CHALMERS | GÖTEBORGS UNIVERSITET

PREPRINT

Reversed Galton-Watson Processes in the Linear Fractional Case

F. C. KLEBANER
S. SAGITOV

Department of Mathematical Sciences
Division of Mathematical Statistics
CHALMERS UNIVERSITY OF TECHNOLOGY
GÖTEBORG UNIVERSITY
Göteborg Sweden 2005

Preprint 2005:5

Reversed Galton-Watson Processes in the Linear Fractional Case

F. C. Klebaner
S. Sagitov

CHALMERS | GÖTEBORGS UNIVERSITET

Mathematical Statistics
Department of Mathematical Sciences
Chalmers University of Technology and Göteborg University
SE-412 96 Göteborg, Sweden
Göteborg, February 2005

Preprint
ISSN 1652-9715/No. 2005:5

Matematiska Vetenskaper
Göteborg 2005

Reversed Galton-Watson processes in the linear fractional case*

F.C. Klebaner
School of Mathematical Sciences
Monash University
Clayton, Vic 3800
Australia

S. Sagitov†
School of Mathematical Sciences
Chalmers University
SE-412 96 Gothenburg
Sweden

February 17, 2005

Abstract

The subject of this paper is a Galton-Watson (GW) process with the linear fractional offspring distribution. Four ways of constructing a Markov chain to model the time-reversed process are presented and analyzed. The approach by Esty is related to the construction based on quasi-stationary distributions. We establish a surprising fact that it has the same transition probabilities as the Q-process associated with the linear fractional GW process, Theorem 5.1. Then we discuss two closely related definitions: the uniform prior reverse, and the minimal reverse, studied earlier by us in the case of geometric reproduction. The fourth approach is by Asmussen and Sigman [2]. A remarkable fact is that all four constructions of the reverse process have the transition probabilities of the GW process with a dual reproduction law and some kind of immigration. In particular, we generalize a result of [5] in which only a geometric offspring distribution was considered.

*Research supported by the Australian Research Council
†Partially supported by the Bank of Sweden Tercentenary Foundation
AMS 1991 subject classifications. 60J80
Key words and phrases.Galton-Watson process, reverse Galton-Watson process, dual

1 Introduction

In population genetics it is important to look back from now and model random fluctuations in population's development. Our approach is to consider the time-reversed Galton-Watson Process.

In general, for a process $\{Z_n\}$ defined on the time interval $n \leq N$, the time-reversed process is Y_n^N is the process Z_{N-n}. As usual, we are interested in the probabilistic structure, and understand equality of processes $Y_\bullet^N \stackrel{d}{=} Z_{N-\bullet}$ in distribution.

It is easy to see that if $\{Z_n\}_{n\in\mathbb{Z}}$ is a stationary Markov Chain with transition probabilities (P_{ij}) and a stationary distribution $\{\pi_j\}$ then for any N the time-reversed process $\{Y_n^N\}$, is also a Markov chain with the same stationary distribution and transition probabilities obtained by the Bayes formula

$$P(Y_{n+1}^N = j | Y_n^N = i) = P(Z_{N-(n+1)} = j | Z_{N-n} = i) = \frac{\pi_j P_{ji}}{\pi_i}. \tag{1}$$

Thus the distribution of the reversed process is independent of N, and one can define the time-reversed process as a Markov chain $\{Y_n\}$ with transition probabilities

$$P(Y_{n+1} = j | Y_n = i) = \frac{\pi_j P_{ji}}{\pi_i}. \tag{2}$$

If $\{Z_n\}$ is non-stationary, then the finite dimensional distributions of the time-reversed process $Y_n^N = Z_{N-n}$ depend on N. Since it is not clear what N should be, this is inconvenient. It is possible eliminate N by taking it to be large, $N \to \infty$. The limit in distribution of finite dimensional distributions of $\{Y_n^N\}$ may not exist for a non-stationary process $\{Z_n\}$. However, for Markov chains with a quasi-stationary distribution, it is possible to think that for large N the chain is in a quasi-stationary regime, and one can use such a distribution to define the time-reversed process by analogy with (2). The process we are interested in, the Galton-Watson process $\{Z_n\}$ is non-stationary but possesses quasi-stationary distributions (see e.g. [3], Ch 2.), and for any such distribution $\{\eta_j\}_{j\geq 1}$ the time-reversed chain can be defined by analogy with (2)

$$P(Y_{n+1} = j | Y_n = i) = \frac{\eta_j P_{ji}}{\eta_i}. \tag{3}$$

This approach is given in Alsmeyer and Rösler [1]. For critical Galton-Watson process the quasi-stationary distribution is unique, but there are infinitely

---

Galton-Watson process, linear fractional generating function, Q-process.

many quasi-stationary distributions in the sub-critical and supercritical cases. For such processes it is not clear which quasi-stationary distribution to take.

There are other approaches to time reversal, some are based on the limiting procedure $N \to \infty$, such as in Esty [4], and some follow different ideas. This paper deals with four constructions, all different from the above mentioned approach based on quasi-stationarity. Three methods are found in the literature and we suggest another one, with each method having its own logic.

We give explicit constructions of the time-reversed processes for the Galton-Watson process with a linear fractional offspring distribution. It is a remarkable result that the time-reversed processes for principally different constructions turn out to be again Galton-Watson processes, with a linear fractional reproduction but with immigration. This finding generalizes our earlier result in [5], where only a geometric offspring distribution was considered. Construction of the time-reversed process allows to answer some questions on the original branching population. For example, the question of the age of the population can be answered by considering the hitting time of one by the time-reversed process, see [5].

The paper is organized as follows. First we present the necessary results on the GW process with a linear fractional reproduction needed in the sequel. These results include explicit formulae for the transition probabilities and generating functions, the duality between the subcritical and supercritical cases, eternal particles and immigration. Then we give different constructions of the reversed GW processes.

2 Linear fractional or $\mathrm{G}(p_0, p)$ reproduction

With any non-negative integer valued random variable ξ there is an associated GW process $\{Z_n\}_{n\geq 0}$ which is a time homogeneous Markov chain with transition probabilities

$$\mathrm{P}(Z_{n+1} = j | Z_n = i) = \mathrm{P}(S_i = j), \ i \geq 0, \ j \geq 0. \tag{4}$$

Here and elsewhere

$$S_i = \xi_1 + \ldots + \xi_i, \text{ where } \xi_1, \ldots, \xi_i \text{ are independent copies of } \xi. \tag{5}$$

If ξ is interpreted as the number of offspring of a single particle, then Z_n gives the size of the n-th generation in a population of independently multiplying particles.

Denote by $h(s) = \mathrm{E}(s^{\xi})$ the probability generating function of ξ, $0 \le s \le 1$. The generating function of Z_n is given by the n-th iterate $h_n(s)$ of h

$$\mathrm{E}(s^{Z_n}|Z_0 = i) = (h_n(s))^i, \ h_1(s) = h(s), \ h_n(s) = h(h_{n-1}(s)).$$

Depending on the offspring mean $m = \mathrm{E}(\xi)$ the extinction probability $Q = \lim_{n\to\infty}(Z_n = 0|Z_0 = 1)$ is either one, if $m \le 1$ and $\mathrm{P}(\xi = 1) < 1$, or less than one, if $m > 1$. The exact value of Q is found as the smallest root of the equation $h(s) = s$.

Consider an offspring distribution governed by two parameters $0 < p_0 < 1$ and $0 < p < 1$, given by

$$\mathrm{P}(\xi = 0) = p_0, \ \mathrm{P}(\xi = i) = q_0 p q^{i-1}, \ i \ge 1, \tag{6}$$

where $q_0 = 1 - p_0$ and $q = 1 - p$. In the special case $p_0 = p$ this is the geometric distribution $\mathrm{G}(p)$. In the literature distribution (6) is called either a modified geometric distribution, or zero-modified geometric distribution. Often it is referred to as a linear fractional distribution due to the form of its generating function

$$h(s) = p_0 + \frac{q_0 p s}{1 - qs} = \frac{p_0 + (p - p_0)s}{1 - qs}. \tag{7}$$

In this paper refer to the distribution (6) as the $\mathrm{G}(p_0, p)$-distribution and to the corresponding GW process as the $\mathrm{G}(p_0, p)$ reproduction.

The key feature of the $\mathrm{G}(p_0, p)$ reproduction is an explicit formula [3, p. 7] for the iterations

$$h_n(s) = 1 - m^n p_n + \frac{m^n p_n^2 s}{1 - q_n s},$$

where $m = q_0/p$ and

$$p_n = \begin{cases} \frac{1-p_0/q}{m^n - p_0/q} & \text{if } p_0 + p \ne 1 \\ \frac{1-p}{1-p+np} & \text{if } p_0 + p = 1 \end{cases}, \ q_n = 1 - p_n, \ n \ge 1, \tag{8}$$

so that in particular $p_1 = p$ and $q_1 = q$. Thus given $Z_0 = 1$, the size of the n-th generation Z_n has the zero-modified geometric distribution $\mathrm{G}(1 - m^n p_n, p_n)$ with the following counterpart of (6)

$$\mathrm{P}(Z_n = 0|Z_0 = 1) = 1 - m^n p_n, \ \mathrm{P}(Z_n = i|Z_0 = 1) = m^n p_n^2 q_n^{i-1}. \tag{9}$$

More generally, in the case of $j \geq 1$ initial particles we have

$$\mathrm{P}(Z_n = 0 | Z_0 = j) = (1 - m^n p_n)^j, \; j \geq 1,$$
$$\mathrm{P}(Z_n = i | Z_0 = j) = \sum_{l=1}^{i} \binom{i-1}{l-1} \binom{j}{l} p_n^{2l} q_n^{i-l} (1 - m^n p_n)^{j-l} m^{nl}, \; i \geq 1.$$

These formulae follow from the next property of the sum (5) in the case (6):

$$\mathrm{P}(S_j = 0) = p_0^j, \; \mathrm{P}(S_j = i) = \sum_{l=1}^{i} \binom{i-1}{l-1} \binom{j}{l} p^l q^{i-l} p_0^{j-l} q_0^l, \; i \geq 1, \; j \geq 1. \quad (10)$$

Here and elsewhere we put $\binom{j}{l} = 0$ for $j < l$. To verify (10) notice that the number of positive summands ξ_k in S_j is a binomially distributed random variable $L \sim \mathrm{Bin}(j, q_0)$ with $\mathrm{P}(L = l) = \binom{j}{l} p_0^{j-l} q_0^l$, $0 \leq l \leq j$. Given $L = l$, the distribution of $(S_j - l) \sim \mathrm{NB}(l, p)$ is Negative Binomial with $\mathrm{P}(S_j - L = k | L = l) = \binom{k+l-1}{k} p^l q^k$, $k \geq 0$. Applying the law of total probabilities we arrive at (10).

The extinction probability of the GW process with $\mathrm{G}(p_0, p)$ reproduction equals $Q = \min(\frac{p_0}{q}, 1)$ in accordance with the first part of (9). Notice that Q is the smallest of the two roots $\frac{p_0}{q}$ and 1 of the equation $h(s) = s$. An important parameter $b = h'(Q)$, needed later, can be computed explicitly as well,

$$b = \begin{cases} p/q_0 & \text{if } p_0 + p < 1 \\ q_0/p & \text{if } p_0 + p \geq 1 \end{cases}.$$

3 Dual reproduction laws

Definition 3.1 *Consider a reproduction law with the offspring distribution generating function $h(s)$ and mean m. A dual reproduction law is defined by the transformed generating function*

$$h^*(s) = h(Rs)/R, \; 0 \leq s \leq 1,$$

where R satisfies $h(R) = R$ and $R \neq 1$ if $m \neq 1$.

This definition is symmetric in that $h(s) = h^*(R^* s)/R^*$. A critical reproduction law is self-dual. The dual offspring mean is $m^* = h'(R)$. In the supercritical case $R = Q$ with $m^* = b$, and the dual law governs the subcritical part of the well known decomposition due to Harris-Sevastyanov transformation [3, p. 47-53]. The dual to a subcritical reproduction law is a supercritical law with the extinction probability $Q^* = 1/R$ and $b^* = m$.

In the special case of the $\mathrm{G}(p_0, p)$ reproduction the dual law is the $\mathrm{G}(q, q_0)$ reproduction with

$$R = p_0/q,\ m^* = p/q_0 = 1/m,\ Q^* = 1/Q,\ b^* = 1/b,$$

and generating function

$$h^*(s) = q + \frac{q_0 ps}{1 - p_0 s}. \tag{11}$$

The next observation is fundamental to our forthcoming analysis.

Lemma 3.1 *Given (5) and (6),*

$$\sum_{j=0}^{\infty} s^j \mathrm{P}(S_j = i) = s\,(h^*(s))'\,(h^*(s))^{i-1},\ i \geq 1. \tag{12}$$

Proof If $i \geq 1$ and $0 < s < 1/p_0$, then according to (10)

$$\sum_{j=0}^{\infty} s^j \mathrm{P}(S_j = i) = \sum_{l=1}^{i}\sum_{j=l}^{\infty} \binom{i-1}{l-1}\binom{j}{l} p^l q^{i-l} s^j p_0^{j-l} q_0^l.$$

Since

$$\sum_{j=l}^{\infty} \binom{j}{l} x^{j-l} = \sum_{k=0}^{\infty} \binom{k+l}{k} x^k = (1-x)^{-l-1},\ 0 < x < 1,$$

it follows

$$\sum_{j=0}^{\infty} s^j \mathrm{P}(S_j = i) = \frac{pq_0 s}{(1-p_0 s)^2} \sum_{l=1}^{i} \binom{i-1}{l-1} p^{l-1} q^{i-l} \left(\frac{q_0 s}{1-p_0 s}\right)^{l-1}$$

and after some algebra we obtain

$$\sum_{j=0}^{\infty} s^j \mathrm{P}(S_j = i) = \frac{pq_0 s}{(1-p_0 s)^2} \left(q + \frac{pq_0 s}{1-p_0 s}\right)^{i-1},\ i \geq 1, \tag{13}$$

which in view of (11) is equivalent to (12).

An alternative way of verifying (13) is to use the joint generating function

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} s^j \mathrm{P}(S_j = i)u^i = \sum_{j=0}^{\infty} s^j \left(h(u)^j - p_0^j\right)$$
$$= \frac{1-qu}{(1-p_0s)(1-qu)-pq_0us} - \frac{1}{1-p_0s} = \frac{pq_0us}{((1-p_0s)(1-qu)-q_0pus)\,(1-p_0s)}$$
$$= \sum_{i=1}^{\infty} \frac{pq_0s}{(1-p_0s)^2}\left(q + \frac{pq_0s}{1-p_0s}\right)^{i-1} u^i.$$

□

Figure 1 reflects our vision of two dual random walks with jump distributions $\mathrm{G}(p_0, p)$ and $\mathrm{G}(q, q_0)$. It is useful for graphical illustrations of the proofs of the forthcoming propositions.

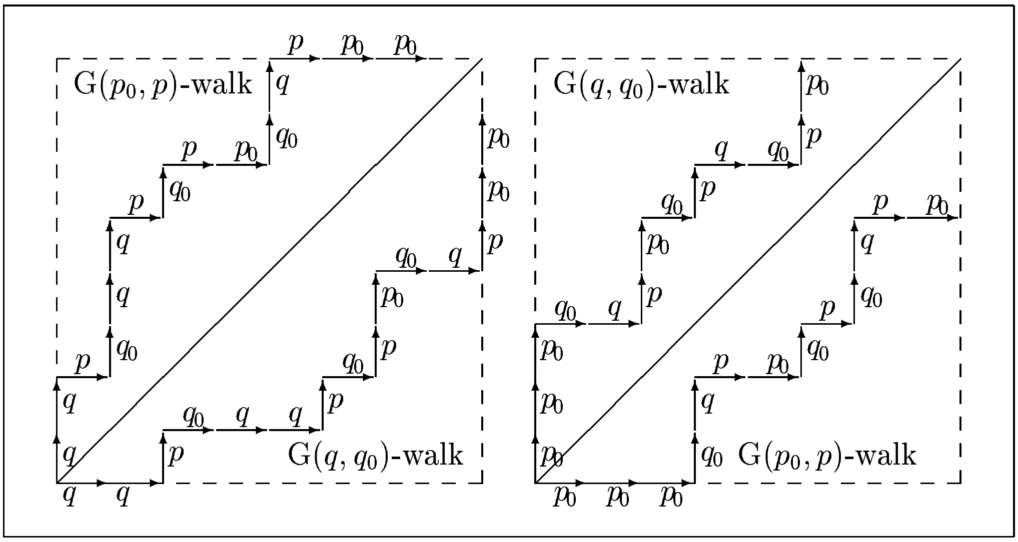

Figure 1: Two examples of dual pairs of random walk trajectories.

4 Eternal particles and immigration

Definition 4.1 *A GW process with an eternal particle, a* $\widehat{\mathrm{GW}}$ *process* $\{\hat{Z}_n\}_{n\geq 0}$ *is defined by*

$$\mathrm{P}(\hat{Z}_{n+1} = j|\hat{Z}_n = i) = \mathrm{P}(S_{i-1} + 1 + \eta = j),\ i \geq 0,\ j \geq 0, \tag{14}$$

where η *is independent of* S_{i-1} *and stands for the number of offspring of the eternal particle.*

Comparing with (4) we see that the eternal particle does not contribute to the sum S_{i-1} and counted separately to produce the next generation size j. If $g(s) = \mathrm{E}(s^\eta)$, then the generating function of the GW process with an eternal particle becomes (compare with (12))

$$\mathrm{E}(s^{\hat{Z}_{n+1}}|\hat{Z}_n = i) = sg(s)h^{i-1}(s).$$

Notice that $\{\hat{Z}_n - 1\}_{n\geq 0}$ is known as a Galton-Watson process with immigration (GWI) process with the immigration generating function $g(s)$. Sometimes we will refer to a $\widehat{\mathrm{GW}}$ process as a $\widehat{\mathrm{GW}}$ process with F_1 reproduction and F_2 immigration to specify the offspring distribution F_1 for ordinary particles and the offspring distribution F_2 for the eternal particle.

An important example of a $\widehat{\mathrm{GW}}$ process is the so called Q-process $\{\hat{Z}_n\}_{n\geq 0}$ associated with the GW process $\{Z_n\}_{n\geq 0}$ introduced in [6]. This is the GW process $\{Z_n\}_{n\geq 0}$ conditioned on not being extinct in the distant future and on being extinct in the even more distant future [3, p. 59]. It is a Markov chain with transition probabilities

$$\mathrm{P}(\hat{Z}_{n+1} = j|\hat{Z}_n = i) = \mathrm{P}(S_i = j)\frac{jQ^{j-i}}{ib}.$$

The corresponding generating function is given by

$$\begin{aligned}
\frac{1}{ibQ^i}\sum_{j=1}^{\infty} s^j \mathrm{P}(S_i = j) jQ^j &= \frac{sQ}{ibQ^i}\sum_{j=1}^{\infty}(sQ)^{j-1} j\mathrm{P}(S_i = j) \\
&= \frac{s}{ibQ^{i-1}}\left(h^i(x)\right)'_{x=sQ} = \frac{s}{b}h'(sQ)\left(\frac{h(sQ)}{Q}\right)^{i-1}.
\end{aligned}$$

The last representation shows that in the subcritical case when $Q = 1$, the Q-process is a $\widehat{\mathrm{GW}}$ process governed by the reproduction generating function $h(s)$ and the immigration generating function $g(s) = h'(s)/m$. In the supercritical and critical cases, the Q-process is again a $\widehat{\mathrm{GW}}$ process but now the reproduction generating function is $h^*(s)$ and the immigration generating function is $g(s) = h'(sQ)/b = (h^*(s))'/m^*$.

Turning to the case of the $\mathrm{G}(p_0, p)$ reproduction we find that in the supercritical and critical cases we have $\frac{h'(sQ)}{b} = \left(\frac{q_0}{1-p_0 s}\right)^2$, which means that the immigration number η has a negative binomial distribution $\mathrm{NB}(2, q_0)$. We summarise our findings in the next proposition.

Proposition 4.1 *In the supercritical and critical cases the Q-process associated with the* $\mathrm{G}(p_0, p)$ *reproduction is a* $\widehat{\mathrm{GW}}$ *process with the* $\mathrm{G}(q, q_0)$ *reproduction and* $\mathrm{G}(q_0) + \mathrm{G}(q_0)$ *immigration.*

In the subcritical case the Q-process of the $\mathrm{G}(p_0, p)$ *reproduction is a* $\widehat{\mathrm{GW}}$ *process with* $\mathrm{G}(p_0, p)$ *reproduction and* $\mathrm{G}(q) + \mathrm{G}(q)$ *immigration.*

5 Reverse GW process

Let $\{Z_n\}_{n\geq 0}$ be a GW process with an arbitrary reproduction law $\{\mathrm{P}(\xi = k)\}_{k\geq 0}$ with $\mathrm{P}(\xi = 1) > 0$. In [4] a reverse GW process $\{Y_n\}_{n\geq 0}$ was introduced in terms of the finite-dimensional distributions by using the following limit procedure

$$\mathrm{P}(Y_{n_1} = i_1, \ldots, Y_{n_k} = i_k | Y_0 = i_0) = \lim_{N\to\infty} \mathrm{P}(Z_{N-n_1} = i_1, \ldots, Z_{N-n_k} = i_k | Z_N = i_0). \tag{15}$$

(Here we restrict the definition of the reverse chain $\{Y_n\}_{n\geq 0}$ to the state space of positive integers.) According to [4], the process $\{Y_n\}_{n\geq 0}$ is well defined, and it is a time-homogeneous Markov chain with transition probabilities

$$\mathrm{P}(Y_{n+1} = j | Y_n = i) = \frac{a_j}{a_i b} \cdot \mathrm{P}(S_j = i), \; i \geq 1, \; j \geq 1, \tag{16}$$

where

$$a_j = \lim_{N\to\infty} \frac{\mathrm{P}(Z_N = j | Z_0 = 1)}{\mathrm{P}(Z_N = 1 | Z_0 = 1)}, \; j \geq 1.$$

Theorem 5.1 *The reverse process of the GW process with the* $G(p_0, p)$ *reproduction is a Markov chain with the same transition probabilities as the Q-process associated with the* $G(p_0, p)$ *reproduction.*

Proof Using (9) and (8) we obtain

$$a_j = \begin{cases} 1 & \text{if } p_0 + p \leq 1 \\ \left(\frac{q}{p_0}\right)^{j-1} & \text{if } p_0 + p > 1 \end{cases}.$$

In view of Lemma 3.1 this implies, on one hand,

$$\sum_{j=0}^{\infty} \frac{q_0}{p} \mathrm{P}(S_j = i) s^j = s \left(\frac{q_0}{1 - p_0 s}\right)^2 \left(q + \frac{p q_0 s}{1 - p_0 s}\right)^{i-1}. \tag{17}$$

and on the other hand,

$$
\begin{aligned}
\sum_{j=0}^{\infty} \frac{p}{q_0}\left(\frac{q}{p_0}\right)^{j-i} \mathrm{P}(S_j=i)s^j &= \frac{p}{q_0}\left(\frac{q}{p_0}\right)^{-i} \sum_{j=1}^{\infty}\left(\frac{qs}{p_0}\right)^{j} \mathrm{P}(S_j=i) \\
&= \frac{p}{q_0}\left(\frac{q}{p_0}\right)^{-i} \frac{pq_0qs}{p_0(1-qs)^2}\left(q+\frac{pq_0qs}{p_0(1-qs)}\right)^{i-1} \\
&= s\left(\frac{p}{1-qs}\right)^{2}\left(p_0+\frac{pq_0s}{1-qs}\right)^{i-1}. \qquad (18)
\end{aligned}
$$

According to (17), the reverse process to the supercritical or critical GW process with $\mathrm{G}(p_0,p)$ reproduction is a $\widehat{\mathrm{GW}}$ process with the $\mathrm{G}(q,q_0)$ reproduction and $\mathrm{G}(q_0)+\mathrm{G}(q_0)$ immigration. While according to (18), the reverse process to the subcritical GW process with $\mathrm{G}(p_0,p)$ reproduction is again a $\widehat{\mathrm{GW}}$ process but now with the $\mathrm{G}(p_0,p)$ reproduction and $\mathrm{G}(q)+\mathrm{G}(q)$ immigration. This is exactly the description of the Q-process associated with $\mathrm{G}(p_0,p)$ reproduction given in the Proposition 4.1.

$\square$

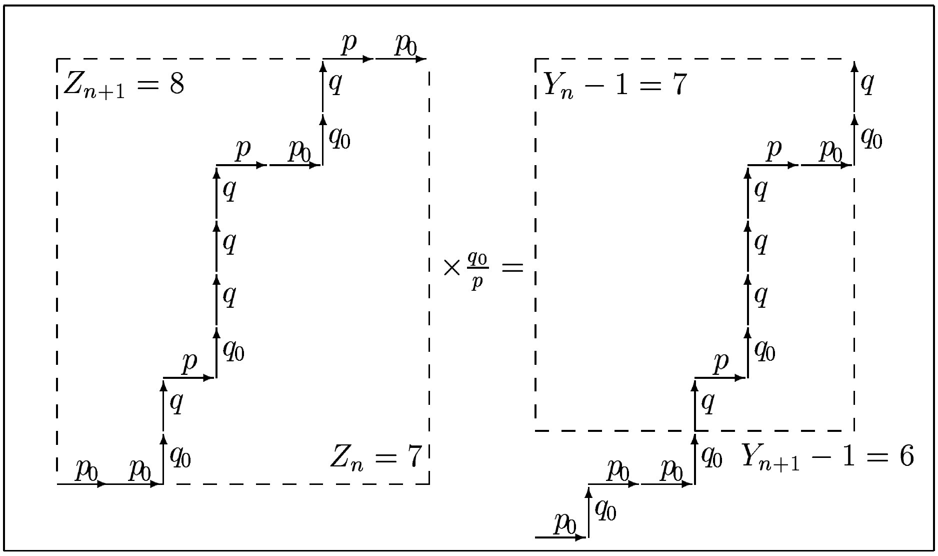

Figure 2: Picture proof of Theorem 5.1 in the supercritical and critical cases.

The assertion of Theorem 5.1 is somewhat counter intuitive. We illustrate it with Figure 2, where the left panel represents a one-generation transition in the forward GW process and the right panel gives the corresponding change in the reverse process. The left panel depicts a path of the $\mathrm{G}(p_0,p)$ random walk (cf. Figure 1) with seven steps leading from level zero to level eight. This corresponds to seven particles producing eight offspring.

In the right panel the $G(p_0, p)$ path from the left panel is rearranged into a $G(q, q_0)$ path by

1. removing the last p arrow in the left path,
2. placing an additional q_0 arrow to the begining of the right path,
3. moving the last p_0 arrow in the left path to the begining of the right path.

The first two changes are meant to reflect the factor $\frac{a_j}{a_i b}$ in (16) which equals $\frac{q_0}{p}$ in the supercritical and critical cases. Notice that the origin of the coordinate system on the right panel is shifted so that the reverse transition is now from seven to six particles. This is done to discount the eternal particle from the picture. It remains to recognize that the resulting $G(q, q_0)$ path corresponds to the $G(q, q_0)$ reproduction with $G(q_0) + G(q_0)$ immigration.

Remark. Observe that the results in [4] concerning the asymptotic behaviour of the reverse GW process in the particular case of zero-modified geometric distribution can be deduced from the known limit theorems for the GWI processes.

6 Uniform prior reverse GW process

Putting $s = 1$ into (13) yields $\sum_{k\geq 1} \mathrm{P}(S_k = i) = p/q_0$. Thus the relation (17) implies that in the supercritical and critical cases the reverse chain has the transition probabilities

$$\mathrm{P}(Y_{n+1} = j | Y_n = i) = \frac{\mathrm{P}(S_j = i)}{\sum_{k\geq 1} \mathrm{P}(S_k = i)}, \ i \geq 1, \ j \geq 1. \tag{19}$$

In view of (4) the reverse process $\{Y_n\}_{n\geq 0}$ becomes the time reversed chain of the original $\{Z_n\}_{n\geq 0}$ with a uniform prior as the stationary distribution. (Remark 2.1 in [5] incorrectly attributes this description to the backward GW process described in the next section.) This observation justifies another definition of the reversed GW process, which does not require a limit procedure like (15).

Definition 6.1 *A Markov chain $\{Y_n\}_{n\geq 0}$ with the transition probabilities (19) is called a uniform prior reverse of the GW process $\{Z_n\}_{n\geq 0}$.*

Notice that the sum in the denominator of (19) is the mean number of visits of the level i by the random walk $\{S_n\}_{n\geq 0}$. It is positive and finite for any $i \geq 1$ if, for example, $\mathrm{P}(\xi = j) > 0$ for $1 \leq j < j_0$, $j_0 \leq \infty$.

Applying the results of the previous sections we conclude that

Proposition 6.1 *The uniform prior reverse of the GW process with $G(p_0, p)$ reproduction is a $\widehat{\mathrm{GW}}$ process with the $G(q, q_0)$ reproduction and $G(q_0) + \mathrm{G}(q_0)$ immigration.*

In the subcritical case the uniform prior reverse drastically differs from the reverse process obtained in the previous section. This is because the conditioning in (15) puts a non-uniform prior distribution in the subcritical case - it does not allow the GW process to visit high levels in the past.

It is straightforward to extend Definition 6.1 to the cases of a GWI process and a $\widehat{\mathrm{GW}}$ process. The random walk should be set as $S_j + \eta$ in the former case and as $S_{j-1} + \eta + 1$ in the latter. We use this opportunity to see if the double reverse in the sense of Definition 6.1 brings us back to the $\mathrm{G}(q, q_0)$ reproduction. The answer to this question is negative.

Proposition 6.2 *The uniform prior reverse of the $\widehat{\mathrm{GW}}$ with the $G(q, q_0)$ reproduction and $G(q_0) + \mathrm{G}(q_0)$ immigration is a Markov chain of $G(q, q_0)$ reproduction conditioned on non-extinction every next generation.*

PROOF If $\eta_1 \sim \mathrm{G}(p)$ and $\eta_2 \sim \mathrm{G}(p)$ are independent, then

$$\mathrm{P}(S_{j-1} + \eta_1 + \eta_2 + 1 = i) = \sum_{k=0}^{i-1} \mathrm{P}(S_{j-1} = i - k - 1) \binom{k+1}{k} q^k p^2.$$

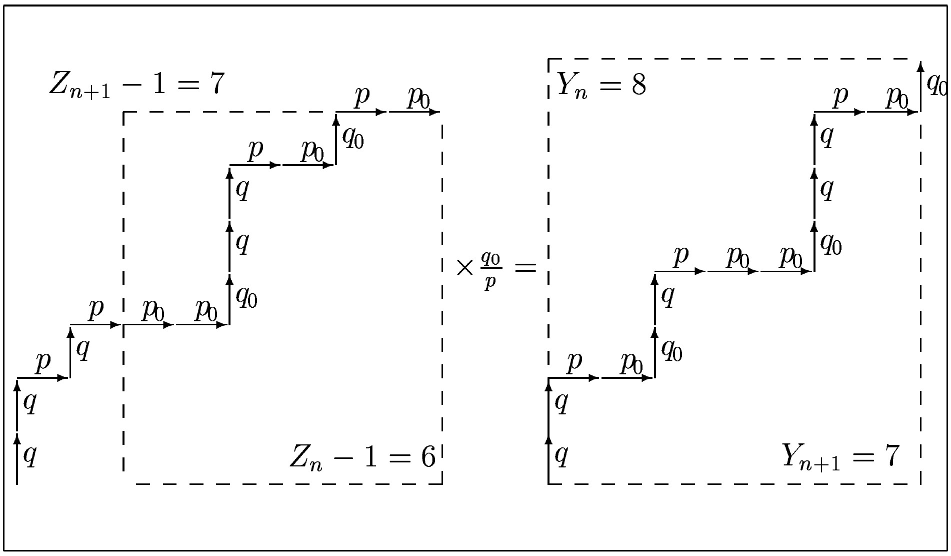

Figure 3: Illustration of the proof of Proposition 6.2

It follows from (13) and (10) that

$$
\begin{aligned}
&\sum_{j=0}^{\infty} s^j \mathrm{P}(S_{j-1} + \eta_1 + \eta_2 + 1 = i) = \\
&= p^2 s \sum_{k=0}^{i-2} (k+1) q^k \sum_{j=0}^{\infty} s^j \mathrm{P}(S_j = i-k-1) + i q^{i-1} p^2 s \sum_{j=0}^{\infty} s^j \mathrm{P}(S_j = 0) \\
&= \frac{p^3 q_0 s^2}{(1-p_0 s)^2} \sum_{k=0}^{i-2} (k+1) q^k \left(q + \frac{p q_0 s}{1 - p_0 s} \right)^{i-k-2} + i q^{i-1} p^2 s \sum_{j=0}^{\infty} s^j p_0^j \\
&= \frac{p^3 q_0 s^2}{(1-p_0 s)^2} \left(q + \frac{p q_0 s}{1-p_0 s} \right)^{i-2} \sum_{k=0}^{i-2} (k+1) \left(\frac{q}{q + \frac{p q_0 s}{1-p_0 s}} \right)^k + \frac{i q^{i-1} p^2 s}{1 - p_0 s}.
\end{aligned}
$$

Now since

$$
\sum_{k=0}^{i-2} (k+1) x^k = \frac{d}{dx} \left(\sum_{k=0}^{i-1} x^k \right) = \frac{d}{dx} \left(\frac{1-x^i}{1-x} \right) = \frac{1 - i x^{i-1} + (i-1) x^i}{(1-x)^2}
$$

we get after some algebra

$$
\sum_{j=0}^{\infty} s^j \mathrm{P}(S_{j-1} + \eta_1 + \eta_2 + 1 = i) = \frac{p}{q_0} \left\{ \left(q + \frac{p q_0 s}{1 - p_0 s} \right)^i - q^i \right\}
$$

This implies

$$
\frac{q_0}{p(1-q^i)} \sum_{j=0}^{\infty} s^j \mathrm{P}(S_{j-1} + \eta_1 + \eta_2 + 1 = i) = \frac{\left(q + \frac{p q_0 s}{1-p_0 s} \right)^i - q^i}{1 - q^i}.
$$

□

We finish this section with a proposition ensuring that the double uniform prior reverse of the GWI process with $\mathrm{G}(p_0, p)$ reproduction and $\mathrm{G}(p)$ immigration brings us back to the original GWI process.

Proposition 6.3 *The uniform prior reverse of the GWI with the* $\mathrm{G}(p_0, p)$ *reproduction and G(p) immigration is the GWI with the* $\mathrm{G}(q, q_0)$ *reproduction and G(q_0) immigration.*

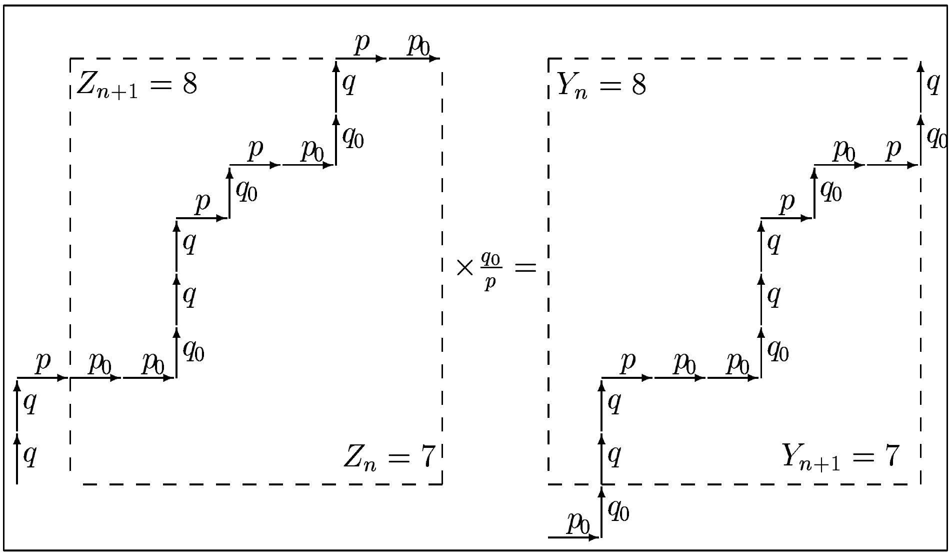

Figure 4: Illustration of the proof of Proposition 6.3.

Proof If $\eta \sim \mathrm{G}(p)$, then

$$\mathrm{P}(S_j + \eta = i) = \sum_{k=0}^{i} \mathrm{P}(S_j = i - k) q^k p.$$

It follows from (13) and (10) that

$$\begin{aligned}
\sum_{j=0}^{\infty} s^j \mathrm{P}(S_j + \eta = i) &= p \sum_{k=0}^{i-1} q^k \sum_{j=0}^{\infty} s^j \mathrm{P}(S_j = i - k) + q^i p \sum_{j=0}^{\infty} s^j \mathrm{P}(S_j = 0) \\
&= \frac{p^2 q_0 s}{(1 - p_0 s)^2} \left(q + \frac{p q_0 s}{1 - p_0 s} \right)^{i-1} \sum_{k=0}^{i-1} \left(\frac{q}{q + \frac{p q_0 s}{1 - p_0 s}} \right)^k + q^i p \sum_{j=0}^{\infty} s^j p_0^j \\
&= \frac{p}{1 - p_0 s} \left(q + \frac{p q_0 s}{1 - p_0 s} \right)^i . \qquad (20)
\end{aligned}$$

This implies $\sum_{j=1}^{\infty} \mathrm{P}(S_j + \eta = i) = p/q_0$ and thereby the assertion

$$\sum_{j=0}^{\infty} s^j \frac{q_0}{p} \mathrm{P}(S_j + \eta = i) = \frac{q_0}{1 - p_0 s} \left(q + \frac{p q_0 s}{1 - p_0 s} \right)^i .$$

□

7 Minimal reverse GW process

Another reverse procedure, similar to Definition 6.1, was suggested earlier in [5]. It is based on $\tau_i = \inf\{j : S_j = i\}$, the first time the random walk $\{S_n\}_{n \geq 1}$ visits the level i. If the random walk jumps over the level i without visiting it, we put $\tau_i = \infty$.

Definition 7.1 *The minimal reverse GW process $\{X_n\}_{n \geq 0}$ is defined as a Markov chain with transition probabilities*

$$\mathrm{P}(X_{n+1} = j | X_n = i) = \mathrm{P}(\tau_i = j | \tau_i < \infty), \;\; i \geq 1, \;\; j \geq 1.$$

In [5] we used the term “backward GW process”. Here we use the name “minimal reverse GW process” to reflect the fact that the minimal reverse chain jumps to a lower or equal level in comparison with the uniform prior reverse if started at the same level. This fact is confirmed by the following proposition (cf Proposition 6.1)

Proposition 7.1 *The minimal reverse of the $G(p_0, p)$ reproduction is the* $\widehat{\mathrm{GW}}$ *with the $G(q, q_0)$ reproduction and $G(q_0)$ immigration.*

PROOF Observe that

$$\begin{aligned} \mathrm{P}(\tau_i = j) &= \mathrm{P}(S_j = i, S_{j-1} \neq i) = \mathrm{P}(S_j = i) - \mathrm{P}(S_{j-1} = S_j = i) \\ &= \mathrm{P}(S_j = i) - \mathrm{P}(S_{j-1} = i)\mathrm{P}(\xi_j = 0) \end{aligned}$$

and according to (13) we obtain

$$\begin{aligned} \sum_{j=0}^{\infty} s^j \mathrm{P}(\tau_i = j) &= \sum_{j=0}^{\infty} s^j \mathrm{P}(S_j = i) - p_0 s \sum_{j=1}^{\infty} s^{j-1} \mathrm{P}(S_{j-1} = i) \\ &= \frac{p q_0 s}{1 - p_0 s} \left(q + \frac{p q_0 s}{1 - p_0 s} \right)^{i-1} . \end{aligned}$$

This implies $\mathrm{P}(\tau_i < \infty) = p$ and thereby the assertion

$$\sum_{j=1}^{\infty} s^j \mathrm{P}(\tau_i = j | \tau_i < \infty) = \frac{q_0 s}{1 - p_0 s} \left(q + \frac{p q_0 s}{1 - p_0 s} \right)^{i-1}.$$

□

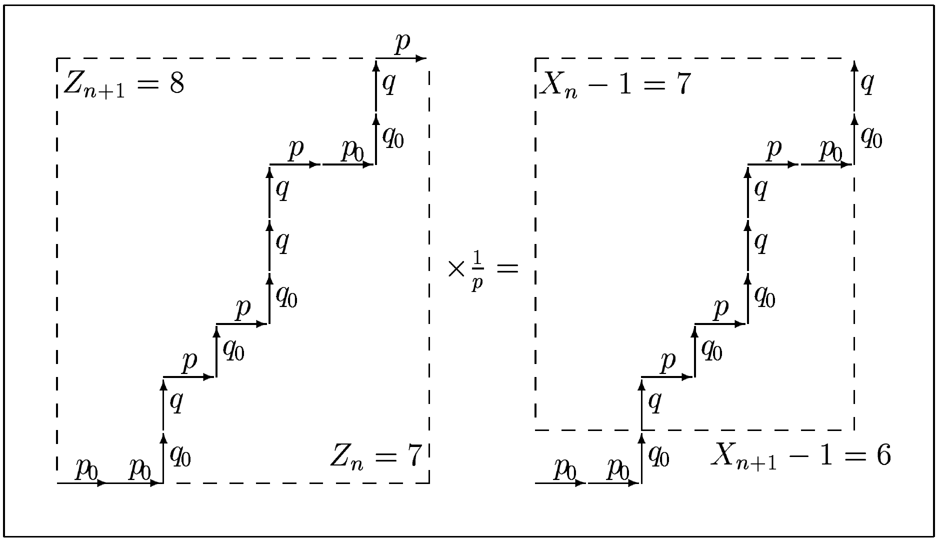

Figure 5: Picture proof of Proposition 8.1.

The next proposition addresses the question of what reproduction process should be reversed in the sense of Definition 7.1 to produce the $\mathrm{G}(p_0, p)$ reproduction. The answer is the GWI process with the $\mathrm{G}(q, q_0)$ reproduction and $\mathrm{G}(q_0)$ immigration.

Proposition 7.2 *The minimal reverse of the GWI with the $G(p_0, p)$ reproduction and $G(p)$ immigration is the GW process with the $G(q, q_0)$ reproduction.*

Proof The proof is similar to the previous one, only now instead of (13) we use (20) and get

$$\sum_{j=0}^{\infty} s^j \mathrm{P}(\tau_i^* = j) = p \left(q + \frac{p q_0 s}{1 - p_0 s} \right)^{i}.$$

This implies $\mathrm{P}(\tau_i^* < \infty) = p$ and thereby the assertion

$$\sum_{j=1}^{\infty} s^j \mathrm{P}(\tau_i = j | \tau_i < \infty) = \left(q + \frac{p q_0 s}{1 - p_0 s} \right)^{i}.$$

□

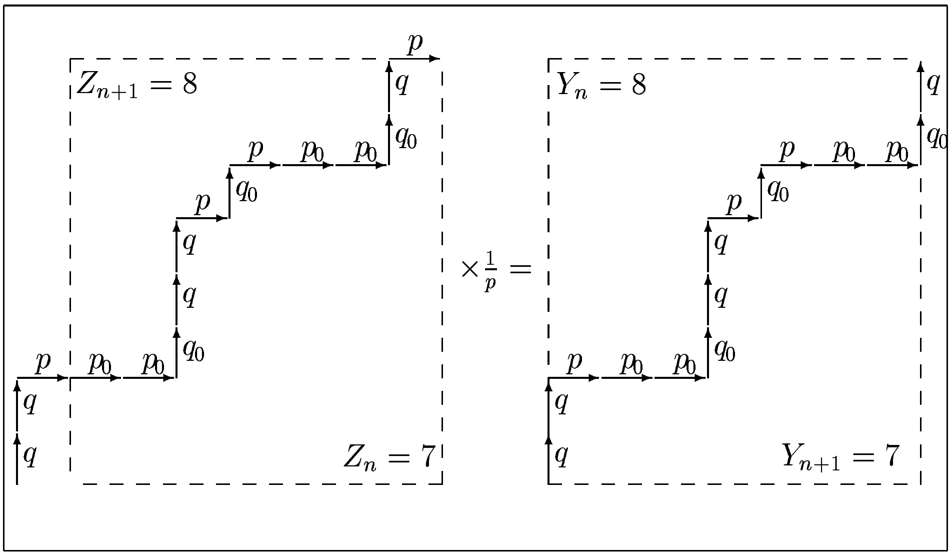

Figure 6: Picture proof of Proposition 7.2.

8 The dual GW process

The dual GW $\{V_n\}_{n\geq 0}$ was introduced in [2]. This definition implies the following formula for the transition probabilities

$$\mathrm{P}(V_{n+1} = j|V_n = i) = \mathrm{P}(S_{j+1} > i \geq S_j),\ i \geq 0,\ j \geq 0.$$

Proposition 8.1 *The dual to the GW process with the $G(p_0, p)$ reproduction is a GWI process with the $G(q, q_0)$ reproduction and $G(q_0)$ immigration.*

Proof Since

$$\mathrm{P}(V_{n+1} = j|V_n = i) = \mathrm{P}(S_{j+1} > i) - \mathrm{P}(S_j > i)$$

and

$$\begin{aligned}\sum_{j=0}^{\infty} s^j \mathrm{P}(S_j > i) &= \sum_{k=i+1}^{\infty} \sum_{j=0}^{\infty} s^j \mathrm{P}(S_j = k) \\ &= \frac{q_0 s}{(1-s)(1-p_0 s)} \left(q + \frac{p q_0 s}{1 - p_0 s}\right)^i\end{aligned}$$

we obtain

$$\sum_{j=0}^{\infty} s^j \mathrm{P}(V_{n+1} = j|V_n = i) = \frac{1-s}{s} \sum_{j=0}^{\infty} s^j \mathrm{P}(S_j > i) = \frac{q_0}{1 - p_0 s} \left(q + \frac{p q_0 s}{1 - p_0 s}\right)^i.$$

□

9 References

1. Alsmeyer G. and Rösler U. (2002) Asexual Versus Promiscuous Bisexual Galton-Watson Processes: The Extinction Probability Ratio. *Ann. Appl. Probab.* 12 , 125-142.

2. Asmussen S. and Sigman K. (1996) Monotone stochastic recursions and their duals. *Probab. Eng. Inf. Sc.* 10, 1-20.

3. Athreya K.B., Ney P.E. *Branching Processes*, Springer-Verlag, 1972.

4. Esty W.W. (1975) The Reverse Galton-Watson Process. *J. Appl. Prob.* 12, 574-580.

5. Klebaner F.C. and Sagitov S. (2002) The age of a Galton-Watson population with a geometric offspring distrubution. *J. Appl. Prob.* 39, 816-828.

6. Lamperti, J. and Ney, P. (1968) Conditioned branching processes and their limiting diffusions. *Theory Prob. Appl.* 13, 128-139.