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L^2-ESTIMATES FOR THE d-EQUATION AND WITTEN'S PROOF OF THE MORSE INEQUALITIES.

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ABSTRACT. This is an introduction to Witten's analytic proof of the Morse inequalities. The text is directed primarily to readers whose main interest is in complex analysis, and the similarities to Hörmander's L^2-estimates for the $\bar\partial$-equation is used as motivation. We also use the method to prove L^2-estimates for the d-equation with a weight $e^{-t\phi}$ where ϕ is a nondegenerate Morse function.

1. INTRODUCTION

The aim of these notes is primarily to give an easy introduction to Witten's proof of the Morse inequalities, see [8]. There are already excellent such accounts (see e g [5]), and the main particularity with this presentation is that it emphasises the relation between Witten's proof and the theory of L^2-estimates for the $\bar\partial$-equation. It is thus written with a mind to a reader whose main interest is in complex analysis, and we shall also take the opportunity to state and prove some weighted L^2-estimates for the d-equation that follow from Witten's method.

The simplest case of Hörmander's L^2-estimates for the $\bar\partial$-equation deal with the equation

$$\bar\partial u = f$$

where f is a $\bar\partial$-closed $(0,q)$-form in a bounded pseudoconvex domain Ω in $\mathbb{C}^n$. The theorem says that this equation can be solved with a solution u that satisfies the estimate

$$\int_\Omega |u|^2 e^{-\phi} \le C \int_\Omega |f|^2 e^{-\phi} \tag{1.1}$$

where C is a constant depending only on the diameter of Ω, and ϕ is any plurisubharmonic function.

This theorem follows from an a priori estimate for a dual problem and the plurisubharmonicity of ϕ enters in this dual estimate through the complex hessian

$$\left(\partial^2\phi/\partial z_j \partial \bar z_k\right) = (\phi_{jk})$$

which is positively semidefinite. It is quite clear that this proof can be adapted to the d-equation (even with some simplifications), and that in that way one obtains L^2-estimates for the d-equation in convex domains and with a convex weight function.

This is however not satisfactory since the condition on a domain for solvability of the d-equation is a purely toplogical one for which convexity is sufficient but far from necessary. It is also not clear why the weight function should be required to be convex. We shall see that the Morse inequalities give us a clue as to what the "right" condition on the weight function should be.

Let us first change the setting somewhat and let Ω be a compact manifold. Let ϕ be a smooth function on Ω. At each critical point of ϕ, i e a point p such that $d\phi(p) = 0$, the hessian of ϕ is a well defined quadratic form on the tangent space of Ω. If this quadratic form is nondegenerate at each critical point, ϕ is said to be a nondegenerate *Morse function*, and the index of a critical point is by definition the number of negative eigenvalues of the quadratic form. Let m_q be the number of critical points of index q - we assume there are only a finite number of critical points (which is automatically fulfilled if Ω is compact). We also let b_q be the q:th Betti number of Ω, i e the dimension of the q:th de Rham cohomology group with real coefficients. Recall that the de Rham cohomology group is by definition the space of d-closed q-forms modulo the space of exact forms. The Betti number b_q is thus zero precisely when the equation

$$du = f \tag{1.2}$$

is solvable for each closed q-form f. The weak Morse inequalities now state that

$$b_q \leq m_q \tag{1.3}$$

for each q. In particular it follows that if m_q is zero, i e if ϕ has no critical points of index q, then (1.2) is always solvable. The strong Morse inequalities say that for any q

$$b_q - b_{q-1} + b_{q-2} ... \leq m_q - m_{q-1} + m_{q-2} .... \tag{1.4}$$

Witten's proof of the Morse inequalities is based on the representation of a cohomology class by a harmonic form, i e one choses in each cohomology class the unique element of minimal (L^2) norm. For this one has to choose a Riemannian metric, and the main idea in the proof is to perturb the L^2-norms by introducing a weight factor, $e^{-\phi}$. Here ϕ is the Morse function in question and t is a large parameter. For these weighted norms one derives an identity, similar to but different from, the d-version of the Kodaira-Nakano-Hörmander identity for $\bar\partial$. In case $m_q = 0$, i e when ϕ has no critical points of index q the identity shows that there can be no harmonic q-forms if t is large enough, so the cohomology must vanish, and we shall see that the identity also leads to L^2-estimates for solutions to (1.2). In the general case the identity implies that the harmonic forms must, if t is large enough, be very concentrated near the critical points of index q with at most one harmonic form concentrated near each such critical point. This leads to the weak Morse inequalities. In general equality does not hold in these inequalities for the reason that at some critical points there may be

no corresponding harmonic forms. It turns out however that if instead of harmonic forms one studies spaces of eigenforms with small eigenvalues - "low-energy forms"- then there will be exactly one such form concentrated near each critical point of index q. The space of low-energy forms also gives a complex with the same cohomology as the space of all forms, and this is what eventually leads to the strong Morse inequalities.

In the case of non compact manifolds, like (domains in) $\mathbb{R}^n$, it turns out that similar arguments can be carried through provided that the manifold is endowed with a Riemannian metric which is complete, and that the Morse function satisfies appropriate conditions at infinity. We will not discuss the Morse inequalities themselves for open manifolds, but we will give a variant of the L^2-estimates for the case when solvability of (1.2) is predicted by the inequalities, i e when the weight function has no critical points of index q. I do not know if such L^2-estimates hold without the completeness assumption, but I believe they don't, since the estimates for noncomplete metrics seem to imply strong conditions on the boundary behaviour of solutions

This paper is organized as follows. In the next section we give an extremely simple motivating example. After that we compare the Witten identity and the d-version of the Hörmander identity for 1-forms in domains in $\mathbb{R}^n$. Section 4 is devoted to the proof of the weak Morse inequalities on compact manifolds , and L^2 estimates for (1.2) on complete Riemannian manifolds. The final section gives a short account of Witten's derivation of the strong Morse inequalties.

The content of these notes was presented at a minicourse in Toulouse in January -05. I would like to thank the organizers and participants in the course for creating such a stimulating atmosphere, and for "encouraging" me to write down the notes of the course. In the course we also discussed the holomorphic Morse inequalities of Demailly, [4], following the approach of Berman, [2]. We have not included this part of the course in these notes and instead refer to the original papers [4], [2] and the survey [1]. It should however be mentioned that the presentation of Witten's method in sections 4 and 6 is influenced by the scaling method of [2].

2. A MOTIVATING EXAMPLE IN $\mathbb{R}$

In this section we study weighted estimates for the d-equation for 1-forms on the real line. This amounts to solving what is arguably the simplest of all differential equations

$$du/dx = f, \tag{2.1}$$

with weighted estimates. To simplify even further we shall, instead of L^2-estimates discuss weighted L^1-estimates of the form

$$\int |u_t| e^{-t\phi} \leq C \int |f| e^{-t\phi}, \tag{2.2}$$

where t is a large positive parameter and the constant C does not depend on t. For what smooth (Morse) functions ϕ is it possible to find solutions u_t such that this estimate holds?

Let first f be a point mass δ_p at a point p. It is clear that if we can solve this equation with a solution $u_{(p)}$ satisfying an estimate (2.2), then

$$u = \int u_{(p)} f(p) dp$$

will be a good solution for an arbitrary f in $L^1(e^{-t\phi})$. With no loss of generality we may assume that $\phi(p) = 0$, so that the right hand side in (2.2) is just a constant. The general solution to (2.1) is constant for $x < p$ and $x > p$ and has a jump discontinuity of size 1 at p. To satisfy (2.2) locally we separate 4 different cases in the behaviour of ϕ , depending on whether ϕ is increasing, decreasing, has a local minimum or a local maximum at p.

In the first case our estimate will be satisfied locally if we choose $u_{(p)}$ equal to 0 to the left of p. In the second case we take $u_{(p)} = 0$ to the right of p. If ϕ has a local minimum at p both of the two previous alternatives work, whereas if ϕ has a local maximum at p no solution to (2.1) satisfies (2.2) even locally.

The condition on (a smooth Morse function) that we arrive at in order to have (2.2) satisfied locally is thus that there be no local maxima, or equivalently no critical point of index 1. This condition is precisely what is predicted by the Morse inequalities, and it is easy to see that if it is satisfied we also have a global weighted estimate.

3. THE d-EQUATION FOR 1-FORMS IN $\mathbb{R}^n$

Let us first recall the fundamental Hörmander identity for compactly supported $(0,1)$ forms in $\mathbb{C}^n$. We let

$$\alpha = \sum \alpha_j d\bar{z}_j$$

be a smooth compactly supported form, and ϕ be a smooth weight function. The formal adjoint of $\bar{\partial}$ in $L^2(e^{-\phi})$ is

$$\vartheta\alpha = \sum \delta_j \alpha_j,$$

where

$$\delta_j v = -e^{\phi}\partial/\partial z_j(e^{-\phi}v) = -v_j + \phi_j v.$$

Then we have

$$\int \left(\sum \phi_{j\bar{k}}\alpha_j\bar{\alpha}_k + \sum |\partial\alpha_j/\partial\bar{z}_k|^2\right) e^{-\phi} = \int \left(|\vartheta\alpha|^2 + |\bar{\partial}\alpha|^2\right) e^{-\phi} \tag{3.1}$$

In particular, if the complex Hessian of ϕ is uniformly bounded from below, and if $\bar{\partial}\alpha = 0$ we obtain an estimate

$$\int |\alpha|^2 e^{-\phi} \leq C \int |\vartheta\alpha|^2 e^{-\phi}.$$

Roughly speaking, this means that the adjoint of the $\bar\partial$-operator is strongly injective on the space of $\bar\partial$-closed forms. Via a functional analysis argument this leads to the surjectivity of the $\bar\partial$-operator itself, which means that the $\bar\partial$-equation is solvable with an estimate. We skip over here the complications in this argument that come from the fact that ϑ is only the formal adjoint, and that we at first only have the inequality for smooth compactly supported forms. In the case of the $\bar\partial$-equation in all of $\mathbb{C}^n$ it is relatively easy to overcome this complication by approximating a general element in the domain of the adjoint to $\bar\partial$ with test forms.

We now state and prove the corresponding identity for real forms. Here we let

$$\partial_j = \partial/\partial x_j,$$

$\phi_j = \partial_j\phi$, (ϕ_{jk}) be the Hessian of ϕ, and denote by

$$\delta\alpha = -e^{\phi}\sum \partial_j(e^{-\phi}\alpha_j) =: \sum \delta_j\alpha_j = -\sum \partial_j\alpha_j - \phi_j\alpha_j$$

the formal adjoint of d in $L^2(e^{-\phi})$.

Proposition 3.1. *Let α be a smooth compactly supported form in $\mathbb{R}^n$. Then*

$$\int \Big(\sum \phi_{jk}\alpha_j\alpha_k + \sum |\partial_k\alpha_j|^2\Big)\, e^{-\phi} = \int \big(|\delta\alpha|^2 + |d\alpha|^2\big)\, e^{-\phi} \tag{3.2}$$

Proof. Integrating by parts we get

$$\int |\delta\alpha|^2 e^{-\phi} = -\int \sum \partial_j(e^{-\phi}\alpha_j)\delta\alpha = \int \sum \alpha_j\partial_j\delta_k\alpha_k e^{-\phi} +$$

$$= \int \sum \phi_{jk}\alpha_j\alpha_k e^{-\phi} + \int \sum \alpha_j\delta_k\partial_j\alpha_k e^{-\phi} =$$

$$= \int \sum \phi_{jk}\alpha_j\alpha_k e^{-\phi} + \int \sum \partial_k\alpha_j\partial_j\alpha_k e^{-\phi}.$$

Formula (3.2) now follows since

$$\sum \partial_k\alpha_j\partial_j\alpha_k = \sum |\partial_k\alpha_j|^2 - |d\alpha|^2$$

□

This identity can be used to obtain solvability of the d-equation in convex domains and estimates for the solution with a convex weight function. To obtain a more general result we shall rewrite the identity.

First replace ϕ by 2ϕ and substitute αe^{ϕ} for α. Then we introduce the twisted d and δ operators by

$$d_{-\phi}\alpha = e^{-\phi}d(e^{\phi}\alpha)$$

and

$$\delta_{\phi}\alpha = e^{\phi}\delta(e^{-\phi}\alpha).$$

Formula (3.2) then says that

$$\int \sum \phi_{jk}\alpha_j\alpha_k + \sum |e^{-\phi}\partial_k(e^{\phi}\alpha_j)|^2 = \int |\delta_{\phi}\alpha|^2 + |d_{-\phi}\alpha|^2 \tag{3.3}$$

The second term in the integrand on the left hand side is

$$\sum(\partial_k\alpha_j+\phi_k\alpha_j)^2=\sum|\partial_k\alpha_j|^2+2\sum\phi_k\partial_k\alpha_j\alpha_j+\sum\phi_k^2|\alpha|^2=$$
$$=\sum|\partial_k\alpha_j|^2+\sum\phi_k\partial_k\alpha_j^2+|d\phi|^2|\alpha|^2.$$

Integrating by parts again, the second term

$$\sum\phi_k\partial_k\alpha_j^2$$

gives a contribution that equals

$$-\int\Delta\phi|\alpha|^2.$$

Summing up we have therefore proved the next proposition.

Proposition 3.2. *Under the same hypotheses as in the previous proposition we have*

(3.4)
$$\int\sum|\partial_k\alpha_j|^2+|d\phi|^2|\alpha|^2+2\sum\phi_{jk}\alpha_j\alpha_k-\Delta\phi|\alpha|^2=\int|\delta_\phi\alpha|^2+|d_{-\phi}\alpha|^2$$

Proposition 3.2 is a special case of the formula used by Witten in his proof of the Morse inequalities. To get solvability of the d-equation from it we need to choose ϕ so that the left hand side dominates the L^2-norm of α.

Replace ϕ by $t\phi$ where t is a large parameter. Then the gradient term in the left hand side of (3.4) grows quadratically in t, whereas the term that contain second order derivatives of ϕ grow only linearly. For large t the (nonnegative) gradient term therefore dominates the two last terms outside the critical points of ϕ, so we now need to study more closely the behaviour of the integrand near the critical points.

Let $x=0$ be a critical point of ϕ. After an orthogonal change of coordinates we may assume that near 0

(3.5)
$$\phi(x)=\phi(0)+1/2\sum\lambda_jx_j^2+O(x^3).$$

Then

$$|d\phi|^2(x)=\sum\lambda_j^2x_j^2+O(x^3),$$

and

$$2\sum\phi_{jk}\alpha_j\alpha_k-\Delta\phi|\alpha|^2=\sum(2\lambda_j-\lambda)\alpha_j^2+O(x),$$

with $\lambda=\sum\lambda_j$. For the estimates we now need a version of the Heisenberg uncertainty inequality.

Lemma 3.3. *Let u be a smooth function on $\mathbb{R}^n$ and let λ_k be real numbers. Then*

$$\int|du|^2+\sum\lambda_k^2x_k^2u^2dx\geq\sum|\lambda_k|\int u^2dx$$

If the left hand side is finite, equality holds if and only if u is a Gaussian function

$$u=Ce^{-\sum|\lambda_k|x_k^2/2}.$$

Proof. We first prove the statement when $n = 1$, and assume to start with that u has compact support. Then

$$\int u^2 dx = \int u^2 \frac{dx}{dx} dx = -\int 2xu'u dx.$$

Multiplying by $|\lambda|$ we find

$$|\lambda| \int u^2 dx = \int 2|\lambda| xu'u dx \leq \int \lambda^2 x^2 u^2 + (u')^2 dx,$$

and the inequality follows. The same argument works if u' and xu lie in L^2 and we see that equality can hold only if

$$-|\lambda| xu = u'$$

which means that $u = Ce^{-|\lambda| x^2/2}$. This proves the Lemma for $n = 1$.

In higher dimensions the inequality follows from the one-variable case if we write

$$|du|^2 = \sum (\partial_k u)^2,$$

and apply the one-variable result in each variable separately. Equality holds iff u is a Gaussian function in each variable separately, and so is Gaussian. □

For the applications we need a weaker local form of the lemma.

Lemma 3.4. *Let $a > 0$. Then for each $\epsilon > 0$ there is a finite constant C, depending only on ϵ and a, such that for any smooth function u on $[-a, a]^n$ we have*

$$\int_{[-a,a]^n} |du|^2 + \sum_k \lambda_k^2 x_k^2 u^2 dx \geq (1 - \epsilon) \sum (|\lambda_k| - C) \int u^2 dx$$

Proof. Again it it suffices to prove the theorem for $n = 1$. Let χ be a smooth positive function between 0 and 1, which equals 1 on $[-a/2, a/2]$ and has compact support in $[-a, a]$, and apply the previous lemma to χu. We then get

$$|\lambda| \int_{-a}^{a} \chi^2 u^2 dx \leq \int_{-a}^{a} \lambda^2 x^2 u^2 \chi^2 + (u')^2 + (\chi')^2 u^2 + 2\chi' \chi u' u dx \leq$$

$$\leq \int_{-a}^{a} \lambda^2 x^2 u^2 \chi^2 + (1 + \epsilon)(u')^2 + C u^2 dx.$$

On the other hand

$$\int_{-a}^{a} (1-\chi^2) \lambda^2 x^2 u^2 dx \geq \lambda^2/4a^2 \int_{-a}^{a} u^2 (1-\chi^2) dx \geq (|\lambda| - a^2) \int_{-a}^{a} u^2 (1-\chi^2) dx.$$

Adding this to the previous inequality we get the claim of the lemma.

□

Now we return to the critical point $x = 0$ and assume that ϕ is given by (3.5) near 0. Replace ϕ by $t\phi$ where t is a large parameter. Considering the expression in the left hand side of (3.4) it follows from the previous lemma that, with $\lambda = \sum \lambda_k$,

$$\int_{[-a,a]^n} \sum |\partial_k \alpha_j|^2 + t^2 |d\phi|^2 |\alpha|^2 + 2t \sum \phi_{jk} \alpha_j \alpha_k - t\Delta\phi |\alpha|^2 \geq$$

$$\geq \int_{[-a,a]^n} \epsilon |x|^2 |\alpha|^2 + t \sum (2\lambda_j - \lambda + (1-\epsilon) \sum |\lambda_k|) \alpha_j^2,$$

modulo an error term of size

$$\int C|\alpha|^2 + t^2 O(a^3)|\alpha|^2 + tO(a)|\alpha|^2.$$

Here C depends only on a and ϵ and is in particular independent of t. We have assumed that ϕ is a Morse function so the critical point is non degenerate which means that all the numbers λ_j are different from 0. Since

$$(2\lambda_j - \lambda + \sum |\lambda_k|) = 2(\lambda_j - \sum_{\lambda_k < 0} \lambda_k),$$

this expression is strictly greater than 0 except if λ_j is negative and all the other λ_ks are positive. In particular, if it is negative 0 must be a critical point of index 1.

Assume this is not the case. Then

$$(2\lambda_j - \lambda + (1-\epsilon) \sum |\lambda_k|)$$

is still positive if ϵ is small enough. If we choose a small enough and then t large enough all the error terms are absorbed and we conclude that

$$\int_{[-a,a]^n} \sum |\partial_k \alpha_j|^2 + t^2 |d\phi|^2 |\alpha|^2 + 2t \sum \phi_{jk} \alpha_j \alpha_k - t\Delta\phi |\alpha|^2 \geq$$

$$t\delta \int_{[-a,a]^n} |\alpha|^2$$

for some positive δ.

Now let ϕ be a nondegenerate Morse function in $\mathbb{R}^n$ and assume moreover that $|d\phi|$ is bounded from below at infinity. Then ϕ has only a finite number of critical points, and we make the hypothesis that none of the critical points is of index 1. We also assume that ϕ satisfies the technical condition, (C) that

$$|D^2\phi| \leq C|d\phi|^2,$$

where D^2 stands for any second order derivative, outside of a compact subset. Repeating the argument above for each of the critical points we conclude from Proposition 3.2 that

$$\int |\delta_{t\phi}\alpha|^2 + |d_{-t\phi}\alpha|^2 \geq t\delta \int |\alpha|^2$$

for any smooth compactly supported 1-form. Substituting back $e^{-t\phi}$ for α we get equivalently that

$$\int (|\delta_{2t\phi}\alpha|^2 + |d\alpha|^2)e^{-2t\phi} \geq t\delta \int |\alpha|^2 e^{-2t\phi}.$$

One can now apply the argument from [6] to obtain the following theorem on solvability of the d-equation . We do not give the details here since we come back to this kind of argument for arbitrary complete Riemannian metrics in the next section.

Theorem 3.5. *Let ϕ be a nondegenerate Morse function in $\mathbb{R}^n$ whose gradient is bounded from below at infinity. Assume ϕ satisfies the technical assumption (C) above. Assume ϕ has no critical point of index 1. Then we can, for any d-closed 1-form f in $\mathbb{R}^n$ and any $t > t_0$ solve the equation*

$$du = f$$

with a function u that satisfies the estimate

$$\int u^2 e^{-2t\phi} \leq C/t \int |f|^2 e^{-2t\phi}.$$

In particular, if u lies in $L^2(e^{-2t\phi}) \cap L^1(e^{-2t\phi})$, then

$$\int (u - u_t)^2 e^{-2t\phi} \leq C/t \int |du|^2 e^{-2t\phi}$$

holds if

$$u_t = \int u e^{-2t\phi} / \int e^{-2t\phi}.$$

The last statement is a weakened version of the so called Brascamp-Lieb inequality ([3]) for non-convex weights. It follows from the first statement since $u - u_t$ is the L^2-minimal solution to

$$d(u - u_t) = du.$$

4. Weak Morse inequalities

Let at first Ω be a compact differentiable manifold of dimension n, equipped with a Riemannian metric. We denote by

$$H^q(X, \mathbb{R})$$

the de Rham cohomology groups of X of order q, i e the quotient between the space of d-closed q-forms on X and its subspace of exact forms. The Riemannian metric induces a norm and a scalar product on the space of q-forms, that can be expressed in terms of the Hodge *-operator as

$$\langle \alpha, \beta \rangle = \int \alpha \wedge *\beta.$$

We denote by $\delta = d^*$ the formal adjoint of the d-operator with respect to this scalar product, defined by

$$\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle,$$

for smooth forms α and β. In classical Hodge theory, see eg [7], one chooses in each cohomology class the unique representative of minimal L^2-norm. If f is such a representative of minimal norm it follows that the function

$$|f + sd\alpha|^2$$

has a minimum for $s = 0$, which means that f must be orthogonal to the space of exact forms, so

$$\delta f = 0,$$

by the definition of the δ-operator. If we introduce the Laplace operator on forms by

$$\Delta f = d\delta + \delta d,$$

it therefore follows that a minimal representative must be *harmonic*, i e solve $\Delta f = 0$. From the ellipticity of the Δ it follows that f is smooth. Conversely, by Pythagoras' theorem, any smooth closed form satisfying $\delta f = 0$ must be a representative of minimal norm of its cohomology class. It is also easy to see that if $\Delta f = 0$ then

$$\langle \Delta f, f \rangle = \int |df|^2 + |\delta f|^2 = 0$$

so f must be a closed form of minimal norm in its cohomology class. Summing up, there is a natural isomorphism between the de Rham cohomology groups and the space of harmonic forms, $\mathcal{H}^2(X)$.

Now let ϕ be a Morse function on X and consider the weighted norms

$$|\alpha|_t^2 = \int |\alpha|^2 e^{-2t\phi}.$$

The basic idea in Witten's proof can be described as choosing a representative of minimal weighted norm for large t. Such a minimal representative satisfies the two equations

$$df = 0$$

and

$$\delta_{2t} f := \delta_{2t\phi} f := e^{2t\phi} \delta(e^{-2t\phi} f) = 0.$$

Here δ_{2t} is the formal adjoint of d with respect to the weighted scalar product. Just as in the previous section it is convenient to substitute $g = fe^{t\phi}$ for f and we then obtain a form that satisfies

$$\delta_t g = 0$$

and

$$d_{-t} g := e^{-t\phi} d(e^{t\phi} g) = 0.$$

Introducing the perturbed Laplace operator

$$\Delta_t := \delta_t d_{-t} + d_{-t} \delta_t,$$

we see, as in the unweighted case, that this is equivalent to the single equation

$$\Delta_t g = 0.$$

Just as in classical Hodge theory we therefore get an isomorphism between the cohomology groups H^q and the spaces

$$\mathcal{H}_t^q := \{g \quad q - \text{form}; \Delta_t g = 0\}.$$

Alternatively, we may view this as introducing a new complex with the coboundary operator d_{-t} and taking a representative of minimal unweighted norm in each cohomology class of the new complex. (Note that δ_t is the adjoint of d_{-t} with respect to the unweighted L^2-norms.)

We shall now derive an integral identity for the expression

$$\int \langle \Delta_t \alpha, \alpha \rangle = \int |\delta_t \alpha|^2 + |d_{-t}\alpha|^2$$

that generalizes formula (3.4). For this we expand the operator Δ_t in powers of t at a point p in X. We may take local coordinates, x, near p, such that $x(p) = 0$ and the Riemannian metric on X is euclidean to first order at p. This means that the metric is given in terms of the coordinates by a matrix $(g_{ij}(x))$ that is the identity matrix when $x = 0$ and satisfies $dg_{ij} = 0$ at $x = 0$. Let ω_j denote the operator (on forms) of interior multiplication with the form dx_j, and let ω_j^* be the dual operator of exterior multiplication with dx_j. Denoting $\partial_j \phi = \phi_j$ we get

$$d_{-t} = d + t \sum \phi_j \omega_j^* = d + t d\phi \wedge$$

and

$$\delta_t = \delta + t \sum \phi_j \omega_j = \delta + t (d\phi \wedge)^*.$$

Moreover

$$d = \sum \omega_j^* \partial_j$$

and

$$\delta = -\sum \partial_j \omega_j.$$

From the definition of the perturbed Laplacian we get at $x = 0$

$$\Delta_t = \Delta + t^2 |d\phi|^2 + t M_\phi. \tag{4.1}$$

Here

$$M_\phi = \sum \phi_{jk} [\omega_j^*, \omega_k], \tag{4.2}$$

with $[\omega_j^*, \omega_k] = \omega_j^* \omega_k - \omega_k \omega_j^*$ In the computations one uses that at $x = 0$ the operators ∂_j and ω_k commute and that

$$\omega_j \omega_k^* + \omega_k^* \omega_j = \delta_{jk}.$$

The expression for the operator M_ϕ, holds under the assumption that the coordinates are chosen so that the metric equals the euclidean metric to first order, but M_ϕ nevertheless of course defines a global zeroth order operator.

Integrating (4.1) we get

$$\int |d_{-t}\alpha|^2 + |\delta_t \alpha|^2 =$$

$$= \langle \Delta_t \alpha, \alpha \rangle = \int \Delta\alpha \cdot \alpha + t^2 |d\phi|^2 |\alpha|^2 + t M_\phi(\alpha) \cdot \alpha.$$

The same relation holds even if X is not compact provided that α has compact support. We shall now use this formula to determine all the Δ_t-harmonic forms in $\mathbb{R}^n$ in the model case when $\phi = 1/2 \sum \lambda_j x_j^2$. In the computations we write with multiindex notation

$$\alpha = \sum \alpha_J dx_J,$$

$$\lambda_J = \sum_J \lambda_j,$$

and we will need an explicit formula for $M_\phi \alpha \cdot \alpha$ that follows from (4.2):

$$M_\phi \alpha \cdot \alpha = \sum (\lambda_J - \lambda_{J^c}) |\alpha_J|^2. \tag{4.3}$$

Theorem 4.1. *Let $\phi = 1/2 \sum \lambda_j x_j^2$ where all λ_j are different from 0 and let Δ_t be the corresponding laplace operator. Let α be a q-form in $L^2(\mathbb{R}^n)$ that satisfies*

$$\Delta_t \alpha = 0.$$

If the number of negative λ_j is not equal to q, then $\alpha = 0$. If the number of negative λ_j is equal to q and, say, the first q λ_j are negative, then

$$\alpha = C e^{-1/2 \sum |\lambda_j| x_j^2} dx_1 \wedge ... dx_q,$$

for some constant C.

Proof. Let χ be a smooth function with compact support in the ball with radius 2 which equals 1 in the ball with radius 1. Let

$$\chi_R := \chi(\cdot/R)^2.$$

Then by (4.1)

$$0 = \langle \Delta_t \alpha, \chi_R \alpha \rangle =$$

$$= \int \chi_R (\Delta\alpha \cdot \alpha + t^2 \sum \lambda_j x_j^2 |\alpha|^2 + t M_\phi \alpha \cdot \alpha).$$

Since, in $\mathbb{R}^n$ with the euclidean metric

$$\int \Delta\alpha \cdot \alpha \chi_R = \int \sum |\partial_k \alpha_J|^2 \chi_R$$

up to an error which is

$$\int O(|d\chi_R||\alpha|(\sum |\partial_k \alpha_J|).$$

we get, by the formula for M_ϕ, that

$$0 = \int \sum |\partial_k \alpha_J|^2 + t^2 \sum \lambda_j x_j^2 |\alpha_J|^2 + \sum (\lambda_J - \lambda_{J^c}) |\alpha_J|^2$$

modulo the same error. Since $|d\chi_R|^2 \leq C\chi_R/R^2$ the error term vanishes as R tends to infinity. Apply Lemma 3.3 to $u = \alpha_J$ for each J. We then get

$$0 \geq \int \sum (\lambda_J - \lambda_{J^c} + \sum |\lambda_j|)|\alpha_J|^2.$$

The coefficient of each $|\alpha_J|^2$ equals twice the sum of all positive λ_j with j in J, minus the sum of all negative λ_j with j outside of J. This number is always nonnegative and equals 0 only when there are precisely q negative eigenvalues, all of them lying in J. If α is not identically equal to 0 the number of negative eigenvalues, i e the index of the critical point 0, must therefore equal q and $\alpha = \alpha_J dx_J$. Finally J must consist of all the indices corresponding to negative λ_j. Moreover, equality must hold in (3.3), so α_J is a Gaussian function of the type claimed.

□

Actually, the only consequence of Theorem 4.1 that we will need is that the dimension of the solution space is 1.

We now turn to the proof of the weak Morse inequalities. Assume α is a q-form on X such that $\Delta_t \alpha = 0$. It follows from (4.1) that

$$\int |d\alpha|^2 + |\delta\alpha|^2 + t^2|d\phi|^2|\alpha|^2 + tM_\phi \alpha \cdot \alpha = 0, \tag{4.4}$$

so

$$t^2 \int |d\phi|^2|\alpha|^2 \leq -t \int M_\phi \alpha \cdot \alpha \leq Ct \int |\alpha|^2. \tag{4.5}$$

Let p be a critical point of ϕ and choose coordinates centered at p with respect to which the metric is Euclidean to first order and ϕ has the local form (3.5). Then $|d\phi|^2 > c|x|^2$ near p. Let p_1, ...p_N be the critical points of ϕ and let B_j be balls in coordinates centered at p_j with radius $A/\sqrt{t}$. It follows that, if t is large enough,

$$t^2 \int_{(\cup B_j)^c} |d\phi|^2|\alpha|^2 \geq cA^2 t \int_{(\cup B_j)^c} |\alpha|^2.$$

Hence, by (4.5),

$$\int_{(\cup B_j)^c} |\alpha|^2 \leq \frac{C}{A^2} \int |\alpha|^2,$$

so

$$\int_{\cup B_j} |\alpha|^2 \geq (1 - \frac{C}{A^2}) \int |\alpha|^2. \tag{4.6}$$

This means that if t is large we can choose A so large that at least half of the mass of α is concentrated in the union of the B_js, i e very near the critical points.

Let F_t be the scaling map in $\mathbb{R}^n$

$$F_t(x) = x/\sqrt{t}.$$

For each j between 1 and N we restrict α to B_j and let

$$\alpha_j = F_t^*(\alpha|_{B_j}),$$

so α_j is a form in the Euclidean ball with radius A. Moreover, α_j is harmonic for the operator Δ_t in $\mathbb{R}^n$ defined by the weight function

$$\phi_t(x) = t(\phi(x/\sqrt{t}) - \phi(0)),$$

(the constant term of ϕ plays no role in the computation of Δ_t) and the Riemannian metric

$$tF_t^*(ds^2) =: ds_t^2 = \sum g_{ij}(x/\sqrt{t})dx_i dx_j.$$

Finally, we normalize by putting

$$f_j = t^{(q-n/2)/2}\alpha_j,$$

to get a form whose norm with respect to ds_t^2 equals the norm of α in B_j. Let $\Psi_{A,t}$ be the map

$$\Psi_{A,t}(\alpha) = f = (f_1, ...f_N).$$

Thus $\Psi_{A,t}$ maps forms on X that are harmonic with respect to Δ_t to forms on a disjoint union of N balls of radius A in $\mathbb{R}^n$ that are harmonic with respect to the corresponding operator defined by the weight ϕ_t and the metric ds_t^2. By (4.6), $\Psi_{A,t}$ is almost an isometry if A is sufficiently large, hence in particular injective.

Now take for any large t an element $\alpha_{(t)}$ in $\mathcal{H}_t^q$ of norm 1, let $f_{(t)} = \Psi_{A,t}(\alpha_{(t)})$ and let t tend to infinity. Since the norms of $f_{(t)}$ are uniformly bounded, we take a subsequence that converges weakly to a limit f_∞. By an identity of the form (4.4) on the union of the balls in $\mathbb{R}^n$ we have a control of the first order derivatives of $f_{(t)}$, so by the Rellich lemma we even have strong convergence on every compact part. Therefore the limit form is nonzero. Moreover, since ds_t^2 tends to the euclidean metric as t tends to infinity, and ϕ_t tends to a quadratic form $1/2\sum \lambda_j x_j^2$, f_∞ is harmonic with respect to a Laplacian like in Theorem 4.1. Here the choice of A is arbitrary so f_∞ actually extends to such a harmonic form in the disjoint union of N copies of $\mathbb{R}^n$. This means that f_∞ lies in a space of dimension at most $N = m_q$.

We can for any t choose an ordered orthormal basis of $\mathcal{H}_t^q$ and apply this argument to each element in the basis. Since $\mathcal{H}_t^q$ is for any t isomorphic to the cohomology group H^q, the number of elements in each such basis equals h^q. The forms that we get after applying the limit procedure described above must be linearily independent, since any fixed nonzero linear combination of them tends to a nonzero limit. Hence $h^q \leq m_q$, so the weak Morse inequalities follow.

5. L^2-ESTIMATES

5.1. **Compact manifolds.** It follows from the weak Morse inequalities of the previous section that if a compact manifold has a Morse function with no critical points of index q, then the equation $du = f$ is always solvable for any d-closed q-form f. In this section we shall make this statement more precise by proving the following variant of Theorem 3.5.

Theorem 5.1. *Let X be a compact n-dimensional Riemannian manifold and let ϕ be a non-degenerate Morse function on X. Let q be an integer between 1 and n such that ϕ has no critical point of index q. Then, for t sufficiently large and for any d-closed q-form f there is a $(q-1)$-form u such that $du = f$ and*

$$\int_X |u|^2 e^{-t\phi} \leq C/t \int_X |f|^2 e^{-t\phi},$$

where C is a constant independent of t.

The main step of the proof consist in establishing the next lemma.

Lemma 5.2. *Under the hypotheses of Theorem 5.1 there is a constant C, independent of t, such that for any smooth q-form α, on X*

$$\int_X |\alpha|^2 \leq C/t \int_X |d_{-t}\alpha|^2 + |\delta_t \alpha|^2.$$

Proof. It follows from (4.1) that

$$\int_X |d_{-t}\alpha|^2 + |\delta_t\alpha|^2 = \int \Delta_t \alpha \cdot \alpha = \int |d\alpha|^2 + |\delta\alpha|^2 + t^2 |d\phi|^2 |\alpha|^2 + t M_\phi \alpha \cdot \alpha.$$

Let p_j be the critical points of ϕ and let B_j be small balls in local coordinates centered at each p_j. Denote by X the union of the balls B_j and let χ be a cutoff function which is equal to zero outside of X and equal to 1 near the points p_j. The integral in the right hand side above can now be decomposed into two terms,

$$I := \int \chi \left(|d\alpha|^2 + |\delta\alpha|^2 + t^2 |d\phi|^2 |\alpha|^2 + t M_\phi \alpha \cdot \alpha \right),$$

and a similar integral, II, with χ replaced by $1 - \chi$. For large t (not depending on α) II is evidently positive and dominates a multiple of

$$t^2 \int (1-\chi) |\alpha|^2.$$

To analyse the first term we notice first that there is no loss of generality in assuming that the metric on X is euclidean with respect to the chosen coordinates in B_j and so is Euclidean in all of X. Integrating by parts we find that

$$I = \int \chi \left((\alpha, \Delta\alpha) + t^2 |d\phi|^2 |\alpha|^2 + t M_\phi \alpha \cdot \alpha \right)$$

plus an error term that can be estimated by

$$\int |d\chi||\nabla\alpha||\alpha|. \tag{5.1}$$

(Here $\nabla\alpha$ is the gradient of α taken componentwise with respect to the given coordinates.) But, in Euclidean coordinates the Laplacian Δ on a form is just the Laplacian taken on each component. Integrating by parts again we therfore get

$$I = \int \chi \left(|\nabla\alpha|^2 + t^2|d\phi|^2|\alpha|^2 + tM_\phi\alpha\cdot\alpha\right)$$

plus an error term of the same type as before. The error term (5.1) is smaller than

$$\epsilon \int |d\chi|^2|\nabla\alpha|^2 + 1/\epsilon \int |\alpha|^2.$$

By a simple, wellknown and useful trick one can choose the cutoff function χ so that $|d\chi|^2 \leq C\chi$ (the trick consist in replacing χ by χ^2). Putting all this together we then get for t large that that

$$\int_X |d_{-t}\alpha|^2 + |\delta_t\alpha|^2 \geq$$

$$\geq ct^2 \int (1-\chi)|\alpha|^2 + \int \chi \left((1-\epsilon)(|\nabla\alpha|^2 + t^2|d\phi|^2|\alpha|^2) + tM_\phi\alpha\cdot\alpha\right) \geq$$

$$ct^2 \int_{U^c} |\alpha|^2 + \int_U \left((1-\epsilon)(|\nabla\alpha|^2 + t^2|d\phi|^2|\alpha|^2) + tM_\phi\alpha\cdot\alpha\right),$$

where we take U to be a neighbourhood of the critical points where χ equals one. Then U is a union of neighbourhoods U_j of p_j that we can take to be little cubes as in Lemma 3.4. By lemma 3.4 and the discussion immediately after we find that

$$\int_{U_j} |\nabla\alpha_J|^2 + t^2|d\phi|^2|\alpha_J|^2 \geq (t(1-\epsilon)\sum|\lambda_k| - C)\int_{U_j}|\alpha_J|^2,$$

if α_J is the coefficient of one component of α with respect to the coordinates near p_j. On the other hand

$$M_\phi\alpha\cdot\alpha = \sum(\lambda_J - \lambda_{J^c})|\alpha_J|^2 + O(|x||\alpha|).$$

Hence

$$\int_{U_j} \left((1-\epsilon)(|\nabla\alpha|^2 + t^2|d\phi|^2|\alpha|^2) + tM_\phi\alpha\cdot\alpha\right) \geq \tag{5.2}$$

$$\geq t\int_{U_j} \sum(\lambda_J - \lambda_{J^c} + (1-\epsilon)\sum|\lambda_k|)|\alpha_J|^2.$$

Precisely as in the discussion of weak Morse inequalties in the previous section we write

$$\sum(\lambda_J - \lambda_{J^c} + \sum|\lambda_k|)$$

as twice the sum of all λ_j with j in J and λ_j positive minus the sum of all λ_j with j not in J and λ_j negative. This expression is always nonnegative and can be zero only if the multiindex J consists precisely of all j with λ_j negative. Since we have assumed that there are no critical points of index q this does not happen at any p_j so the expression is strictly positive. Hence

$$\sum(\lambda_J - \lambda_{J^c} + (1-\epsilon)\sum|\lambda_k|)$$

is also positive for ϵ sufficiently small, and the lemma follows.

□

It is now easy to deduce Theorem 5.1. The statement of the theorem is equivalent to saying that we can solve

$$d_{-t}u = f$$

with an estimate

$$\int |u|^2 \leq C/t \int |f|^2$$

if $d_{-t}f = 0$ and t is large enough. For this we use the following consequence of Lemma 5.2.

Lemma 5.3. *Let f be a d_{-t} closed q-form on X. Under the hypotheses of Theorem 5.1 there is a constant C such that*

$$|\int (f,\alpha)|^2 \leq C/t \int |\delta_t\alpha|^2 \int |f|^2$$

for any smooth q-form α on X.

Proof. Decompose $\alpha = \alpha_1 + \alpha_2$ where $d_{-t}\alpha_1 = 0$ and α_2 is orthogonal to the kernel of d_{-t}. Then α_2 is in particular orthogonal to the range to d_{-t} so $\delta_t\alpha_2 = 0$. Moreover, $d_{-t}\alpha_2 = d_{-t}\alpha$ is smooth, so $\Delta_t\alpha_2$ is smooth. Hence α_2 and therefore also α_1 are smooth. Hence it suffices to prove the statement of the lemma for α_1, i e we may assume that $d_{-t}\alpha = 0$. But then by Lemma 5.2

$$|\int (f,\alpha)|^2 \leq \int |\alpha|^2 \int |f|^2 \leq C/t \int |\delta_t\alpha|^2 \int |f|^2.$$

□

Now define a linear functional on the space of forms $\delta_t\alpha$ where α is a smooth q-form by

$$L(\delta_t\alpha) = \int (f,\alpha).$$

By the last lemma L is well defined and has norm less than $\sqrt{C/t}\|f\|$. The Riesz representation theorem then shows that there is a $(q-1)$-form u with norm less than the norm of L satisfying

$$\int (f,\alpha) = \int (u,\delta_t).$$

Then $d_{-t}u = f$ which proves Theorem 5.1.

5.2. **Open manifolds.** We next discuss briefly the case of open manifolds. For the estimates of Lemma 5.2 to work we then need to assume that $|d\phi|$ be bounded from below by a positive constant at infinity. We also need that the second derivatives be bounded from above. Under these assumptions, Lemma 5.2 can be proved just as for compact manifolds if α is of compact support.

The next step is to generalize Lemma 5.3. It is enough to have the lemma for forms α of compact support, but a difficulty now arises when we decompose α into α_1 and α_2 since these forms in general will not have compact support anymore. To handle this difficulty one assumes that in addition the metric on X is *complete*. Under this extra hypothesis one can then prove that

$$\int |\alpha_1|^2 \leq C/t \int |\delta_t \alpha_1|^2$$

by applying Lemma 5.2 to $\chi_\nu \alpha_1$, where χ_ν is a sequence of compactly supported cutoff functions that are equal to 1 on larger and larger compact sets that eventually cover all of X. Since the metric is complete χ_ν can be chosen with gradients uniformly bounded which implies that the error terms appearing in the approximation of α by $\chi_\nu \alpha$ goes to zero. The rest of the argument runs as before and we obtain a variant of Theorem 5.1 for open manifolds with complete metrics and nondegenerate Morse functions with gradients bounded from below at infinity.

Theorem 5.4. *Let X be a complete n-dimensional manifold and let ϕ be a non-degenerate Morse function on X. Assume the gradient of ϕ is uniformly bounded from below at infinity and that the second derivatives of ϕ are bounded from above by $|d\phi|^2$ at infinity. Let q be an integer between 1 and n such that ϕ has no critical point of index q. Then for all t sufficiently large and for any d-closed form f of degree q there is a solution to the equation $du = f$ satisfying the estimate*

$$\int |u|^2 e^{-t\phi} \leq C/t \int |f|^2 e^{-t\phi}.$$

Note that any open manifold can be given a complete Riemannian metric. If the manifold has an exhaustion function, ψ, which is a nondegenerate Morse function whithout critical points of index q we can compose ψ with a rapidly increasing functions to obtain another function ϕ satisfying the conditions of the theorem. Theorem 5.4 therefore implies solvability of the d-equation in such manifolds.

It is also worth noticing that there is no assumption on the boundary behaviour of ϕ explicitly in the assumptions of Theorem 5.4; the assumption is only on the derivatives of ϕ. Therefore the theorem also gives solvability in cases that are rather unrelated to Morse theory. A case in point is when M is an arbitrary compact manifold and $X = M \times (-1, 1)$. Let t be the projection from X to $(-1, 1)$ and let ϕ be a strictly increasing function of t. Then ϕ has no critical points at all and if we give X a complete metric and

make the derivative of ϕ sufficiently large at the end points of the interval $(-1,1)$, we get solvability of the d-equation for all closed forms f that lie in the corresponding weighted L^2-space. This may look surprising at first sight since the cohomology of M is arbitrary, but is explained by the fact that the completeness of the metric forces f to tend to zero as t goes to 1 or -1, so f is homotopic to 0 and therfore exact.

6. Strong Morse inequalities.

In this section we again let X be a compact manifold (without boundary). The proof of the weak Morse inequalities in section 3 depended on an estimate of the dimension of the space of Δ_t-harmonic forms on X. The proof of the strong Morse inequalities uses that the same estimate holds for the larger space of *low energy forms*.

Any smooth form on X can be written

$$\alpha = \sum \alpha_j$$

where the α_j are eigenforms for Δ_t, i e

$$\Delta_t \alpha_j = E_j \alpha_j.$$

Choose a function $\epsilon(t)$ slowly tending to zero. We shall say that α is a low energy form if all the eigenvalues (or energy levels) E_j appearing in the decomposition above satisfy

$$E_j \leq \epsilon(t) t.$$

If α is a low energy form the identity 4.4 can be replaced by an inequality

$$\int |d\alpha|^2 + |\delta\alpha|^2 + t^2 |d\phi|^2 |\alpha|^2 + t M_\phi \alpha \cdot \alpha = \tag{6.1}$$

$$= \int (\Delta_t \alpha, \alpha) = \sum \|\alpha_j\|^2 E_j \leq \epsilon(t) t \|\alpha\|^2.$$

Just as insection 4 it follows from this (as soon as, say, $\epsilon(t) \leq 1$) that most of the mass of α is concentrated in the union of balls B_j with radius $A/\sqrt{t}$ around the critical points

$$\int_{\cup B_j} |\alpha|^2 \geq (1 - \frac{C}{A^2}) \|\alpha\|^2.$$

Arguing as in section 4, we see that the dimension of the space of low energy forms of degree q does not exceed m_q, the number of critical points of index q. This is because rescaling will give us forms on $\mathbb{R}^n$ that are combination of eigenforms of the scaled laplacian with all eigenvalues at most $\epsilon(t)$, so in the limit we get harmonic forms again.

The point of using low energy forms instead of harmonic forms is that now we may arrange things so that equality holds in this estimate. To see this, fix a critical point p_j of index q and choose local coordinates x centered at p_j so that $\phi = \phi(0) + 1/2 \sum \lambda_j x_j^2$ to second order. Let γ be a harmonic q-form of norm 1 for the weight function $1/2 \sum \lambda_j x_j^2$ in $\mathbb{R}^n$. Let χ be a

cutoff function in $\mathbb{R}^n$ which is equal to 1 if $|x| < 1$ and equal to zero if $|x| > 2$. Let

$$\gamma_{A,t} = \chi(x\sqrt{t}/A)\gamma.$$

We then use the (inverse of) the scaling map F_t to define a form

$$\alpha_{A,t} = F_t^{-1^*}(\gamma_{A,t})$$

in B_j. Choose finally $A = A_t$ tending to infinity. One can then check that

$$\|\Delta_t \alpha_{A_t,t}\| \leq t\epsilon(t)^2 \|\alpha_{A_t,t}\|$$

where $\epsilon(t)$ tends to zero. (Notice that by the explicit form of γ from Theorem 4.1, γ is dominated by $e^{-cA_t^2}$ where the derivative of $\chi(x/A_t)$ is nonzero.) Expanding $\alpha_{A_t,t}$ in eigenforms of the Δ_t-operator, this means that

$$\sum E_j^2 \|\alpha_j\|^2 \leq \epsilon(t)^4 t^2 \sum \|\alpha_j\|^2. \tag{6.2}$$

Finally we let α_t be the projection of $\alpha_{A_t,t}$ on eigenforms with eigenvalues $E_j \leq \epsilon(t)t$. From 6.2 we see that

$$\|\alpha_t\| \geq (1 - \epsilon(t))\|\alpha_{A_t,t}\| \sim 1$$

The upshot of this is that for each critical point p_j of index equal to q, and for t large, we have found a low energy form on X concentrated near p_j. Therfore the dimension of the space of low energy forms is for large t at least (and, as we know, at most) equal to m_q. This is, together with the next lemma from linear algebra, the crucial step in Witten's proof of the strong Morse inequalities

Lemma 6.1. *Let*

$$0 \xrightarrow{d} E^0 \xrightarrow{d} E^1 \xrightarrow{d} ...E^q \xrightarrow{d} ..$$

be a complex of finite dimensional vector spaces, and let H^q be the corresponding cohomology groups. Let e_q be the dimension of E_q and let h_q be the dimension of H^q. Then, for any $q \geq 0$,

$$h_q - h_{q-1} + h_{q-2}... \leq e_q - e_{q-1} + e_{q-2}...$$

Proof. Let Z^q be the kernel of d as a map from E^q, and let z_q be the dimension of Z_q. Then, since the dimension of $E^{q-1}/Z^{q-1} = e_{q-1} - z_{q-1}$ is the dimension of the range of d in E^q,

$$h_q = z_q - (e_{q-1} - z_{q-1}).$$

Hence

$$h_q - h_{q-1} + h_{q-2}... = z_q - e_{q-1} + e_{q-2} - .. \leq e_q - e_{q-1} + e_{q-2}...$$

□

We shall apply the lemma with $d = d_{-t}$ and $E^q = E^q(t)$, the space of low energy forms. Since d_{-t} commutes with Δ_t, d_{-t} maps E^q to E^{q+1}, so we do get a complex. Moreover d_{-t} maps the orthogonal complement of E^q into the orthogonal complement of E^{q+1} so the orthogonal complements

also give a complex. This latter complex is exact, since elements there in particular are orthogonal to harmonic forms. Therefore the complex E^q defines the same cohomology as the full d_{-t}-complex. Since now $e_q = m_q$, the number of critical points of index q, Lemma 6.2 gives that

$$h_q - h_{q-1} + h_{q-2} ... \leq m_q - m_{q-1} + m_{q-2} ...$$

so the proof of the strong Morse inequalities is complete.

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