic map from U to Ω and define ξ_t by is action on a local section to E

$$\langle \xi_t, h_t \rangle = h_t(f(t)).$$

Since the right hand side here is a holomorphic function of t, ξ is indeed a holomorphic section to E^*. The norm of ξ at a point is given by

$$\|\xi_t\|^2 = \sup_{\|h_t\| \leq 1} |h_t(f(t))| = K_t(f(t), f(t)),$$

where $K_t(z, z)$ is the Bergman kernel function for A_t^2. It therefore follows from Theorem 1.1 that $K_t(z, z)$ is plurisubharmonic in D, which is the starting point of the results in [1].

In [1] it is proved that this subharmonicity property of the Bergman kernel persists if D is a general pseudoconvex domain in $\mathbb{C}^m \times \mathbb{C}^n$, for general plurisubharmonic weight functions. In this case the spaces A_t^2 are the Bergman spaces for the slices of D, $D_t = \{z; (t, z) \quad \text{lies in} \quad D\}$. This more general case should also lead to a positively curved vector bundle. The main problem in proving such an extension of Theorem 1.1 is not to prove the inequalities involved, but rather to define the right notion of vector bundle in this case. In general, the spaces A_t^2 will not be identical as vector spaces, so the bundle in question is not locally trivial.

There is however a natural analog of Theorem 1.1 for holomorphic fibrations with compact fibers. Consider a complex manifold X of dimension $n + m$ which is fibered over another connected complex m-dimensional manifold Y. We then have a holomorphic map, p, from X to Y with surjective differential, and all the fibers $X_t = p^{-1}(t)$ are assumed compact. This implies, see [23], that the fibers are all diffemorphic, but they are in general not biholomorphic to each other. We also need to assume that the total space has a Kähler metric, ω. Let L be a holomorphic, hermitian line bundle over the total space X. Our substitute for the Bergman spaces A_t^2 is now the space of global sections over each fiber to $L \otimes K_{X_t}$,

$$E_t = \Gamma(X_t, L|X_t \otimes K_{X_t}),$$

where K_{X_t} is the canonical bundle of, i e the bundle of forms of bidegree $(n, 0)$ on each fiber. We assume that L is semipositive so that the hermitian metric on L can be, and is, chosen so that it has nonnegative curvature form. Fix a point y in Y and choose local coordinates $t = (t_1, ...t_m)$ near y with $t(y) = 0$. We consider the coordinates as functions on X by identifying t with $t \circ p$, and let $dt = dt_1 \wedge ...dt_m$. The canonical bundle of a fiber X_t can then be identified with the restriction of K_X, the canonical bundle of the total space, to X_t by mapping a local section u to K_{X_t} to $u \wedge dt$. This map is clearly injective, and it is also surjective since any section w to K_X can locally be written

$$w = u \wedge dt,$$

and the restriction of u to X_t is independent of the choice of u. With this identification, any global holomorphic section of $L \otimes K_{X_t}$ over a fiber can be extended to a holomorphic section of $L \otimes K_{X_s}$, for s near t. When L is trivial this follows from the Kähler assumption, by invariance of Hodge numbers, see [23]. When L is semipositive it follows from a variant of the Ohsawa-Takegoshi extension theorem, that we discuss in an appendix . Starting from a basis for $\Gamma(X_t, L|X_t \otimes K_{X_t})$ we therefore get a local holomorphic frame for E, so E has a natural structure as a holomorphic vector bundle. Moreover, elements of E_t can be naturally integrated over the fiber and we obtain in this way a metric, $\|\cdot\|$ on E in complete analogy with the plane case. We then get the same conclusion as before:

Theorem 1.2. *If the total space X is Kähler and L is (semi)positive over X, then $(E, \|\cdot\|)$ is (semi)positive in the sense of Nakano.*

This can be compared to a result of Kollar, [17], section 3, , who proved positivity properties for E when L is trivial and X projective. Kollar's results are however formulated in an algebraic way and not in terms of an explicit bound for the curvature tensor and the precise relation to Theorem 1.2 is not clear to me. Related work is also due to Tsuji, see [21] and the references therein. Tsuji's method is more analytical, but still assumes L to be trivial as well as some additional assumption on the fibration.

Not surprisingly, the curvature of the bundle E in Theorem 1.2 can be zero at some point and in some direction only if the curvature of the line bundle L also degenerates. In section 5 we shall prove a result that indicates that conditions for degeneracy of the curvature of E are much more restrictive than that: When X is a product, null vectors for the curvature can only come from infinitesimal automorphisms of the fiber.

In section 6 we discuss some, largely philosophical, relations between Theorem 1.2 and recent work on the variation of Kähler metrics. This corresponds to the case when X is a product $U \times Z$ with one-dimensional base U, and when L is the pull-back of a bundle on Z under the second projection map. The variation of the metric on L that we get from the fibration then gives a path in the space of Kähler metrics on Z and the lower bound that we get for the curvature operator in this case is precisely the Toepliz operator defined by the geodesic curvature of this path.

Another example of the situation in Theorem 1.2 arises naturally if we start with a (finite rank) holomorphic vector bundle V over Y and let $\mathbb{P}(V)$ be the associated bundle of projective spaces *of the dual bundle* V^*. This is then clearly an - even locally trivial - holomorphic fibration and there is a naturally defined line bundle L over the total space

$$L = O_{\mathbb{P}(V)}(1),$$

that restricts to the hyperplane section bundle over each fiber. The global holomorphic sections of this bundle over each fiber are now the linear forms on V^*, i e the elements of V. In other words, $\tilde{E}$ is isomorphic to V. To be able to integrate over the fibers, we need to take tensor products with the canonical bundle, and we therefore replace L by

$$L^{r+1} = O_{\mathbb{P}(V)}(r+1)$$

(with r being the rank of V). Since the canonical bundle of a fiber is $O(-r)$ we see that on each fiber $L \otimes K_{X_t} = O(1)$ so its space of global sections is again equal to V_t. Define as before

$$E = \Gamma(X_t, L^{r+1}|X_t \otimes K_{X_t}).$$

One can then verify that, globally, E is isomorphic to $V \otimes \det V$. The condition that L is positive is now equivalent to $O_{\mathbb{P}(V)}(1)$ being positive which is the same as saying that V is *ample* in the sense of Hartshorne, [12]. We therefore obtain the following result as a corollary of Theorem 1.2.

Theorem 1.3. *Let V be a (finite rank) holomorphic vector bundle over a complex manifold which is ample in the sense of Hartshorne. Then $V \otimes \det V$ has a smooth hermitian metric which is strictly positive in the sense of Nakano.*

Replacing $O_{\mathbb{P}(V)}(r+1)$ by $O_{\mathbb{P}(V)}(r+m)$, we also get that $S^m(V) \otimes \det V$ is Nakano-positive for any non-negative m, where $S^m(V)$ is the m:th symmetric power of V.

It is a well known conjecture of Griffiths, [10], that an ample vector bundle is positive in the sense of Griffiths. Theorem 1.3 can perhaps be seen as indirect evidence for this conjecture, since by a a theorem of Demailly, [7], $V \otimes \det V$ is Nakano positive if V itself is Griffiths positive. It seems that not so much is known about Griffiths' conjecture in general, except that it does hold when Y is a compact curve (see [22], [4]).

After the first version of this mansucript was completed I received a preprint by C Mourogane and S Takayama, [16]. There they prove that $V \otimes \det V$ is positive in the sense of Griffiths, assuming the base manifold is projective. The method of proof is quite different from this paper, as is the metric they find.

The proof of Theorem 1.1 is based on regarding the bundle E as a holomorphic subbundle of the hermitian bundle F with fibers

$$F_t = L^2(\Omega, e^{-\phi^t}) =: L^2_t.$$

By definition, the curvature of F is a $(1,1)$-form

$$\sum \Theta^F_{jk} dt_j \wedge d\bar{t}_k$$

whose coefficients are operators on F_t. By direct and simple computation,

$$\Theta^F$$

is the operator of multiplication with $\partial_t \bar{\partial}_t \phi$, so this is positive as soon as ϕ is plurisubharmonic of t for z fixed. By a formula of Griffiths, the curvature of the holomorphic subbundle E is obtained from the curvature of F by subtracting the *second fundamental form* of E, and the crux of the proof is to control this term by the curvature of F. For this we note that the second fundamental form is given by the square of the norm of an element in the orthogonal complement of A^2_t in L^2_t. This element is therefore the minimal solution of a certain $\bar\partial$-equation, and the needed inequality follows from an application of Hörmander's L^2-estimate.

We have not been able to generalize this proof to the situation of Theorem 1.2. The proof does generalize to the case of a holomorphically trivial fibration, but in the general case we have not been able to find a good choice of the bundle F with easily computed Chern connection and curvature operator. We therefore compute directly the Chern connection of the bundle E itself, and compute the curvature from there, much as one proves Griffiths' formula. In these computations appears also the Kodaira-Spencer class of the fibration, [23]. This class plays somewhat the role of another second fundamental form, but this time of a quotient bundle, arising when we restrict $(n, 0)$-forms to the fiber. The Kodaira-Spencer class therefore turns out to give a positive contribution to the curvature. This proof could also be adapted to give Theorem 1.1 by using fiberwise complete Kähler metrics, but we have chosen not to do so since the first proof seems conceptually clearer.

Finally, I would like to thank Sebastien Boucksom for pointing out the relation between Theorem 1.1 and the Griffiths conjecture, Jean-Pierre Demailly for encouraging me to treat also the case of a general non-trivial fibration and Yum-Tong Siu for helpful discussions and for mentioning to me the work of Kollar. Last but not least, thanks are due to H Yamaguchi, whose work on plurisubharmonicity of the Robin function [24] and Bergman kernels, [15] was an important source of inspiration for this work.

2. CURVATURE OF FINITE AND INFINITE RANK BUNDLES

Let E be a holomorphic vector bundle with a hermitian metric over a complex manifold Y. By definition this means that there is a holomorphic projection map p from E to Y and that every point in Y has a neighbourhood U such that $p^{-1}(U)$ is isomorphic to $U \times W$, where W is a vector space equipped with a smoothly varying hermitian metric. In our applications it is important to be able to allow this vector space to have infinite dimension, in which case we assume that the metrics are also complete, so that the fibers are Hilbert spaces.

Let $t = (t_1, ...t_m)$ be a system of local coordinates on Y. The Chern connection, D_{t_j} is now given by a collection of differential operators acting on smooth sections to $U \times W$ and satisfying

$$\partial_{t_j}(u, v) = (D_{t_j}u, v) + (u, \bar\partial_{t_j}v),$$

with $\partial_{t_j} = \partial/\partial_{t_j}$ and $\bar{\partial}_{t_j} = \partial/\partial \bar{t}_j$. The curvature of the Chern connection is a $(1,1)$-form of operators

$$\Theta = \sum \Theta_{jk} dt_j \wedge d\bar{t}_k,$$

where the coefficients Θ_{jk} are densily defined operators on W. By definition these coefficients are the commutators

$$\Theta_{jk} = [D_{t_j}, \bar{\partial}_{t_k}].$$

The vector bundle is said to be positive in the sense of Griffiths if for any section u to W and any vector v in $\mathbb{C}^m$

$$\sum (\Theta_{jk} u, u) v_j \bar{v}_k \geq \delta \|u\|^2 |v|^2$$

for som positive δ. E is said to be positive in the sense of Nakano if for any m-tuple $(u_1, ...u_m)$ of sections to W

$$\sum (\Theta_{jk} u_j, u_k) \geq \delta \sum \|u_j\|^2$$

Taking $u_j = uv_j$ we see that Nakano positivity implies positivity in the sense of Griffiths.

The dual bundle of E is the vector bundle E^* whose fiber at a point t in Y is the Hilbert space dual of E_t. There is therefore a natural antilinear isometry between E^* and E, which we will denote by J. If u is a local section to E, ξ is a local section to E^*, and $\langle \cdot, \cdot \rangle$ denotes the pairing between E^* and E we have

$$\langle \xi, u \rangle = (u, J\xi).$$

Under the natural holomorphic structure on E^* we then have

$$\bar{\partial}_{t_j} \xi = J^{-1} D_{t_j} J \xi,$$

and the Chern connection on E^* is given by

$$D^*_{t_j} \xi = J^{-1} \bar{\partial}_{t_j} J \xi.$$

It follows that

$$\bar{\partial}_{t_j} \langle \xi, u \rangle = \langle \bar{\partial}_{t_j} \xi, u \rangle + \langle \xi, \bar{\partial}_{t_j} u \rangle,$$

and

$$\partial_{t_j} \langle \xi, u \rangle = \langle D^*_{t_j} \xi, u \rangle + \langle \xi, D_{t_j} u \rangle,$$

and hence

$$0 = \left[\partial_{t_j}, \bar{\partial}_{t_j}\right] \langle \xi, u \rangle = \langle \Theta^*_{jk} \xi, u \rangle + \langle \xi, \Theta_{jk} u \rangle,$$

if we let Θ^* be the curvature of E^*. If ξ_j is an r-tuple of sections to E^*, and $u_j = J\xi_j$, we thus see that

$$\sum (\Theta^*_{jk} \xi_j, \xi_k) = -\sum (\Theta_{jk} u_k, u_j).$$

Notice that the order between u_k and u_j in the right hand side is opposite to the order between the ξs in the left hand side. Therefore E^* is negative in the sense of Griffiths iff E is positive in the sense of Griffiths, but we can not draw the same conclusion in the the case of Nakano positivity.

If u is a holomorphic section to E we also find that

$$\frac{\partial^2}{\partial t_j \partial \bar{t}_k}(u,u) = (D_{t_j}u, D_{t_k}u) - (\Theta_{jk}u, u)$$

and it follows after a short computation that E is (strictly) negative in the sense of Griffiths if and only if $\log \|u\|^2$ is (strictly) plurisubharmonic for any nonvanishing holomorphic section u.

We next briefly recapitulate the Griffiths formula for the curvature of a subbundle. Assume E is a holomorphic subbundle of the bundle F, and let π be the fiberwise orthogonal projection from F to E. We also let $\pi_\perp$ be the orthogonal projection on the orthogonal complement of E. By the definition of Chern connection we have

$$D^E = \pi D^F.$$

Let $\bar{\partial}_{t_j}\pi$ be defined by

$$\bar{\partial}_{t_j}(\pi u) = (\bar{\partial}_{t_j}\pi)u + \pi(\bar{\partial}_{t_j}u). \tag{2.1}$$

Computing the commutators occuring in the definition of curvature we see that

$$\Theta^E_{jk}u = -(\bar{\partial}_{t_k}\pi)D^F_{t_j}u + \pi\Theta^F_{jk}u, \tag{2.2}$$

if u is a section to E. By (2.1) $(\bar{\partial}\pi)v = 0$ if v is a section to E, so

$$(\bar{\partial}\pi)D^F u = (\bar{\partial}\pi)\pi_\perp D^F u. \tag{2.3}$$

Since $\pi\pi_\perp = 0$ it also follows that

$$(\bar{\partial}\pi)\pi_\perp D^F u = -\pi\bar{\partial}(\pi_\perp D^E u),$$

so if v is also a section to E,

$$((\bar{\partial}_{t_k}\pi)D^F_{t_j}u, v) = -(\bar{\partial}_{t_k}(\pi_\perp D^F_{t_j}u), v) =$$

$$= ((\pi_\perp D^F_{t_j}u), D^F_{t_k}v) = (\pi_\perp(D^F_{t_j}u), \pi_\perp(D^F_{t_k}v)).$$

Combining with (2.2) we finally get that if u and v are both sections to E then

$$(\Theta^F_{jk}u, v) = (\pi_\perp(D^F_{t_j}u), \pi_\perp(D^F_{t_k}v)) + (\Theta^E_{jk}u, v), \tag{2.4}$$

which is the starting point for the proof of Theorem 1.1.

For the proof of Theorem 1.2 we finally describe another way of computing the curvature form of a vector bundle. Fix a point y in Y and choose local coordinates t centered at y. Any point u_0 in the fiber E_0 over y can be extended to a holomorphic section u of E near 0. Modifying u by a linear combination $\sum t_j v_j$ for suitably chosen local holomorphic sections v_j we can also arrange things so that $Du = 0$ at $t = 0$. Let u and v be two local sections with this property and compute

$$\partial_{\bar{t}_k}\partial_{t_j}(u,v) = \partial_{\bar{t}_k}(D_{t_j}u, v) = (\partial_{\bar{t}_k}D_{t_j}u, v) = -(\Theta_{jk}u, v).$$

Let u_j me an m-tuple of holomorphic sections to E, satisfying $Du_j = 0$ at 0. Put

$$T_u = \sum (u_j, u_k) \widehat{dt_j \wedge d\bar{t}_k}.$$

Here $\widehat{dt_j \wedge d\bar{t}_k}$ denotes the wedge product of all dt_i and $d\bar{t}_i$ except dt_j and $d\bar{t}_k$, with a sign chosen so that T_u is a positive form. Then

$$i\partial\bar{\partial}T_u = -\sum (\Theta_{jk} u_j, u_k) dV_t,$$

so E is Nakano-positive at a given point if and only if this expression is negative for any choice of holomorphic sections u_j satisfying $Du_j = 0$ at the point.

3. THE PROOF OF THEOREM 1.1

We consider the setup described before the statement of Theorem 1.1 in the introduction. Thus E is the vector bundle over U whose fibers are the Bergman spaces A_t^2 equipped with the weighted L^2 metrics induced by $L^2(\Omega, e^{-\phi^t})$. We also let F be the vector bundle with fiber $L^2(\Omega, e^{-\phi^t})$, so that E is a trivial subbundle of the trivial bundle F with a metric induced from a nontrivial metric on F. From the definition of the Chern connection we see that

$$D_{t_j}^F = \partial_{t_j} - \phi_j,$$

where the last term in the right hand side should be interpreted as the operator of multiplication by the (smooth) function $-\phi_j = -\partial_{t_j}\phi^t$. (In the sequel we use the letters j, k for indices of the t-variables,and the letters λ, μ for indices of the z-variables.) For the curvature of F we therefore get

$$\Theta_{jk}^F = \phi_{jk},$$

the operator of multiplication with the Hessian of ϕ with respect to the t-variables. We shall now apply formula (2.4), so let u_j be smooth sections to E. This means that u_j are functions that depend smoothly on t and holomorphically on z. To verify the positivity of E in the sense of Nakano we need to estimate from below the curvature of E acting on the k-tuple u,

$$\sum (\Theta_{jk}^E u_j, u_k).$$

By (2.4) this means that we need to estimate from above

$$\sum (\pi_\perp(\phi_j u_j), \pi_\perp(\phi_k u_k)) = \|\pi_\perp(\sum \phi_j u_j)\|^2.$$

Put $w = \pi_\perp(\sum \phi_j u_j)$. For fixed t, w solves the $\bar{\partial}_z$-equation

$$\bar{\partial} w = \sum u_j \phi_{j\lambda} d\bar{z}_\lambda,$$

since the u_js are holomorphic in z. Moreover, since w lies in the orthogonal complement of A^2, w is the minimal solution to this equation.

We shall next apply Hörmander's weighted L^2-estimates for the $\bar{\partial}$-equation. The precise form of these estimates that we need says that if f is a $\bar{\partial}$-closed form in a pseududoconvex domain Ω, and if ψ is a smooth strictly plurisubharmonic weight function, then the minimal solution w to the equation $\bar\partial v = f$ satisfies

$$\int_\Omega |w|^2 e^{-\psi} \leq \int_\Omega \sum \psi^{\lambda\mu} f_\lambda \bar{f}_\mu e^{-\psi},$$

where $(\psi^{\lambda\mu})$ is the inverse of the complex Hessian of ψ (see [6]).

In our case this means that

$$\int_\Omega |w|^2 e^{-\phi^t} \leq \int_\Omega \sum \phi^{\lambda\mu} \phi_{j\lambda} u_j \overline{\phi_{k\mu} u_k} e^{-\phi^t}.$$

Inserting this estimate in formula (2.4) together with the formula for the curvature of F we find

$$\sum (\Theta^E_{jk} u_j, u_k) \geq \int_\Omega \sum_{jk} \left(\phi_{jk} - \sum_{\lambda\mu} \phi^{\lambda\mu} \phi_{j\lambda} \bar\phi_{k\mu} \right) u_j \bar{u}_k e^{-\phi^t}. \tag{3.1}$$

We claim that the expression

$$D_{jk} =: \left(\phi_{jk} - \sum_{\lambda\mu} \phi^{\lambda\mu} \phi_{j\lambda} \bar\phi_{k\mu} \right),$$

in the integrand is a positive definite matrix at any fixed point . By a linear change of variables in t we may of course assume that the vector u that D acts on equals $(1, 0...0)$. Let $\Phi = i\partial\bar\partial\phi$ where the $\partial\bar\partial$-operator acts on t_1 and the z-variables, the remaining t-variables being fixed. Then

$$\Phi = \Phi_{11} + i\alpha \wedge d\bar{t}_1 + i dt_1 \wedge \bar\alpha + \Phi',$$

where Φ_{11} is of bidegree $(1,1)$ in t_1, α is of bidegree $(1,0)$ in z, and Φ' is of bidegree $(1,1)$ in z. Then

$$\Phi_{m+1} = \Phi^{m+1}/(m+1)! = \Phi_{1,1} \wedge \Phi'_m - i\alpha \wedge \bar\alpha \wedge \Phi'_{m-1} \wedge i dt_1 \wedge d\bar{t}_1.$$

Both sides of this equation are forms of maximal degree that can be written as certain coefficients multiplied by the Euclidean volume form of $\mathbb{C}^{m+1}$. The coefficient of the left hand side is the hessian of ϕ with respect to t_1 and z together. Similarily, the coefficient of the first term on the right hand side is ϕ_{11} times the hessian of ϕ with respect to the z-variables only. Finally, the coefficient of the last term on the right hand side is the norm of the $(0,1)$ form in z

$$\bar\partial_z \partial_{t_1} \phi$$

measured in the metric defined by Φ', multiplied by the volume form of the same metric. Dividing by the coefficient of Φ'_m we thus see that the matrix D acting on a vector u as above equals the hessian of ϕ with respect to t_1 and z divided by the hessian of ϕ with respect to the z-variables only. This expression is therefore positive so the proof of Theorem 1.1 is complete.

4. KÄHLER FIBRATIONS WITH COMPACT FIBERS

Let X be a Kähler manifold of dimension $m+n$, fibered over a complex m-dimensional manifold Y. This means that we have a holomorphic map p from X to Y with surjective differential at all points. All our computations will be local, so we may as well assume that $Y = U$ is a ball or polydisk in $\mathbb{C}^m$. For each t in U we let

$$X_t = p^{-1}(t)$$

be the fiber of X over t. We shall assume that all fibers are compact.

Next, we let L be a holomorphic hermitian line bundle over X. Our standing assumption on L is that it is semipositive, i e that it is equipped with a smooth hermitian metric of nonnegative curvature. For each fiber X_t we are interested in the space of holomorphic L-valued $(n,0)$-forms on X_t,

$$\Gamma(X_t, L|_{X_t} \otimes K_{X_t}) =: E_t.$$

For each t, E_t is a finite dimensional vector space and we claim that

$$E := \bigcup \{t\} \times E_t$$

has a natural structure as a holomorphic vector bundles.

To see this we first note that K_{X_t} is isomorphic to $K_X|_{X_t}$, the restriction of the canonical bundle of the total space to X_t, via the map that sends a section u to K_{X_t} to

$$\tilde{u} := u \wedge dt,$$

where $dt = dt_1 \wedge ...dt_m$. It is clear that this map is injective. Conversely, any local section $\tilde{u}$ to K_X can be locally represented as $\tilde{u} := u \wedge dt$, and even though u is not uniquely determined, the restriction of u to each fiber is uniquely determined. The semipositivity of L, and the assumption that X is Kähler, implies that any section u to K_{X_t} for one fixed t can be locally extended in the sense that there is a holomorphic section $\tilde{u}$ to K_X over $p^{-1}(W)$ for some neighbourhood W of t whose restriction to X_t maps to u under the isomorphism above. In case L is trivial this follows from the fact that Hodge numbers are locally constant, see [23]. For general semipositive bundles L it follows from a result of Ohsawa-Takegoshi type, that will be discussed in an appendix.

Taking a basis for E_t for one fixed t and extending as above we therefore get a local frame for the bundle E. We define a complex structure on E by saying that an $(n,0)$-form over $p^{-1}(W)$, u, whose restriction to each fiber is holomorphic, defines a holomorphic section to E if $u \wedge dt$ is a holomorphic section to K_X. The frame we have constructed is therefore holomorphic.

Note that this means that u is holomorphic if and only if $\bar\partial u$ can be written

$$\bar\partial u = \sum \eta^j \wedge dt_j, \tag{4.1}$$

with η^j smooth forms of bidegree $(n-1,1)$. Again, the η^j are not uniquely determined, but their restrictions to fibers are.

Even though we will not use it, it is worth mentioning the connection between the forms η_j^k and the Kodaira-Spencer class of the fibration, see [23]. The Kodaira-Spencer class at a point t in the base, is a map from the holomorphic tangent space of U to the first Dolbeault cohomology group,

$$H^{1,0}(X_t, T^{1,0}(X_t)),$$

of X_t with values in the holomorphic tangent space of X_t, i e, as t varies it is a $(1,0)$-form, $\sum \theta_j dt_j$ on U with values in $H^{1,0}(X_t, T^{1,0}(X_t))$.The classes θ_j can be represented by $\bar\partial$-closed $(0,1)$-forms on X_t whose coefficients are vector fields of type $(1,0)$. Letting the vectorfield act on forms by contraction we obtain a map from (p,q)-forms on X_t to $(p-1,q+1)$-forms. The forms

$$\eta^j$$

is what we obtain when we let this map operate on u. Different choices of extensions of u from X_t correspond to different representatives of the same cohomology class.

Let now u be a smooth local section to E. This means that u can be represented by a smooth L-valued form of bidegree $(n,0)$ over $p^{-1}(W)$ for some W open in U, such that the restriction of u to each fiber is holomorphic. Then

$$\bar\partial u = \sum d\bar t_j \wedge \nu^j,$$

where ν^j define sections to E. We define the $(0,1)$-part of the connection D on E by letting

$$D^{0,1}u = \sum \nu^j d\bar t_j.$$

Sometimes we write

$$\nu^j = \bar\partial_{t_j} u$$

with the understanding that this refers to the $\bar\partial$ operator on E. Note that $D^{0,1}u = 0$ for $t = t_0$ if and only if each ν^j vanishes when restricted to X_{t_0}, i e if $\bar\partial u \wedge dt = 0$, which is consistent with the definition of holomorphicity given earlier. Note also that if we chose another $(n,0)$-form u' to represent the same section to E, then $u - u'$ vanishes when restricted to each fiber. Hence $u - u' = \sum a_j \wedge dt_j$ and it follows that $D^{0,1}$ is well defined.

The bundle E has a naturally defined hermitian metric, induced by the metric on L. To define the metric, let u_t be an element of E_t. Locally, with respect to a local trivialization of L, u_t is given by a scalar valued $(n,0)$-form, u', and the metric on L is given by a smooth weight function ϕ'. Put

$$[u_t, u_t] = c_n u' \wedge \overline{u'} e^{-\phi'},$$

with $c_n = i^{n^2}$ is chosen to make this (n,n)-form positive. Clearly this definition is independent of the trivialization, so $[u_t, u_t]$ is globally defined. The metric on E_t is now defined as

$$\|u_t\|^2 = \int_{X_t} [u_t, u_t],$$

and the associated scalar product is

$$(u_t, v_t)_t = \int_{X_t} [u_t, v_t].$$

In the sequel we will, abusively, write $[u, v] = c_n u \wedge \bar{v} e^{-\phi}$. When t varies we suppress the dependence on t and get a smooth hermitian metric on E. For local sections u and v to E the scalar product is then a function of t and it will be convenient to write this function as

$$(u, v) = p_*([u, v]) = p_*(c_n u \wedge \bar{v} e^{-\phi}).$$

Here p_* denotes the direct image, or push-forward, of a form, defined by

$$\int_U p_*(\alpha) \wedge \beta = \int_X \alpha \wedge p^*(\beta),$$

if α is a form on X and β is a form on U.

With the metric and the $\bar{\partial}$ operator defined on E we can now proceed to find the $(1, 0)$-part of the Chern connection. Let u be a form on X with values in L. Locally, with respect to a trivialization of L, u is given by a scalar valued form u' and the metric on L is given by a function ϕ'. Let

$$\partial^{\phi'} u' = e^{\phi'} \partial(e^{-\phi'} u').$$

One easily verifies that this expression is invariantly defined, and we will, somewhat abusively, write $\partial^{\phi'} u' = \partial^{\phi} u$, using ϕ to indicate the metric on L. Let now in particular u be of bidegree $(n, 0)$ and such that the restrictions of u to fibers are holomorphic. As $\partial^{\phi} u$ is of bidegree $(n + 1, 0)$ we can write

$$\partial^{\phi} u = \sum dt_j \wedge \mu^j,$$

where μ^j are smooth $(n, 0)$-forms whose restrictions to fibers are uniquely defined. These restrictions are in general not holomorphic so we let

$$P(\mu^j)$$

be the orthogonal projection of μ^j on the space of holomorphic forms on each fiber. We claim that the $(1, 0)$-part of the Chern connection on E is given by

$$D^{1,0} u = \sum P(\mu^j) dt_j.$$

To prove this it suffices, by the definition of Chern connection, to verify that

$$\partial_{t_j}(u, v) = (P(\mu^j), v) + (u, \bar{\partial}_{t_j} v) = (\mu^j, v) + (u, \bar{\partial}_{t_j} v) \tag{4.2}$$

if u and v are smooth sections to E. But

$$\partial(u, v) = \partial p_*([u, v]) =$$

$$= c_n (p_*(\partial^{\phi} u \wedge \bar{v}\, e^{-\phi}) + (-1)^n p_*(u \wedge \overline{\bar{\partial} v}\, e^{-\phi})) =$$

$$c_n (p_*(\sum \hat{u}_j \wedge \bar{v} \wedge dt_j\, e^{-\phi}) + p_*(u \wedge \bar{\nu}^j \wedge dt_j\, e^{-\phi})).$$

This equals

$$\sum ((\mu^j, v) + (u, \nu^j)) dt_j,$$

so we have proved 4.2. We will write $P(\mu^j) = D_{t_j} u$.

We are now ready to verify the Nakano positivity of the bundle E. For this we will use the recipe given at the end of section 2. Let u_j be an m-tuple of holomorphic sections to E that satisfy $D^{1,0} u_j = 0$ at a given point that we take to be equal to 0. Let

$$T_u = \sum (u_j, u_k) \widehat{dt_j \wedge d\bar{t}_k}.$$

Here $\widehat{dt_j \wedge d\bar{t}_k}$ denotes the product of all differentials dt_i and $d\bar{t}_i$, except dt_j and $d\bar{t}_k$ multiplied by a number of modulus 1, so that T_u is nonnegative. We need to verify that

$$i\partial\bar{\partial} T_u$$

is negative. Represent the u_js by smooth forms on X, and put

$$\hat{u} = \sum u_j \wedge \widehat{dt_j}.$$

Then, with $N = n + m - 1$,

$$T_u = c_N p_*(\hat{u} \wedge \overline{\hat{u}}\, e^{-\phi})$$

Thus

$$\bar{\partial} T_u = c_N (p_*(\bar{\partial}\hat{u} \wedge \overline{\hat{u}} e^{-\phi}) + (-1)^N p_*(\hat{u} \wedge \overline{\partial^\phi \hat{u}} e^{-\phi})).$$

Since each u_j is holomorphic we have seen that

$$\bar{\partial} u_j = \sum \eta_j^l \wedge dt_l,$$

so the first term on the right hand side vanishes identically for reasons of bidegree. Thus

$$\partial\bar{\partial} T_u = c_N ((-1)^N p_*(\partial^\phi \hat{u} \wedge \overline{\partial^\phi \hat{u}} e^{-\phi}) + p_*(\hat{u} \wedge \overline{\bar{\partial}\partial^\phi \hat{u}}))$$

We rewrite the last term, using

$$\bar{\partial}\partial^\phi + \partial^\phi\bar{\partial} = \partial\bar{\partial}\phi.$$

Since

$$p_*(\hat{u} \wedge \overline{\bar{\partial}\hat{u}})$$

vanishes identically we find that

$$(-1)^N p_*(\hat{u} \wedge \overline{\partial^\phi \bar{\partial}\hat{u}}) + p_*(\bar{\partial}\hat{u} \wedge \overline{\bar{\partial}\hat{u}}) = 0,$$

so all in all

$$\partial\bar{\partial} T_u = \tag{4.3}$$

$$c_N \left((-1)^N p_*(\partial^\phi \hat{u} \wedge \overline{\partial^\phi \hat{u}} e^{-\phi}) - p_*(\hat{u} \wedge \overline{\hat{u}} \wedge \partial\bar{\partial}\phi) + (-1)^N p_*(\bar{\partial}\hat{u} \wedge \overline{\bar{\partial}\hat{u}}) \right).$$

Now recall that

$$\partial^\phi u_j = \sum dt_k \wedge \mu_j^k.$$

Since we have chosen the u_j so that $D^{1,0}u_j = 0$ at $t = 0$ it follows that the restrictions of all μ_j^k to X_0 are orthogonal to the space of holomorphic L-valued $(n, 0)$-forms. Thus

$$\partial^\phi \hat{u} = (-1)^n \sum \mu_j^j dt =: \mu dt,$$

where μ is also orthogonal to holomorphic forms on X_0 Let dV_t be the (Euclidean) volume element in the t-variables.If we multiply 4.3 by i, the first term in the right hand side is

$$\int_{X_0} |\mu|^2 \, dV_t.$$

This therefore gives a positive contribution in the computation of $i\partial\bar{\partial}T_u$, and it plays a role of the second fundamental form for the subbundle of holomorphic forms in section 3. We will control it in the same way as before, using that μ is the L^2-minimal solution to a certain $\bar{\partial}$-equation.

To compute the contribution coming from the third term in 4.3 we use that u_j are holomorphic sections to E. This means that

$$\bar{\partial} u_j = \sum \eta_j^k \wedge dt_k,$$

where η_j^k are of bidegree $(n-1, 1)$. Hence

$$\bar{\partial} \hat{u} = \sum \eta_j^j dt =: \eta dt.$$

What we get from the third term is thus

$$c_n \int_{X_0} \eta \wedge \bar{\eta} \, dV_t.$$

The quadratic form in η appearing here is indefinite. To see that we can choose η so that this form is negative we need a lemma.

Lemma 4.1. *Let ω be the Kähler form on X. All $\eta_j^k \wedge \omega$, and hence $\eta \wedge \omega$ are $\bar{\partial}$-exact on X_0.*

Proof. Let τ be a $\bar{\partial}$-closed L^{-1}-valued $(0, n-2)$-form on X_0, and extend τ smoothly to X. By Serre duality it is enough to prove that

$$\int_{X_0} \eta_j^k \wedge \omega \wedge \tau = 0.$$

But

$$\bar{\partial} u_j = \sum \eta_j^k dt_k,$$

so

$$\bar{\partial} u_j \wedge \omega = \sum \eta_j^k \wedge \omega \wedge dt_k.$$

It is therefore enough to prove that

$$p_*(\bar{\partial} u_j \wedge \omega \wedge \tau) = 0$$

for $t = 0$. We use

$$p_*(\bar{\partial} u_j \wedge \omega \wedge \tau) = \bar{\partial} p_*(u_j \wedge \omega \wedge \tau) + (-1)^n p_*(u_j \wedge \omega \wedge \bar{\partial}\tau).$$

The first term on the right hand side here vanishes identically since $p_*(u_j \wedge \omega \wedge \tau)$ is zero for degree reasons. The second term vanishes for $t = 0$ since it is a form of bidegree $(1,0)$ in t and $\bar\partial\tau$ is zero on X_0 so $\bar\partial\tau \wedge d\bar t = 0$ for $t = 0$. □

This means that we can write

$$\eta_j^k \wedge \omega = \bar\partial \xi_j^k,$$

on X_0 with ξ_j^k of bidegree $(n,1)$. Moreover, ξ_j^k can be written $\gamma_j^k \wedge \omega$ for some $(n-1,0)$-forms on X_0 that we extend smoothly to X.

Now replace our choice of the $(n,0)$-forms u_j by

$$u_j - \sum_k \gamma_j^k \wedge dt_k.$$

This does not change the u_j considered as sections to E, but it has the effect of changing η_j^k to

$$\eta_j^k - \bar\partial\gamma_j^k,$$

so after this change we get that all forms $\eta_j^k \wedge \omega$ vanish on X_0. It also changes the μ_j^k, but since $D^{1,0}u_j$ is unchanged the new μ_j^k are also orthogonal to the space of holomorphic forms on X_0 - which is easy to see directly too.

The upshot of this is that we may now assume that $\eta \wedge \omega = 0$ on X_0, so η is a primitive form on X_0. For primitive forms

$$c_n \int_{X_0} \eta \wedge \bar\eta = -\int_{X_0} |\eta|^2.$$

With this we have obtained that, for $t = 0$,

$$i\partial\bar\partial T_u = \left(\int_{X_0} |\mu|^2 - \int_{X_0} |\eta|^2\right) dV_t - p_*(\hat u \wedge \overline{\hat u} \wedge i\partial\bar\partial\phi), \tag{4.4}$$

and we are now ready for the final estimate.

Recall that $\partial^\phi \hat u = \mu \wedge dt$ and that $\bar\partial \hat u = \eta \wedge dt$. Hence

$$\partial\bar\partial\phi \wedge \hat u = \bar\partial\partial^\phi \hat u + \partial^\phi \bar\partial \hat u = (\bar\partial\mu + \partial^\phi\eta) \wedge dt.$$

This means that, on X_0

$$\bar\partial\mu = -\partial^\phi\eta + R,$$

where R satisfies

$$R \wedge dt = i\partial\bar\partial\phi \wedge \hat u.$$

Since μ is orthogonal to the space of holomorphic forms on X_0 it follows that on X_0 μ can be written

$$\mu = \bar\partial^*\alpha,$$

where α is a $\bar\partial$-closed form of bidegree $(n,1)$. Letting χ be equal to the Hodge-* of α, so that $\chi \wedge \omega = \alpha$, we get

$$\mu = \partial^\phi\chi.$$

Then, with $M = n + m$,

$$\int_{X_0} |\mu|^2 \, dV_t = c_M p_*(\mu \wedge dt \wedge \bar{\mu} \wedge d\bar{t} \, e^{-\phi}) =$$

$$= c_M (-1)^M p_*(\chi \wedge dt \wedge \overline{\bar{\partial}\mu} \wedge d\bar{t} \, e^{-\phi}) =$$

$$= c_M (-1)^M \left(-p_*(\chi \wedge dt \wedge \overline{\partial^\phi \eta} \wedge d\bar{t} e^{-\phi}) + p_*(\chi \wedge dt \wedge \overline{\hat{u}} \wedge i\partial\bar{\partial}\phi e^{-\phi}) \right),$$

at $t = 0$. The first term here is, up to a sign equal to

$$\int_{X_0} \bar{\partial}\chi \wedge \bar{\eta} e^{-\phi} \, dV_t,$$

where the integral, by the Cauchy inequality, is dominated by

$$1/2 \int_{X_0} |\bar{\partial}\chi|^2 + 1/2 \int_{X_0} |\eta|^2.$$

For the second term we use the Cauchy inequality for the quadratic form

$$(\alpha, \beta) = \alpha \wedge \bar{\beta} \wedge i\partial\bar{\partial}\phi,$$

and find that it can be dominated by

$$1/2 \int_{X_0} \chi \wedge \bar{\chi} \wedge i\partial\bar{\partial}\phi \, dV_t + 1/2 p_*(\hat{u} \wedge \overline{\hat{u}} \wedge i\partial\bar{\partial}\phi).$$

Collecting, we have proved that

$$\int_{X_0} |\mu|^2 \, dV_t \leq \tag{4.5}$$

$$\leq 1/2 \left(\int_{X_0} |\bar{\partial}\chi|^2 + \int_{X_0} |\eta|^2 + \int_{X_0} \chi \wedge \bar{\chi} \wedge i\partial\bar{\partial}\phi \right) dV_t + 1/2 p_*(\hat{u} \wedge \overline{\hat{u}} \wedge i\partial\bar{\partial}\phi).$$

We now apply the Hörmander, or in this case of compact manifolds, Kodaira-Nakano, identity:

$$\int_{X_0} \chi \wedge \bar{\chi} \wedge i\partial\bar{\partial}\phi \, e^{-\phi} + \int_{X_0} |\bar{\partial}\chi|^2 = \int_{X_0} |\mu|^2.$$

Inserting in 4.5, we get after simplification that

$$\left(\int_{X_0} |\mu|^2 - \int_{X_0} |\eta|^2 \right) dV_t - p_*(\hat{u} \wedge \overline{\hat{u}} \wedge i\partial\bar{\partial}\phi) \leq 0. \tag{4.6}$$

By 4.4, this means that $i\partial\bar{\partial}T_u \leq 0$, so E is at least seminegative in the sense of Nakano. If $i\partial\bar{\partial}\phi$ is strictly positive, equality can hold only if $\chi \wedge dt$ is equal to $\hat{u}$, which is only possible if u_j vanish on X_0. Therefore E is strictly positive if L is strictly positive so we have proved Theorem 1.2.

In the next section we shall see that even when L is only semipositive, equality can hold in our estimates only in very special cases.

5. SEMIPOSITIVE VECTOR BUNDLES.

In this section we will discuss when equality holds in the inequalities of Theorem 1.2, i e when the bundle E is not strictly positive. As we have already seen in the last section, this can only happen if the line bundle L is not strictly positive. More specifically, it requires that equality holds in both applications of the Cauchy inequality at the end of the previous section. In the first case, this implies that $\eta = \bar\partial\chi$, so in particular η is $\bar\partial$-exact on X_0.

For simplicity we assume from now that the base domain U is one-dimensional, so that we do not need to discuss degeneracy in different directions. Then, if $\eta = \bar\partial\chi$, we may change our form

$$\hat{u} = u_1 =: u$$

to $u - \chi \wedge dt$. Since the restriction of $\chi \wedge dt$ to each fiber vanishes, this new form still represents the same section to E and now $\bar\partial(u - \chi \wedge dt) = 0$ (on X_0). In other words, we may assume that $\eta = 0$ on X_0. But, then it also follows that $\bar\partial\chi = 0$, so χ is a holomorphic L-valued $(n-1, 0)$-form on X_0.

To continue to analyse this situation one step further we now assume that the fibration we consider is locally holomorphically trivial, i e that $X = U \times Z$, where Z is a compact n-dimensional complex manifold. Moreover, we assume that the curvature of our metric ϕ on L, Θ^L, is strictly positive along each fiber $X_t \simeq Z$. Then we can also assume that we have chosen our Kähler metric ω on X so that $\omega_t := \omega|X_t = i\Theta^L|X_t$ on each fiber.

Since X now is a global product we can decompose Θ^L according to its degree in t and z, where z is a ny local coordinate on Z. In particular, there is a well defined $(0, 1)$-form θ^L on X such that $dt \wedge \theta^L$ is the component of Θ^L of degree 1 in dt. Expressed in invariant language,

$$\theta^L = \delta_{\partial/\partial t}\Theta^L,$$

where $\delta_\cdot$ means contraction with a vector field. Using the formulas

$$\partial\delta_V + \delta_V\partial = \mathcal{L}_V, \tag{5.1}$$

where $\mathcal{L}$ is the Lie derivative, and

$$\bar\partial\delta_V + \delta_V\bar\partial = 0, \tag{5.2}$$

if V is a holomorhic vector field we see that θ^L is $\bar\partial$-closed, and that

$$\partial\theta^L = \mathcal{L}_{\partial/\partial t}\Theta^L.$$

There is a unique $(1, 0)$ vector field V_t on each X_t, defined by

$$\delta_{V_t}\omega_t = \theta^L.$$

Our key obeservation is contained in the next lemma.

Lemma 5.1. *Assume that for some $u \neq 0$ in E_0, $(\Theta^E u, u) = 0$. Then V_0 is a holomorphic vector field on X_0.*

Proof. Recall that, with the notation of the previous section,

$$(\bar\partial\mu + \partial^\phi\eta)\wedge dt = \partial\bar\partial\phi\wedge u.$$

Since $\eta = 0$ on X_0 it follows that $\partial^\phi\eta\wedge dt\wedge d\bar t = 0$ for $t = 0$. Choose local coordinates z on $Z \approx X_0$ and write

$$u = u_0 dz + v\wedge dt,$$

where $\bar\partial v = 0$ on X_0, since $\eta = 0$ on X_0. Then

$$\partial\bar\partial\phi\wedge u_0 dz\wedge d\bar t = (\bar\partial\mu - \partial\bar\partial\phi\wedge v)\wedge dt\wedge d\bar t.$$

But on X_0

$$\bar\partial\mu = \bar\partial\partial^\phi\chi = \partial\bar\partial\phi\wedge\chi,$$

since $\bar\partial\chi = 0$, so

$$idt\wedge\theta^L\wedge u_0 dz\wedge d\bar t = i(\chi - v)\wedge\Theta^L\wedge dt\wedge d\bar t = (\chi - v)\wedge\omega_t\wedge dt\wedge d\bar t$$

for $t = 0$. Thus, restricted to X_0

$$i(-1)^{n+1}\theta^L\wedge u_0 dz = (\chi - v)\wedge\omega_0.$$

As $\omega_0\wedge u_0 dz = 0$

$$\delta_{V_0}(\omega_0)\wedge u_0 dz + \omega_0\wedge\delta_{V_0}(u_0 dz) = 0$$

so finally we obtain that

$$(-1)^n\delta_{V_0}(u_0 dz) = (\chi - v)$$

on X_0. Since $\bar\partial\chi = \bar\partial v = 0$ the right hand side is a holomorphic form on $X_)$. Since u_0 is also holomorphic it follows that V must be holomorphic too, except possibly where u vanishes. But since V is smooth, V must actually be holomorphic everywhere by Riemann's theorem on removable singularities. □

The proof of the Lemma may seem a bit obscure, but the idea behind it is more easily explained if we assume that the line bundle L is the pull-back of a fixed bundle on Z. Then we can choose from the start an extension of u - namely the pull back under the projection on the central fiber- such that η and v are zero. The proof of Theorem 1.2 then reduces to an estimate of μ, an L^2-minimal solution to a certain $\bar\partial$-equation, just as in the case of Theorem 1.1. Lemma 5.1 is then a consequence of the next proposition that we state separately, since we feel it has an independent interest.

Proposition 5.2. *Let L be a positive line bundle over a compact complex manifold Z. Give Z the Kähler metric defined by the curvature form of L. Let μ be the L^2-minimal solution to $\bar\partial\mu = f$, where f is an L-valued $(n,1)$-form on Z. Then equality holds in Hörmander's estimate, i e*

$$\int_Z |\mu|^2 = \int_Z |f|^2$$

if and only if $\gamma = *f$ is a holomorphic form.

Proof. Let ϕ be the metric on L. As in the proof in the previous section

$$\mu = \partial^{\phi}\gamma,$$

for some $\bar\partial$-closed $(n-1,0)$-form γ. Thus

$$\int_Z |\mu|^2 = \int_Z f \wedge \bar\gamma e^{-\phi} \leq \|f\|\|\gamma\|.$$

By the Hörmander-Kodaira-Nakano identity

$$\int_Z \chi \wedge \bar\chi \wedge i\partial\bar\partial\phi\, e^{-\phi} + \int_Z |\bar\partial\chi|^2 = \int_Z |\mu|^2.$$

The left hand side here is the norm squared of γ so it follows that

$$\|\mu\|^2 \leq \|f\|^2$$

with equality only if $\bar\partial\gamma = 0$. In that case

$$f = \bar\partial\partial^{\phi}\gamma = \partial\bar\partial\phi \wedge \gamma$$

so $\gamma = *f$. The argument is easily seen to be reversible. □

We are now ready to state the main theorem of this section.

Theorem 5.3. *Assume that Z has no nonzero global holomorphic vector field. Suppose that*

(i)X is locally a product $U \times Z$ where U is an open set in $\mathbb{C}$,

(ii) L is semipositive on X,

and that

(iii) L restricted to each fiber is strictly positive. Let ω_t the Kähler metric be the fiber X_t induced by the curvature of L. Then if for each t in U there is some element u_t in E_t such that

$$(\Theta^E u_t, u_t) = 0.$$

$$\omega_t = \omega_0$$

for t in U.

Proof. By Lemma 5.1 the restriction of θ^L to each fiber X_t is zero. Hence

$$\partial\theta^L = \mathcal{L}_{\partial/\partial t}\Theta^L$$

also vanishes on fibers, which means that

$$\frac{d}{dt}\omega_t = 0.$$

□

6. THE SPACE OF KÄHLER METRICS.

In this section we will specify the situation even more, and assume that $X = U \times Z$ is a product, and that moreover the line bundle L is the pullback of a bundle on Z under the projection on the second factor. Intuitively this means that not only are all fibers the same, but also the line bundle on them, so it is only the metric that varies. Fix one metric ϕ_0 on L, that we can take to be the pullback of a metric on the bundle on Z, i e independent of the t-variable. Then any other metric on L can be written

$$\phi = \phi_0 + \psi,$$

where ψ is a function on X. We also continue to assume that U is a domain in $\mathbb{C}$. Let u be an element in E_t.

In this situation we have an explicit lower bound for the curvature form operating on u, generalizing 3.1:

$$(\Theta^E u, u) \geq \int_{X_t} \left(\psi_{t\bar{t}} - |\bar{\partial}_z \psi_t|^2_\phi\right) [u, u]. \tag{6.1}$$

Here the expression $|f|_\phi$ means the norm of the form f with respect to the metric $i\partial\bar{\partial}\phi$ on X_t. This can be proved, either by adapting the method of section 3 - note that we may replace any t-derivative of ϕ by the corresponding derivative of ψ since ϕ_0 is independent of t - or from the more complicated proof in section 5. In the last case the inequality we use is

$$\int |\mu|^2 \leq \int |\bar{\partial}\mu|^2_\phi$$

if μ is orthogonal to the space of holomorphic sections of $L \otimes K_{X_t}$. From the proof(s) it is also clear that for 6.1 to hold it is not necessary that the metric ϕ be semipositive on X - it is enough that the restriction to each fiber be positive. Then 6.1 still holds even though this of course does no longer imply that the curvature on E is nonnegative. Moreover, Theorem 5.2 also holds if we replace the hypothesis

$$(\Theta^E u, u) = 0,$$

by the hypothesis that equality holds in 6.1.

The expression occuring in the integrand in 6.1,

$$C(\psi) = \left(\psi_{t\bar{t}} - |\bar{\partial}_z \psi_t|^2_\phi\right)$$

plays a crucial role in the recent work on variations of Kähler metrics on compact manifolds, see [19], [14], [8], [9],[18] and [5], to quote just a few. Fixing a line bundle L on Z, these papers consider the space $\mathcal{K}(L)$ of all Kähler metrics whose Kähler form is cohomologous to the Chern class of L. This means precisely that the Kähler form can be written

$$i\partial\bar{\partial}\phi = i\partial\bar{\partial}\phi_0 + i\partial\bar{\partial}\psi,$$

for some function ψ. and so the set up we described above, where ψ depends on t corresponds to a path in $\mathcal{K}(L)$.

The tangent space of $\mathcal{K}(L)$ at a point ϕ is a space of functions $\dot{\psi}$ and a Riemannian metric on the tangent space is given by the L^2-norm

$$|\dot{\psi}|^2 = \int_Z |\dot{\psi}|^2 (i\partial\bar{\partial}\phi)^n/n!.$$

In this way, $\mathcal{K}(L)$ becomes an infinite-dimensional Riemannian manifold.

Now consider our space X above and let $U = \{|\mathrm{Re}\, t| < 1\}$ be a strip, and consider functions ψ that depend only on $\mathrm{Re}\, t$. Then

$$4C(\psi) = \ddot{\psi} - |\bar{\partial}_z \dot{\psi}|^2_\phi$$

if we use dots to denote derivatives with respect to $\mathrm{Re}\, t$. The link between Theorem 1.2 and the papers quoted above lies in the fact that, by the results in [19], the right hand side here is the geodesic curvature of the path in $\mathcal{K}(L)$ determined by ψ.

A basic idea in the papers quoted above is to consider the spaces

$$E_t = \Gamma(X_t, K_{X_t} \otimes L)$$

with the induced L^2-metric as a finite dimensional approximation or "quantization" of the manifold Z with metric $\phi_t = \phi_0 + \psi(t, \cdot)$. (Actually this is not quite true. In the papers quoted above one does not take the tensor product with the canonical bundle, but instead integrates with respect to the volume element $(i\partial\bar{\partial}\phi_t)^n/n!$.) Here one also replaces L by L^k - with k^{-1} playing the role of Planck's constant - and studies the asymptotic behaviour as k goes to infinity. Under this "quantization" map, functions on Z correspond to the induced Toepliz operator on E_t.

The inequality 6.1 can now be formulated as saying that "the curvature of the quantization is greater than the quantization of the curvature", i e that the curvature operator of the vector bundle corresponding to a path in $\mathcal{K}(L)$ is greater than the Toepliz operator defined by the geodesic curvature of the path. Moreover, Theorem 5.2 implies that if Z has no nonzero global holomorphic vector fields, then equality holds only for a constant path.

7. BUNDLES OF PROJECTIVE SPACES

Let V be a holomorphic line bundle of finite rank r over a complex manifold Y, and let V^* be its dual bundle. We let $\mathbb{P}(V)$ be the fiber bundle over Y whose fiber at each point t of the base is the projective space of lines in V_t^*, $\mathbb{P}(V_t^*)$. Then $\mathbb{P}(V)$ is a holomorphically locally trivial fibration. There is a naturally defined line bundle $O_{\mathbb{P}(V)}(1)$ over $\mathbb{P}(V)$ whose restriction to any fiber $\mathbb{P}(V_t^*)$ is the hyperplane section bundle. One way to define this bundle is to first consider the tautological line bundle $O_{\mathbb{P}(V)}(-1)$. The total space of this line bundle, with the zero section removed, is just the total space of V^* with the zero section removed, and the projection to $\mathbb{P}(V)$ is the map that sends a nonzero point in V_t^* to its image in $\mathbb{P}(V_t^*)$. The bundle $O_{\mathbb{P}(V)}(1)$ is then defined as the dual of $O_{\mathbb{P}(V)}(-1)$. The global holomorphic sections of this bundle over any fiber are in one to one correspondence with the linear forms on V_t^*, i e the elements of V. More generally, $O_{\mathbb{P}(V)}(1)^l = O_{\mathbb{P}(V)}(l)$

has as global holomorphic sections over each fiber the homogenuous polynomials on V_t^* of degree l, i e the elemets of the l:th symmetric power of V. We shall apply Theorem 1.2 to the line bundles

$$L(l) =: O_{\mathbb{P}(V)}(l).$$

Let $E(l)$ be the vector bundle whose fiber over a point t in Y is the space of global holomorphic sections of $L(l) \otimes K_{\mathbb{P}(V_t^*)}$. If $l < r$ there is only the zero section, so we assume from now on that l is greater than or equal to r.

We claim that

$$E(r) = \det V,$$

the determinant bundle of V. To see this, note that $L(r) \otimes K_{\mathbb{P}(V_t^*)}$ is trivial on each fiber, since the canonical bundle of $(r-1)$-dimensional projective space is $O(-r)$. The space of global sections is therefore one dimensional. A convenient basis element is

$$\sum_1^r z_j \widehat{dz_j},$$

if z_j are coordinates on V_t^*. Here $\widehat{dz_j}$ is the wedge product of all differentials dz_k except dz_j with a sign chosen so that $dz_j \wedge \widehat{dz_j} = dz_1 \wedge ... dz_r$. If we make a linear change of coordinates on V_t^*, this basis element gets multiplied with the determinant of the matrix giving the change of coordinates, so the bundle of sections must transform as the determinant of V. Since

$$L(r+1) \otimes K_{\mathbb{P}(V_t^*)} = O_{\mathbb{P}(V)}(1) \otimes L(r) \otimes K_{\mathbb{P}(V_t^*)},$$

it also follows that

$$E(r+1) = V \otimes \det V.$$

In the same way

$$E(r+m) = S^m(V) \otimes \det V,$$

where $S^m(V)$ is the mth symmetric power of V.

Let us now assume that V is ample in the sense of Hartshorne, see [12]. By a theorem of Hartshorne, [12], V is ample if and only if $L(1)$ is ample, i e has a metric with strictly positive curvature. Theorem 1.2 then implies that the L^2-metric on each of the bundles $E(r+m)$ for $m \geq 0$ has curvature which is strictly positive in the sense of Nakano, so we obtain:

Theorem 7.1. *Let V be a vector bundle (of finite rank) over a complex manifold. Assume V is ample in the sense of Hartshorne. Then for any $m \geq 0$ the bundle*

$$S^m(V) \otimes \det V$$

has an hermitian metric with curvature which is (strictly) positive in the sense of Nakano.

8. APPENDIX

In this section we will state and prove an extension result of Ohsawa-Takegoshi type which in particular implies that the bundles E that we have discussed in this paper really are vector bundles. The proof follows the method of [2], combined with ideas from [6], that allow us to include the case of singular metrics. See also [20] for a closely related result.

Theorem 8.1. *Let X be a Kähler manifold fibered over the unit ball U in $\mathbb{C}^m$, with compact fibers X_t. Let L be a holomorphic line bundle on X with a hermitian metric - possibly singular - with semipositive curvature. Let u be a holomorphic section to $K_{X_0} \otimes L$ over X_0 such that*

$$\int_{X_0} [u, u] \leq 1.$$

Then there is a holomorphic section, $\tilde{u}$ to K_X over X such that $\tilde{u} = u \wedge dt$ for $t = 0$ and

$$\int_X [\tilde{u}, \tilde{u}] \leq C$$

where C is an absolute constant.

Proof. We assume $m = 1$. The general case follows in the same way, extending with respect to one variable at the time. At first we also assume that the metric on L is smooth. The proof follows closely the method in [2] so we will be somewhat sketchy.

Let $f = u \wedge [X_0]/(2\pi i)$, where $[X_0]$ is the current of integration on X_0. Then $\bar{\partial} f = 0$ and if v is any solution to $\bar{\partial} v = f$ then $\tilde{u} = tv$ is a section to K_X that extends u in the sense described. To find a v with L^2-estimates we need to estimate

$$\int_X (f, \alpha)$$

for any compactly supported test form α of bidegree $(n+1, 0)$ on X. For α given, decompose $\alpha = \alpha^1 + \alpha^2$, where α^1 is $\bar{\partial}$-closed, and α^2 is orthogonal to the kernel of $\bar{\partial}$. This means that α^2 can be written

$$\alpha^2 = \bar{\partial}^* \beta$$

for some β. By the regularity of the $\bar{\partial}$-Neumann problem α^i and β are all smooth up to the boundary. We first claim that

$$\int (f, \alpha^2) = 0. \tag{8.1}$$

This is not surprising since f is $\bar{\partial}$-closed, but it is not quite evident since f is not in L^2. To prove it, extend u smoothly to X. Then $\bar{\partial} u \wedge d\bar{t} = 0$ for $t = 0$. Let χ be a smooth cut-off function equal to one near the origin in $\mathbb{R}$, and put

$$\chi_\epsilon(t) = \chi(|t|^2/\epsilon).$$

Then

$$f = \chi_\epsilon f = \bar{\partial}(u \wedge dt/t)\chi_\epsilon - \bar{\partial} u \wedge dt/t \chi_\epsilon =$$

$$= \bar\partial(u\wedge dt/t\chi_\epsilon) - u\wedge dt/t\wedge\bar\partial\chi_\epsilon - \bar\partial u\wedge dt/t\chi_\epsilon =: I + II + III.$$

Clearly the scalar product between I and α^2 vanishes. It is also clear that the scalar product between III and α^2 goes to zero as ϵ goes to zero. The scalar product between II and α^2 equals, up to signs

$$\int \chi'(\bar\partial u\wedge dt\wedge d\bar t, \beta)/\epsilon,$$

which is easily seen to tend to zero as well since $\bar\partial u\wedge d\bar t$ vanishes for $t=0$. Hence 8.1 follows. Therefore

$$|\int_X (f,\alpha)|^2 = |\int_X (f,\alpha^1)|^2 \leq \int_{X_0} \gamma\wedge\bar\gamma e^{-\phi},$$

where γ is the Hodge-* of α^1. The form γ satifies $\omega\wedge\gamma = \alpha^1$ and $\bar\partial^*\alpha = \bar\partial^*\alpha^1 = \partial^\phi\gamma$. To estimate this we apply the Siu $\partial\bar\partial$-Bochner formula (see [2]): If w is any nonnegative function smooth up to the boundary of X, then

$$-\int i\partial\bar\partial w\wedge c_n\gamma\wedge\bar\gamma e^{-\phi} + \int i\partial\bar\partial\phi\wedge c_n\gamma\wedge\bar\gamma e^{-\phi} \leq \tag{8.2}$$

$$\leq 2c_n\mathrm{Re}\int \bar\partial\partial^\phi\gamma\wedge\bar\gamma e^{-\phi}w = \int |\bar\partial^*\alpha|^2 w + \int(\bar\partial^*\alpha, \bar\partial w\wedge\gamma).$$

Now choose $w = (1/2\pi)\log(1/|t|^2)$. If $i\partial\bar\partial\phi$ is nonnegative we then find that

$$\int_{X_0}\gamma\wedge\bar\gamma e^{-\phi} \leq C\int |\bar\partial^*\alpha|^2(\log(1/|t|^2+1/|t|)+\int_X idt\wedge d\bar t\wedge\gamma\wedge\bar\gamma e^{-\phi}(1/|t|).$$

To take care of the last term we repeat the last argument once more, this time choosing $w = (1-|t|)$ and finally obtain an estimate

$$|\int_X (f,\alpha)|^2 \leq C\int |\bar\partial^*\alpha|^2(1/|t|).$$

This implies that there is some function v on X such that

$$\int_X (f,\alpha) = \int (v, \bar\partial^*\alpha),$$

for all test forms α, and satisfying

$$\int |v|^2|t| \leq C.$$

Then $\tilde u := tv$ satisfies the conclusion of the theorem.

We now turn to the modifications needed when the metric ϕ is not smooth. Since X has a complete Kähler metric, we can apply the results and methods of [6]. There is then, possibly after shrinking the domain slightly, a decreasing sequence of smooth metrics, ϕ_k that tend to ϕ pointwise. Moreover, the curvature of ϕ_k satisfies an estimate

$$i\partial\bar\partial\phi_k \geq -\lambda_k\omega$$

where λ_k is a decreasing sequence of continuous functions that tend to zero almost everywhere. We can then repeat the arguments above, replacing ϕ with ϕ_k. The estimate we obtain is

$$|\int_X (f,\alpha)|_k^2 \leq C\int |\bar\partial^*\alpha|_k^2(1/|t|) + \int |\alpha|_k^2\lambda_k/|t|.$$

Here the subscripts indicate the dependence of our metric on k. This implies that there are functions v_k and q_k such that v_k satisfies the same estimate as before, with ϕ replaced by ϕ_k,

$$\int |t||q_k|_k^2 \leq C$$

and $\bar\partial v_k = f + \lambda_k^{1/2} q_k$. Since

$$\int \lambda_k^{1/2}|q_k|_k \leq C\int \lambda_k/|t|$$

the sequence q_k goes to zero in L^1_{loc}. An appropriate weak limit of the sequence v_k then solves the equation $\bar\partial v = f$ and satisfies the same estimate as before.

□

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