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PREPRINT 2005:51

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Göteborg Sweden 2005

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Göteborg, December 2005

Preprint 2005:51
ISSN 1652-9715

Matematiskt centrum
Göteborg 2005

THE MAXIMUM POLYGON AREA AND ITS RELATION TO THE ISOPERIMETRIC INEQUALITY

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1. INTRODUCTION

The isoperimetric inequality states that all closed curves in the plane of length L enclosing an area A satisfy $A \leq L^2/(4\pi)$, with equality only for the circle. It is relatively easy to prove by variational techniques that a curve of given length enclosing a maximal area must be a circle and historically the main obstacle to a complete proof of the inequality was the existence of a maximizer. For a modern overview of the isoperimetric theorem and its history, we refer to e.g. Ref. [1].

In more recent times, several shorter and less complicated proofs of the isoperimetric inequality have been published, both geometric [2] and analytic [3]. In this note we add to this collection and give another simple proof of the isoperimetric inequality, based on a polygon area theorem, which we will state and prove below. One main point of the proof is that it uses only the existence of a maximizer in a finite dimensional situation.

2. THE POLYGON AREA THEOREM

The polygon area theorem states that a closed polygon with n sides of given length has maximal area only when the polygon is inscribed in a circle. We shall give a proof of this theorem using induction. The induction step in the proof is very simple if we accept as obvious the fact that a maximizing polygon is necessarily convex. We have, however, not found any direct argument why this must be so (except that its "obvious"...). Therefore we shall state the theorem in a slightly more involved way, using areas counted with multiplicity.

We give all the line segments an orientation and consider only polygonal paths, γ, such that the orientations of the line segments that make up the path are consistent. The curve γ is then also oriented. We now define the

Date: December 5, 2005.

functional to be maximized

$$A_\gamma = \int_\gamma x dy.$$

If γ is a positively oriented simple curve it follows of course from Stokes theorem that A_γ equals the area enclosed by γ. The theorem we have in mind can now be stated as follows.

Theorem 2.1. *Let γ be a closed oriented curve composed of given line segments $l_1, \dots l_N$, and assume that γ maximizes the functional A_γ among all such curves. Then γ is simple, convex, positively oriented and has all its vertices on one circle.*

Proof. We argue by induction, first assuming as known the case $N = 4$. (Note that when $N = 3$, there is nothing to prove.) Let γ be a polygonal curve as above, and consider two adjacent line segments in γ, for simplicity denoted l_1 and l_2, with l_1 preceeding l_2. Let the starting point of l_1 be a and the end point of l_2 b. Finally, we denote by α the angle between the vectors l_1 and l_2 (in that order). We first claim that $0 < \alpha < \pi$. To see this, form another path γ' by replacing l_1 and l_2 in γ by their reflections in the line segment $[a, b]$. Then

$$A_{\gamma'} = A_\gamma + \int_P x dy,$$

where P is a parallellogram formed by l_1, l_2 and their reflections in $[a, b]$. It is easy to verify that if α were negative, then the last integral would be greater than 0, contradicting the maximality of γ. Thus all angles of γ are positive, which means that we turn to the left as we move along the curve, in the sense of the orientation. This also almost implies that the curve is convex, except that we do not yet know that γ is simple.

Next we consider the polygon γ'' obtained from γ by replacing l_1 and l_2 by the single oriented line segment $[a, b]$. The polygon γ'' then consists of $N - 1$ line segments. Moreover

$$A_\gamma = A''_\gamma + \int_T x dy,$$

where T is a triangle formed by l_1, l_2 and $[a, b]$. Since the last integral is a fixed number, γ'' must give the maximal area among all other polygons formed using the same line segments. Therefore, by induction, γ'' is simple and convex. Since by the above α is nonnegative it follows that γ'' and $l_1 \cup l_2$ lie on different sides of the line through a and b, so γ is also a simple curve. As noted above, this implies that γ is also convex. Moreover, by induction, all the vertices of γ'' lie on one circle.

Similarily, any other subset of all but one of the vertices of γ lie on one circle. If $N > 4$, any two circles arising in this way have at least three points in common, and hence must be equal. Thus, all the vertices of γ lie on one single circle.

For completeness we also prove Theorem 2.1 for $N = 4$ (this is known as Brahmagupta's theorem). We use the notation above. Let l_3 and l_4 be the remaining two line segments, which must lie on the other side of $[a, b]$, and let β be the angle between l_3 and l_4. We have

$$2A_\gamma = |l_1||l_2|\sin\alpha + |l_3||l_4|\sin\beta.$$

This quantity must be maximal subject to the side condition $|a - b|^2 =$

$$|l_1|^2 + |l_2|^2 - 2|l_1||l_2|\cos\alpha = |l_3|^2 + |l_4|^2 - 2|l_3||l_4|\cos\beta,$$

where α and β are variable.

It follows from Lagrange's theorem that the vectors $(\cos\alpha, \cos\beta)$ and $(\sin\alpha, -\sin\beta)$ are parallell. Hence

$$\sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin(\alpha+\beta) = 0$$

so $\alpha + \beta = \pi$, which, by elementary geometry, is exactly the condition that all the vertices lie on one circle.

$\square$

3. The isoperimetric inequality

With this theorem, the isoperimetric inequality can be proven in a straightforward manner. Consider a closed curve C with length L and area A. Approximate C with a polygon p with very short elements, so that the length and area of p, L_p and A_p, are very close to the length and area of C. The polygon $p*$ which has the same sides as p, but maximum area, is according to the polygon area theorem inscribed in a circle so its area A_{p*} is close to that of the circle, i.e. $A_{p*} \approx L_p^2/4\pi$. We thus have:

$$A \approx A_p < A_{p*} \approx \frac{L_{p*}^2}{4\pi} \approx \frac{L^2}{4\pi} \tag{3.1}$$

which is the isoperimetric inequality.

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