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ABSTRACT. We present methods and explicit formulas for describing simple weight modules over twisted generalized Weyl algebras. When a certain commutative subalgebra is finitely generated over an algebraically closed field we obtain a classification of a class of locally finite simple weight modules as those induced from simple modules over a subalgebra isomorphic to a tensor product of noncommutative tori. As an application we describe simple weight modules over the quantized Weyl algebra.

1. INTRODUCTION

Inspired by [10], Bavula defined in [2], [1] the notion of a generalized Weyl algebra (GWA) which is a class of algebras which include $U(\mathfrak{sl}(2))$, $U_q(\mathfrak{sl}(2))$, down-up algebras, and the Weyl algebra, as examples. In addition to various ring theoretic properties, the simple modules were also described for some GWAs in [2]. In [6] all simple and indecomposable weight modules of GWAs of rank (or degree) one were classified.

Higher rank GWAs were defined in [2] as tensor products of rank one GWAs. This has some consequences on the side of representations. In [3] the authors studied indecomposable weight modules over certain higher rank GWAs.

In [8], with the goal to enrich the representation theory in the higher rank case, the authors defined the twisted generalized Weyl algebras (TGWA). This is a class of algebras which include all higher rank GWAs (if a certain subring R has no zero divisors) and also many algebras which can be viewed as twisted tensor products of rank one GWAs, for example certain Mickelsson step algebras and extended Orthogonal Gelfand-Zetlin algebras [7]. Under a technical assumption on the algebra formulated using a biserial graph, some torsion-free simple weight modules were described in [8]. Simple graded weight modules were studied in [7] using an analogue of the Shapovalov form.

In this paper we describe a more general class of locally finite simple weight modules over TGWAs using the well-known technique of considering the maximal graded subalgebra which preserves the weight spaces. It is known that under quite general assumptions (see Theorem 18 in [5]) any simple weight module over a TGWA is a unique quotient of a module which is induced from a simple module over this subalgebra. Our main results are the description of this subalgebra under various assumptions (Theorem 4.5 and Theorem 4.8) and the explicit formulas (Theorem 5.4) of the associated module of the TGWA. In contrast to [8], we do not assume that the orbits are torsion-free and we allow the modules to have some inner breaks, as long as they do not have any so called *proper* inner breaks (see Definition 3.7). The weight spaces will not in general be one-dimensional in our case, which was the case in [8], [7].

Moreover, as an application we classify the simple weight modules without proper inner breaks over a quantized Weyl algebra of rank two (Theorem 6.14).

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The paper is organized as follows. In Section 2 the definitions of twisted generalized Weyl constructions and algebras are given together with some examples. Weight modules and the subalgebra $B(\omega)$ are defined.

In Section 3 we first prove some simple facts and then define the class of simple weight modules with no proper inner breaks. We also show that this class properly contains all the modules studied in [8].

Section 4 is devoted to the description of the subalgebra $B(\omega)$. When the ground field is algebraically closed and a certain subalgebra R is finitely generated, we show that it is isomorphic to a tensor product of noncommutative tori for which the finite-dimensional irreducible representations are easy to describe.

In Section 5 we specify a basis and give explicit formulas for the irreducible quotient of the induced module.

Finally, in Section 6 we consider as an example the quantized Weyl algebra and determine certain important subsets of $\mathbb{Z}^n$ related to $B(\omega)$ and the support of modules as solutions to some systems of equations. In the rank two case we describe all simple weight modules with finite-dimensional weight spaces and no proper inner breaks.

2. Definitions

2.1. **The TGWC and TGWA.** Fix a positive integer n and set $\underline{n} = \{1, 2, \ldots, n\}$. Let K be a field, and let R be a commutative unital K-algebra, $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ be an n-tuple of pairwise commuting K-automorphisms of R, $\boldsymbol{\mu} = (\mu_{ij})_{i,j\in\underline{n}}$ be a matrix with entries from $K^* := K\backslash\{0\}$ and $\boldsymbol{t} = (t_1, \ldots, t_n)$ be an n-tuple of nonzero elements from R. The *twisted generalized Weyl construction* (TGWC) A' obtained from the data $(R, \boldsymbol{\sigma}, \boldsymbol{t}, \boldsymbol{\mu})$ is the unital K-algebra generated over R by $X_i, Y_i, (i \in \underline{n})$ with the relations

$$X_i r = \sigma_i(r) X_i, \qquad Y_i r = \sigma_i^{-1}(r) Y_i, \qquad \text{for } r \in R, i \in \underline{n}, \tag{2.1}$$

$$Y_i X_i = t_i, \qquad X_i Y_i = \sigma_i(t_i), \qquad \text{for } i \in \underline{n}, \tag{2.2}$$

$$X_i Y_j = \mu_{ij} Y_j X_i, \qquad \text{for } i, j \in \underline{n}, i \neq j. \tag{2.3}$$

From the relations (2.1)–(2.3) follows that A' carries a $\mathbb{Z}^n$-gradation $\{A'_g\}_{g\in\mathbb{Z}^n}$ which is uniquely defined by requiring

$$\deg X_i = e_i, \quad \deg Y_i = -e_i, \quad \deg r = 0, \quad \text{for } i \in \underline{n}, r \in R,$$

where $e_i = (0, \ldots, \overset{i}{1}, \ldots, 0)$. The *twisted generalized Weyl algebra* (TGWA) $A = A(R, \boldsymbol{\sigma}, \boldsymbol{t}, \boldsymbol{\mu})$ of *rank* n is defined to be A'/I, where I is the sum of all graded two-sided ideals of A' intersecting R trivially. Since I is graded, A inherits a $\mathbb{Z}^n$-gradation $\{A_g\}_{g\in\mathbb{Z}^n}$ from A'.

Note that from relations (2.1)–(2.3) follows the identity

$$X_i X_j t_i = X_j X_i \mu_{ji} \sigma_j^{-1}(t_i) \tag{2.4}$$

which holds for $i, j \in \underline{n}, i \neq j$. Multiplying (2.4) from the left by $\mu_{ij} Y_j$ we obtain

$$X_i\big(t_i t_j - \mu_{ij}\mu_{ji}\sigma_i^{-1}(t_j)\sigma_j^{-1}(t_i)\big) = 0 \tag{2.5}$$

for $i, j \in \underline{n}, i \neq j$. One can show that the algebra A', hence A, is nontrivial if one assumes that $t_i t_j = \mu_{ij}\mu_{ji}\sigma_i^{-1}(t_j)\sigma_j^{-1}(t_i)$ for $i, j \in \underline{n}, i \neq j$. Analogous identities exist for Y_i.

2.2. **Examples.** Some of the first motivating examples of a *generalized Weyl algebra* (GWA), i.e. a TGWC of rank 1, are $U(\mathfrak{sl}(2))$, $U_q(\mathfrak{sl}(2))$ and of course the Weyl algbra A_1. We refer to [2] for details.

We give some examples of TGWAs of higher rank.

2.2.1. *Quantized Weyl algebras.* Let $\Lambda = (\lambda_{ij})$ be an $n \times n$ matrix with nonzero complex entries such that $\lambda_{ij} = \lambda_{ji}^{-1}$. Let $\bar{q} = (q_1, \dots, q_n)$ be an n-tuple of elements of $\mathbb{C}\backslash\{0, 1\}$. The n:th quantized Weyl algebra $A_n^{\bar{q},\Lambda}$ is the $\mathbb{C}$-algebra with generators x_i, y_i, $1 \le i \le n$, and relations

$$x_i x_j = q_i \lambda_{ij} x_j x_i, \qquad y_i y_j = \lambda_{ij} y_j y_i, \tag{2.6}$$

$$x_i y_j = \lambda_{ji} y_j x_i, \qquad x_j y_i = q_i \lambda_{ij} y_i x_j, \tag{2.7}$$

$$x_i y_i - q_i y_i x_i = 1 + \sum_{k=1}^{i-1} (q_k - 1) y_i x_i, \tag{2.8}$$

for $1 \le i < j \le n$. Let $R = \mathbb{C}[t_1, \dots, t_n]$ be the polynomial algebra in n variables and σ_i the $\mathbb{C}$-algebra automorphisms defined by

$$\sigma_i(t_j) = \begin{cases} t_j, & j < i, \\ 1 + q_i t_i + \sum_{k=1}^{i-1} (q_k - 1) t_k, & j = i, \\ q_i t_j, & j > i. \end{cases} \tag{2.9}$$

One can check that the σ_i commute. Let $\boldsymbol{\mu} = (\mu_{ij})_{i,j \in \underline{n}}$ be defined by $\mu_{ij} = \lambda_{ji}$ and $\mu_{ji} = q_i \lambda_{ij}$ for $i < j$. Let also $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ and $\boldsymbol{t} = (t_1, \dots, t_n)$. One can show that the maximal graded ideal of the TGWC $A'(R, \boldsymbol{\sigma}, \boldsymbol{t}, \boldsymbol{\mu})$ is generated by the elements

$$X_i X_j - q_i \lambda_{ij} X_j X_i, \; Y_i Y_j - \lambda_{ij} Y_j Y_i, \quad 1 \le i < j \le n.$$

Thus $A_n^{\bar{q},\Lambda}$ is isomorphic to the TGWA $A(R, \boldsymbol{\sigma}, \boldsymbol{t}, \boldsymbol{\mu})$ via $x_i \mapsto X_i$, $y_i \mapsto Y_i$.

2.2.2. *Q_{ij}-CCR.* Let $(Q_{ij})_{i,j=1}^d$ be an $d \times d$ matrix with complex entries such that $Q_{ij} = Q_{ji}^{-1}$ if $i \ne j$ and A_d be the algebra generated by elements a_i, a_i^*, $1 \le i \le d$ and relations

$$a_i^* a_i - Q_{ii} a_i a_i^* = 1, \qquad a_i^* a_j = Q_{ij} a_j a_i^*,$$
$$a_i a_j = Q_{ji} a_j a_i, \qquad a_i^* a_j^* = Q_{ij} a_j^* a_i^*,$$

where $1 \le i, j \le d$ and $i \ne j$. Let $R = \mathbb{C}[t_1, \dots, t_d]$ and define the autormorphisms σ_i of R by $\sigma_i(t_j) = t_j$ if $i \ne j$ and $\sigma_i(t_i) = 1 + Q_{ii} t_i$. Let $\mu_{ij} = Q_{ji}$ for all i, j. Then A_d is isomorphic to the TGWA $A(R, (\sigma_1, \dots, \sigma_n), (t_1, \dots, t_n), \boldsymbol{\mu})$.

2.2.3. *Mickelsson and OGZ algebras.* In both the above examples the generators X_i and X_j commute up to a multiple of the ground field. This need not be the case as shown in [7], where it was shown that Mickelsson step algebras and extended orthogonal Gelfand-Zetlin algebras are TGWAs.

2.3. **Weight modules.** Let A be a TGWC or a TGWA. Let $\mathrm{Max}(R)$ denote the set of all maximal ideals in R. A module M over A is called a *weight module* if

$$M = \oplus_{\mathfrak{m} \in \mathrm{Max}(R)} M_{\mathfrak{m}},$$

where

$$M_{\mathfrak{m}} = \{v \in M \mid \mathfrak{m} v = 0\}.$$

The *support*, $\mathrm{supp}(M)$, of M is the set of all $\mathfrak{m} \in \mathrm{Max}(R)$ such that $M_{\mathfrak{m}} \ne 0$. A weight module is *locally finite* if all the weight spaces $M_{\mathfrak{m}}$, $\mathfrak{m} \in \mathrm{supp}(M)$, are finite-dimensional over the ground field K.

Since the σ_i are pairwise commuting, the free abelian group $\mathbb{Z}^n$ acts on R as a group of K-algebra automorphisms by

$$g(r) = \sigma_1^{g_1} \sigma_2^{g_2} \dots \sigma_n^{g_n}(r) \tag{2.10}$$

for $g = (g_1, \dots, g_n) \in \mathbb{Z}^n$ and $r \in R$. Then $\mathbb{Z}^n$ also acts naturally on $\mathrm{Max}(R)$ by $g(\mathfrak{m}) = \{g(r) \mid r \in \mathfrak{m}\}$. Note that

$$X_i M_{\mathfrak{m}} \subseteq M_{\sigma_i(\mathfrak{m})} \quad \text{and} \quad Y_i M_{\mathfrak{m}} \subseteq M_{\sigma_i^{-1}(\mathfrak{m})} \tag{2.11}$$

for any $\mathfrak{m} \in \mathrm{Max}(R)$. If $a \in A$ is homogenous of degree $g \in \mathbb{Z}^n$, then by using (2.1) and (2.11) repeatedly one obtains the very useful identities

$$a \cdot r = g(r) \cdot a, \quad r \cdot a = a \cdot (-g)(r), \tag{2.12}$$

for $r \in R$ and

$$a M_{\mathfrak{m}} \subseteq M_{g(\mathfrak{m})} \tag{2.13}$$

for $\mathfrak{m} \in \mathrm{Max}(R)$.

2.4. **Subalgebras leaving the weight spaces invariant.** Let $\omega \subseteq \mathrm{Max}(R)$ be an orbit under the action of $\mathbb{Z}^n$ on $\mathrm{Max}(R)$ defined in (2.10). Let

$$\mathbb{Z}^n_\omega = \mathbb{Z}^n_{\mathfrak{m}} = \{g \in \mathbb{Z}^n \mid g(\mathfrak{m}) = \mathfrak{m}\} \tag{2.14}$$

where $\mathfrak{m}$ is some point in ω. Since $\mathbb{Z}^n$ is abelian, $\mathbb{Z}^n_\omega$ does not depend on the choice of $\mathfrak{m}$ from ω. Define

$$B(\omega) = \oplus_{g \in \mathbb{Z}^n_\omega} A_g. \tag{2.15}$$

Since A is $\mathbb{Z}^n$-graded and since $\mathbb{Z}^n_\omega$ is a subgroup of $\mathbb{Z}^n$, $B(\omega)$ is a subalgebra of A and $R = A_0 \subseteq B(\omega)$. Let $\mathfrak{m} \in \omega$ and suppose that M is a simple weight A-module with $\mathfrak{m} \in \mathrm{supp}(M)$. Since M is simple we have $\mathrm{supp}(M) \subseteq \omega$. Using (2.13) it follows that $B(\omega) M_{\mathfrak{m}} \subseteq M_{\mathfrak{m}}$ and by definition $M_{\mathfrak{m}}$ is annihilated by $\mathfrak{m}$ hence also by the two-sided ideal $(\mathfrak{m})$ in $B(\omega)$ generated by $\mathfrak{m}$. Thus $M_{\mathfrak{m}}$ is naturally a module over the algebra

$$B_{\mathfrak{m}} := B(\omega)/(\mathfrak{m}). \tag{2.16}$$

By Proposition 7.2 in [7] (see also Theorem 18 in [5] for a general result), $M_{\mathfrak{m}}$ is a simple $B_{\mathfrak{m}}$-module, and any simple $B_{\mathfrak{m}}$-module occurs as a weight space in a simple weight A-module. Moreover, two simple weight A-modules M, N are isomorphic if and only if $M_{\mathfrak{m}}$ and $N_{\mathfrak{m}}$ are isomorphic as $B_{\mathfrak{m}}$-modules. Therefore we are led to study the algebra $B_{\mathfrak{m}}$ and simple modules over it.

3. Preliminaries

3.1. **Reduced words.** Let $L = \{X_i\}_{i \in \underline{n}} \cup \{Y_i\}_{i \in \underline{n}}$. By a *word* $(a; Z_1, \dots, Z_k)$ in A we will mean an element a in A which is a product of elements from the set L, together with a fixed tuple $(Z_1, \dots, Z_k)$ of elements from L such that $a = Z_1 \cdot \ldots \cdot Z_k$. When referring to a word we will often write $a = Z_1 \dots Z_k \in A$ to denote the word $(a; Z_1, \dots, Z_k)$ or just write $a \in A$, suppressing the fixed representation of a as a product of elements from L.

Set $X_i^* = Y_i$ and $Y_i^* = X_i$. For a word $a = Z_1 \dots Z_k \in A$ we define

$$a^* := Z_k^* \cdot Z_{k-1}^* \cdot \ldots \cdot Z_1^*.$$

In the special case when $\mu_{ij} = \mu_{ji}$ for all i, j then by (2.1)-(2.3) there is an anti-involution $*$ on A' defined by $X_i^* = Y_i$, and $r^* = r$ for $r \in R$. Since $I^* = I$ this anti-involution carries over to A.

Definition 3.1. A word $Z_1 \dots Z_k$ will be called *reduced* if

$$Z_i \neq Z_j^* \text{ for } i, j \in \underline{k}$$

and

$$Z_i \in \{X_r\}_{r \in \underline{n}} \Longrightarrow Z_j \in \{X_r\}_{r \in \underline{n}} \; \forall j \geq i.$$

For example $Y_1Y_2Y_1X_3$ is reduced whereas $Y_1Y_2X_1$ and $Y_1X_2Y_3$ are not. The following Lemma and Corollary explains the importance of the reduced words.

Lemma 3.2. *Any word b in A can be written $b = a \cdot r = r' \cdot a$, where a is a reduced word, and $r, r' \in R$.*

Proof. If a and r has been found we can take $r' = (\deg a)(r)$, according to (2.12). Thus we concentrate on finding a and r. Let $b = Z_1 \dots Z_k$ be an arbitrary word in A. We prove the statement by induction on k. If $k = 1$, then b is necessarily reduced so take $a = b, r = 1$. When $k > 1$, use the induction hypothesis to write

$$Z_1 \dots Z_{k-1} = Y_{i_1} \dots Y_{i_l} X_{j_1} \dots X_{j_m} \cdot r',$$

where $1 \leq i_u, j_v \leq n$ and $i_u \neq j_v$ for any u, v. Consider first the case when $Z_k = Y_j$ for some $j \in \underline{n}$. Then

$$Z_1 \dots Z_k = Y_{i_1} \dots Y_{i_l} X_{j_1} \dots X_{j_m} Y_j \cdot \sigma_j(r').$$

If $j_v \neq j$ for $v = 1, \dots, m$ we are done because using relation (2.3) repeatedly we obtain,

$$Z_1 \dots Z_k = Y_{i_1} \dots Y_{i_l} Y_j X_{j_1} \dots X_{j_m} \cdot \mu\sigma_j(r')$$

for some $\mu \in K^*$. Otherwise, let $v \in \{1, \dots, m\}$ be maximal such that $j_v = j$. Then

$$\begin{aligned} Z_1 \dots Z_k &= Y_{i_1} \dots Y_{i_l} X_{j_1} \dots X_{j_v} Y_j X_{j_{v+1}} \dots X_{j_m} \mu\sigma_j(r') = \\ &= Y_{i_1} \dots Y_{i_l} X_{j_1} \dots X_{j_{v-1}} X_{j_{v+1}} \dots X_{j_m} w(t_j)\mu\sigma_j(r') \end{aligned}$$

for some $\mu \in K^*$ and some $w \in W$. It remains to consider the case $Z_k = X_j$ for some $j \in \underline{n}$. But using that

$$Y_{i_1} \dots Y_{i_l} X_{j_1} \dots X_{j_m} = X_{j_1} \dots X_{j_m} Y_{i_1} \dots Y_{i_l} \mu$$

for some $\mu \in K^*$, it is clear that this case is analogous. □

Corollary 3.3. *Each A_g, $g \in W$, is generated as a right (and also as a left) R-module by the reduced words of degree g.*

Lemma 3.4. *Suppose $*$ defines an anti-involution on A. Let $\mathfrak{p}$ be a prime ideal of R. Let $g \in \mathbb{Z}^n$ and let $a \in A_g$. If $ba \notin \mathfrak{p}$ for some $b \in A_{-g}$ then $a^*a \notin \mathfrak{p}$.*

Proof. Since $\mathfrak{p}$ is prime, and $ba \in R$ we have

$$\mathfrak{p} \not\ni (ba)^2 = (ba)^* ba = a^* b^* ba = a^* a \cdot (-\deg a)(b^* b)$$

so in particular $a^* a \notin \mathfrak{p}$. □

Remark 3.5. If we assume a and b to be words in the formulation of Lemma 3.4, one can easily show that the statement remains true without the restriction on $*$ to be an anti-involution.

3.2. **Inner breaks and canonical modules.** Let A be a TGWC or a TGWA and let M be a simple weight module over A. In [8] Remark 1 it was noted that the problem of describing simple weight modules over a TGWC is wild in general. This is a motivation for restricting attention to some subclass which has nice properties. In [8] the following definition was made.

Definition 3.6. The support of M has *no inner breaks* if for all $\mathfrak{m} \in \operatorname{supp}(M)$,

$$t_i \in \mathfrak{m} \Longrightarrow \sigma_i(\mathfrak{m}) \notin \operatorname{supp}(M), \text{ and}$$
$$\sigma_i(t_i) \in \mathfrak{m} \Longrightarrow \sigma_i^{-1}(\mathfrak{m}) \notin \operatorname{supp}(M).$$

We introduce the following property.

Definition 3.7. We say that M has *no proper inner breaks* if for any $\mathfrak{m} \in \operatorname{supp}(M)$ and any word a with $aM_\mathfrak{m} \neq 0$ we have $a^*a \notin \mathfrak{m}$.

Observe that whether or not $a^*a \in \mathfrak{m}$ for a word a does not depend on the particular representation of a as a product of generators. Note also that to prove that a simple weight module M has no proper inner breaks, it is sufficient to find for any $\mathfrak{m} \in \operatorname{supp}(M)$ and any word a with $aM_\mathfrak{m} \neq 0$ a word $b \in A$ of degree $-\deg a$ such that $ba \notin \mathfrak{m}$ because then $a^*a \notin \mathfrak{m}$ automatically by Remark 3.5. In fact one can show that a simple weight module M has no proper inner breaks if (and only if) there exists an $\mathfrak{m} \in \operatorname{supp}(M)$ such that for any reduced word $a \in A$ with $aM_\mathfrak{m} \neq 0$ and $aM_\mathfrak{m} \subseteq M_\mathfrak{m}$ there is a word b of degree $-\deg a$ such that $ba \notin \mathfrak{m}$. However we will not use this result.

The choice of terminology in Definition 3.7 is motivated by the following proposition.

Proposition 3.8. *If M has no inner breaks, then M has no proper inner breaks either.*

Proof. Let $\mathfrak{m} \in \operatorname{supp}(M)$ and $a = Z_1 \dots Z_k \in A$ be a word such that $aM_\mathfrak{m} \neq 0$. Thus $Z_i \dots Z_k M_\mathfrak{m} \neq 0$ for $i = 1, \dots, k+1$ so (2.13) implies that

$$(\deg Z_i \dots Z_k)(\mathfrak{m}) \in \operatorname{supp}(M).$$

If M has no inner breaks, it follows that $Z_i^* Z_i \notin (\deg Z_{i+1} \dots Z_k)(\mathfrak{m})$ for $i = 1, \dots, k$. Now using (2.12),

$$a^*a = Z_k^* \dots Z_1^* Z_1 \dots Z_k = Z_k^* \dots Z_2^* Z_2 \dots Z_k (-\deg Z_2 \dots Z_k)(Z_1^* Z_1) =$$

$$= \dots = \prod_{i=1}^{k} (-\deg Z_{i+1} \dots Z_k)(Z_i^* Z_i) \notin \mathfrak{m}. \tag{3.1}$$

Thus M has no proper inner breaks. □

In [8], a simple weight module M was defined to be *canonical* if for any $\mathfrak{m}, \mathfrak{n} \in \operatorname{supp}(M)$ there is an automorphism σ of R of the form

$$\sigma = \sigma_{i_1}^{\varepsilon_1} \cdot \ldots \cdot \sigma_{i_k}^{\varepsilon_k}, \quad \varepsilon_j = \pm 1 \text{ and } 1 \leq i_j \leq n, \text{ for } j = 1, \dots, k,$$

such that $\sigma(\mathfrak{m}) = \mathfrak{n}$ and such that for each $j = 1, \dots, k$,

$$t_{i_j} \notin \sigma_{i_{j+1}}^{\varepsilon_{j+1}} \dots \sigma_{i_k}^{\varepsilon_k}(\mathfrak{m}) \quad \text{if } \varepsilon_j = 1, \text{ and} \tag{3.2}$$

$$\sigma_{i_j}(t_{i_j}) \notin \sigma_{i_{j+1}}^{\varepsilon_{j+1}} \dots \sigma_{i_k}^{\varepsilon_k}(\mathfrak{m}) \quad \text{if } \varepsilon_j = -1. \tag{3.3}$$

This definition can be reformulated as follows.

Proposition 3.9. *M is canonical iff for any $\mathfrak{m}, \mathfrak{n} \in \operatorname{supp}(M)$ there is a word $a \in A$ such that $aM_\mathfrak{m} \subseteq M_\mathfrak{n}$ and $a^*a \notin \mathfrak{m}$.*

Proof. Suppose M is canonical, and let $\mathfrak{m}, \mathfrak{n} \in \operatorname{supp}(M)$. Let σ be as in the definition of canonical module. Define $a = Z_1 \dots Z_k$ where $Z_j = X_{i_j}$ if $\varepsilon_j = 1$ and $Z_j = Y_{i_j}$ otherwise. Using (2.13) we see that $aM_\mathfrak{m} \subseteq M_\mathfrak{n}$. Also, (3.2) and (3.3) translates into

$$Z_j^* Z_j \notin (\deg Z_{j+1} \dots Z_k)(\mathfrak{m})$$

for $j = 1, \dots, k$. Using the calculation (3.1) and that $\mathfrak{m}$ is prime we deduce that $a^*a \notin \mathfrak{m}$.

Conversely, given a word $a = Z_1 \dots Z_k \in A$ with $aM_\mathfrak{m} \subseteq M_\mathfrak{n}$ and $a^*a \notin \mathfrak{m}$, we define $\varepsilon_i = 1$ if $Z_i = X_i$ and $\varepsilon_i = -1$ otherwise. Then from $a^*a \notin \mathfrak{m}$ follows that $\sigma := \sigma_{i_1}^{\varepsilon_1} \cdot \ldots \cdot \sigma_{i_k}^{\varepsilon_k}$ satisfies (3.2) and (3.3) by the same reasoning as above. □

Corollary 3.10. *If M has no proper inner breaks, then M is canonical.*

Proof. We only need to note that since M is a simple weight module there is for each $\mathfrak{m}, \mathfrak{n} \in \operatorname{supp}(M)$ a word a such that $0 \neq aM_{\mathfrak{m}} \subseteq M_{\mathfrak{n}}$. □

Under the assumptions in [8] any canonical module has no inner breaks (see [8], Proposition 1). However we have the following example of a TGWA A and a simple weight module M over A which has no proper inner breaks, and thus is canonical by Corollary 3.10, but nonetheless has an inner break.

Example 3.11. Let $R = \mathbb{C}[t_1, t_2]$ and define the $\mathbb{C}$-algebra automorphisms σ_1 and σ_2 of R by $\sigma_i(t_j) = -t_j$ for $i, j = 1, 2$. Let $\boldsymbol{\mu} = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Let $A' = A'(R, \boldsymbol{t}, \boldsymbol{\sigma}, \boldsymbol{\mu})$ be the associated TGWC, where $\boldsymbol{t} = (t_1, t_2), \boldsymbol{\sigma} = (\sigma_1, \sigma_2)$. Then one can check that $I = \langle X_1X_2 + X_2X_1, Y_1Y_2 + Y_2Y_1 \rangle$. Let M be a vector space over $\mathbb{C}$ with basis $\{v, w\}$ and define an A'-module structure on M by letting $X_1M = Y_1M = 0$ and

$$\begin{aligned} X_2v &= w, & X_2w &= v, \\ Y_2v &= w, & Y_2w &= -v. \end{aligned}$$

It is easy to check that the required relations are satisfied and that $IM = 0$, hence M becomes an A-module. Let $\mathfrak{m} = (t_1, t_2 + 1)$ and $\mathfrak{n} = (t_1, t_2 - 1)$. Then

$$M = M_{\mathfrak{m}} \oplus M_{\mathfrak{n}}, \quad \text{where } M_{\mathfrak{m}} = \mathbb{C}v, M_{\mathfrak{n}} = \mathbb{C}w$$

so M is a weight module. Any proper nonzero submodule of M would also be a weight module by standard results. That no such submodule can exist is easy to check, so M is simple. One checks that M has no proper inner breaks. But $t_1 \in \mathfrak{m}$ and $\sigma_1(\mathfrak{m}) = \mathfrak{n} \in \operatorname{supp}(M)$ so $\mathfrak{m}$ is an inner break.

4. The weight space preserving subalgebra and its irreducible representations

In this section, let A be a TGWC, $\mathfrak{m} \in \operatorname{Max}(R)$ and let ω be the $\mathbb{Z}^n$-orbit of $\mathfrak{m}$. Recall the set $\mathbb{Z}^n_\omega$ defined in (2.14). Define the following subsets of $\mathbb{Z}^n$:

$$(4.1) \qquad \tilde{G}_{\mathfrak{m}} = \{g \in \mathbb{Z}^n \mid a^*a \notin \mathfrak{m} \text{ for some word } a \in A_g\} \quad \text{and} \quad G_{\mathfrak{m}} = \tilde{G}_{\mathfrak{m}} \cap \mathbb{Z}^n_\omega.$$

Let also $\varphi_{\mathfrak{m}} : A \to A/(\mathfrak{m})$ denote the canonical projection, where $(\mathfrak{m})$ is the two-sided ideal in A generated by $\mathfrak{m}$, and let $R_{\mathfrak{m}} = R/\mathfrak{m}$ be the residue field of R at $\mathfrak{m}$.

Lemma 4.1. *Let $g \in \tilde{G}_{\mathfrak{m}}$. Then*

$$(4.2) \qquad \varphi_{\mathfrak{m}}(A_g) = R_{\mathfrak{m}} \cdot \varphi_{\mathfrak{m}}(a) = \varphi_{\mathfrak{m}}(a) \cdot R_{\mathfrak{m}}$$

for any word $a \in A_g$ with $a^*a \notin \mathfrak{m}$.

Proof. Let $b \in A_g$ be any element and $a \in A_g$ a word such that $a^*a \notin \mathfrak{m}$, We must show that there is an $r \in R$ such that $\varphi_{\mathfrak{m}}(b) = \varphi_{\mathfrak{m}}(r)\varphi_{\mathfrak{m}}(a)$. Since $a^*a \notin \mathfrak{m}$ and $\mathfrak{m}$ is maximal, $1 - r_1a^*a \in \mathfrak{m}$ for some $r_1 \in R$. Set $r = br_1a^*$. Then $r \in R$ and

$$b - ra = b(1 - r_1a^*a) \in (\mathfrak{m}).$$

The last equality in (4.2) is immediate using (2.12). □

The following result was proved in [8] Lemma 8 for simple weight modules with so called regular support which in particular means that they have no inner breaks. It is still true in the more general situation when M has no proper inner breaks. Recall the ideal I from the definition of a *TGWA*.

Proposition 4.2. *Suppose A is a TGWC. If M is a simple weight A-module with no proper inner breaks, then $IM = 0$. Hence M is naturally a module over the associated TGWA A/I.*

Proof. Since I is graded and M is a weight modules, it is enough to show that $(I \cap A_g)M_{\mathfrak{m}} = 0$ for any $g \in \mathbb{Z}^n$ and any $\mathfrak{m} \in \mathrm{supp}(M)$. Assume that $a \in I \cap A_g$ and $av \neq 0$ for some $v \in M_{\mathfrak{m}}$. Then $a_1 v \neq 0$ for some word a_1 in a. Since M has no proper inner breaks, $a_1^* a_1 \notin \mathfrak{m}$ so by Lemma 4.1 there is an $r \in R$ such that $av = a_1 rv$. Thus $0 \neq a_1^* a_1 rv = a_1^* av$ which implies that $a_1^* a \in R \backslash \{0\}$. This contradicts that $a \in I$. $\square$

We fix now for each $g \in \tilde{G}_{\mathfrak{m}}$ a word $a_g \in A_g$ such that $a_g^* a_g \notin \mathfrak{m}$. For $g = 0$ we choose $a_g = 1$.

Lemma 4.3. *For any $g \in \tilde{G}_{\mathfrak{m}}, h \in G_{\mathfrak{m}}$ we have*
a) $(a_g a_h^*)^* a_g a_h^* \notin \mathfrak{m}$ so in particular $g - h \in \tilde{G}_{\mathfrak{m}}$ and $G_{\mathfrak{m}}$ is a subgroup of $\mathbb{Z}^n_\omega$,
b) $\varphi_{\mathfrak{m}}(A_g)\varphi_{\mathfrak{m}}(A_h) = \varphi_{\mathfrak{m}}(A_g A_h) = \varphi_{\mathfrak{m}}(A_{g+h})$,
c) $A_{g+h} M_{\mathfrak{m}} = A_g M_{\mathfrak{m}}$.

Proof. a) We have

$$(a_g a_h^*)^* a_g a_h^* = a_h a_g^* a_g a_h^* = a_h a_h^* h(a_g^* a_g). \tag{4.3}$$

Now $a_g^* a_g \notin \mathfrak{m}$ so $h(a_g^* a_g) \notin h(\mathfrak{m}) = \mathfrak{m}$. And

$$\mathfrak{m} \not\ni (a_h^* a_h)^2 = a_h^* (a_h a_h^*) a_h = a_h^* a_h \cdot (-h)(a_h a_h^*)$$

so $a_h a_h^* \notin h(\mathfrak{m}) = \mathfrak{m}$. Since $\mathfrak{m}$ is maximal the right hand side of (4.3) does not belong to $\mathfrak{m}$. Since $\deg(a_g a_h^*) = g - h$ we obtain $g - h \in \tilde{G}_{\mathfrak{m}}$. If in addition $g \in G_{\mathfrak{m}}$ then $g - h \in \mathbb{Z}^n_\omega$ also since $\mathbb{Z}^n_\omega$ is a group. Thus $g - h \in G_{\mathfrak{m}}$ so $G_{\mathfrak{m}}$ is a subgroup of $\mathbb{Z}^n_\omega$.

b) Since $\varphi_{\mathfrak{m}}$ is a homomorphism, the first equality holds. By part a), $-h \in G_{\mathfrak{m}}$ so by part a) again, $(a_g a_{-h}^*)^* a_g a_{-h}^* \notin \mathfrak{m}$. Hence by Lemma 4.1, we have

$$\varphi_{\mathfrak{m}}(A_{g+h}) = R_{\mathfrak{m}} \cdot \varphi_{\mathfrak{m}}(a_g a_{-h}^*) \subseteq \varphi_{\mathfrak{m}}(A_g A_h).$$

The reverse inclusion holds since $\{A_g\}_{g \in \mathbb{Z}^n}$ is a gradation of A.

c) By part a), $g + h = g - (-h) \in \tilde{G}_{\mathfrak{m}}$. Thus by part b),

$$A_{g+h} M_{\mathfrak{m}} = \varphi_{\mathfrak{m}}(A_{g+h}) M_{\mathfrak{m}} = \varphi_{\mathfrak{m}}(A_g A_h) M_{\mathfrak{m}} = A_g A_h M_{\mathfrak{m}} \subseteq A_g M_{h(\mathfrak{m})} = A_g M_{\mathfrak{m}}.$$

By part a), the same calculation holds if we replace g by $g + h$ and and h by $-h$, which gives the opposite inclusion. $\square$

Lemma 4.4. *Let $g \in \mathbb{Z}^n \backslash \tilde{G}_{\mathfrak{m}}$. Then $A_g M_{\mathfrak{m}} = 0$ for any simple weight module M over A with no proper inner breaks.*

Proof. Let $a \in A_g$ be any word. Then $a^* a \in \mathfrak{m}$ and hence if M is a simple weight module over A with no proper inner breaks, $aM_{\mathfrak{m}} = 0$. Since the words generate A_g as a left R-module, it follows that $A_g M_{\mathfrak{m}} = 0$. $\square$

4.1. **General case.** Recall that $(\mathfrak{m})$ denotes the two-sided ideal in A generated by $\mathfrak{m}$. Since $(\mathfrak{m})$ is a graded ideal in A, there is an induced $\mathbb{Z}^n$-gradation of the quotient $A/(\mathfrak{m})$ and $\varphi_{\mathfrak{m}}(A_g) = (A/(\mathfrak{m}))_g$. Corresponding to the decomposition $\mathbb{Z}^n_\omega$ into the subset $G_{\mathfrak{m}}$ and its complement are two K-subspaces of the algebra $B_{\mathfrak{m}} = B(\omega)/\big(B(\omega) \cap (\mathfrak{m})\big)$ which will be denoted by $B_{\mathfrak{m}}^{(1)}$ and $B_{\mathfrak{m}}^{(0)}$ respectively. In other words, $B_{\mathfrak{m}} = B_{\mathfrak{m}}^{(1)} \oplus B_{\mathfrak{m}}^{(0)}$, where

$$B_{\mathfrak{m}}^{(1)} = \bigoplus_{g \in G_{\mathfrak{m}}} (A/(\mathfrak{m}))_g \quad \text{and} \quad B_{\mathfrak{m}}^{(0)} = \bigoplus_{g \in \mathbb{Z}^n_\omega \backslash G_{\mathfrak{m}}} (A/(\mathfrak{m}))_g.$$

By Lemma 4.3a), $G_{\mathfrak{m}}$ is a subgroup of the free abelian group $\mathbb{Z}^n$, hence is free abelian itself of rank $k \leq n$. Let $s_1, \dots, s_k$ denote a basis for $G_{\mathfrak{m}}$ over $\mathbb{Z}$ and let $b_i = \varphi_{\mathfrak{m}}(a_{s_i})$ for $i = 1, \dots, k$. Note also that $R_{\mathfrak{m}}$ is an extension field of K and that

$\mathbb{Z}^n_\omega$ acts naturally on R_m as a group of K-automorphisms. Let $\{\rho_j\}_{j\in J}$ be a basis for R_m over K.

Theorem 4.5. *a)* $B^{(0)}_\mathrm{m} M_\mathrm{m} = 0$ *for any simple weight module* M *over* A *with no proper inner breaks, and*
b) the b_i *are invertible and as a* K*-linear space,* $B^{(1)}_\mathrm{m}$ *has a basis*

$$\{\rho_j b_1^{l_1} \dots b_k^{l_k} \mid j \in J \text{ and } l_i \in \mathbb{Z} \text{ for } 1 \le i \le k\} \tag{4.4}$$

and the following commutation relations hold

$$b_i \lambda = s_i(\lambda) b_i, \quad i = 1, \dots, k, \lambda \in R_\mathrm{m}, \tag{4.5}$$

$$b_i b_j = \lambda_{ij} b_j b_i, \quad i, j = 1, \dots, k \tag{4.6}$$

for some nonzero $\lambda_{ij} \in R_\mathrm{m}$.

Proof. a) Let $g \in \mathbb{Z}^n_\omega \backslash G_\mathrm{m}$. By Lemma 4.4, $A_g M_\mathrm{m} = 0$ and thus $\varphi_\mathrm{m}(A_g) M_\mathrm{m} = 0$.

b) Since $s_i \in G_\mathrm{m}$, $\varphi_\mathrm{m}(a^*_{s_i}) b_i \in R_\mathrm{m}\backslash\{0\}$ and by Lemma 4.3a) with $g = 0$ and $h = s_i$ we have $b_i \varphi_\mathrm{m}(a^*_{s_i}) \in R_\mathrm{m}\backslash\{0\}$. So the b_i are invertible. The relation (4.5) follows from (2.12). Next we prove (4.6). From Lemma 4.3a) and Lemma 4.1 we have $\varphi(A_{s_i+s_j}) = R_\mathrm{m} b_i b_j$. Switching i and j it follows that (4.6) must hold for some nonzero $\lambda_{ij} \in R_\mathrm{m}$.

Finally we prove that (4.4) is a basis for $B^{(1)}_\mathrm{m}$ over K. Linear independence is clear. Let $g \in G_\mathrm{m}$ and write $g = \sum_i l_i s_i$. By repeated use of Lemma 4.3b) we obtain that

$$\varphi_\mathrm{m}(A_g) = \varphi_\mathrm{m}(A_{\mathrm{sgn}(l_1)s_1})^{|l_1|} \dots \varphi_\mathrm{m}(A_{\mathrm{sgn}(l_k)s_k})^{|l_k|}.$$

For $l_i = 0$ the factor should be interpreted as R_m. By Lemma 4.1,

$$\varphi_\mathrm{m}(A_{\pm s_i})^l = R_\mathrm{m} b_i^{\pm l}$$

for $l > 0$ so using (4.5) we get

$$\varphi_\mathrm{m}(A_g) = R_\mathrm{m} b_1^{l_1} \dots b_k^{l_k}.$$

The proof is finished. □

4.2. **Restricted case.** In this subsection we will assume that K is algebraically closed. Moreover we will assume that the K-algebra inclusion $K \hookrightarrow R_\mathrm{m}$ is onto which is the case when R is finitely generated as a K-algebra by the (weak) Nullstellensatz. Then $\mathbb{Z}^n_\omega$ acts trivially on R_m. The structure of $B^{(1)}_\mathrm{m}$ given in Theorem 4.5 is then simplified in the following way.

Corollary 4.6. *Let* $k = \operatorname{rank} G_\mathrm{m}$ *and let* $b_i = \varphi_\mathrm{m}(a_{s_i})$ *for* $i = 1, \dots, k$ *where* $\{s_1, \dots, s_k\}$ *is a* $\mathbb{Z}$*-basis for* G_m*. Then* $B^{(1)}_\mathrm{m}$ *is the* K*-algebra with invertible generators* $b_1, \dots, b_k$ *and the relation*

$$b_i b_j = \lambda_{ij} b_j b_i, \quad 1 \le i, j \le k.$$

Using the normal form of a skew-symmetric integral matrix we will now show that $B^{(1)}_\mathrm{m}$ can be expressed as a tensor product of noncommutative tori. Consider the matrix $(\lambda_{ij})_{1\le i,j\le k}$ from (4.6).

Claim 4.7. *If* $B^{(1)}_\mathrm{m}$ *has a nontrivial irreducible finite-dimensional representation, then all the* λ_{ij} *are roots of unity.*

Proof. Indeed, let N be a finite-dimensional simple module over $B^{(1)}_\mathrm{m}$ and let $i \in \{1, \dots, k\}$. Since K is algebraically closed, b_i has an eigenvector $0 \ne v \in N$ with eigenvalue μ, say. Since b_i is invertible, $\mu \ne 0$. Let $j \ne i$ and consider the vector

$b_j v$. It is also nonzero, since b_j is invertible, and it is an eigenvector of b_i with eigenvalue $\lambda_{ij}\mu$. Repeating the process, we obtain a sequence

$$\mu,\ \lambda_{ij}\mu,\ \lambda_{ij}^2\mu,\ \ldots$$

of eigenvalues of b_i. Since N is finite-dimensional, they cannot all be pairwise distinct, and thus λ_{ij} is a root of unity. □

For $\lambda \in K$, let T_λ denote the K-algebra with two invertible generators a and b satisfying $ab = \lambda ba$. T_λ (or its C^*-analogue) is sometimes referred to as a noncommutative torus.

Theorem 4.8. *Let $k = \operatorname{rank} G_{\mathfrak{m}}$. If all the λ_{ij} in (4.6) are roots of unity, then there is a root of unity λ, an integer r with $0 \le r \le \lfloor k/2 \rfloor$ and positive integers $p_i, i = 1, \ldots, r$ with $1 = p_1 | p_2 | \ldots | p_r$ such that*

$$B_{\mathfrak{m}}^{(1)} \simeq T_{\lambda^{p_1}} \otimes T_{\lambda^{p_2}} \otimes \cdots \otimes T_{\lambda^{p_r}} \otimes L$$

where L is a Laurent polynomial algebra over K in $k - 2r$ variables.

Proof. If $k = 1$, then $B_{\mathfrak{m}}^{(1)} \simeq K[b_1, b_1^{-1}]$ and $r = 0$. If $k > 1$, let p be the smallest positive integer such that $\lambda_{ij}^p = 1$ for all i, j. Using that K is algebraically closed, we fix a primitive p:th root of unity $\varepsilon \in K$. Then there are integers θ_{ij} such that

$$\lambda_{ij} = \varepsilon^{\theta_{ij}}$$

and

$$\theta_{ji} = -\theta_{ij}. \tag{4.7}$$

Equation (4.7) means that $\Theta = (\theta_{ij})$ is a $k \times k$ skew-symmetric integer matrix. Next, consider a change of generators of the algebra $B_{\mathfrak{m}}^{(1)}$:

$$b_i \mapsto b_i' = b_1^{u_{i1}} \cdots b_k^{u_{ik}} \tag{4.8}$$

Such a change of generators can be done if we are given an invertible $k \times k$ integer matrix $U = (u_{ij})$. The new commutation relations are

$$\begin{aligned} b_i' b_j' &= b_1^{u_{i1}} \cdots b_k^{u_{ik}} b_1^{u_{j1}} \cdots b_k^{u_{jk}} = \\ &= \lambda_{11}^{u_{1i}u_{1j}} \ldots \lambda_{k1}^{u_{ki}u_{1j}} \cdot \ldots \\ &\quad \cdot \lambda_{1k}^{u_{1i}u_{kj}} \ldots \lambda_{kk}^{u_{ki}u_{kj}} \cdot b_j' b_i' = \\ &= \varepsilon^{\sum_{p,q} \theta_{pq} u_{pi} u_{qj}} b_j' b_i' \end{aligned}$$

Hence $\Theta' = U^T \Theta U$. By Theorem IV.1 in [9] there is a U such that Θ' has the skew normal form

$$\begin{pmatrix} 0 & \theta_1 & & & & & & & \\ -\theta_1 & 0 & & & & & & & \\ & & 0 & \theta_2 & & & & & \\ & & -\theta_2 & 0 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 0 & \theta_r & \\ & & & & & -\theta_r & 0 & \\ & & & & & & & \mathbf{0} \end{pmatrix}$$

where $r \le \lfloor k/2 \rfloor$ is the rank of Θ, the θ_i are nonzero integers, $\theta_i | \theta_{i+1}$ and $\mathbf{0}$ is a $k - 2r$ by $k - 2r$ zero matrix. Set $\lambda = \varepsilon^{\theta_1}$ and $p_i = \theta_i / \theta_1$ for $i = 1, \ldots, r$. The claim follows. □

The following result, describing simple modules over the tensor product of noncommutative tori, is more or less well-known, but we provide a proof for convenience.

Proposition 4.9. *Let M be a finite dimensional simple module over*

$$T := T_{\lambda_1} \otimes \cdots \otimes T_{\lambda_r},$$

where the λ_i are roots of unity in K. Then there are simple modules M_i over T_{λ_i} such that, as T-modules,

$$M \simeq M_1 \otimes \cdots \otimes M_r.$$

Proof. Denote the generators of T_{λ_i} by a_i and b_i. We will view T_{λ_i} as subalgebras of T. Since the elements $a_i, i = 1, \ldots, r$ commute and M is finite dimensional and K is algebraically closed, there is a nonzero common eigenvector $w \in M$ of the a_i:

$$a_i w = \mu_i w, \quad i = 1, \ldots, r, \tag{4.9}$$

where $\mu_i \in K^*$ because a_i is invertible. Let n_i be the order of λ_i. Then $b_i^{n_i}$ acts as a scalar by Schur's Lemma. By simplicity of M, any element of M has the form (using the commutation relations and (4.9))

$$\sum_{j \in \mathbb{Z}^r,\ 0 \le j_i < n_i} \rho_j b_1^{j_1} \ldots b_r^{j_r} \cdot w, \tag{4.10}$$

where $\rho_j \in K$. This shows that

$$\dim_K M \le n_1 \cdot \ldots \cdot n_r.$$

But the terms in (4.10) all belong to different weight spaces with respect to the commutative subalgebra generated by $a_1, \ldots, a_r$:

$$a_i \cdot b_1^{j_1} \ldots b_r^{j_r} w = \lambda_i^{j_i} \mu_i \cdot b_1^{j_1} \ldots b_r^{j_r} w, \quad i = 1, \ldots, r,$$

and

$$(\lambda_1^{j_1} \mu_1, \ldots, \lambda_r^{j_r} \mu_r) \neq (\lambda_1^{l_1} \mu_1, \ldots, \lambda_r^{l_r} \mu_r)$$

if $j, l \in \mathbb{Z}^r, 0 \le j_i, l_i < n_i$ and $j \neq l$. Hence by standard results they must be linearly independent. Thus

$$\dim_K M = n_1 \cdot \ldots \cdot n_r. \tag{4.11}$$

Next, set $M_i = T_{\lambda_i} \cdot w$. Then $M_i = \oplus_{j=0}^{n_i - 1} K b_i^j \cdot w$ and

$$\dim_K M_i = n_i. \tag{4.12}$$

Finally, define

$$\psi : M_1 \otimes \ldots \otimes M_r \to M$$

by

$$\psi(w \otimes \ldots \otimes w) = w$$

and by requiring that ψ is a T-module homomorphism. This is possible since $M_1 \otimes \ldots \otimes M_r$ is generated by $w \otimes \ldots \otimes w$ as a T-module. Then ψ is surjective, since M is simple. Also the dimensions on both sides agree, so ψ is an isomorphism of T-modules. □

5. Explicit formulas for the induced modules

In this section we show explicitly how one can obtain simple weight modules with no proper inner breaks over a TGWA (equivalently over a TGWC by Proposition 4.2) from the structure of its weight spaces as $B(\omega)$-modules.

Since the $B(\omega)$-modules were described in the restricted case in Subsection 4.2, we obtain in particular a description of all simple weight modules over A with no proper inner breaks and finite-dimensional weight spaces if R is finitely generated over an algebraically closed field K.

5.1. **A basis for M.** Let $\{v_i\}_{i\in I}$ be a basis for $M_\mathfrak{m}$ over K. By Lemma 4.3a), $\tilde{G}_\mathfrak{m}$ is the union of some cosets in $\mathbb{Z}^n/G_\mathfrak{m}$. Let $S \subseteq \mathbb{Z}^n$ be a set of representatives of these cosets. For $g \in \tilde{G}_\mathfrak{m}$, choose $r_g \in R$ such that $a_g' := r_g a_g^*$ satisfies $\varphi_\mathfrak{m}(a_g')\varphi_\mathfrak{m}(a_g) = 1$.

Theorem 5.1. *The set $C = \{a_g v_i \mid g \in S, i \in I\}$ is a basis for M over K.*

Proof. First we show that C is linearly independent over K. Assume that

$$\sum_{g,i} \lambda_{gi} a_g v_i = 0.$$

Then $\sum_i \lambda_{gi} a_g v_i = 0$ for each g since the elements belong to different weight spaces. Hence $0 = a_g' \sum_i \lambda_{gi} a_g v_i = \sum_i \lambda_{gi} v_i$ for each g. Since v_i is a basis over K, all the λ_{gi} must be zero.

Next we prove that C spans M over K. Since M is simple and $M_\mathfrak{m} \neq 0$,

$$M = AM_\mathfrak{m} = \sum_{g\in\mathbb{Z}^n} A_g M_\mathfrak{m} = \sum_{g\in\tilde{G}_\mathfrak{m}} A_g M_\mathfrak{m} = \sum_{h\in S}\sum_{g\in h+G_\mathfrak{m}} A_g M_\mathfrak{m} = \sum_{h\in S} A_h M_\mathfrak{m}$$

by Lemma 4.4 and Lemma 4.3c). □

Corollary 5.2. $\operatorname{supp}(M) = \{g(\mathfrak{m}) \mid g \in S\}$ *and* $g(\mathfrak{m}) \neq h(\mathfrak{m})$ *if* $g, h \in S$, $g \neq h$.

Corollary 5.3. $\dim M = |S| \cdot \dim M_\mathfrak{m}$ *with natural interpretation of* ∞.

5.2. **The action of A.** Our next step is to describe the action of the X_i, Y_i on the basis C for M. Let $\zeta : \tilde{G}_\mathfrak{m} \to S$ be the function defined by requiring $g - \zeta(g) \in G_\mathfrak{m}$.

Theorem 5.4. *Let $g \in S$ and let $v \in M_\mathfrak{m}$. Then*

$$X_i a_g v = \begin{cases} a_h \cdot b_{g,i} v & \text{if } g + e_i \in \tilde{G}_\mathfrak{m}, \\ 0 & \text{otherwise}, \end{cases}$$

where $h = \zeta(g + e_i)$ and

$$b_{g,i} = (-h)(X_i a_g a_{g+e_i-h}' a_h') \cdot a_{g+e_i-h}$$

and

$$Y_i a_g v = \begin{cases} a_k \cdot c_{g,i} v & \text{if } g - e_i \in \tilde{G}_\mathfrak{m}, \\ 0 & \text{otherwise}, \end{cases}$$

where $k = \zeta(g - e_i)$ and

$$c_{g,i} = (-k)(Y_i a_g a_{g-e_i-k}' a_k') \cdot a_{g-e_i-k}.$$

Remark 5.5. Note that

$$\deg X_i a_g a_{g+e_i-h}' a_h' = \deg Y_i a_g a_{g-e_i-k}' a_k' = 0$$

so the action of $\mathbb{Z}^n$ on these elements is well defined. Thus we see that $\deg b_{g,i} \in G_\mathfrak{m}$ and $\deg c_{g,i} \in G_\mathfrak{m}$, i.e. that $b_{g,i}$ and $c_{g,i}$ belong to $B(\omega)$. Therefore the action of these elements on a basis element v_i of $M_\mathfrak{m}$ can be determined if we know the structure of $M_\mathfrak{m}$ as an $B(\omega)$-module. In the restricted case this was described in Subsection 4.2. Expanding the result in the basis $\{v_i\}$ again and acting by a_h or a_k we obtain a linear combination of basis elements from the set C.

Proof. Assume $g + e_i \in \tilde{G}_\mathfrak{m}$. Let $h = \zeta(g + e_i)$. Then

$$\begin{aligned} X_i a_g v &= X_i a_g a_{g+e_i-h}' a_{g+e_i-h} v = \\ &= (X_i a_g a_{g+e_i-h}' a_h') a_h a_{g+e_i-h} v = \\ &= a_h \cdot (-h)(X_i a_g a_{g+e_i-h}' a_h') \cdot a_{g+e_i-h} v. \end{aligned}$$

If $g + e_i \notin \tilde{G}_\mathfrak{m}$, then $X_i a_g v = 0$ by Lemma 4.4.

Assume $g - e_i \in \tilde{G}_{\mathfrak{m}}$. Let $k = \zeta(g - e_i)$. Then

$$\begin{aligned} Y_i a_g v &= Y_i a_g a'_{g-e_i-k} a_{g-e_i-k} v = \\ &= (Y_i a_g a'_{g-e_i-k} a'_k) a_k a_{g-e_i-k} v = \\ &= a_k \cdot (-k)(Y_i a_g a'_{g-e_i-k} a'_k) \cdot a_{g-e_i-k} v. \end{aligned}$$

If $g - e_i \notin \tilde{G}_{\mathfrak{m}}$, then $Y_i a_g v = 0$ by Lemma 4.4. □

Note that we do not need the technical assumptions in the proof of Theorem 1 in [8] under which the exact formulas for simple weight modules were obtained.

6. Application to quantized Weyl algebras

In this final part we will apply the methods developed in the previous sections to the problem of describing representations of the quantized Weyl algebra, defined in Section 2.2. As mentioned there, it is naturally a TGWA.

First we find the isotropy group and the set $\tilde{G}_{\mathfrak{m}}$ expressed as solution of systems of linear equations (see Proposition 6.3 and Proposition 6.4). These sets are directly related to the structure of the subalgebra $B(\omega)$ (Theorem 4.5) and the support of a module (Corollary 5.2).

Then in Section 6.2 we give a complete classification of all locally finite simple weight modules with no proper inner breaks over a quantized Weyl algebra of rank two. The parameters q_1 and q_2 are allowed to be any numbers from $\mathbb{C}\backslash\{0,1\}$. Example 6.7 shows that the assumption that the modules have no proper inner breaks is not superfluous.

6.1. The isotropy group and $\tilde{G}_{\mathfrak{m}}$. Let $R = \mathbb{C}[t_1, \dots, t_n]$ and fix $\mathfrak{m} = (t_1 - \alpha_1, \dots, t_n - \alpha_n) \in \mathrm{Max}(R)$. Let ω be the orbit of $\mathfrak{m}$ under the action (2.10) of $\mathbb{Z}^n$. Set $[k]_q = \sum_{j=0}^{k-1} q^i$ for $k \in \mathbb{Z}$ and $q \in \mathbb{C}$. Recall the definition (2.9) of the automorphisms σ_i of R.

Proposition 6.1. *Let $(g_1, \dots, g_n) \in \mathbb{Z}^n$. Then*

$$\begin{aligned} \sigma_1^{g_1} \dots \sigma_n^{g_n}(\mathfrak{m}) = \\ \Big([g_1]_{q_1} + q_1^{g_1} t_1 - \alpha_1,\quad & [g_2]_{q_2}\big(1 + (q_1 - 1)\alpha_1\big) + q_1^{g_1} q_2^{g_2} t_2 - \alpha_2, \dots \\ \dots, [g_j]_{q_j}\big(1 + \sum_{r=1}^{j-1}(q_r - 1)\alpha_r\big) &+ q_1^{g_1} \dots q_j^{g_j} t_j - \alpha_j, \dots \\ \dots, [g_n]_{q_n}\big(1 + \sum_{r=1}^{n-1}(q_r - 1)\alpha_r\big) &+ q_1^{g_1} \dots q_n^{g_n} t_n - \alpha_n\Big). \end{aligned}$$

Proof. Induction. □

For notational brevity we set $\beta_i = (q_i - 1)\alpha_i$ and $\gamma_i = 1 + \beta_1 + \beta_2 + \dots + \beta_i$. We also set $\gamma_0 = 1$. The numbers γ_i will play an important role in the next statements. By a *j-break* we mean an ideal $\mathfrak{n} \in \mathrm{Max}(R)$ such that $t_j \in \mathfrak{n}$.

Corollary 6.2. *For $j = 1, \dots, n$ we have*

$$t_j \in \sigma_1^{g_1} \dots \sigma_n^{g_n}(\mathfrak{m}) \Longleftrightarrow \gamma_j = q_j^{g_j} \gamma_{j-1}.$$

Thus ω contains a j-break iff $\gamma_j = q_j^k \gamma_{j-1}$ for some integer k.

Proof. By Proposition 6.1,

$$t_j \in \sigma_1^{g_1} \dots \sigma_n^{g_n}(\mathfrak{m})$$

iff

$$[g_j]_{q_j}\big(1+\sum_{r=1}^{j-1}(q_r-1)\alpha_r\big)=\alpha_j.$$

Multiply both sides with q_j-1 to get

$$(q_j^{g_j}-1)(1+\beta_1+\ldots+\beta_{j-1})=\beta_j.$$

□

The next Proposition describes the isotropy subgroup $\mathbb{Z}^n_\omega$ defined in (2.14).

Proposition 6.3. *We have*

$$\mathbb{Z}^n_\omega=\{g\in\mathbb{Z}^n\mid (q_1^{g_1}\ldots q_j^{g_j}-1)\gamma_j=0\ \forall j=1,\ldots,n\}. \tag{6.1}$$

Proof. From Proposition 6.1, $\sigma_1^{g_1}\ldots\sigma_n^{g_n}(\mathfrak{m})=\mathfrak{m}$ iff

$$\begin{aligned}
\alpha_1&=[g_1]_{q_1}+q_1^{g_1}\alpha_1\\
\alpha_2&=[g_2]_{q_2}\big(1+(q_1-1)\alpha_1\big)+q_1^{g_1}q_2^{g_2}\alpha_2\\
&\ \vdots\\
\alpha_n&=[g_n]_{q_n}\big(1+(q_1-1)\alpha_1+\ldots+(q_{n-1}-1)\alpha_{n-1}\big)+q_1^{g_1}\ldots q_n^{g_n}\alpha_n
\end{aligned}$$

Multiply the i:th equation by q_i-1. Then the system can be written

$$\begin{aligned}
\beta_1&=q_1^{g_1}-1+q_1^{g_1}\beta_1\\
\beta_2&=(q_2^{g_2}-1)(1+\beta_1)+q_1^{g_1}q_2^{g_2}\beta_2\\
&\ \vdots\\
\beta_n&=(q_n^{g_n}-1)(1+\beta_1+\ldots+\beta_{n-1})+q_1^{g_1}\ldots q_n^{g_n}\beta_n
\end{aligned}$$

or equivalently

$$\begin{aligned}
1+\beta_1&=q_1^{g_1}(1+\beta_1)\\
1+\beta_1+\beta_2&=q_2^{g_2}(1+\beta_1)+q_1^{g_1}q_2^{g_2}\beta_2\\
&\ \vdots\\
1+\beta_1+\ldots+\beta_n&=q_n^{g_n}(1+\beta_1+\ldots+\beta_{n-1})+q_1^{g_1}\ldots q_n^{g_n}\beta_n
\end{aligned}$$

Now for i from 1 to $n-1$, replace the expression $1+\beta_1+\ldots+\beta_i$ in the right hand side of the $i+1$:th equation by the right hand side of the i:th equation. After simplification, the claim follows. □

Note that it follows from (6.1) that the subgroup

$$Q=\{g\in\mathbb{Z}^n\mid q_j^{g_j}=1 \text{ for } j=1,\ldots,n\} \tag{6.2}$$

of $\mathbb{Z}^n$ is always contained in $\mathbb{Z}^n_\omega$ for any orbit ω. Moreover $\mathbb{Z}^n_\omega=Q$ if ω (viewed as a subset of $\mathbb{C}^n$) does not intersect the union of the hyperplanes in $\mathbb{C}^n$ defined by the equations $1+(q_1-1)x_1+\ldots+(q_j-1)x_j=0$ $(1\le j\le n)$.

Another case of interest is when for any j, $q_1^{g_1}\ldots q_j^{g_j}=1$ implies $g_1=\ldots=g_j=0$. If for instance the q_j are pairwise distinct prime numbers this hold. Then $\mathbb{Z}^n_\omega=\{0\}$ unless $1+\beta_1+\ldots+\beta_j=0$ for all j, i.e. unless ω contains the point

$$\mathfrak{n}_0=(t_1-(1-q_1)^{-1},t_2,\ldots,t_n).$$

So in this very special case we have $\omega=\{\mathfrak{n}_0\}$ and $\mathbb{Z}^n_\omega=\mathbb{Z}^n$.

We now turn to the set $\tilde{G}_\mathfrak{m}$ defined in (4.1) which can here be described explicitly in terms of $\mathfrak{m}$ in the following way.

Proposition 6.4.

$$\tilde{G}_{\mathfrak{m}} = \tilde{G}_{\mathfrak{m}}^{(1)} \times \ldots \times \tilde{G}_{\mathfrak{m}}^{(n)},$$

where

$$\begin{aligned}\tilde{G}_{\mathfrak{m}}^{(j)} &= \{k \geq 0 \mid \gamma_j \neq q_j^i \gamma_{j-1} \ \forall i = 0, 1, \ldots, k-1\} \cup \\ &\cup \{k < 0 \mid \gamma_j \neq q_j^i \gamma_{j-1} \ \forall i = -1, -2, \ldots, k\}.\end{aligned}$$

Proof. From the relations of the algebra follows that the subspace spanned by the words in A_g is one-dimensional. Thus $g \in \tilde{G}_{\mathfrak{m}}$ iff

$$Z_n^{-g_n} \ldots Z_1^{-g_1} Z_1^{g_1} \ldots Z_n^{g_n} \notin \mathfrak{m} \tag{6.3}$$

where $Z_i^k = X_i^k$ if $k \geq 0$ and $Z_i^k = Y_i^{-k}$ if $k < 0$. Since $\sigma_i(t_j) = t_j$ for $j < i$, (6.3) is equivalent to

$$Z_n^{-g_n} Z_n^{g_n} \ldots Z_1^{-g_1} Z_1^{g_1} \notin \mathfrak{m}.$$

Since $\mathfrak{m}$ is prime, this holds iff $Z_j^{-g_j} Z_j^{g_j} \notin \mathfrak{m}$ for each j. If $g_j = 0$ this is true. If $g_j > 0$ we have

$$Z_j^{-g_j} Z_j^{g_j} = Y_j^{g_j} X_j^{g_j} = Y_j^{g_j-1} X_j^{g_j-1} \sigma_j^{-g_j+1}(t_j) = \ldots = t_j \sigma_j^{-1}(t_j) \ldots \sigma_j^{-g_j+1}(t_j),$$

while if $g_j < 0$

$$Z_j^{-g_j} Z_j^{g_j} = X_j^{-g_j} Y_j^{-g_j} = X_j^{-g_j-1} Y_j^{-g_j-1} \sigma_j^{-g_j}(t_j) = \ldots = \sigma_j(t_j) \ldots \sigma_j^{-g_j}(t_j).$$

Since $\mathfrak{m}$ is prime, $g \in \tilde{G}_{\mathfrak{m}}$ iff for all $j = 1, \ldots, n$

$$t_j \notin \sigma_j^i(\mathfrak{m}), \quad i = 0, \ldots, g_j - 1 \text{ if } g_j \geq 0,$$

and

$$t_j \notin \sigma_j^i(\mathfrak{m}), \quad i = -1, -2 \ldots, g_j \text{ if } g_j < 0.$$

The claim now follows from Corollary 6.2. □

Corollary 6.5. *If $\{1, \alpha_1, \alpha_2, \ldots, \alpha_n\}$ is linearly independent over $\mathbb{Q}(q_1, \ldots, q_n)$, then $\tilde{G}_{\mathfrak{m}} = \mathbb{Z}^n$.*

6.2. **Description of simple weight modules over rank two algebras.** Assume from now on that A is a quantized Weyl algebra of rank two. In this section we will obtain a list of all locally finite simple weight A-modules with no proper inner breaks.

We consider first some families of ideals in $\mathrm{Max}(R)$. Define for $\lambda \in \mathbb{C}$,

$$\begin{aligned}\mathfrak{n}_\lambda^{(1)} &= \big(t_1 - (1-\lambda)(1-q_1)^{-1}, t_2 - \lambda(1-q_2)^{-1}\big), \\ \mathfrak{n}_\lambda^{(2)} &= \big(t_1 - (1-q_1)^{-1}, t_2 - \lambda\big),\end{aligned}$$

and set $\mathfrak{n}_0 = \mathfrak{n}_0^{(1)} = \mathfrak{n}_0^{(2)}$. The following lemma will be useful.

Lemma 6.6. *For $\lambda \in \mathbb{C}$ and integers k, l we have*

$$\sigma_1^k \sigma_2^l(\mathfrak{n}_\lambda^{(1)}) = \mathfrak{n}_{\lambda q_1^{-k}}^{(1)}, \tag{6.4}$$

$$\sigma_1^k \sigma_2^l(\mathfrak{n}_\lambda^{(2)}) = \mathfrak{n}_{\lambda q_1^{-k} q_2^{-l}}^{(2)}. \tag{6.5}$$

Proof. Follows from Proposition 6.1 or by direct calculation using the definition (2.9) of the σ_i. □

The following example shows the existence of locally finite simple weight modules M over A which have some proper inner breaks.

Example 6.7. Assume that $q_1\lambda_{12}$ is a root of unity of order r. Let M be a vector space of dimension r and let $\{v_0, v_1, \dots, v_{r-1}\}$ be a basis for M. Define an action of A on M as follows.

$$X_1 v_k = \begin{cases} v_{k+1}, & k < r-1 \\ v_0, & k = r-1 \end{cases} \qquad X_2 v_k = (q_1\lambda_{12})^{-k} v_k$$

$$Y_1 v_k = \begin{cases} (1-q_1)^{-1} v_{k-1}, & k > 0 \\ (1-q_1)^{-1} v_{r-1}, & k = 0 \end{cases} \qquad Y_2 v_k = 0$$

It is easy to check that (2.6)–(2.8) hold so this defines a module over A. It is immediate that $M = M_{\mathfrak{m}}$ where $\mathfrak{m} = \mathfrak{n}_0 = (t_1 - (1-q_1)^{-1}, t_2)$ so M is a weight module and M is simple by standard arguments. However, recalling Definition 3.7, M has some proper inner breaks in the sense that $\mathfrak{m} \in \operatorname{supp}(M)$, $X_2 M_{\mathfrak{m}} \neq 0$ but $Y_2 X_2 M_{\mathfrak{m}} = 0$.

We will describe the isotropy groups of the different ideals in $\operatorname{Max}(R)$. Let K_1 and K_2 denote the kernels of the group homomorphisms from $\mathbb{Z}\times\mathbb{Z}$ to the multiplicative group $\mathbb{C}\backslash\{0\}$ which map (k,l) to q_1^k and $q_1^k q_2^l$ respectively. Then $Q = K_1 \cap K_2$ where Q was defined in (6.2). For $\mathfrak{m} \in \operatorname{Max}(R)$, recall that $\mathbb{Z}^2_{\mathfrak{m}} = \{g \in \mathbb{Z}^2 \mid g(\mathfrak{m}) = \mathfrak{m}\}$. The following corollary describes the isotropy group $\mathbb{Z}^2_{\mathfrak{m}}$ of any $\mathfrak{m} \in \operatorname{Max}(R)$.

Corollary 6.8. *Let $\lambda \in \mathbb{C}\backslash\{0\}$ and $\mathfrak{n} \in \operatorname{Max}(R)\backslash\{\mathfrak{n}^{(i)}_\mu \mid \mu \in \mathbb{C}, i = 1, 2\}$. Then we have the following equalities in the lattice of subgroups of $\mathbb{Z}^2$.*

$$\begin{tikzcd}[arrows=dash] & \mathbb{Z}^2_{\mathfrak{n}_0} = \mathbb{Z}^2 \arrow[dl] \arrow[dr] & \\ \mathbb{Z}^2_{\mathfrak{n}^{(1)}_\lambda} = K_1 \arrow[dr] & & \mathbb{Z}^2_{\mathfrak{n}^{(2)}_\lambda} = K_2 \arrow[dl] \\ & \mathbb{Z}^2_{\mathfrak{n}} = Q & \end{tikzcd}$$

Proof. The family of ideals $\{\mathfrak{n}^{(1)}_\lambda \mid \lambda \in \mathbb{C}\}$ are precisely those for which $\gamma_2 = 0$. And $\{\mathfrak{n}^{(2)}_\lambda \mid \lambda \in \mathbb{C}\}$ are exactly those such that $\gamma_1 = 0$. Thus the claim follows from Proposition 6.3. $\square$

Let M be a simple weight A-module with no proper inner breaks and finite dimensional weight spaces, $\mathfrak{m} = (t_1 - \alpha_1, t_2 - \alpha_2) \in \operatorname{supp} M$ and let ω be the orbit of $\mathfrak{m}$. We consider four main cases separately: $\mathfrak{m} = \mathfrak{n}_0$, $\mathfrak{m} = \mathfrak{n}^{(1)}_\lambda$ for some $\lambda \neq 0$, $\mathfrak{m} = \mathfrak{n}^{(2)}_\lambda$ for some $\lambda \neq 0$ and $\mathfrak{m} \notin \{\mathfrak{n}^{(i)}_\mu \mid \mu \in \mathbb{C}, i = 1, 2\}$. Some of these cases will contain subcases. In each case we will proceed along the following steps, which also illustrate the procedure for a general TGWA.

(1) Find the sets $\mathbb{Z}^n_{\mathfrak{m}}$ and $\tilde{G}_{\mathfrak{m}}$ using Corollary 6.8 and Proposition 6.4. Write down $G_{\mathfrak{m}} = \mathbb{Z}^n_{\mathfrak{m}} \cap \tilde{G}_{\mathfrak{m}}$ and choose a basis $\{s_1, \dots, s_k\}$ for $G_{\mathfrak{m}}$ over $\mathbb{Z}$.
(2) For each $g \in \tilde{G}_{\mathfrak{m}}$, choose a word a_g of degree g such that $a_g^* a_g \notin \mathfrak{m}$.
(3) Using Corollary 4.6, describe $B^{(1)}_{\mathfrak{m}}$ and the finite-dimensional simple $B^{(1)}_{\mathfrak{m}}$-module $M_{\mathfrak{m}}$.
(4) Choose a set of representatives S for $\tilde{G}_{\mathfrak{m}}/G_{\mathfrak{m}}$. By Theorem 5.1 we know then a basis C for M.
(5) Calculate the action of X_i, Y_i on the basis using either relations (2.6)–(2.8) or Theorem 5.4.

We will use the following notation: $Z_j^k = X_j^k$ if $k \geq 0$ and $Z_j^k = Y_j^{-k}$ if $k < 0$. Note that the k in Z_j^k should only be regarded as an upper index, not as a power. The choice of a_g in step two above is more or less irrelevant for a quantized Weyl algebra because each A_g is one-dimensional. Therefore we will always choose $a_g = Z_1^{g_1} Z_2^{g_2}$ where $g = (g_1, g_2)$.

6.3. **The case $\mathfrak{m} = \mathfrak{n}_0$.** Here $\alpha_1 = (1-q_1)^{-1}$, $\alpha_2 = 0$ so that $\gamma_1 = \gamma_2 = 0$. By Corollary 6.8 we have $\mathbb{Z}^2_{\mathfrak{m}} = \mathbb{Z}^2$ and from Proposition 6.4 one obtains that $\tilde{G}_{\mathfrak{m}} = \mathbb{Z} \times \{0\}$. Thus $G_{\mathfrak{m}} = \mathbb{Z} \times \{0\} = \mathbb{Z} \cdot s_1$ with $s_1 = (1,0)$. Since $G_{\mathfrak{m}}$ has rank one, Corollary 4.6 implies that $B_{\mathfrak{m}}^{(1)}$ is isomorphic to the Laurent polynomial algebra $\mathbb{C}[T, T^{-1}]$ in one variable. Therefore $M_{\mathfrak{m}}$ is one-dimensional, say $M_{\mathfrak{m}} = \mathbb{C}v_0$ and $b_1 = \varphi_{\mathfrak{m}}(Z_1^1) = \varphi_{\mathfrak{m}}(X_1)$, hence X_1, acts in $M_{\mathfrak{m}}$ as some nonzero scalar ρ. And

$$Y_1 v_0 = \rho^{-1} Y_1 X_1 v_0 = \rho^{-1}(1-q_1)^{-1} v_0.$$

Here $S = \{(0,0)\}$ and $C = \{v_0\}$ is a basis for M with the following action:

$$\begin{aligned} X_1 v_0 &= \rho v_0, & X_2 v_0 &= 0, \\ Y_1 v_0 &= \rho^{-1}(1-q_1)^{-1} v_0, & Y_2 v_0 &= 0. \end{aligned} \tag{6.6}$$

That $Z_2^{\pm 1} v_0 = 0$ follows from Theorem 5.4 since $(0, \pm 1) \notin \tilde{G}_{\mathfrak{m}}$.

6.4. **The case $\mathfrak{m} = \mathfrak{n}_\lambda^{(1)}$, $\lambda \neq 0$.** Here $\alpha_1 = (1-\lambda)(1-q_1)^{-1}$ and $\alpha_2 = \lambda(1-q_1)^{-1}$ so $\gamma_1 = \lambda$ and $\gamma_2 = 0$. By Proposition 6.4, $\tilde{G}_{\mathfrak{m}}^{(2)} = \mathbb{Z}$ and

$$\tilde{G}_{\mathfrak{m}}^{(1)} = \{k \geq 0 \mid \lambda \neq q_1^i \; \forall i = 0, 1, \dots, k-1\} \cup \{k < 0 \mid \lambda \neq q_1^i \; \forall i = -1, -2, \dots, k\}.$$

We consider four subcases according to whether ω contains a 1-break or not and whether q_1 is a root of unity or not.

6.4.1. *The case $\mathfrak{m} = \mathfrak{n}_\lambda^{(1)}$, $\lambda \neq 0$, ω contains a 1-break and q_1 is a root of unity.* By Corollary 6.2 $\lambda = q_1^k$ for some $k \in \mathbb{Z}$. Let o_1 be the order of q_1. Then $\mathbb{Z}^2_{\mathfrak{m}} = K_1 = (o_1 \mathbb{Z}) \times \mathbb{Z}$. We can further assume that $k \in \{0, 1, \dots, o_1 - 1\}$.

Note that $X_1^k M_{\mathfrak{m}} \neq 0$ because $\deg X_1^k = (k, 0) \in \tilde{G}_{\mathfrak{m}}$ so $Y_1^k X_1^k \notin \mathfrak{m}$. Hence $\sigma_1^k(\mathfrak{m}) \in \operatorname{supp}(M)$. By Lemma 6.6, $\sigma_1^k(\mathfrak{m}) = \mathfrak{n}^{(1)}_{q_1^k q_1^{-k}} = \mathfrak{n}_1^{(1)}$. We can thus change notation and let $\mathfrak{m} = \mathfrak{n}_1^{(1)}$. Then by Proposition 6.4 we have

$$\tilde{G}_{\mathfrak{m}} = \{0, -1, -2, \dots, -o_1 + 1\} \times \mathbb{Z}.$$

And $G_{\mathfrak{m}} = \tilde{G}_{\mathfrak{m}} \cap \mathbb{Z}^2_{\mathfrak{m}} = \{0\} \times \mathbb{Z}$. By Corollary 4.6, $B_{\mathfrak{m}}^{(1)}$ is a Laurent polynomial algebra in one variable. Thus $M_{\mathfrak{m}}$ is one dimensional with a basis vector, say v_0. X_2 acts by some nonzero scalar ρ on v_0 and $Y_2 X_2 v_0 = (1-q_2)^{-1} v_0$. X_1 and $Y_1^{o_1}$ act as zero on $M_{\mathfrak{m}}$ by Lemma 4.4 because their degrees $(1,0)$ and $(-o_1, 0)$ does not belong to $\tilde{G}_{\mathfrak{m}}$.

As a set of representatives for $\tilde{G}_{\mathfrak{m}}/G_{\mathfrak{m}}$ we choose

$$S = \{(0,0), (-1,0), (-2,0), \dots, (-o_1+1, 0)\}.$$

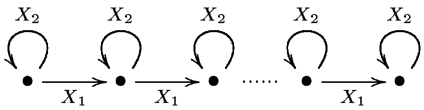
By Corollary 5.2 we obtain that

$$\operatorname{supp}(M) = \{\mathfrak{n}_1^{(1)}, \mathfrak{n}^{(1)}_{q_1^{-1}}, \dots, \mathfrak{n}^{(1)}_{q_1^{-o_1+1}}\}.$$

By 5.1, the set

$$C = \{v_j := Y_1^j v_0 \mid j = 0, 1, \dots, o_1 - 1\}$$

is a basis for M. The following picture shows the support of the module and how the X_i act on it. Since the Y_i just act in the opposite direction of the X_i we do not draw their arrows.



Using Lemma 6.6,

$$X_1 v_j = X_1 Y_1^j v_0 = Y_1^{j-1} \sigma_1^j(t_1) v_0 = [j]_{q_1} v_{j-1}$$

and from relations (2.6)–(2.8) follow that

$$X_2 v_j = q_1^j \lambda_{12}^j Y_1^j X_2 v_0 = \rho \lambda_{12}^j q_1^j v_j,$$

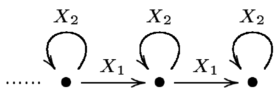
$$Y_2 v_j = \lambda_{21}^j Y^j Y_2 v_0 = (1-q_2)^{-1} \rho^{-1} \lambda_{21}^j v_j.$$

Thus the action on the basis $\{v_0, \dots, v_{o_1 - 1}\}$ is

$$\begin{aligned} X_1 v_j &= \begin{cases} 0, & j = 0, \\ [j]_{q_1} v_{j-1}, & 0 < j \le o_1 - 1, \end{cases} \\ Y_1 v_j &= \begin{cases} v_{j+1}, & 0 \le j < o_1 - 1, \\ 0, & j = o_1 - 1, \end{cases} \\ X_2 v_j &= \rho \lambda_{12}^j q_1^j v_j, \\ Y_2 v_j &= (1-q_2)^{-1} \rho^{-1} \lambda_{21}^j v_j. \end{aligned} \tag{6.7}$$

6.4.2. *The case* $\mathfrak{m} = \mathfrak{n}_\lambda^{(1)}$, $\lambda \neq 0$, ω *contains a* 1*-break and* q_1 *is not a root of unity.* Now there is a unique integer $k \in \mathbb{Z}$ such that $\lambda = q_1^k$. If $k \ge 0$, then $\tilde{G}_\mathfrak{m}^{(1)}$ is the set of all integers $\le k$ while if $k < 0$, then $\tilde{G}_\mathfrak{m}^{(1)}$ is all integers $\ge k+1$.

If $k \ge 0$, $X_1^k M_\mathfrak{m} \neq 0$ because $(k, 0) \in \tilde{G}_\mathfrak{m}$ so $Y_1^k X_1^k \notin \mathfrak{m}$. Therefore $\sigma_1^k(\mathfrak{m}) = \mathfrak{n}_1^{(1)} \in \operatorname{supp}(M)$. We change notation and let $\mathfrak{m} = \mathfrak{n}_1^{(1)}$. Then $\tilde{G}_\mathfrak{m}^{(1)} = \{\dots, -2, -1, 0\}$ and $G_\mathfrak{m} = \{0\} \times \mathbb{Z}$. We choose $S = \{(i, 0) \mid i \le 0\}$. $Y_2 X_2 = (1-q_2)^{-1}$ on $M_\mathfrak{m}$ so $M_\mathfrak{m} = \mathbb{C} v_0$, for a basis vector v_0, and $X_2 v_0 = \rho v_0$ for some $\rho \in \mathbb{C}^*$. The set $C = \{v_j := Y_1^j v_0 \mid j \le 0\}$ is a basis for M and we have the following picture of $\operatorname{supp}(M)$.



One easily obtains the following action on the basis $\{v_j \mid j \le 0\}$:

$$\begin{aligned} X_1 v_j &= \begin{cases} 0, & j = 0, \\ [j]_{q_1} v_{j-1}, & j \ge 1, \end{cases} \\ Y_1 v_j &= v_{j+1}, \\ X_2 v_j &= \rho \lambda_{12}^j q_1^j v_j, \\ Y_2 v_j &= (1-q_2)^{-1} \rho^{-1} \lambda_{21}^j v_j. \end{aligned} \tag{6.8}$$

The case $k < 0$ is analogous and yields a lowest weight representation with $\mathfrak{m} = \mathfrak{n}_{q_1^{-1}}^{(1)}$ as its lowest weight. A basis for M is then

$$C = \{v_j := X_1^j v_0 \mid j \ge 0\},$$

where $M_{\mathfrak{m}} = \mathbb{C}v_0$ and the action is given by

$$
\begin{aligned}
X_1 v_j &= v_{j+1}, \\
Y_1 v_j &= \begin{cases} 0, & j = 0, \\ [-j]_{q_1} v_{j-1}, & j > 0, \end{cases} \\
X_2 v_j &= (q_1\lambda_{12})^{-j}\rho v_j, \\
Y_2 v_j &= \lambda_{12}^{j}(1-q_2)^{-1}\rho^{-1} v_j.
\end{aligned}
\tag{6.9}
$$

6.4.3. *The case* $\mathfrak{m} = \mathfrak{n}_\lambda^{(1)}$, $\lambda \neq 0$, ω *contains no* 1*-break and* q_1 *is a root of unity.* By Corollary 6.2, $\lambda \neq q_1^k$ for all $k \in \mathbb{Z}$. So by Proposition 6.4, $\tilde{G}_{\mathfrak{m}} = \mathbb{Z}^2$. $G_{\mathfrak{m}} = (o_1\mathbb{Z}) \times \mathbb{Z}$ and we can choose $S = \{0, 1, \ldots, o_1 - 1\} \times \{0\}$. From

$$X_1^{o_1} X_2 = (q_1\lambda_{12})^{o_1} X_2 X_1^{o_1} = \lambda_{12}^{o_1} X_2 X_1^{o_1}$$

and Corollary 4.6 follows that $B_{\mathfrak{m}}^{(1)} \simeq T_{\lambda_{12}^{o_1}}$. It can only have finite-dimensional irreducible representations if $\lambda_{12}^{o_1}$ is a root of unity. Assuming this, any such representation is r-dimensional, where r is the order of $\lambda_{12}^{o_1}$, and is parametrized by $\mathbb{C}^* \times \mathbb{C}^* \ni (\rho, \mu)$ with basis

$$M_{\mathfrak{m}} = \operatorname{Span}\{v_j := X_2^j v_0 \mid j = 0, 1, \ldots, r-1\},$$

where $X_1^{o_1} v_0 = \rho v_0$ and relations

$$
\begin{aligned}
X_1^{o_1} v_j &= \lambda_{12}^{o_1 j}\rho v_j, \\
X_2 v_j &= \begin{cases} v_{j+1}, & 0 \le j < r-1, \\ \mu v_0, & j = r-1. \end{cases}
\end{aligned}
$$

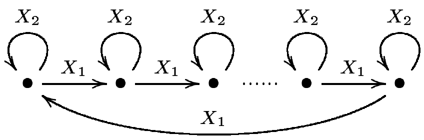
Therefore by Theorem 5.1,

$$M = \operatorname{Span}\{w_{ij} = X_1^i v_j \mid 0 \le i < o_1, 0 \le j < r\}.$$

Using the commutation relations and the formulas in Lemma 6.6 we can write down the action as follows.

$$
\begin{aligned}
X_1 w_{ij} &= \begin{cases} w_{i+1,j}, & 0 \le i < o_1 - 1, \\ \lambda_{12}^{o_1 j}\rho w_{0,j}, & i = o_1 - 1, \end{cases} \\
Y_1 w_{ij} &= \begin{cases} (1-\lambda)(1-q_1)^{-1}\lambda_{12}^{-o_1 j}\rho^{-1} w_{o_1-1,j}, & i = 0, \\ (1-\lambda q_1^{-i})(1-q_1)^{-1} w_{i-1,j}, & 0 < i \le o_1 - 1, \end{cases} \\
X_2 w_{ij} &= \begin{cases} q_1^{-i}\lambda_{21}^{i} w_{i,j+1}, & 0 \le j < r-1, \\ q_1^{-i}\lambda_{21}^{i}\mu w_{i,0}, & j = r-1, \end{cases} \\
Y_2 w_{ij} &= \begin{cases} \lambda_{12}^{i}\mu^{-1}\lambda(1-q_2)^{-1} w_{i,r-1}, & j = 0, \\ \lambda_{12}^{i}\lambda(1-q_2)^{-1} w_{i,j-1}, & 0 < j \le r-1. \end{cases}
\end{aligned}
\tag{6.10}
$$

The action can be illustrated in the following way.



6.4.4. *The case* $\mathfrak{m} = \mathfrak{n}_\lambda^{(1)}$, $\lambda \neq 0$, ω *contains no* 1*-break and* q_1 *is not a root of unity.* By Corollary 6.2, $\lambda \neq q_1^k$ for all $k \in \mathbb{Z}$. Now $\mathbb{Z}_\mathfrak{m}^2 = \{0\} \times \mathbb{Z}$ so $G_\mathfrak{m} = \{0\} \times \mathbb{Z}$. $M_\mathfrak{m}$ is one-dimensional with basis v_0, say, and $X_2 = \rho$ on $M_\mathfrak{m}$ while $Y_2X_2 = \lambda(1-q_2)^{-1} \neq 0$ on $M_\mathfrak{m}$. We choose $S = \mathbb{Z} \times \{0\}$. Then a basis for M is

$$C = \{v_j := X_1^j v_0 \mid j \geq 0\} \cup \{v_j := \zeta_j Y_1^{-j} v_0 \mid j < 0\},$$

where we determine ζ_j by requiring that $X_1 v_j = v_{j+1}$ for all j. Explicitly we have for $j < 0$,

$$\zeta_j = \frac{(1-q_1)^{-j}}{(1-\lambda q_1^{-j})(1-\lambda q_1^{-j-1})\dots(1-\lambda q_1)}.$$

Using the commutation relations and the formulas in Lemma 6.6 we get the action on $M = \mathrm{Span}\{v_j \mid j \in \mathbb{Z}\}$.

$$\begin{aligned} X_1 v_j &= v_{j+1}, & X_2 v_j &= q_1^{-j}\lambda_{12}^{-j}\rho v_j, \\ Y_1 v_j &= \frac{1-\lambda q_1^{-j+1}}{1-q_1} v_{j-1}, & Y_2 v_j &= \lambda_{12}^j \lambda (1-q_2)^{-1}\rho^{-1} v_j, \end{aligned} \tag{6.11}$$

and a corresponding diagram

$$\begin{tikzcd} \cdots\cdots \bullet \arrow[loop above, "X_2"] \arrow[r, "X_1"] & \bullet \arrow[loop above, "X_2"] \arrow[r, "X_1"] & \bullet \arrow[loop above, "X_2"] \cdots\cdots \end{tikzcd}.$$

6.5. **The case** $\mathfrak{m} = \mathfrak{n}_\lambda^{(2)}$, $\lambda \neq 0$. Here $\gamma_1 = 0$ while $\gamma_2 = \lambda(q_2 - 1)$. By Corollary 6.2, ω does not contain any breaks. We have $\tilde{G}_\mathfrak{m} = \mathbb{Z}^2$ and $G_\mathfrak{m} = \mathbb{Z}_\mathfrak{m}^2 = K_2$.

We will need some lemmas in order to proceed.

Lemma 6.9. *For* $k, l \in \mathbb{Z}$ *we have*

$$Z_1^k Z_2^l = q_1^{k\bar{l}} \lambda_{12}^{kl} Z_2^l Z_1^k, \tag{6.12}$$

where $\bar{l} = \max\{0, l\}$.

Proof. Relations (2.6)–(2.8) can be rewritten in the more compact form

$$Z_1^k Z_2^l = q_1^{k\delta_{l,1}} \lambda_{12}^{kl} Z_2^l Z_1^k, \quad k, l = \pm 1,$$

where $\delta_{l,1}$ is the Kronecker symbol. After repeated application of this, (6.12) follows. □

By Lemma 6.6 we have for $k, l \in \mathbb{Z}$,

$$\sigma_1^k \sigma_2^l (t_1) = (1-q_1)^{-1} \mod \mathfrak{m}, \tag{6.13}$$

$$\sigma_1^k \sigma_2^l (t_2) = \lambda q_1^k q_2^l \mod \mathfrak{m}. \tag{6.14}$$

Lemma 6.10. *Let* $k, l \in \mathbb{Z}$ *and let* $m = \min\{|k|, |l|\}$. *Then, as operators on* $M_\mathfrak{m}$, *we have*

$$Z_1^k Z_1^l = \begin{cases} Z_1^{k+l}, & kl \geq 0, \\ (1-q_1)^{-m} Z_1^{k+l}, & kl < 0, \end{cases} \tag{6.15}$$

$$Z_2^k Z_2^l = \begin{cases} Z_2^{k+l}, & kl \geq 0, \\ \lambda^m q_2^{(1-2l+(\mathrm{sgn}\, l)m)m/2} Z_2^{k+l}, & kl < 0. \end{cases} \tag{6.16}$$

Proof. Direct calculation using (6.13) and (6.14). For example if $k > 0$ and $l < 0$ we have

$$\begin{aligned} Z_2^k Z_2^l &= X_2^k Y_2^{-l} = X_2^{k-1}\sigma_2(t_2)Y_2^{-l-1} = \\ &= X_2^{k-1}Y_2^{-l-1}\sigma_2^{-l}(t_2) = X_2^{k-1}Y_2^{-l-1}\lambda q_2^{-l} = \ldots = \\ &= \lambda q_2^{-l}\lambda q_2^{-l-1}\ldots\lambda q_2^{-l-(m-1)}Z_2^{k+l} = \\ &= \lambda^m q_2^{-lm-m(m-1)/2}Z_2^{k+l}. \end{aligned}$$

□

Lemma 6.11. *Let $k, l \in \mathbb{Z}$ and let $m = \min\{|k|, |l|\}$. Then, as operators on $M_{\mathfrak{m}}$,*

$$Z_1^k Z_1^l = Z_1^l Z_1^k, \tag{6.17}$$

and

$$Z_2^k Z_2^l = c(k,l) Z_2^l Z_2^k, \tag{6.18}$$

where

$$c(k,l) = \begin{cases} 1, & kl \geq 0, \\ q_2^{(k-l)m-(\operatorname{sgn} k - \operatorname{sgn} l)m^2/2}, & kl < 0. \end{cases} \tag{6.19}$$

Proof. Follows directly from Lemma 6.10. □

Lemma 6.12. *Let $g = (g_1, g_2) \in \mathbb{Z}^2 = \tilde{G}_{\mathfrak{m}}$ and set $r_g = \varphi_{\mathfrak{m}}(a_g^* a_g)^{-1}$ where $\varphi_{\mathfrak{m}}$ is the projection $R \to R/\mathfrak{m} = K$. Then*

$$r_g = (1-q_1)^{|g_1|}(\lambda^{-1}q_2^{(g_2-1)/2})^{|g_2|} \tag{6.20}$$

and $(a_g)^{-1} = r_g a_g^* = r_g Z_2^{-g_2} Z_1^{-g_1}$ as operators on $M_{\mathfrak{m}}$.

Proof. We have

$$a_g^* a_g = (Z_1^{g_1} Z_2^{g_2})^* Z_1^{g_1} Z_2^{g_2} = Z_2^{-g_2} Z_1^{-g_1} Z_1^{g_1} Z_2^{g_2} = Z_1^{-g_1} Z_1^{g_1} Z_2^{-g_2} Z_2^{g_2},$$

by Lemma 6.9. Thus by Lemma 6.10,

$$\varphi_{\mathfrak{m}}(a_g^* a_g) = (1-q_1)^{-|g_1|}\lambda^{|g_2|}q_2^{(1-2g_2+g_2)|g_2|/2}$$

which proves the formula. The last statement is immediate. □

We consider the three subcases corresponding to the rank of the free abelian group K_2.

6.5.1. *The case* $\mathfrak{m} = \mathfrak{n}_\lambda^{(2)}, \lambda \neq 0, \operatorname{rank} K_2 = 0$. $G_{\mathfrak{m}} = K_2 = \{0\}$ so $B_{\mathfrak{m}}^{(1)} = R$ which is commutative, hence $M_{\mathfrak{m}} = \mathbb{C}v_0$ for some v_0, and $S = \mathbb{Z}^2$. Thus $C = \{a_g v_0 \mid g \in \mathbb{Z}^2\}$ is a basis for M and using Lemma 6.10 and Lemma 6.9 we obtain that the action of X_i is given by

$$\begin{aligned} X_1 a_g v_0 &= \begin{cases} a_{g+e_1}v_0, & g_1 \geq 0, \\ (1-q_1)^{-1}a_{g+e_1}v_0, & g_1 < 0, \end{cases} \\ X_2 a_g v_0 &= \begin{cases} (q_1\lambda_{12})^{-g_1}a_{g+e_2}v_0, & g_2 \geq 0, \\ (q_1\lambda_{12})^{-g_1}\lambda q_2^{-g_2}a_{g+e_2}v_0, & g_2 < 0. \end{cases} \end{aligned} \tag{6.21}$$

The action of Y_i on the basis is deduced uniquely from

$$\begin{aligned} Y_1 X_1 a_g v_0 &= (1-q_1)^{-1}a_g v_0, \\ Y_2 X_2 a_g v_0 &= \lambda q_1^{-g_1} q_2^{-g_2} a_g v_0, \end{aligned} \tag{6.22}$$

which hold by (6.13) and (6.14).

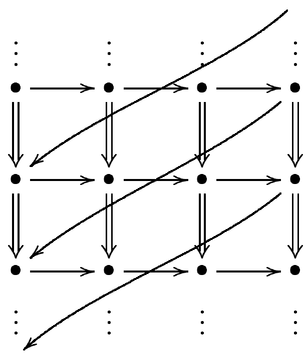


FIGURE 1. Example of a weight diagram for M when $\mathfrak{m} = \mathfrak{n}_\lambda^{(2)}$, $\lambda \neq 0$ and $\operatorname{rank} K_2 = 1$. Here $a = 4$, $b = -2$. The action of X_1 is indicated by $\rightarrow$ arrows, while $\Rightarrow$ arrows are used for X_2.

6.5.2. *The case* $\mathfrak{m} = \mathfrak{n}_\lambda^{(2)}, \lambda \neq 0, \operatorname{rank} K_2 = 1$. Let (a, b) be a basis element. Since $G_\mathfrak{m} = K_2$ which is of rank one, $B_\mathfrak{m}^{(1)} \simeq \mathbb{C}[T, T^{-1}]$ by Corollary 4.6 so $M_\mathfrak{m}$ is one-dimensional. As before we let $M_\mathfrak{m} = \mathbb{C}v_0$. Then $Z_1^a Z_2^b v_0 = \rho v_0$ for some $\rho \in \mathbb{C}^*$.

We assume $a \neq 0$. The case $b \neq 0$ can be treated similarly. By changing basis, we can assume that $a > 0$. Choose $S = \{0, 1, \ldots, a-1\} \times \mathbb{Z}$. The corresponding basis for M is

$$C = \{w_{ij} := X_1^i Z_2^j v_0 \mid 0 \leq i \leq a-1, j \in \mathbb{Z}\}.$$

We now aim to apply Theorem 5.4. If $0 \leq i < a-1$ then clearly $X_1 w_{ij} = w_{i+1,j}$. And

$$X_1 w_{a-1,j} = X_1^a Z_2^j v_0 \in \mathbb{C} Z_2^{j-b} v_0 = \mathbb{C} w_{0,j-b}.$$

We want to compute the coefficient of $w_{0,j-b}$. Similarly to the proof of Theorem 5.4 we have, using Lemma 6.12, Lemma 6.9 and (6.16),

$$\begin{aligned} X_1 w_{a-1,j} = Z_1^a Z_2^j v_0 &= (Z_1^a Z_2^j r_{(a,b)} Z_2^{-b} Z_1^{-a}) Z_1^a Z_2^b v_0 = \\ &= r_{(a,b)} (q_1 \lambda_{12})^{ja} q_1^{a \cdot \overline{-b}} \lambda_{12}^{-ab} Z_2^j Z_2^{-b} Z_1^a Z_1^{-a} \rho v_0 = \\ &= (\lambda^{-1} q_2^{(b-1)/2})^{|b|} q_1^{a(j+\overline{-b})} \lambda_{12}^{a(j-b)} \rho C_0 w_{0,j-b}, \end{aligned}$$

where

$$C_0 = \begin{cases} 1, & b \leq 0, \\ \lambda^{\min\{j,b\}} q_2^{(1+2b-\min\{j,b\})\min\{j,b\}/2}, & b > 0. \end{cases}$$

Using Lemma 6.9 one easily get the action of X_2 on the basis. We conclude that

$$\begin{aligned} X_1 w_{ij} &= \begin{cases} w_{i+1,j}, & 0 \leq i < a-1, \\ (\lambda^{-1} q_2^{(b-1)/2})^{|b|} q_1^{a(j+\overline{-b})} \lambda_{12}^{a(j-b)} \rho C_0 w_{0,j-b}, & i = a-1, \end{cases} \\ X_2 w_{ij} &= \begin{cases} q_1^{-i} \lambda_{21}^i w_{i,j+1}, & j \geq 0, \\ q_1^{-i} \lambda_{21}^i \lambda q_2^j w_{i,j+1}, & j < 0. \end{cases} \end{aligned} \tag{6.23}$$

The action of the Y_i is uniquely determined by

$$\begin{aligned} Y_1 X_1 v_{ij} &= (1 - q_1)^{-1} v_{ij}, \\ Y_2 X_2 v_{ij} &= \lambda q_1^{-i} q_2^{-j} v_{ij}, \end{aligned} \tag{6.24}$$

which hold by (6.13)–(6.14). See Figure 1 for a visual representation.

6.5.3. *The case* $\mathfrak{m} = \mathfrak{n}_\lambda^{(2)}, \lambda \neq 0, \operatorname{rank} K_2 = 2$. Let $s_1 = \mathbf{a} = (a_1, a_2)$, $s_2 = \mathbf{b} = (b_1, b_2)$ be a basis for $G_\mathfrak{m} = K_2$ over $\mathbb{Z}$. We can assume that $a_1, b_1 \geq 0$ and that $d := \left|\begin{smallmatrix} a_1 & b_1 \\ a_2 & b_2 \end{smallmatrix}\right| > 0$.

By Corollary 4.6, $B_\mathfrak{m}^{(1)} \simeq T_\nu$ for some ν which we will now determine. Using Lemma 6.9 and Lemma 6.11 we have, as operators on $M_\mathfrak{m}$,

$$\begin{aligned} Z_1^{a_1} Z_2^{a_2} Z_1^{b_1} Z_2^{b_2} &= q_1^{-b_1 \overline{a_1}} \lambda_{12}^{-b_1 a_2} c(a_2, b_2) Z_1^{b_1} Z_1^{a_1} Z_2^{b_2} Z_2^{a_2} = \\ &= q_1^{a_1 \overline{b_2} - b_1 \overline{a_2}} \lambda_{12}^{a_1 b_2 - b_1 a_2} c(a_2, b_2) Z_1^{b_1} Z_2^{b_2} Z_1^{a_1} Z_2^{a_2}. \end{aligned}$$

We conclude that $B_\mathfrak{m}^{(1)} \simeq T_\nu$ where

$$\nu = \lambda_{12}^d q_1^{a_1 \overline{b_2} - b_1 \overline{a_2}} c(a_2, b_2). \tag{6.25}$$

The function c was defined in (6.19), $d = a_1 b_2 - b_1 a_2$ and $\overline{k} := \max\{0, k\}$ for $k \in \mathbb{Z}$. For $M_\mathfrak{m}$ to be finite-dimensional it is thus necessary that this ν is a root of unity. Assume this and let r denote its order. Then $\dim M_\mathfrak{m} = r$. Let

$$\{v_0, v_1, \dots, v_{r-1}\} \tag{6.26}$$

be a basis such that

$$Z_1^{a_1} Z_2^{a_2} v_j = \nu^j \rho v_j, \tag{6.27}$$

$$Z_1^{b_1} Z_2^{b_2} v_j = \begin{cases} v_{j+1}, & 0 \leq j < r-1, \\ \mu v_0, & j = r-1, \end{cases} \tag{6.28}$$

where $\rho, \mu \in \mathbb{C}^*$.

The next step is to determine a set $S \subseteq \tilde{G}_\mathfrak{m} = \mathbb{Z}^2$ of representatives for the set of cosets $\tilde{G}_\mathfrak{m}/G_\mathfrak{m} = \mathbb{Z}^2/K_2$ which makes it possible to write down the action of the algebra later. We proceed as follows.

Recall that $K_2 = \mathbb{Z} \cdot (a_1, a_2) \oplus \mathbb{Z} \cdot (b_1, b_2)$. Let d_1 be the smallest positive integer such that $(d_1, 0) \in K_2$. We claim that $d_1 = d / \operatorname{GCD}(a_2, b_2)$. Indeed d_1 must be of the form $ka_1 + lb_1$ where $k, l \in \mathbb{Z}$ and $ka_2 + lb_2 = 0$ with $\operatorname{GCD}(k, l) = 1$. For such k, l, $k | b_2$, $l | a_2$ and $b_2/k = -a_2/l =: p > 0$. Then $\operatorname{GCD}(a_2/p, b_2/p) = 1$ which implies that $\operatorname{GCD}(a_2, b_2) = p$. Thus $d_1 = ka_1 + lb_1 = (b_2 a_1 - a_2 b_1)/p = d/\operatorname{GCD}(a_2, b_2)$ as claimed.

Next, let d_2 denote the smallest positive integer such that some K_2-translation of $(0, d_2)$ lies on the x-axis, i.e. such that

$$\big((0, d_2) + K_2\big) \cap \mathbb{Z} \times \{0\} \neq \emptyset.$$

Such an integer exists because if we write $\operatorname{GCD}(a_2, b_2) = ka_2 + lb_2$, then

$$(0, ka_2 + lb_2) - k(a_1, a_2) - l(b_1, b_2) = (-ka_1 - lb_1, 0).$$

On the other hand if $(0, d_2) + k\mathbf{a} + l\mathbf{b} \in \mathbb{Z} \times \{0\}$, i.e. if $d_2 = ka_2 + lb_2$, then $\operatorname{GCD}(a_2, b_2) | d_2$. Therefore $d_2 = \operatorname{GCD}(a_2, b_2)$.

We also see that for any point in $\mathbb{Z}^2$ of the form (x, d_2) there is a $g \in K_2$ such that $(x, d_2) + g \in \mathbb{Z} \times \{0\}$. Also, $(d_1, 0) \in K_2$ so for any point of the form (d_1, y) there is a $g \in K_2$ (namely $(-d_1, 0)$) such that $(d_1, y) + g \in \{0\} \times \mathbb{Z}$.

Suppose now that for some $k, l \in \mathbb{Z}$,

$$k(a_1, a_2) + l(b_1, b_2) \in K_2 \cap \{0, 1, \dots, d_1 - 1\} \times \{0, 1, \dots, d_2 - 1\}.$$

Then we would have $(0, ka_2 + lb_2) - (k\mathbf{a} + l\mathbf{b}) \in \mathbb{Z} \times \{0\}$ and $ka_2 + lb_2 \in \{0, 1, \dots, d_2 - 1\}$ which contradicts the minimality of d_2 unless $ka_2 + lb_2 = 0$. But in this case $(ka_1 + lb_1, 0) \in K_2$ which contradicts the minimality of d_1 unless $ka_1 + lb_1 = 0$. Hence $K_2 \cap \{0, 1, \dots, d_1 - 1\} \times \{0, 1, \dots, d_2 - 1\} = \{(0, 0)\}$. We have shown that

$$S := \{0, 1, \dots, d_1 - 1\} \times \{0, 1, \dots, d_2 - 1\}$$

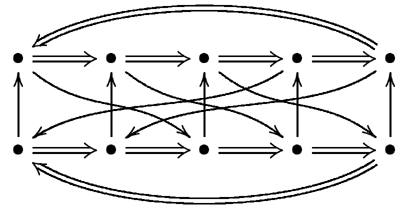


FIGURE 2. An example of the action on $\operatorname{supp}(M)$ when $\mathfrak{m} = \mathfrak{n}_\lambda^{(2)}$, $\lambda \neq 0$ and $\operatorname{rank} K_2 = 2$. Here $\mathbf{a} = (2, -2)$, $\mathbf{b} = (3, 2)$, $d_1 = 5, d_2 = 2$ and $s = 2$. The $\Rightarrow$ arrows indicate the action of X_1 and the $\to$ arrows show the action of X_2.

is a set of representatives for $\mathbb{Z}^2/K_2$. In particular we get from Corollary 5.3 that $\dim M$ is finite and

$$\dim M / \dim M_\mathfrak{m} = |S| = d_1 d_2 = a_1 b_2 - b_1 a_2.$$

We fix now integers a_2', b_2' such that

$$d_2 = \mathrm{GCD}(a_2, b_2) = a_2' a_2 + b_2' b_2 \tag{6.29}$$

and such that $-a_2' a_1 - b_2' b_1 \in \{0, 1, \ldots, d_1 - 1\}$. This can be done because for any $p \in \mathbb{Z}$, $(a_2'', b_2'') := (a_2' + pb_2/d_2, b_2' - pa_2/d_2)$ also satisfies $a_2'' a_2 + b_2'' b_2 = d_2$ but now

$$-a_2'' a_1 - b_2'' b_1 = -(a_2' + pb_2/d_2) a_1 - (b_2' - pa_2/d_2) b_1 = -a_2' a_1 - b_2' b_1 - pd_1.$$

We set

$$s = -a_2' a_1 - b_2' b_1. \tag{6.30}$$

Let $(i, j) \in S$. We have the following reductions in $\mathbb{Z}^2$ modulo K_2.

$$(1,0) + (i,j) = \begin{cases} (i+1, j), & 0 \leq i < d_1 - 1, \\ (0, j), & i = d_1 - 1, \end{cases}$$

$$(0,1) + (i,j) = \begin{cases} (i, j+1), & 0 \leq j < d_2 - 1, \\ (i+s, 0), & j = d_2 - 1, i + s \leq d_1 - 1, \\ (i+s-d_1, 0), & j = d_2 - 1, j + s > d_1 - 1. \end{cases}$$

From this we can understand how the X_i act on the support of M, see Figure 2 for an example. By Theorem 5.1 the set

$$C = \{ w_{ijk} := X_1^i X_2^j v_k \mid 0 \leq i < d_1, 0 \leq j < d_2, 0 \leq k < r \}$$

is a basis for M where v_k is the basis (6.26) for $M_\mathfrak{m}$.

If $0 \leq i < d_1 - 1$ we clearly have $X_1 w_{ijk} = w_{i+1,j,k}$. Suppose $i = d_1 - 1$. Then by Lemma 6.9,

$$X_1 w_{ijk} = X_1^{d_1} X_2^j v_k = q_1^{d_1 j} \lambda_{12}^{d_1 j} X_2^j X_1^{d_1} v_k.$$

Thus we must express $X_1^{d_1}$ in terms of $Z_1^{a_1} Z_2^{a_2}$ and $Z_1^{b_1} Z_2^{b_2}$. Since $(d_1, 0) = b_2/d_2 \mathbf{a} - a_2/d_2 \mathbf{b}$ we have

$$(Z_1^{a_1} Z_2^{a_2})^{b_2/d_2} (Z_1^{b_1} Z_2^{b_2})^{-a_2/d_2} = C_1^{-1} X_1^{d_1} \tag{6.31}$$

as operators on $M_\mathfrak{m}$ for some constant C_1^{-1} which we must calculate.

Lemma 6.13. *The constant C_1 defined in (6.31) is given by*

$$C_1^{-1} = r_{\mathbf{a}}^{\overline{-b_2/d_2}} (q_1^{-a_1 \overline{a_2}} \lambda_{12}^{-a_1 a_2})^{\frac{b_2}{d_2}(\frac{b_2}{d_2} - 1)/2} \cdot r_{\mathbf{b}}^{\overline{a_2/d_2}} (q_1^{-b_1 \overline{b_2}} \lambda_{12}^{-b_1 b_2})^{\frac{a_2}{d_2}(\frac{a_2}{d_2} + 1)/2} \cdot \\ \cdot q_1^{b_1 a_2 \overline{a_2 b_2}/d_2^2} \lambda_{12}^{b_1 a_2^2 b_2 / d_2^2} r_{(0, -b_2 a_2/d_2)}^{-1} C_1', \tag{6.32}$$

where the r_g, $g \in \mathbb{Z}^2$ *are given by (6.20),*

$$C_1' = \begin{cases} (1-q_1)^{-\min\{|a_1b_2/d_2|,|b_1a_2/d_2|\}}, & a_2b_2 > 0, \\ 1, & a_2b_2 \le 0, \end{cases}$$

$\overline{k} = \max\{0, k\}$ *for* $k \in \mathbb{Z}$ *and* $d_2 = \mathrm{GCD}(a_2, b_2)$.

Proof. If $b_2 \ge 0$ for example, we have by Lemma 6.9

$$\begin{aligned} (Z_1^{a_1} Z_2^{a_2})^{b_2/d_2} &= q_1^{-a_1\overline{a_2}} \lambda_{12}^{-a_1a_2} \cdot (q_1^{-a_1\overline{a_2}} \lambda_{12}^{-a_1a_2})^2 \cdot \ldots \\ &\ldots \cdot (q_1^{-a_1\overline{a_2}} \lambda_{12}^{-a_1a_2})^{b_2/d_2-1} Z_1^{a_1b_2/d_2} Z_2^{a_2b_2/d_2} = \\ &= (q_1^{-a_1\overline{a_2}} \lambda_{12}^{-a_1a_2})^{\frac{b_2}{d_2}(\frac{b_2}{d_2}-1)/2} Z_1^{a_1b_2/d_2} Z_2^{a_2b_2/d_2} \end{aligned}$$

When $b_2 < 0$ we get a similar calculation where $r_{\mathbf{a}}^{-b_2/d_2}$ appears by Lemma 6.12. $(Z_1^{b_1} Z_2^{b_2})^{-a_2/d_2}$ can analogously be expressed as a multiple of $Z_1^{-b_1a_2/d_2} Z_2^{-b_2a_2/d_2}$. We then commute $Z_2^{a_2b_2/d_2}$ and $Z_1^{-b_1a_2/d_2}$ using Lemma 6.9. As a last step we use Lemma 6.10 and obtain two more factors. □

We conclude that

$$X_1 w_{ijk} = \begin{cases} w_{i+1,j,k}, & i < d_1 - 1, \\ q_1^{jd_1} \lambda_{12}^{jd_2} C_1 \nu^{b_2/d_2 k_1''} \rho^{b_2/d_2} \mu^{k_1'} w_{0,j,k_1''}, & i = d_1 - 1. \end{cases}$$

Here

$$k - a_2/d_2 = rk_1' + k_1'' \quad \text{with } 0 \le k_1'' < r. \tag{6.33}$$

Next we turn to the description of how X_2 acts on the basis C. If $0 \le j < d_2 - 1$ we have $X_2 w_{ijk} = q_1^{-i} \lambda_{12}^{-i} w_{i,j+1,k}$ by Lemma 6.9. Suppose $j = d_2 - 1$. Then, as in the first step of the proof of Theorem 5.4,

$$X_2 w_{ijk} = q_1^{-i} \lambda_{12}^{-i} X_1^i X_2^{d_2} v_k = q_1^{-i} \lambda_{12}^{-i} X_1^i (X_2^{d_2} r_{(-s,d_2)} Z_2^{-d_2} Z_1^s)(Z_1^{-s} Z_2^{d_2}) v_k. \tag{6.34}$$

By (6.16) and (6.20),

$$\begin{aligned} X_2^{d_2} r_{(-s,d_2)} Z_2^{-d_2} Z_1^s &= r_{(-s,d_2)} r_{(0,-d_2)}^{-1} Z_1^s = \\ &= (1-q_1)^s (\lambda^{-1} q_2^{(d_2-1)/2})^{d_2} (\lambda^{-1} q_2^{(-d_2-1)/2})^{d_2} Z_1^s = \\ &= (1-q_1)^s (\lambda^2 q_2)^{-d_2} Z_1^s. \end{aligned} \tag{6.35}$$

We must express $Z_1^{-s} Z_2^{d_2}$ in the generators of the algebra $B_{\mathfrak{m}}^{(1)}$ in order to calculate its action on v_k.

$$(Z_1^{a_1} Z_2^{a_2})^{a_2'} (Z_1^{b_1} Z_2^{b_2})^{b_2'} = C_2^{-1} Z_1^{-s} Z_2^{d_2}, \tag{6.36}$$

for some $C_2 \in \mathbb{C}^*$ since the degree on both sides are equal by (6.29) and (6.30). Similarly to the proof of Lemma 6.13,

$$\begin{aligned} C_2^{-1} = r_{\mathbf{a}}^{\overline{-a_2'}} (q_1^{-a_1\overline{a_2}} \lambda_{12}^{-a_1a_2})^{a_2'(a_2'-1)/2} \cdot r_{\mathbf{b}}^{\overline{-b_2'}} (q_1^{-b_1\overline{b_2}} \lambda_{12}^{-b_1b_2})^{b_2'(b_2'-1)/2} \cdot \\ \cdot q_1^{-b_1b_2'\overline{a_2a_2'}} \lambda^{-b_1b_2'a_2a_2'} C_2' C_2'', \end{aligned} \tag{6.37}$$

and

$$C_2' = \begin{cases} 1, & a_2'b_2' \ge 0, \\ (1-q_1)^{-\min\{|a_1a_2'|,|b_1b_2'|\}}, & a_2'b_2' < 0, \end{cases}$$

$$C_2'' = \begin{cases} 1, & a_2a_2'b_2b_2' \ge 0, \\ \lambda^{m'} q_2^{(1-2b_2b_2'+(\operatorname{sgn} b_2b_2')m')m'/2}, & a_2a_2'b_2b_2' < 0, \end{cases}$$

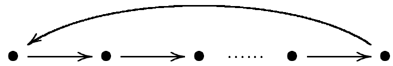


FIGURE 3. Weight diagram when $\mathfrak{m} = \mathfrak{n}_\lambda^{(2)}$, $\lambda \neq 0$, $\operatorname{rank} K_2 = 2$ and $q_1 = q_2$.

where $m' = \min\{|a_2 a_2'|, |b_2 b_2'|\}$. Furthermore, letting

$$b_2' + k = r k_2' + k_2'', \quad \text{where } 0 \le k_2'' < r \tag{6.38}$$

we have by (6.27)–(6.28),

$$(Z_1^{a_1} Z_2^{a_2})^{a_2'} (Z_1^{b_1} Z_2^{b_2})^{b_2'} v_k = \nu^{a_2' k_2''} \rho^{a_2'} \mu^{k_2'} v_{k_2''}. \tag{6.39}$$

If $i + s \le d_1 - 1$ we can now write down the action of X_2 on w_{ijk} by combining (6.34)–(6.37), (6.39) to get a multiple of $w_{i+s,0,k_2''}$. However if $i + s > d_1 - 1$, we must reduce further because then $(i + s, 0) \notin S$. Let

$$k_2'' - a_2/d_2 = r k_3' + k_3'', \quad \text{where } 0 \le k_3'' < r. \tag{6.40}$$

Then by the calculations for the action of $X_1^{d_1}$ on $M_{\mathfrak{m}}$,

$$X_1^{d_1} v_{k_2''} = X_1^{i+s-d_1} X_1^{d_1} v_{k_2''} = C_1 \mu^{k_3'} \nu^{k_3'' b_2/d_2} \rho^{b_2/d_2} w_{i+s-d_1,0,k_3''}.$$

Summing up, M has a basis

$$\{w_{ijk} \mid 0 \le i < d_1, 0 \le j < d_2, 0 \le k < r\}$$

and X_1, X_2 act on this basis as follows.

(6.41)

$$X_1 w_{ijk} = \begin{cases} w_{i+1,j,k}, & i < d_1 - 1, \\ q_1^{jd_1} \lambda_{12}^{jd_2} C_1 \nu^{b_2/d_2 k_1''} \rho^{b_2/d_2} \mu^{k_1'} w_{0,j,k_1''}, & i = d_1 - 1. \end{cases}$$

$$X_2 w_{ijk} = (q_1 \lambda_{12})^{-i} \cdot \begin{cases} w_{i,j+1,k}, \\ \qquad \text{if } 0 \le j < d_2 - 1, \\ (1-q_1)^s (\lambda^2 q_2)^{-d_2} C_2 \nu^{a_2' k_2''} \rho^{a_2'} \mu^{k_2'} w_{i+s,0,k_2''}, \\ \qquad \text{if } j = d_2 - 1 \text{ and } i + s \le d_1 - 1, \\ (1-q_1)^s (\lambda^2 q_2)^{-d_2} C_2 \nu^{a_2' k_2'' + k_3'' b_2/d_2} \rho^{a_2' + b_2/d_2} \mu^{k_2' + k_3'} C_1 w_{i+s-d_1,0,k_3''}, \\ \qquad \text{if } j = d_2 - 1 \text{ and } i + s > d_1 - 1, \end{cases}$$

where C_1 is given by (6.32), C_2 by (6.37) and ν by (6.25). The parameters ρ and μ comes from the action (6.27), (6.28) of $B_{\mathfrak{m}}^{(1)}$ on $M_{\mathfrak{m}}$ and k_i', k_i'' are defined in (6.33), (6.38) and (6.40).

The action of the Y_i is uniquely determined by

$$\begin{aligned} Y_1 X_1 w_{ijk} &= (1 - q_1)^{-1} w_{ijk}, \\ Y_2 X_2 w_{ijk} &= \lambda q_1^{-i} q_2^{-j} w_{ijk}. \end{aligned} \tag{6.42}$$

We remark that the case $q_1 = q_2$ corresponds to $\mathbf{a} = (a_1, a_2) = (1, -1)$. Then $d_2 = 1$, $d_1 = d = |b_1 + b_2|$ and $s = 1$. X_1 and X_2 will act on the support in the same direction, cyclically as in Figure 3. The explicit action can be deduced from the above more general case noting that here $k_2'' = k$, $k_2' = 0$ and

$$k_1' = k_3' = \begin{cases} 0, & k < r - 1, \\ 1, & k = r - 1, \end{cases} \qquad k_1'' = k_3'' = \begin{cases} k, & k < r - 1, \\ 0, & k = r - 1. \end{cases}$$

6.6. **The case $\mathfrak{m} \notin \{\mathfrak{n}_\mu^{(i)} \mid \mu \in \mathbb{C}, i = 1,2\}$.** This is the generic case. We have $\mathbb{Z}^2_{\mathfrak{m}} = Q$ by Corollary 6.8. Our statements here generalize without any problem to the case of arbitrary rank.

Assume first that the q_i are roots of unity of orders o_i ($i = 1,2$) and that ω does not contain any 1-breaks or 2-breaks. Then by Corollary 6.2 and Proposition 6.4 we have $\tilde{G}_{\mathfrak{m}} = \mathbb{Z}^2$. Thus $G_{\mathfrak{m}} = (o_1\mathbb{Z}) \times (o_2\mathbb{Z})$. Moreover,

$$X_1^{o_1} X_2^{o_2} = \lambda_{12}^{o_1 o_2} X_2^{o_2} X_1^{o_1}$$

so $B_{\mathfrak{m}}^{(1)} \simeq T_{\lambda_{12}^{o_1 o_2}}$ by Corollary 4.6. This algebra has only finite dimensional representations if $\lambda_{12}^{o_1 o_2}$ is a root of unity. Assuming this, let r be the order of $\lambda_{12}^{o_1 o_2}$. Then there are $\rho, \mu \in \mathbb{C}^*$ and $M_{\mathfrak{m}}$ has a basis $v_0, v_1, \ldots, v_{r-1}$ such that

$$\begin{aligned} X_1^{o_1} v_i &= \lambda_{12}^{i o_1 o_2} \rho v_i \\ X_2^{o_1} v_i &= \begin{cases} v_{i+1} & 0 \le i < p-1 \\ \mu v_0 & i = p-1 \end{cases} \end{aligned}$$

Choose $S = \{0,1,\ldots,o_1-1\} \times \{0,1,\ldots,o_2-1\}$. The corresponding basis for M is $C = \{w_{ijk} := X_1^i X_2^j v_k \mid 0 \le i < o_1, 0 \le j < o_2, 0 \le k < r\}$. The following formulas are easily deduced using (2.6)–(2.8).

$$\begin{aligned} X_1 w_{ijk} &= \begin{cases} w_{i+1,j,k}, & k < o_1 - 1, \\ \lambda_{12}^{o_1(o_2 k + j)} \rho w_{0jk}, & k = o_1 - 1, \end{cases} \\ X_2 w_{ijk} &= (q_1 \lambda_{12})^{-i} \cdot \begin{cases} w_{i,j+1,l}, & l < o_2 - 1, \\ w_{i,0,l+1}, & l = o_2 - 1, i < r-1, \\ \mu w_{i00}, & l = o_2 - 1, i = r-1. \end{cases} \end{aligned} \tag{6.43}$$

The action of Y_1, Y_2 is determined by

$$\begin{aligned} Y_1 X_1 w_{ijk} &= q_1^{-i}(\alpha_1 - [i]_{q_1}) w_{ijk}, \\ Y_2 X_2 w_{ijk} &= q_1^{-i} q_2^{-j}(\alpha_2 - [j]_{q_2}(1 + (q_1 - 1)\alpha_1)) w_{ijk}. \end{aligned} \tag{6.44}$$

In all other cases one can show using the same argument that $\dim M_{\mathfrak{n}} = 1$ for all $\mathfrak{n} \in \operatorname{supp}(M)$ and that M can be realized in a vector space with basis $\{w_{ij}\}_{(i,j)\in I}$, where $I = I_1 \times I_2$ is one of the following sets

$$\begin{aligned} &\mathbb{N}_{d_1} \times \mathbb{N}_{d_2}, \quad \mathbb{N}_{d_1} \times \mathbb{Z}^{\pm}, \quad \mathbb{Z}^{\pm} \times \mathbb{N}_{d_2}, \quad \mathbb{Z} \times \mathbb{Z}, \\ &\mathbb{Z}^{\pm} \times \mathbb{Z}, \quad \mathbb{Z} \times \mathbb{Z}^{\pm}, \quad \mathbb{Z}^{\pm} \times \mathbb{Z}^{\pm}, \quad \mathbb{Z}^{\pm} \times \mathbb{Z}^{\mp}, \end{aligned}$$

where $\mathbb{N}_d = \{0,1,\ldots,d-1\}$, $\mathbb{Z}^{\pm} = \{k \in \mathbb{Z} \mid \pm k \ge 0\}$ and d_i is the order of q_i if finite. The action of the generators is given by the following formulas.

$$\begin{aligned} X_1 w_{ij} &= \begin{cases} w_{i+1,j}, & (i+1,j) \in I, \\ \rho \lambda_{12}^{d_1 j} w_{0,j}, & (i+1,j) \notin I, I_1 = \mathbb{N}_{d_1} \text{ and } \alpha_1 \neq [i]_{q_1}, \\ 0, & \text{otherwise}, \end{cases} \\ X_2 w_{ij} &= (q_1 \lambda_{12})^{-i} \cdot \begin{cases} w_{i,j+1}, & (i,j+1) \in I, \\ \mu w_{i,0}, & (i,j+1) \notin I, I_2 = \mathbb{N}_{d_2} \\ & \quad \text{and } \alpha_2 \neq [j]_{q_2}(1 + (q_1 - 1)\alpha_1), \\ 0, & \text{otherwise}, \end{cases} \end{aligned} \tag{6.45}$$

$$Y_1 w_{ij} = q_1^{-i+1}(\alpha_1 - [i-1]_{q_1})\cdot$$
$$\cdot\begin{cases} w_{i-1,j}, & (i-1,j)\in I, \\ (\rho\lambda_{12}^{d_1 j})^{-1} w_{d_1-1,j}, & (i-1,j)\notin I, I_1=\mathbb{N}_{d_1} \text{ and } \alpha_1 \neq [i-1]_{q_1}, \\ 0, & \text{otherwise,} \end{cases}$$

$$Y_2 w_{ij} = \lambda_{12}^{-i} q_2^{-j+1}(\alpha_2 - [j-1]_{q_2}(1+(q_1-1)\alpha_1))\cdot \tag{6.46}$$
$$\cdot\begin{cases} w_{i,j+1}, & (i,j+1)\in I, \\ \mu^{-1} w_{i,d_2-1}, & (i,j+1)\notin I, I_1=\mathbb{N}_{d_2} \\ & \quad \text{and } \alpha_2 \neq [j-1]_{q_2}(1+(q_1-1)\alpha_1), \\ 0, & \text{otherwise.} \end{cases}$$

Thus we have proved the following result.

Theorem 6.14. *Let A be a quantized Weyl algebra of rank two with arbitrary parameters $q_1, q_2 \in \mathbb{C}\backslash\{0,1\}$. Then any simple weight A-module with no proper inner breaks is isomorphic to one of the modules defined by formulas (6.6), (6.7), (6.8), (6.9), (6.10), (6.11), (6.21-6.22), (6.23-6.24), (6.41-6.42), (6.43-6.44) or (6.45-6.46).*

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