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Locally Finite Simple Weight Modules over Twisted Generalized Weyl Algebras

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Abstract. We present methods and explicit formulas for describing simple
weight modules over twisted generalized Weyl algebras. When a certain com-
mutative subalgebra is finitely generated over an algebraically closed field we
obtain a classification of a class of locally finite simple weight modules as those
induced from simple modules over a subalgebra isomorphic to a tensor product
of noncommutative tori. As an application we describe simple weight modules
over the quantized Weyl algebra.

1. Introduction

Inspired by [10], Bavula defined in [2], [1] the notion of a generalized Weyl algebra
(GWA) which is a class of algebras which include $U(\mathfrak{sl}(2))$, $U_q(\mathfrak{sl}(2))$, down-up
algebras, and the Weyl algebra, as examples. In addition to various ring theoretic
properties, the simple modules were also described for some GWAs in [2]. In [6] all
simple and indecomposable weight modules of GWAs of rank (or degree) one were
classified.

Higher rank GWAs were defined in [2] as tensor products of rank one GWAs.
This has some consequences on the side of representations. In [3] the authors
studied indecomposable weight modules over certain higher rank GWAs.

In [8], with the goal to enrich the representation theory in the higher rank case, the authors
defined the twisted generalized Weyl algebra (TGRA). This is a class
of algebras which include all higher rank GWAs (if a certain subring $R$ has no zero
divisors) and also many algebras which can be viewed as twisted tensor products
of rank one GWAs, for example certain Mickelson step algebras and extended Orthogonally
Gelfand-Zetlin algebras [7]. Under a technical assumption on the algebra
formulated using a bialgebra graph, some torsion-free simple weight modules were
described in [8]. Simple graded weight modules were studied in [7] using an analogue
of the Shapovalov form.

In this paper we describe a more general class of locally finite simple weight
modules over TGWAs using the well-known technique of considering the maximal
graded subalgebra which preserves the weight spaces. It is known that under
quite general assumptions (see Theorem 18 in [5]) any simple weight module over
a TWA is a unique quotient of a module which is induced from a simple module
over this subalgebra. Our main results are the description of this subalgebra un-
der various assumptions (Theorem 4.5 and Theorem 4.8) and the explicit formulas
(Theorem 5.4) of the associated module of the TGRA. In contrast to [8], we do
not assume that the orbits are torsion-free and we allow the modules to have some
inner breaks, as long as they do not have any so called proper inner breaks (see
Definition 3.7). The weight spaces will not in general be one-dimensional in our
case, which was the case in [8], [7].

Moreover, as an application we classify the simple weight modules without proper
inner breaks over a quantized Weyl algebra of rank two (Theorem 6.14).

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The paper is organized as follows. In Section 2 the definitions of twisted generalized Weyl constructions and algebras are given together with some examples. Weight modules and the subalgebra $B(\omega)$ are defined.

In Section 3 we first prove some simple facts and then define the class of simple weight modules with no proper inner breaks. We also show that this class properly contains all the modules studied in [8]. Section 4 is devoted to the description of the subalgebra $B(\omega)$. When the ground field is algebraically closed and a certain subalgebra $R$ is finitely generated, we show that it is isomorphic to a tensor product of noncommutative tori for which the finite-dimensional irreducible representations are easy to describe.

In Section 5 we specify a basis and give explicit formulas for the irreducible quotient of the induced module.

Finally, in Section 6 we consider as an example the quantized Weyl algebra and determine certain important subsets of $Z^n$ related to $B(\omega)$ and the support of modules as solutions to some systems of equations. In the rank two case we describe all simple weight modules with finite-dimensional weight spaces and no proper inner breaks.

2. Definitions

2.1. The TGWC and TGWA. Fix a positive integer $n$ and set $\mathbb{N} = \{1, 2, \ldots, n\}$. Let $K$ be a field, and let $R$ be a commutative unital $K$-algebra, $\sigma = (\sigma_1, \ldots, \sigma_n)$ be an $n$-tuple of pairwise commuting $K$-automorphisms of $R$, $\mu = (\mu_{ij})_{i,j \in \mathbb{N}}$ be a matrix with entries from $K^* := K \setminus \{0\}$ and $t = (t_1, \ldots, t_n)$ be an $n$-tuple of nonzero elements from $R$. The twisted generalized Weyl construction (TGWC) $A'$ obtained from the data $(R, \sigma, t, \mu)$ is the unital $K$-algebra generated over $R$ by $X_i, Y_i, (i \in \mathbb{N})$ with the relations

$$\begin{align*}
(2.1) & \quad X_i^r = \sigma_i(r)X_i, & Y_i^r = \sigma_i^{-1}(r)Y_i, & \text{for } r \in R, i \in \mathbb{N} \\
(2.2) & \quad Y_iX_i = t_i, & X_iY_i = \sigma_i(t_i), & \text{for } i \in \mathbb{N} \\
(2.3) & \quad X_iX_j = \mu_{ij}^{-1}X_jX_i, & & \text{for } i, j \in \mathbb{N}, i \neq j.
\end{align*}$$

From the relations (2.1)−(2.3) follows that $A'$ carries a $\mathbb{Z}^n$-gradation $\{A'_g\}_{g \in \mathbb{Z}^n}$ which is uniquely defined by requiring

$$\deg X_i = e_i, \quad \deg Y_i = -e_i, \quad \deg r = 0, \quad \text{for } i \in \mathbb{N}, r \in R,$$

where $e_i = (0, \ldots, 1, \ldots, 0)$. The twisted generalized Weyl algebra (TGWA) $A = A(R, \sigma, t, \mu)$ of rank $n$ is defined to be $A'/I$, where $I$ is the sum of all graded two-sided ideals of $A'$ intersecting $R$ trivially. Since $I$ is graded, $A$ inherits a $\mathbb{Z}^n$-gradation $\{A_g\}_{g \in \mathbb{Z}^n}$ from $A'$.

Note that from relations (2.1)−(2.3) follows the identity

$$X_iX_jt_i = X_jX_i\mu_{ji}^{-1}(t_i)$$

which holds for $i, j \in \mathbb{N}, i \neq j$. Multiplying (2.4) from the left by $\mu_{ij}Y_j$ we obtain

$$X_i(t_it_j - \mu_{ij}\mu_{ji}^{-1}(t_j)\sigma_i^{-1}(t_i)) = 0$$

for $i, j \in \mathbb{N}, i \neq j$. One can show that the algebra $A'$, hence $A$, is nontrivial if one assumes that $t_it_j = \mu_{ij}\mu_{ji}^{-1}(t_j)\sigma_i^{-1}(t_i)$ for $i, j \in \mathbb{N}, i \neq j$. Analogous identities exist for $Y_i$.

2.2. Examples. Some of the first motivating examples of a generalized Weyl algebra (GWA), i.e. a TGWC of rank 1, are $U(\mathfrak{sl}(2))$, $U_q(\mathfrak{sl}(2))$ and of course the Weyl algebra $A_1$. We refer to [2] for details.

We give some examples of TGWAs of higher rank.
2.2.1. Quantized Weyl algebras. Let $\Lambda = (\Lambda_{ij})$ be an $n \times n$ matrix with nonzero complex entries such that $\Lambda_{ij} = \lambda_{ji}^{-1}$. Let $\vec{q} = (q_1, \ldots, q_n)$ be an $n$-tuple of elements of $\mathbb{C}\setminus\{0,1\}$. The $n$th quantized Weyl algebra $A_{\mathbb{C}}^{\vec{q}, \Lambda}$ is the $\mathbb{C}$-algebra with generators $x_i, y_i, 1 \leq i \leq n$, and relations
\begin{align}
\tag{2.6}
x_i x_j &= q_i \lambda_{ij} x_j x_i, \\
\tag{2.7}
y_i y_j &= \lambda_{ji} y_j y_i, \\
x_i y_j &= \lambda_{ji} y_i x_j,
\end{align}
for $1 \leq i < j \leq n$. Let $R = \mathbb{C}[t_1, \ldots, t_n]$ be the polynomial algebra in $n$ variables and $\sigma_i$ the $\mathbb{C}$-algebra automorphisms defined by
\begin{equation}
\sigma_i(t_j) = \begin{cases} t_j, & j < i, \\
1 + q_i t_i + \sum_{k=1}^{i-1} (q_k - 1) t_k, & j = i, \\
q_i t_j, & j > i.
\end{cases}
\end{equation}
One can check that the $\sigma_i$ commute. Let $\mu = (\mu_{ij})_{i,j} \in \mathbb{C}$ be defined by $\mu_{ij} = \lambda_{ji}$ and $\mu_{ij} = q_i \lambda_{ji}$ for $i < j$. Let also $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $t = (t_1, \ldots, t_n)$. One can show that the maximal graded ideal of the TGWC $A'(R, \sigma, t, \mu)$ is generated by the elements
\[ X_i X_j - q_i \lambda_{ij} X_j X_i, \quad Y_i Y_j - \lambda_{ij} Y_j Y_i, \quad 1 \leq i < j \leq n. \]
Thus $A_{\mathbb{C}}^{\vec{q}, \Lambda}$ is isomorphic to the TGWA $A(R, \sigma, t, \mu)$ via $x_i \mapsto X_i, y_i \mapsto Y_i$.

2.2.2. Qij-CCR. Let $(Q_{ij})_{i,j=1}^d$ be an $d \times d$ matrix with complex entries such that $Q_{ij} = Q_{ji}^{-1}$ if $i \neq j$ and $A_d$ be the algebra generated by elements $a_i, a_i^*, 1 \leq i \leq d$ and relations
\begin{align*}
\tag{2.8}
a_i^* a_i - Q_{ii} a_i a_i^* &= 1, & a_i^* a_j = Q_{ij} a_j a_i^*, \\
\tag{2.9}
a_i a_j &= Q_{ji} a_j a_i, & a_i^* a_j^* = Q_{ji} a_j^* a_i^*,
\end{align*}
where $1 \leq i, j \leq d$ and $i \neq j$. Let $R = \mathbb{C}[t_1, \ldots, t_d]$ and define the automorphisms $\sigma_i$ of $R$ by $\sigma_i(t_j) = t_j$ if $i \neq j$ and $\sigma_i(t_i) = 1 + Q_{ii} t_i$. Let $\mu_{ij} = Q_{ji}$ for all $i, j$. Then $A_d$ is isomorphic to the TGWA $A(R, (\sigma_1, \ldots, \sigma_n), (t_1, \ldots, t_n), \mu)$.

2.2.3. Mickelsson and OGZ algebras. In both the above examples the generators $X_i$ and $X_j$ commute up to a multiple of the ground field. This need not be the case as shown in [7], where it was shown that Mickelsson step algebras and extended orthogonal Gelfand-Zetlin algebras are TGWAs.

2.3. Weight modules. Let $A$ be a TGWC or a TGWA. Let $\text{Max}(R)$ denote the set of all maximal ideals in $R$. A module $M$ over $A$ is called a weight module if
\[ M = \bigoplus_{m \in \text{Max}(R)} M_m, \]
where
\[ M_m = \{ v \in M \mid mv = 0 \}. \]
The support, $\text{supp}(M)$, of $M$ is the set of all $m \in \text{Max}(R)$ such that $M_m \neq 0$. A weight module is locally finite if all the weight spaces $M_m, m \in \text{supp}(M)$, are finite-dimensional over the ground field $K$.

Since the $\sigma_i$ are pairwise commuting, the free abelian group $\mathbb{Z}^n$ acts on $R$ as a group of $K$-algebra automorphisms by
\begin{equation}
\tag{2.10}
g(r) = \sigma_1^{r_1} \sigma_2^{r_2} \ldots \sigma_n^{r_n}(r)
\end{equation}
for \( g = (g_1, \ldots, g_n) \in \mathbb{Z}^n \) and \( r \in R \). Then \( \mathbb{Z}^n \) also acts naturally on \( \text{Max}(R) \) by \( g(m) = \{ g(r) \mid r \in m \} \). Note that

\[
X_m \mathbb{Z}^n \subseteq M_{\sigma^{-1}(m)} \quad \text{and} \quad Y_m \mathbb{Z}^n \subseteq M_{\sigma^{-1}(m)}
\]

for any \( m \in \text{Max}(R) \). If \( A \) is homogeneous of degree \( g \in \mathbb{Z}^n \), then by using (2.1) and (2.11) repeatedly one obtains the very useful identities

\[
a \cdot r = g(r) \cdot a, \quad r \cdot a = a \cdot (-g)(r),
\]

for \( r \in R \) and

\[
a \mathbb{Z}^n \subseteq M_g(m)
\]

for \( m \in \text{Max}(R) \).

2.4. Subalgebras leaving the weight spaces invariant. Let \( \omega \subseteq \text{Max}(R) \) be an orbit under the action of \( \mathbb{Z}^n \) on \( \text{Max}(R) \) defined in (2.10). Let

\[
\mathbb{Z}^n_\omega = \mathbb{Z}^n_m = \{ g \in \mathbb{Z}^n \mid g(m) = m \}
\]

where \( m \) is some point in \( \omega \). Since \( \mathbb{Z}^n \) is abelian, \( \mathbb{Z}^n_\omega \) does not depend on the choice of \( m \) from \( \omega \). Define

\[
B(\omega) = \odot_{g \in \mathbb{Z}^n_\omega} A_g.
\]

Since \( A \) is \( \mathbb{Z}^n \)-graded and since \( \mathbb{Z}^n_\omega \) is a subgroup of \( \mathbb{Z}^n \), \( B(\omega) \) is a subalgebra of \( A \) and \( R = A_0 \subseteq B(\omega) \). Let \( m \in \omega \) and suppose that \( M \) is a simple weight \( A \)-module with \( m \in \text{supp}(M) \). Since \( M \) is simple we have \( \text{supp}(M) \subseteq \omega \). Using (2.13) it follows that \( B(\omega) \mathbb{Z}^n_m \subseteq M_m \) and by definition \( M_m \) is annihilated by \( m \) hence also by the two-sided ideal \( (m) \) in \( B(\omega) \) generated by \( m \). Thus \( M_m \) is naturally a module over the algebra

\[
B_m := B(\omega)/(m).
\]

By Proposition 7.2 in [7] (see also Theorem 18 in [5] for a general result), \( M_m \) is a simple \( B_m \)-module, and any simple \( B_m \)-module occurs as a weight space in a simple weight \( A \)-module. Moreover, two simple weight \( A \)-modules \( M, N \) are isomorphic if and only if \( M_m \) and \( N_m \) are isomorphic as \( B_m \)-modules. Therefore we are led to study the algebra \( B_m \) and simple modules over it.

3. Preliminaries

3.1. Reduced words. Let \( L = \{ X_i \}_{i \in \mathbb{N}} \cup \{ Y_i \}_{i \in \mathbb{N}} \). By a word \((a; Z_1, \ldots, Z_k)\) in \( A \) we will mean an element \( a \) in \( A \) which is a product of elements from the set \( L \), together with a fixed tuple \((Z_1, \ldots, Z_k)\) of elements from \( L \) such that \( a = Z_1 \cdots Z_k \).

When referring to a word we will often write \( a = Z_1 \cdots Z_k \in A \) to denote the word \((a; Z_1, \ldots, Z_k)\) or just write \( a \in A \), suppressing the fixed representation of \( a \) as a product of elements from \( L \).

Set \( X_i^* = Y_i \) and \( Y_i^* = X_i \). For a word \( a = Z_1 \cdots Z_k \in A \) we define

\[
a^* := Z_k^* \cdots Z_{k-1}^* \cdots Z_1^*.
\]

In the special case when \( \mu_{ij} = \mu_{ji} \) for all \( i, j \) then by (2.1)-(2.3) there is an anti-involution \( * \) on \( A \) defined by \( X_i^* = Y_i \), and \( r^* = r \) for \( r \in R \). Since \( I^* = I \) this anti-involution carries over to \( A \).

Definition 3.1. A word \( Z_1 \cdots Z_k \) will be called reduced if

\[
Z_i \neq Z_j^* \text{ for } i, j \in \mathbb{N}
\]

and

\[
Z_i \in \{ X_r \}_{r \in \mathbb{N}} \Longrightarrow Z_j \in \{ X_r \}_{r \in \mathbb{N}} \forall j \geq i.
\]
For example $Y_1 Y_2 Y_1 X_3$ is reduced whereas $Y_1 Y_2 X_1$ and $Y_1 X_2 Y_3$ are not. The following Lemma and Corollary explains the importance of the reduced words.

**Lemma 3.2.** Any word $b$ in $A$ can be written $b = a \cdot r = r' \cdot a$, where $a$ is a reduced word, and $r, r' \in R$.

**Proof.** If $a$ and $r$ has been found we can take $r' = (\deg a)(r)$, according to (2.12). Thus we concentrate on finding $a$ and $r$. Let $b = Z_1 \cdots Z_k$ be an arbitrary word in $A$. We prove the statement by induction on $k$. If $k = 1$, then $b$ is necessarily reduced so take $a = b, r = 1$. When $k > 1$, use the induction hypothesis to write

$$Z_1 \cdots Z_{k-1} = Y_{i_1} \cdots Y_{i_u} X_{j_1} \cdots X_{j_m} \cdot r',$$

where $1 \leq i_u, j_v \leq n$ and $i_u \neq j_v$ for any $u, v$. Consider first the case when $Z_k = Y_j$ for some $j \in \mathfrak{u}$. Then

$$Z_1 \cdots Z_k = Y_{i_1} \cdots Y_{i_u} X_{j_1} \cdots X_{j_m} Y_j \cdot \sigma_j(r').$$

If $j_v \neq j$ for $v = 1, \ldots, m$ we are done because using relation (2.3) repeatedly we obtain,

$$Z_1 \cdots Z_k = Y_{i_1} \cdots Y_{i_u} Y_j X_{j_1} \cdots X_{j_m} \cdot \mu \sigma_j(r')$$

for some $\mu \in K^*$. Otherwise, let $v \in \{1, \ldots, m\}$ be maximal such that $j_v = j$. Then

$$Z_1 \cdots Z_k = Y_{i_1} \cdots Y_{i_u} X_{j_1} \cdots X_{j_m} Y_j X_{j_{v+1}} \cdots X_{j_{k-1}} Y_{i_v} w(t_j) \mu \sigma_j(r')$$

for some $\mu \in K^*$ and some $w \in W$. It remains to consider the case $Z_k = X_j$ for some $j \in \mathfrak{u}$. But using that

$$Y_{i_1} \cdots Y_{i_u} X_{j_1} \cdots X_{j_m} = X_{j_1} \cdots X_{j_m} Y_{i_1} \cdots Y_{i_u}$$

for some $\mu \in K^*$, it is clear that this case is analogous. □

**Corollary 3.3.** Each $A_g, g \in W$, is generated as a right (and also as a left) $R$-module by the reduced words of degree $g$.

**Lemma 3.4.** Suppose $*$ defines an anti-involution on $A$. Let $p$ be a prime ideal of $R$. Let $g \in \mathbb{Z}^n$ and let $a \in A_g$. If $b a \notin p$ for some $b \in A_{-g}$ then $a^* a \notin p$.

**Proof.** Since $p$ is prime, and $b a \in R$ we have

$$p \not\supset (ba)^2 = (ba)^* ba = a^* b^* ba = a^* a \cdot (\deg a)(b^* b)$$

so in particular $a^* a \notin p$. □

**Remark 3.5.** If we assume $a$ and $b$ to be words in the formulation of Lemma 3.4, one can easily show that the statement remains true without the restriction on $*$ to be an anti-involution.

### 3.2. Inner breaks and canonical modules

Let $A$ be a TGWC or a TGWA and let $M$ be a simple weight module over $A$. In [8] Remark 1 it was noted that the problem of describing simple weight modules over a TGWC is wild in general. This is a motivation for restricting attention to some subclass which has nice properties. In [8] the following definition was made.

**Definition 3.6.** The support of $M$ has no inner breaks if for all $m \in \text{supp}(M)$,

$$t_i m = M \implies \sigma_i(m) \notin \text{supp}(M),$$

and

$$\sigma_i(\bar{t}_i) m = M \implies \sigma^{-1}_i(m) \notin \text{supp}(M).$$

We introduce the following property.
Definition 3.7. We say that $M$ has no proper inner breaks if for any $m \in \text{supp}(M)$ and any word $a$ with $aM_m \neq 0$ we have $a^*a \notin m$.

Observe that whether or not $a^*a \in m$ for a word $a$ does not depend on the particular representation of $a$ as a product of generators. Note also that to prove that a simple weight module $M$ has no proper inner breaks, it is sufficient to find for any $m \in \text{supp}(M)$ and any word $a$ with $aM_m \neq 0$ a word $b \in A$ of degree $-\deg a$ such that $ba \notin m$ because then $a^*a \notin m$ automatically by Remark 3.5. In fact one can show that a simple weight module $M$ has no proper inner breaks if (and only if) there exists an $m \in \text{supp}(M)$ such that for any reduced word $a \in A$ with $aM_m \neq 0$ and $aM_m \subseteq m$ there is a word $b$ of degree $-\deg a$ such that $ba \notin m$. However we will not use this result.

The choice of terminology in Definition 3.7 is motivated by the following proposition.

Proposition 3.8. If $M$ has no inner breaks, then $M$ has no proper inner breaks either.

Proof. Let $m \in \text{supp}(M)$ and $a = Z_1 \ldots Z_k \in A$ be a word such that $aM_m \neq 0$. Thus $Z_i \ldots Z_k M_m \neq 0$ for $i = 1, \ldots, k+1$ so (2.13) implies that

$$(\deg Z_i \ldots Z_k)(m) \in \text{supp}(M).$$

If $M$ has no inner breaks, it follows that $Z_i^* Z_i \notin (\deg Z_{i+1} \ldots Z_k)(m)$ for $i = 1, \ldots, k$. Now using (2.12),

$$a^*a = Z_k^* \ldots Z_1^* Z_k = Z_k^* Z_2 \ldots Z_k(- \deg Z_2 \ldots Z_k)(Z_k^* Z_i) = \ldots = \prod_{i=1}^k (- \deg Z_{i+1} \ldots Z_k)(Z_k^* Z_i) \notin m.$$

Thus $M$ has no proper inner breaks. \hfill \Box

In [8], a simple weight module $M$ was defined to be canonical if for any $m, n \in \text{supp}(M)$ there is an automorphism $\sigma$ of $R$ of the form

$$\sigma = \sigma_{i_1}^+ \ldots \sigma_{i_k}^+,$$

such that $\sigma(m) = n$ and such that for each $j = 1, \ldots, k$,

$$(3.2)\quad t_{ij} \notin \sigma_{i+1}^+ \ldots \sigma_{n}^+(m) \quad \text{if} \quad \varepsilon_j = 1, \quad \text{and}$$

$$(3.3)\quad \sigma_{i_j}(t_{ij}) \notin \sigma_{i+1}^+ \ldots \sigma_{n}^+(m) \quad \text{if} \quad \varepsilon_j = -1.$$

This definition can be reformulated as follows.

Proposition 3.9. $M$ is canonical iff for any $m, n \in \text{supp}(M)$ there is a word $a \in A$ such that $aM_m \subseteq M_n$ and $a^*a \notin m$.

Proof. Suppose $M$ is canonical, and let $m, n \in \text{supp}(M)$. Let $\sigma$ be as in the definition of canonical module. Define $a = Z_1 \ldots Z_k$ where $Z_j = X_{i_j}$ if $\varepsilon_j = 1$ and $Z_j = Y_{i_j}$ otherwise. Using (2.13) we see that $aM_m \subseteq M_n$. Also, (3.2) and (3.3) translates into

$$Z_j^* Z_j \notin (\deg Z_{j+1} \ldots Z_k)(m)$$

for $j = 1, \ldots, k$. Using the calculation (3.1) and that $m$ is prime we deduce that $a^*a \notin m$.

Conversely, given a word $a = Z_1 \ldots Z_k \in A$ with $aM_m \subseteq M_n$ and $a^*a \notin m$, we define $\varepsilon_i = 1$ if $Z_i = X_i$ and $\varepsilon_i = -1$ otherwise. Then from $a^*a \notin m$ follows that $\sigma := \sigma_{i_1}^+ \ldots \sigma_{i_k}^+$ satisfies (3.2) and (3.3) by the same reasoning as above. \hfill \Box

Corollary 3.10. If $M$ has no proper inner breaks, then $M$ is canonical.
Proof. We only need to note that since $M$ is a simple weight module there is for each $m, n \in \text{supp}(M)$ a word $a$ such that $0 \neq aM_m \subseteq M_n$. \hfill \Box

Under the assumptions in [8] any canonical module has no inner breaks (see [8], Proposition 1). However we have the following example of a TGWA $A$ and a simple weight module $M$ over $A$ which has no proper inner breaks, and thus is canonical by Corollary 3.10, but nonetheless has an inner break.

**Example 3.11.** Let $R = \mathbb{C}[t_1, t_2]$ and define the $\mathbb{C}$-algebra automorphisms $\sigma_1$ and $\sigma_2$ of $R$ by $\sigma_i(t_j) = -t_j$ for $i, j = 1, 2$. Let $\mu = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Let $A' = A'(R, t, \sigma, \mu)$ be the associated TGWC, where $t = (t_1, t_2)$, $\sigma = (\sigma_1, \sigma_2)$. Then one can check that $I = \langle x_1 x_2 + x_2 x_1, y_1 y_2 + y_2 y_1 \rangle$. Let $M$ be a vector space over $\mathbb{C}$ with basis $\{v, w\}$ and define an $A'$-module structure on $M$ by letting $X_1 M = Y_1 M = 0$ and

$$
X_2 v = w, \quad X_2 w = v,
$$

$$
Y_2 v = w, \quad Y_2 w = -v.
$$

It is easy to check that the required relations are satisfied and that $IM = 0$, hence $M$ becomes an $A$-module. Let $m = (t_1, t_2 + 1)$ and $n = (t_1, t_1 - 1)$. Then

$$
M = M_m \oplus M_n, \quad \text{where} \quad M_m = \mathbb{C}v, M_n = \mathbb{C}w
$$

so $M$ is a weight module. Any proper nonzero submodule of $M$ would also be a weight module by standard results. That no such submodule can exist is easy to check, so $M$ is simple. One checks that $M$ has no proper inner breaks. But $t_1 \in m$ and $\sigma_1(m) = n \in \text{supp}(M)$ so $m$ is an inner break.

4. The weight space preserving subalgebra and its irreducible representations

In this section, let $A$ be a TGWC, $m \in \text{Max}(R)$ and let $\omega$ be the $\mathbb{Z}^n$-orbit of $m$. Recall the set $Z^*_\omega$ defined in (2.14). Define the following subsets of $\mathbb{Z}^n$:

$$
(4.1) \quad \tilde{G}_m = \{g \in \mathbb{Z}^n \mid a^* a \notin m \text{ for some word } a \in A_g\} \quad \text{and} \quad G_m = \tilde{G}_m \cap Z^*_\omega.
$$

Let also $\varphi_m : A \rightarrow A/(m)$ denote the canonical projection, where $(m)$ is the two-sided ideal in $A$ generated by $m$, and let $R_m = R/m$ be the residue field of $R$ at $m$.

**Lemma 4.1.** Let $g \in \tilde{G}_m$. Then

$$
(4.2) \quad \varphi_m(A_g) = R_m : \varphi_m(a) = \varphi_m(a) : R_m
$$

for any word $a \in A_g$ with $a^* a \notin m$.

**Proof.** Let $b \in A_g$ be any element and $a \in A_g$ a word such that $a^*a \notin m$. We must show that there is an $r \in R$ such that $\varphi_m(b) = \varphi_m(r) \varphi_m(a)$. Since $a^*a \notin m$ and $m$ is maximal, $1 - r_1 a^* a \in m$ for some $r_1 \in R$. Set $r = b r_1 a^*$. Then $r \in R$ and

$$
b - ra = b(1 - r_1 a^* a) \in (m).
$$

The last equality in (4.2) is immediate using (2.12). \hfill \Box

The following result was proved in [8] Lemma 8 for simple weight modules with so called regular support which in particular means that they have no inner breaks. It is still true in the more general situation when $M$ has no proper inner breaks. Recall the ideal $I$ from the definition of a TGWA $A$.

**Proposition 4.2.** Suppose $A$ is a TGWA. If $M$ is a simple weight $A$-module with no proper inner breaks, then $IM = 0$. Hence $M$ is naturally a module over the associated TGWA $A/I$. 
Proof. Since I is graded and \( M \) is a weight modules, it is enough to show that \((I \cap A_g)M_m = 0\) for any \( g \in \mathbb{Z}^n \) and any \( m \in \text{supp}(M)\). Assume that \( a \in I \cap A_g \) and \( av \neq 0 \) for some \( v \in M_m \). Then \( a v \neq 0 \) for some word \( a_1 v \) in \( a \). Since \( M \) has no proper inner breaks, \( a_1^*a_1 \notin m \) so by Lemma 4.1 there is an \( r \in R \) such that \( av = a_1^*arv = a_1^*av \) which implies that \( a_1^*a \in R\{0\} \). This contradicts that \( a \in I \).

We fix now for each \( g \in \mathcal{G}_m \) a word \( a_g \in A_g \) such that \( a_g^*a_g \notin m \). For \( g = 0 \) we choose \( a_g = 1 \).

Lemma 4.3. For any \( g \in \mathcal{G}_m, h \in G_m \) we have
a) \( (a_ga_g^*)^*a_ga_g^* \notin m \) so in particular \( g - h \in \mathcal{G}_m \) and \( G_m \) is a subgroup of \( \mathbb{Z}_w^m \).
b) \( \varphi_m(A_g)\varphi_m(A_h) = \varphi_m(A_gA_h) = \varphi_m(A_{g+h}) \),
c) \( A_{g+h}M_m = A_gM_m \).

Proof. a) We have

\[
(a_ga_g^*)^*a_ga_g^* = a_ga_g^*a_ga_g^* = a_ga_g^*(a_ga_g^*) = a_ga_g^*(-h)(a_ga_g^*)
\]

so \( a_ga_g^* \notin h(m) = m \). Since \( m \) is maximal the right hand side of (4.3) does not belong to \( m \). Since \( \text{deg}(a_ga_g^*) = g - h \) we obtain \( g - h \in \mathcal{G}_m \). If in addition \( g \in G_m \) then \( g - h \in \mathbb{Z}_w^m \) since \( \mathbb{Z}_w^m \) is a group. Thus \( g - h \in G_m \) so \( G_m \) is a subgroup of \( \mathbb{Z}_w^m \).

b) Since \( \varphi_m \) is a homomorphism, the first equality holds. By part a), \(-h \in G_m \) so by part a) again, \( (a_ga_g^*)^*a_ga_g^* \notin m \). Hence by Lemma 4.1, we have

\[
\varphi_m(A_{g+h}) = R_m : \varphi_m(a_ga_g^*) \subseteq \varphi_m(A_gA_h).
\]
The reverse inclusion holds since \( \{A_g\}_{g \in \mathbb{Z}_w} \) is a gradation of \( A \).

c) By part a), \( g + h = g - (-h) \in \mathcal{G}_m \). Thus by part b),

\[
A_{g+h}M_m = \varphi_m(A_{g+h})M_m = \varphi_m(A_gA_h)M_m = A_gA_hM_m \subseteq A_gM_m \subseteq A_gM_{h(m)} \subseteq A_gM_m.
\]

By part a), the same calculation holds if we replace \( g \) by \( g + h \) and \( h \) by \( -h \), which gives the opposite inclusion.

Lemma 4.4. Let \( g \in \mathbb{Z}_w^m \setminus \mathcal{G}_m \). Then \( A_gM_m = 0 \) for any simple weight module \( M \) over \( A \) with no proper inner breaks.

Proof. Let \( a \in A_g \) be any word. Then \( a^*a \in m \) and hence if \( M \) is a simple weight module over \( A \) with no proper inner breaks, \( aM_m = 0 \). Since the words generate \( A_g \) as a left \( R \)-module, it follows that \( A_gM_m = 0 \).

4.1. General case. Recall that \( (m) \) denotes the two-sided ideal in \( A \) generated by \( m \). Since \( (m) \) is a graded ideal in \( A \), there is an induced \( \mathbb{Z}^n \)-gradation of the quotient \( A/(m) \) and \( \varphi_m(A_g) = (A/(m))_g \). Corresponding to the decomposition \( \mathbb{Z}_w^n \) into the subset \( G_m \) and its complement are two \( K \)-subspaces of the algebra \( B_m = B(\omega)/(B(\omega) \cap (m)) \) which will be denoted by \( B_m^{(1)} \) and \( B_m^{(0)} \) respectively. In other words, \( B_m = B_m^{(1)} \oplus B_m^{(0)} \), where

\[
B_m^{(1)} = \bigoplus_{g \in G_m} (A/(m))_g \quad \text{and} \quad B_m^{(0)} = \bigoplus_{g \in \mathbb{Z}_w \setminus G_m} (A/(m))_g.
\]

By Lemma 4.3a), \( G_m \) is a subgroup of the free abelian group \( \mathbb{Z}^n \), hence is free abelian itself of rank \( k \leq n \). Let \( s_1, \ldots, s_k \) denote a basis for \( G_m \) over \( \mathbb{Z} \) and let \( b_i = \varphi_m(a_{s_i}) \) for \( i = 1, \ldots, k \). Note also that \( R_m \) is an extension field of \( K \) and that
$\mathbb{Z}_n^n$ acts naturally on $R_m$ as a group of $K$-automorphisms. Let $\{\rho_j\}_{j \in J}$ be a basis for $R_m$ over $K$.

**Theorem 4.5.** a) $B_m^{(0)}M_m = 0$ for any simple weight module $M$ over $A$ with no proper inner breaks, and

b) the $b_i$ are invertible and as a $K$-linear space, $B_m^{(1)}$ has a basis

$$\{\rho_j b_1^{i_1} \cdots b_k^{i_k} | j \in J \text{ and } i_1, \ldots, i_k \in \mathbb{Z} \text{ for } 1 \leq i \leq k\}$$

and the following commutation relations hold

$$b_i \lambda = s_i(\lambda) b_i, \quad i = 1, \ldots, k, \lambda \in R_m,$$

$$b_i b_j = \lambda_{ij} b_j b_i, \quad i, j = 1, \ldots, k,$$

for some nonzero $\lambda_{ij} \in R_m$.

**Proof.** a) Let $g \in \mathbb{Z}_n^n \setminus G_m$. By Lemma 4.4, $A_g M_m = 0$ and thus $\varphi_m(A_g)M_m = 0$.

b) Since $s_i \in G_m$, $\varphi_m(a_{s_i}) b_i \in R_m \setminus \{0\}$ and by Lemma 4.3a) with $g = 0$ and $h = s_i$ we have $b_i \varphi_m(a_{s_i}) \in R_m \setminus \{0\}$. So the $b_i$ are invertible. The relation (4.5) follows from (2.12). Next we prove (4.6). From Lemma 4.3a) and Lemma 4.1 we have $\varphi(A_{s_i + s_j}) = R_m b_i b_j$. Switching $i$ and $j$ it follows that (4.6) must hold for some nonzero $\lambda_{ij} \in R_m$.

Finally we prove that (4.4) is a basis for $B_m^{(1)}$ over $K$. Linear independence is clear. Let $g \in G_m$ and write $g = \sum_i l_i s_i$. By repeated use of Lemma 4.3b) we obtain that

$$\varphi_m(A_g) = \varphi_m(A_{s_{g_{l_1}}} s_{l_1}) \cdots \varphi_m(A_{s_{g_{l_i}}} s_{l_i}) b_i^{l_i}.$$

For $l_i = 0$ the factor should be interpreted as $R_m$. By Lemma 4.1,

$$\varphi_m(A_{s_{l_i}}) = R_m b_i^{l_i}$$

for $l > 0$ so using (4.5) we get

$$\varphi_m(A_g) = R_m b_1^{l_1} \cdots b_k^{l_k}.$$

The proof is finished. $\square$

### 4.2. Restricted case

In this subsection we will assume that $K$ is algebraically closed. Moreover we will assume that the $K$-algebra inclusion $K \hookrightarrow R_m$ is onto which is the case when $R$ is finitely generated as a $K$-algebra by the (weak) Nullstellensatz. Then $\mathbb{Z}_n^n$ acts trivially on $R_m$. The structure of $B_m^{(1)}$ given in Theorem 4.5 is then simplified in the following way.

**Corollary 4.6.** Let $k = \text{rank} G_m$ and let $b_i = \varphi_m(a_{s_i})$ for $i = 1, \ldots, k$ where $\{s_1, \ldots, s_k\}$ is a $A$-basis for $G_m$. Then $B_m^{(1)}$ is the $K$-algebra with invertible generators $b_1, \ldots, b_k$ and the relation

$$b_i b_j = \lambda_{ij} b_j b_i, \quad 1 \leq i, j \leq k.$$

Using the normal form of a skew-symmetric integral matrix we will now show that $B_m^{(1)}$ can be expressed as a tensor product of noncommutative tori. Consider the matrix $(\lambda_{ij})_{1 \leq i, j \leq k}$ from (4.6).

**Claim 4.7.** If $B_m^{(1)}$ has a nontrivial irreducible finite-dimensional representation, then all the $\lambda_{ij}$ are roots of unity.

**Proof.** Indeed, let $N$ be a finite-dimensional simple module over $B_m^{(1)}$ and let $i \in \{1, \ldots, k\}$. Since $K$ is algebraically closed, $b_i$ has an eigenvector $0 \neq v \in N$ with eigenvalue $\mu$, say. Since $b_i$ is invertible, $\mu \neq 0$. Let $j \neq i$ and consider the vector
It is also nonzero, since $b_j$ is invertible, and it is an eigenvector of $b_i$ with eigenvalue $\lambda_{ij}\mu$. Repeating the process, we obtain a sequence

$$
\mu, \lambda_{ij}\mu, \lambda_{ij}^2\mu, \ldots
$$

of eigenvalues of $b_i$. Since $N$ is finite-dimensional, they cannot all be pairwise distinct, and thus $\lambda_{ij}$ is a root of unity. \hfill \Box

For $\lambda \in K$, let $T_\lambda$ denote the $K$-algebra with two invertible generators $a$ and $b$ satisfying $ab = \lambda ba$. $T_\lambda$ (or its $C^*$-analogue) is sometimes referred to as a noncommutative torus.

**Theorem 4.8.** Let $k = \text{rank} G_m$. If all the $\lambda_{ij}$ in (4.6) are roots of unity, then there is a root of unity $\lambda$, an integer $r$ with $0 \leq r \leq [k/2]$ and positive integers $p_i, i = 1, \ldots, r$ with $1 = p_1|p_2| \ldots |p_r$ such that

$$
B^{(1)}_m \cong T_{\lambda^r_1} \otimes T_{\lambda^r_2} \otimes \cdots \otimes T_{\lambda^r_r} \cong L
$$

where $L$ is a Laurent polynomial algebra over $K$ in $k - 2r$ variables.

**Proof.** If $k = 1$, then $B^{(1)}_m \cong K[b_1, b_1^{-1}]$ and $r = 0$. If $k > 1$, let $p$ be the smallest positive integer such that $\lambda_{ij}^p = 1$ for all $i, j$. Using that $K$ is algebraically closed, we fix a primitive $p$th root of unity $\zeta \in K$. Then there are integers $\theta_{ij}$ such that

$$
\lambda_{ij} = \zeta^{\theta_{ij}}
$$

and

$$
(4.7) \quad \theta_{ji} = -\theta_{ij}.
$$

Equation (4.7) means that $\Theta = (\theta_{ij})$ is a $k \times k$ skew-symmetric integer matrix.

Next, consider a change of generators of the algebra $B^{(1)}_m$:

$$
(4.8) \quad b_i \mapsto b_i^j = b_1^{u_{i1}} \cdots b_k^{u_{ik}}
$$

Such a change of generators can be done if we are given an invertible $k \times k$ integer matrix $U = (u_{ij})$. The new commutation relations are

$$
b_i^jb_j^i = b_1^{u_{i1}} \cdots b_k^{u_{ik}} b_1^{u_{j1}} \cdots b_k^{u_{jk}} = \\
= \lambda_{1i1}^{u_{i1}} \cdots \lambda_{1i1}^{u_{i1}} \cdots \\
= \lambda_{kik}^{u_{ik}} \cdots \lambda_{kik}^{u_{ik}} \cdots \\
= \zeta \sum_{p=1}^{\infty} \theta_{ip} u_{ip} u_{jp} b_i^j b_j^i
$$

Hence $\Theta' = U^T \Theta U$. By Theorem IV.1 in [9] there is a $U$ such that $\Theta'$ has the skew normal form

$$
\begin{pmatrix}
0 & \theta_1 & 0 & \cdots & 0 \\
-\theta_1 & 0 & \theta_2 & \cdots & 0 \\
& \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \theta_r \\
& & & -\theta_r & 0
\end{pmatrix}
$$

where $r \leq [k/2]$ is the rank of $\Theta$, the $\theta_i$ are nonzero integers, $\theta_i/\theta_{i+1}$ and $0$ is a $k - 2r$ by $k - 2r$ zero matrix. Set $\lambda = \zeta^{\theta_1}$ and $p_i = \theta_i/\theta_1$ for $i = 1, \ldots, r$. The claim follows. \hfill \Box

The following result, describing simple modules over the tensor product of noncommutative tori, is more or less well-known, but we provide a proof for convenience.
Proposition 4.9. Let $M$ be a finite dimensional simple module over

$$T := T_{\lambda_1} \otimes \cdots \otimes T_{\lambda_r},$$

where the $\lambda_i$ are roots of unity in $K$. Then there are simple modules $M_i$ over $T_{\lambda_i}$ such that, as $T$-modules,

$$M \simeq M_1 \otimes \cdots \otimes M_r.$$

Proof. Denote the generators of $T_{\lambda_i}$ by $a_i$ and $b_i$. We will view $T_{\lambda_i}$ as subalgebras of $T$. Since the elements $a_i, i = 1, \ldots, r$ commute and $M$ is finite dimensional and $K$ is algebraically closed, there is a nonzero common eigenvector $w \in M$ of the $a_i$:

$$a_i w = \mu_i w, \quad i = 1, \ldots, r,$$

(4.9)

where $\mu_i \in K^*$ because $a_i$ is invertible. Let $n_i$ be the order of $\lambda_i$. Then $b_i^{n_i}$ acts as a scalar by Schur’s Lemma. By simplicity of $M$, any element of $M$ has the form (using the commutation relations and (4.9))

$$\sum_{j \in \mathbb{Z}^{\tau}, 0 \leq j < n_i} \rho_j b_i^j \ldots b_i^j w,$$

(4.10)

where $\rho_j \in K$. This shows that

$$\dim_K M \leq n_1 \cdots n_r.$$

But the terms in (4.10) all belong to different weight spaces with respect to the commutative subalgebra generated by $a_1, \ldots, a_r$:

$$a_i \cdot b_i^j \ldots b_i^j \cdot w = \lambda_i^j \mu_i \cdot b_i^j \ldots b_i^j \cdot w, \quad i = 1, \ldots, r,$$

and

$$(\lambda_i^j \mu_1, \ldots, \lambda_i^j \mu_r) \neq (\lambda_i^j \mu_1, \ldots, \lambda_i^j \mu_r)$$

if $j \in \mathbb{Z}^r, 0 \leq j_i < n_i$ and $j \neq j$. Hence by standard results they must be linearly independent. Thus

$$\dim_K M = n_1 \cdots n_r.$$

(4.11)

Next, set $M_i = T_{\lambda_i} \cdot w$. Then $M_i = \oplus_{j=0}^{n_i-1} K b_i^j \cdot w$ and

$$\dim_K M_i = n_i.$$

(4.12)

Finally, define

$$\psi : M_1 \otimes \cdots \otimes M_r \rightarrow M$$

by

$$\psi (w \otimes \cdots \otimes w) = w$$

and by requiring that $\psi$ is a $T$-module homomorphism. This is possible since $M_1 \otimes \cdots \otimes M_r$ is generated by $w \otimes \cdots \otimes w$ as a $T$-module. Then $\psi$ is surjective, since $M$ is simple. Also the dimensions on both sides agree, so $\psi$ is an isomorphism of $T$-modules. \qed

5. Explicit formulas for the induced modules

In this section we show explicitly how one can obtain simple weight modules with no proper inner breaks over a TGWA (equivalently over a TGWC by Proposition 4.2) from the structure of its weight spaces as $B(\omega)$-modules.

Since the $B(\omega)$-modules were described in the restricted case in Subsection 4.2, we obtain in particular a description of all simple weight modules over $A$ with no proper inner breaks and finite-dimensional weight spaces if $R$ is finitely generated over an algebraically closed field $K$. 


5.1. **A basis for** $M$. Let $\{v_i\}_{i \in I}$ be a basis for $M_m$ over $K$. By Lemma 4.3a), $\tilde{G}_m$ is the union of some cosets in $\mathbb{Z}^n/G_m$. Let $S \subseteq \mathbb{Z}^n$ be a set of representatives of these cosets. For $g \in \tilde{G}_m$, choose $r_g \in R$ such that $a_g := r_g a_g^M$ satisfies $\varphi_m(d_g) \varphi_m(a_g) = 1$.

**Theorem 5.1.** The set $C = \{a_g v_i \mid g \in S, i \in I\}$ is a basis for $M$ over $K$.

**Proof.** First we show that $C$ is linearly independent over $K$. Assume that

$$\sum_{g,i} \lambda_{g,i} a_g v_i = 0.$$

Then $\sum_g \lambda_{g,i} a_g v_i = 0$ for each $g$ since the elements belong to different weight spaces. Hence $0 = a_g^M \sum_i \lambda_{g,i} a_g v_i = \sum_i \lambda_{g,i} v_i$ for each $g$. Since $v_i$ is a basis over $K$, all the $\lambda_{g,i}$ must be zero.

Next we prove that $C$ spans $M$ over $K$. Since $M$ is simple and $M_m \neq 0$,

$$M = AM_m = \sum_{g \in \mathbb{Z}^n} A_g M_m = \sum_{g \in G_m} A_g M_m = \sum_{h \in S} \sum_{g \in h + G_m} A_g M_m = \sum_{h \in S} A_h M_m$$

by Lemma 4.4 and Lemma 4.3c).

**Corollary 5.2.** $\supp(M) = \{g(m) \mid g \in S\}$ and $g(m) \neq h(m)$ if $g, h \in S, g \neq h$.

**Corollary 5.3.** $\dim M = |S| \cdot \dim M_m$ with natural interpretation of $\infty$.

5.2. **The action of** $A$. Our next step is to describe the action of the $X_i, Y_i$ on the basis $C$ for $M$. Let $\zeta : \tilde{G}_m \to S$ be the function defined by requiring $g - \zeta(g) \in G_m$.

**Theorem 5.4.** Let $g \in S$ and let $v \in M_m$. Then

$$X_i a_g v = \begin{cases} a_h \cdot b_{g,i} v & \text{if } g + e_i \in \tilde{G}_m, \\ 0 & \text{otherwise}, \end{cases}$$

where $h = \zeta(g + e_i)$ and

$$b_{g,i} = (-h)(X_i a_g a_{g+e_i}^d) : a_{g+e_i}^-h.$$

and

$$Y_i a_g v = \begin{cases} a_h \cdot c_{g,i} v & \text{if } g - e_i \in \tilde{G}_m, \\ 0 & \text{otherwise}, \end{cases}$$

where $k = \zeta(g - e_i)$ and

$$c_{g,i} = (-k)(Y_i a_g a_{g-e_i}^d) : a_{g-e_i}^-k.$$

**Remark 5.5.** Note that

$$\deg X_i a_g a_{g+e_i}^d : a_{g+e_i}^-h a_h = \deg Y_i a_g a_{g-e_i}^-k a_k = 0$$

so the action of $\mathbb{Z}^n$ on these elements is well defined. Thus we see that $\deg b_{g,i} \in G_m$ and $\deg c_{g,i} \in G_m$, i.e. that $b_{g,i}$ and $c_{g,i}$ belong to $B(\omega)$. Therefore the action of these elements on a basis element $v_i$ of $M_m$ can be determined if we know the structure of $M_m$ as an $B(\omega)$-module. In the restricted case this was described in Subsection 4.2. Expanding the result in the basis $\{v_i\}$ again and acting by $a_h$ or $a_h$ we obtain a linear combination of basis elements from the set $C$.

**Proof.** Assume $g + e_i \in \tilde{G}_m$. Let $h = \zeta(g + e_i)$. Then

$$X_i a_g v = X_i a_g a_{g+e_i}^d : a_{g+e_i}^-h v =$$

$$= (X_i a_g a_{g+e_i}^d) a_h a_{g+e_i}^-h v =$$

$$= a_h \cdot (-h)(X_i a_g a_{g+e_i}^d) a_{g+e_i}^-h v.$$

If $g + e_i \notin \tilde{G}_m$, then $X_i a_g v = 0$ by Lemma 4.4.
Assume $g - e_i \in \mathcal{G}_m$. Let $k = \zeta(g - e_i)$. Then
\[
Y_i a_g v = Y_i a_g a_{g-e_i-k} a_{g-e_i-k} v = \\
= (Y_i a_g a_{g-e_i-k} a_{g-e_i-k}^k) a_k a_{g-e_i-k} v = \\
= a_k \cdot (-k) (Y_i a_g a_{g-e_i-k} a_{g-e_i-k}^k) a_k a_{g-e_i-k} v.
\]
If $g - e_i \notin \mathcal{G}_m$, then $Y_i a_g v = 0$ by Lemma 4.4.

Note that we do not need the technical assumptions in the proof of Theorem 1 in [8] under which the exact formulas for simple weight modules were obtained.

6. Application to quantized Weyl algebras

In this final part we will apply the methods developed in the previous sections to the problem of describing representations of the quantized Weyl algebra, defined in Section 2.2. As mentioned there, it is naturally a TGWA.

First we find the isotropy group and the set $\mathcal{G}_m$ expressed as solution of systems of linear equations (see Proposition 6.3 and Proposition 6.4). These sets are directly related to the structure of the subalgebra $\mathcal{B}(\omega)$ (Theorem 4.5) and the support of a module (Corollary 5.2).

Then in Section 6.2 we give a complete classification of all locally finite simple weight modules with no proper inner breaks over a quantized Weyl algebra of rank two. The parameters $q_1$ and $q_2$ are allowed to be any numbers from $\mathbb{C} \setminus \{0,1\}$. Example 6.7 shows that the assumption that the modules have no proper inner breaks is not superfluous.

6.1. The isotropy group and $\mathcal{G}_m$. Let $R = \mathbb{C}[t_1, \ldots, t_n]$ and fix $m = (t_1 - \alpha_1, \ldots, t_n - \alpha_n) \in \text{Max}(R)$. Let $\omega$ be the orbit of $m$ under the action (2.10) of $\mathbb{Z}^n$. Set $[k]_q = \sum_{j=0}^{k-1} q^j$ for $k \in \mathbb{Z}$ and $q \in \mathbb{C}$. Recall the definition (2.9) of the automorphisms $\sigma_i$ of $R$.

Proposition 6.1. Let $(g_1, \ldots, g_n) \in \mathbb{Z}^n$. Then

\[
\sigma_1^{g_1} \cdots \sigma_n^{g_n}(m) = \\
= \left([g_1]_{q_1} + q_1^{g_1} t_1 - \alpha_1, \ 2\cdot q_2 \cdot (1 + (q_1 - 1)\alpha_1) + q_2^{g_2} t_2 - \alpha_2, \ldots \\
\ldots, [g_j]_{q_j} \cdot (1 + \sum_{r=1}^{j-1} (q_r - 1)\alpha_r) + q_j^{g_1} \cdots q_j^{g_j} t_j - \alpha_j, \ldots \\
\ldots, [g_n]_{q_n} \cdot (1 + \sum_{r=1}^{n-1} (q_r - 1)\alpha_r) + q_n^{g_1} \cdots q_n^{g_n} t_n - \alpha_n \right).
\]

Proof. Induction. □

For notational brevity we set $\beta_i = (q_i - 1)\alpha_i$ and $\gamma_i = 1 + \beta_1 + \beta_2 + \ldots + \beta_i$. We also set $\gamma_0 = 1$. The numbers $\gamma_i$ will play an important role in the next statements. By a $j$-break we mean an ideal $n \in \text{Max}(R)$ such that $t_j \in n$.

Corollary 6.2. For $j = 1, \ldots, n$ we have
\[
t_j \in \sigma_1^{g_1} \cdots \sigma_n^{g_n}(m) \iff \gamma_j = q_j^{g_j} \gamma_{j-1}.
\]

Thus $\omega$ contains a $j$-break iff $\gamma_j = q_j^{g_j} \gamma_{j-1}$ for some integer $k$.

Proof. By Proposition 6.1,
\[
t_j \in \sigma_1^{g_1} \cdots \sigma_n^{g_n}(m)
\]
iff
\[ [g_j]_{q_j} \left( 1 + \sum_{r=1}^{j-1} (q_r - 1) \alpha_r \right) = \alpha_j. \]

Multiply both sides with \( q_j - 1 \) to get
\[ (q_j^{q_j} - 1)(1 + \beta_1 + \ldots + \beta_{j-1}) = \beta_j. \]

\[ \square \]

The next Proposition describes the isotropy subgroup \( Z^n_\omega \) defined in (2.14).

**Proposition 6.3.** We have
\[(6.1) \quad Z^n_\omega = \{ g \in \mathbb{Z}^n \mid (q_1^{q_1} \ldots q_j^{q_j} - 1) \gamma_j = 0 \ \forall j = 1, \ldots, n \}. \]

**Proof.** From Proposition 6.1, \( \sigma_1 \sigma_2 \ldots \sigma_n(m) = m \) iff
\[ \alpha_1 = [g_1]_{q_1} + q_1^{q_1} \alpha_1 \]
\[ \alpha_2 = [g_2]_{q_2} \left( 1 + (q_1 - 1) \alpha_1 \right) + q_1^{q_1} q_2^{q_2} \alpha_2 \]
\[ \vdots \]
\[ \alpha_n = [g_n]_{q_n} \left( 1 + (q_1 - 1) \alpha_1 + \ldots + (q_{n-1} - 1) \alpha_{n-1} \right) + q_1^{q_1} \ldots q_n^{q_n} \alpha_n \]

Multiply the \( i \)-th equation by \( q_i - 1 \). Then the system can be written
\[ \beta_1 = q_1^{q_1} - 1 + q_1^{q_1} \beta_1 \]
\[ \beta_2 = (q_2^{q_2} - 1)(1 + \beta_1) + q_1^{q_1} q_2^{q_2} \beta_2 \]
\[ \vdots \]
\[ \beta_n = (q_n^{q_n} - 1)(1 + \beta_1 + \ldots + \beta_{n-1}) + q_1^{q_1} \ldots q_n^{q_n} \beta_n \]

or equivalently
\[ 1 + \beta_1 = q_1^{q_1} (1 + \beta_1) \]
\[ 1 + \beta_1 + \beta_2 = q_2^{q_2} (1 + \beta_1) + q_1^{q_1} q_2^{q_2} \beta_2 \]
\[ \vdots \]
\[ 1 + \beta_1 + \ldots + \beta_n = q_n^{q_n} (1 + \beta_1 + \ldots + \beta_{n-1}) + q_1^{q_1} \ldots q_n^{q_n} \beta_n \]

Now for \( i \) from 1 to \( n - 1 \), replace the expression \( 1 + \beta_1 + \ldots + \beta_i \) in the right hand side of the \( i + 1 \)-th equation by the right hand side of the \( i \)-th equation. After simplification, the claim follows. \( \square \)

Note that it follows from (6.1) that the subgroup
\[(6.2) \quad Q = \{ g \in \mathbb{Z}^n \mid q_j^{q_j} = 1 \ \text{for} \ j = 1, \ldots, n \}\]
of \( \mathbb{Z}^n \) is always contained in \( Z^n_\omega \) for any orbit \( \omega \). Moreover \( Z^n_\omega = Q \) if \( \omega \) (viewed as a subset of \( \mathbb{C}^n \)) does not intersect the union of the hyperplanes in \( \mathbb{C}^n \) defined by the equations \( 1 + (q_1 - 1)x_1 + \ldots + (q_j - 1)x_j = 0 \ (1 \leq j \leq n) \).

Another case of interest is when for any \( j \), \( q_1^{q_1} \ldots q_j^{q_j} = 1 \) implies \( g_1 = \ldots = g_j = 0 \). If for instance the \( q_j \) are pairwise distinct prime numbers this hold. Then \( Z^n_\omega = \{ 0 \} \) unless \( 1 + \beta_1 + \ldots + \beta_j = 0 \) for all \( j \), i.e. unless \( \omega \) contains the point
\[ n_0 = (t_1 - (1 - q_1)^{-1}, t_2, \ldots, t_n). \]

So in this very special case we have \( \omega = \{ n_0 \} \) and \( Z^n_\omega = \mathbb{Z}^n \).

We now turn to the set \( \hat{G}_m \) defined in (4.1) which can here be described explicitly in terms of \( m \) in the following way.
Proposition 6.4. 
\[ \tilde{G}_m = \tilde{G}_m^{(1)} \times \ldots \times \tilde{G}_m^{(n)} \]
where
\[ \tilde{G}_m^{(j)} = \{k \geq 0 \mid \gamma_j \neq q_k^{\gamma_{j-1}} \forall i = 0, 1, \ldots, k - 1\} \cup \{k < 0 \mid \gamma_j \neq q_k^{\gamma_{j-1}} \forall i = -1, -2, \ldots, k\} \]

Proof. From the relations of the algebra follows that the subspace spanned by the words in \( A_g \) is one-dimensional. Thus \( g \in \tilde{G}_m \) iff
\[ Z_n^{-\tilde{g}_n} \ldots Z_1^{-\tilde{g}_1} Z_1^{\tilde{g}_1} \ldots Z_n^{\tilde{g}_n} \notin m \]
where \( \tilde{Z}_n = \tilde{X}_n \) if \( k \geq 0 \) and \( \tilde{Z}_n = \tilde{Y}_n^{-k} \) if \( k < 0 \). Since \( \sigma_i(t_j) = t_j \) for \( j < i \), (6.3) is equivalent to
\[ Z_n^{-\tilde{g}_n} Z_n^{\tilde{g}_n} \ldots Z_1^{-\tilde{g}_1} Z_1^{\tilde{g}_1} \notin m. \]
Since \( m \) is prime, this holds iff \( Z_j^{-\tilde{g}_j} Z_j^{\tilde{g}_j} \notin m \) for each \( j \). If \( g_j = 0 \) this is true. If \( g_j > 0 \) we have
\[ Z_j^{-\tilde{g}_j} Z_j^{\tilde{g}_j} = Y_j^{\tilde{g}_j} X_j^{\tilde{g}_j} = Y_j^{\tilde{g}_j-1} X_j^{\tilde{g}_j-1} \sigma_j^{-\tilde{g}_j+1} (t_j) = \ldots = t_j \sigma_j^{-1} (t_j) \ldots \sigma_j^{-\tilde{g}_j+1} (t_j), \]
while if \( g_j < 0 \)
\[ Z_j^{-\tilde{g}_j} Z_j^{\tilde{g}_j} = X_j^{-\tilde{g}_j} Y_j^{-\tilde{g}_j} = X_j^{-\tilde{g}_j-1} Y_j^{-\tilde{g}_j-1} \sigma_j^{-\tilde{g}_j} (t_j) = \ldots = \sigma_j (t_j) \ldots \sigma_j^{-\tilde{g}_j} (t_j). \]
Since \( m \) is prime, \( g \in \tilde{G}_m \) iff for all \( j = 1, \ldots, n \)
\[ t_j \notin \sigma_j^{\tilde{g}_j} (m), \quad i = 0, \ldots, g_j - 1 \text{ if } g_j \geq 0, \]
and
\[ t_j \notin \sigma_j^{\tilde{g}_j} (m), \quad i = -1, -2, \ldots, g_j \text{ if } g_j < 0. \]
The claim now follows from Corollary 6.2. \( \square \)

Corollary 6.5. If \( \{1, \alpha_1, \alpha_2, \ldots, \alpha_n\} \) is linearly independent over \( \mathbb{Q}(q_1, \ldots, q_n) \), then \( \tilde{G}_m = \mathbb{Z}^n \).

6.2. Description of simple weight modules over rank two algebras. Assume from now on that \( A \) is a quantized Weyl algebra of rank two. In this section we will obtain a list of all locally finite simple weight \( A \)-modules with no proper inner breaks.

We consider first some families of ideals in \( \text{Max}(R) \). Define for \( \lambda \in \mathbb{C} \),
\[ n_\lambda^{(1)} = (t_1 - (1 - \lambda)(1 - q_1)^{-1}, t_2 - \lambda(1 - q_2)^{-1}), \]
\[ n_\lambda^{(2)} = (t_1 - (1 - q_1)^{-1}, t_2 - \lambda), \]
and set \( n_0 = n_0^{(1)} = n_0^{(2)} \). The following lemma will be useful.

Lemma 6.6. For \( \lambda \in \mathbb{C} \) and integers \( k, l \) we have
\[ \sigma_k^\lambda \sigma_l^\lambda (n_\lambda^{(1)}) = n_{\lambda q_l^{-k}}^{(1)}, \]
\[ \sigma_k^\lambda \sigma_l^\lambda (n_\lambda^{(2)}) = n_{\lambda q_l^{-k}}^{(2)}. \]

Proof. Follows from Proposition 6.1 or by direct calculation using the definition (2.9) of the \( \sigma_i \). \( \square \)

The following example shows the existence of locally finite simple weight modules \( M \) over \( A \) which have some proper inner breaks.
Example 6.7. Assume that $q_1 \lambda_{12}$ is a root of unity of order $r$. Let $M$ be a vector space of dimension $r$ and let $\{v_0, v_1, \ldots, v_{r-1}\}$ be a basis for $M$. Define an action of $A$ on $M$ as follows:

$$
X_1 v_k = \begin{cases} 
  v_{k+1}, & k < r - 1 \\
  v_0, & k = r - 1
\end{cases} \quad X_2 v_k = (q_1 \lambda_{12})^{-k} v_k
$$

$$
Y_1 v_k = \begin{cases} 
  (1 - q_1)^{-1} v_{k-1}, & k > 0 \\
  (1 - q_1)^{-1} v_{r-1}, & k = 0
\end{cases} \quad Y_2 v_k = 0
$$

It is easy to check that (2.6)–(2.8) hold so this defines a module over $A$. It is immediate that $M = M_m$ where $m = n_0 = (t_1 - (1 - q_1)^{-1})^2$ so $M$ is a weight module and $M$ is simple by standard arguments. However, recalling Definition 3.7, $M$ has some proper inner breaks in the sense that $m \in \text{supp}(M)$, $X_2 M_m \neq 0$ but $Y_2 X_2 M_m = 0$.

We will describe the isotropy groups of the different ideals in Max($R$). Let $K_1$ and $K_2$ denote the kernels of the group homomorphisms from $Z \times Z$ to the multiplicative group $C \setminus \{0\}$ which map $(k, l)$ to $q_1^k q_2^l$ and $(q_1^k, q_2^l)$ respectively. Then $Q = K_1 \cap K_2$ where $Q$ was defined in (6.2). For $m \in \text{Max}(R)$, recall that $Z_m^2 = \{g \in Z^2 \mid g(m) = m\}$. The following corollary describes the isotropy group $Z_m^2$ of any $m \in \text{Max}(R)$.

Corollary 6.8. Let $\lambda \in C \setminus \{0\}$ and $n \in \text{Max}(R) \setminus \{n^{(i)}_\mu \mid \mu \in C, i = 1, 2\}$. Then we have the following equalities in the lattice of subgroups of $Z^2$.

\[
\begin{array}{ccc}
Z_{n_0}^2 &=& Z^2 \\
Z_{n^{(1)}_\lambda}^2 &=& K_1 \\
Z_{n^{(2)}_\lambda}^2 &=& K_2 \\
Z_n^2 &=& Q
\end{array}
\]

Proof. The family of ideals $\{n^{(1)}_\lambda \mid \lambda \in C\}$ are precisely those for which $\gamma_2 = 0$. And $\{n^{(2)}_\lambda \mid \lambda \in C\}$ are exactly those such that $\gamma_1 = 0$. Thus the claim follows from Proposition 6.3.

Let $M$ be a simple weight $A$-module with no proper inner breaks and finite dimensional weight spaces, $m = (t_1 - \alpha_1, t_2 - \alpha_2) \in \text{supp} M$ and let $\omega$ be the orbit of $m$. We consider four main cases separately: $m = n_0$, $m = n^{(1)}_\lambda$ for some $\lambda \neq 0$, $m = n^{(2)}_\lambda$ for some $\lambda \neq 0$ and $m \notin \{n^{(i)}_\mu \mid \mu \in C, i = 1, 2\}$. Some of these cases will contain subcases. In each case we will proceed along the following steps, which also illustrate the procedure for a general T-GWA.

1. Find the sets $Z_m$ and $\tilde{G}_m$ using Corollary 6.8 and Proposition 6.4. Write down $G_m = Z_m \cap \tilde{G}_m$ and choose a basis $\{s_1, \ldots, s_k\}$ for $G_m$ over $Z$.
2. For each $g \in \tilde{G}_m$, choose a word $a_g$ of degree $g$ such that $a_g^\omega a_g \notin m$.
3. Using Corollary 4.6, describe $B^{(1)}_m$ and the finite-dimensional simple $B^{(1)}_m$-module $M_m$.  
4. Choose a set of representatives $S$ for $\tilde{G}_m/G_m$. By Theorem 5.1 we know then a basis $C$ for $M$.
5. Calculate the action of $X_1$, $Y_1$ on the basis using either relations (2.6)–(2.8) or Theorem 5.4.
We will use the following notation: $Z_j^k = X_j^k$ if $k \geq 0$ and $Z_j^{-k} = Y_j^{-k}$ if $k < 0$. Note that the $k$ in $Z_j^k$ should only be regarded as an upper index, not as a power. The choice of $a_g$ in step two above is more or less irrelevant for a quantized Weyl algebra because each $A_g$ is one-dimensional. Therefore we will always choose $a_g = Z_0^1 Z_2^2$ where $g = (q_1, q_2)$.

6.3. The case $m = n_0$. Here $\alpha_1 = (1 - q_1)^{-1}$, $\alpha_2 = 0$ so that $\gamma_1 = \gamma_2 = 0$. By Corollary 6.8 we have $Z_m^2 = Z^2$ and from Proposition 6.4 one obtains that $\tilde{G}_m = Z \times \{0\}$. Thus $G_m = Z \times \{0\} = Z \cdot s_1$ with $s_1 = (1,0)$. Since $G_m$ has rank one, Corollary 4.6 implies that $B_m^{(1)}$ is isomorphic to the Laurent polynomial algebra $\mathbb{C}[T, T^{-1}]$ in one variable. Therefore $M_m$ is one-dimensional, say $M_m = \mathbb{C}v_0$ and $b_1 = \varphi_m(Z_1^1) = \varphi_m(X_1)$, hence $X_1$, acts in $M_m$ as some nonzero scalar $\rho$. And

\[
Y_1 v_0 = \rho^{-1} Y_1 X_1 v_0 = \rho^{-1} (1 - q_1)^{-1} v_0.
\]

Here $S = \{(0,0)\}$ and $C = \{v_0\}$ is a basis for $M$ with the following action:

\[
\begin{align*}
X_1 v_0 &= \rho v_0, & X_2 v_0 &= 0, \\
Y_1 v_0 &= \rho^{-1} (1 - q_1)^{-1} v_0, & Y_2 v_0 &= 0.
\end{align*}
\]

That $Z_m^{\pm 1} v_0 = 0$ follows from Theorem 5.4 since $(0, \pm 1) \notin \tilde{G}_m$.

6.4. The case $m = n_1^{\lambda}$, $\lambda \neq 0$. Here $\alpha_1 = (1 - \lambda)(1 - q_1)^{-1}$ and $\alpha_2 = \lambda(1 - q_1)^{-1}$ so $\gamma_1 = \lambda$ and $\gamma_2 = 0$. By Proposition 6.4, $\tilde{G}_m^{(1)} = Z$ and

\[
\tilde{G}_m^{(1)} = \{k \geq 0 \mid \lambda \neq q_1^k \forall i = 0, 1, \ldots, k - 1 \} \cup \{k < 0 \mid \lambda \neq q_1^k \forall i = -1, -2, \ldots, k\}.
\]

We consider four subcases according to whether $\omega$ contains a 1-break or not and whether $q_1$ is a root of unity or not.

6.4.1. The case $m = n_1^{\lambda}$, $\lambda \neq 0$, $\omega$ contains a 1-break and $q_1$ is a root of unity. By Corollary 6.2, $\lambda = q_1^k$ for some $k \in \mathbb{Z}$. Let $q_1$ be the order of $q_1$. Then $Z_m^{2} = K_1 = (\alpha_1 Z) \times Z$. We can further assume that $k \in \{0, 1, \ldots, \alpha_1 - 1\}$.

Note that $X_1^k M_m \neq 0$ because $deg X_1^k = (k, 0) \in \tilde{G}_m$ so $Y_1^k X_1^k \notin \tilde{m}$. Hence $o_1^j(m) \in \text{supp}(M)$. By Lemma 6.6, $o_1^j(m) = n_1^{(1)}$. We can thus change notation and let $m = n_1^{(1)}$. Then by Proposition 6.4 we have

\[
\tilde{G}_m = \{0, -1, -2, \ldots, -\alpha_1 + 1\} \times Z.
\]

And $G_m = \tilde{G}_m \cap Z_m^2 = \{0\} \times Z$. By Corollary 4.6, $B_m^{(1)}$ is a Laurent polynomial algebra in one variable. Thus $M_m$ is one dimensional with a basis vector, say $v_0$. $X_2$ acts by some nonzero scalar $\rho$ on $v_0$ and $Y_2 X_2 v_0 = (1 - q_2)^{-1} v_0$. $X_1$ and $Y_1^{\alpha_1}$ act as zero on $M_m$ by Lemma 4.4 because their degrees $(1, 0)$ and $(-\alpha_1, 0)$ does not belong to $\tilde{G}_m$.

As a set of representatives for $\tilde{G}_m / G_m$ we choose

\[
S = \{(0,0), (-1,0), (-2,0), \ldots, (-\alpha_1 + 1,0)\}.
\]

By Corollary 5.2 we obtain that

\[
\text{supp}(M) = \{n_1^{(1)}, n_{\alpha_1}^{(1)}, \ldots, n_{\alpha_1+1}^{(1)}\}.
\]

By 5.1, the set

\[
C = \{v_j := Y_1^j v_0 \mid j = 0, 1, \ldots, \alpha_1 - 1\}
\]
is a basis for $M$. The following picture shows the support of the module and how the $X_i$ act on it. Since the $Y_i$ just act in the opposite direction of the $X_i$ we do not draw their arrows.

Using Lemma 6.6,

$$X_1v_j = X_1Y_1^jv_0 = Y_1^{j-1}\sigma_1^j(t_1)v_0 = [j]_{q_1}v_{j-1}$$
and from relations (2.6)-(2.8) follow that

$$X_2v_j = q_1^j\lambda_2^jY_1^jX_2v_0 = \rho\lambda_2^jg^jv_j,$$
$$Y_2v_j = \lambda_2^jY_2v_0 = (1 - q_2)^{-1}\rho^{-1}\lambda_2^j v_j.$$

Thus the action on the basis $\{v_0, \ldots, v_{o_1-1}\}$ is

$$X_1v_j = \begin{cases} 0, & j = 0, \\ [j]_{q_1}v_{j-1}, & 0 < j \leq o_1 - 1, \end{cases}$$
$$Y_1v_j = \begin{cases} v_{j+1}, & 0 \leq j < o_1 - 1, \\ 0, & j = o_1 - 1, \end{cases}$$
$$X_2v_j = \rho\lambda_2^jg^jv_j,$$
$$Y_2v_j = (1 - q_2)^{-1}\rho^{-1}\lambda_2^j v_j.$$  

6.4.2. The case $m = n_1^{(1)}$, $\lambda \neq 0$, $\omega$ contains a 1-break and $q_1$ is not a root of unity.

Now there is a unique integer $k \in \mathbb{Z}$ such that $\lambda = q_1^k$. If $k \geq 0$, then $G_m^{(1)}$ is the set of all integers $\leq k$ while if $k < 0$, then $G_m^{(1)}$ is all integers $\geq k + 1$.

If $k \geq 0$, $X_1^kM_m \neq 0$ because $(k,0) \in G_m$ so $Y_1^kX_1^k \notin m$. Therefore $\sigma_1^k(m) = n_1^{(1)} \in \text{supp}(M)$. We change notation and let $m = n_1^{(1)}$. Then $G_m^{(1)} = \{ \ldots, -2, -1, 0 \}$ and $G_m = \{0\} \times \mathbb{Z}$. We choose $S = \{(i,0) \mid i \leq 0\}$. $Y_2X_2 = (1 - q_2)^{-1}$ on $M_m$ so $M_m = \mathbb{C}v_0$, for a basis vector $v_0$, and $X_2v_0 = \rho_0v_0$ for some $\rho \in \mathbb{C}^*$. The set $C = \{v_j := Y_1^jv_0 \mid j \leq 0\}$ is a basis for $M$ and we have the following picture of $\text{supp}(M)$.

One easily obtains the following action on the basis $\{v_j \mid j \leq 0\}$:

$$X_1v_j = \begin{cases} 0, & j = 0, \\ [j]_{q_1}v_{j-1}, & j \geq 1, \end{cases}$$
$$Y_1v_j = v_{j+1},$$
$$X_2v_j = \rho\lambda_2^jg^jv_j,$$
$$Y_2v_j = (1 - q_2)^{-1}\rho^{-1}\lambda_2^j v_j.$$  

The case $k < 0$ is analogous and yields a lowest weight representation with $m = n_1^{(1)}_{q_1^{-1}}$ as its lowest weight. A basis for $M$ is then

$$C = \{v_j := X_1^jv_0 \mid j \geq 0\},$$
where $M_m = \mathbb{C} v_0$ and the action is given by

$$
X_1 v_j = v_{j+1},
Y_1 v_j = \begin{cases} 0, & j = 0, \\
[-j]_{q_1} v_{j-1}, & j > 0,
\end{cases}
$$

(6.9)

$$
X_2 v_j = (q_1 \lambda_{12})^{-j} \rho v_j,
Y_2 v_j = \lambda_{12}^j (1 - q_2)^{-j} \rho^{-1} v_j.
$$

6.4.3. The case $m = n^{(1)}_\lambda$, $\lambda \neq 0$, $\omega$ contains no 1-break and $q_1$ is a root of unity. By Corollary 6.2, $\lambda \neq q_1^k$ for all $k \in \mathbb{Z}$. So by Proposition 6.4, $\tilde{G}_m = \mathbb{Z}^2$. $G_m = (\alpha_1 \mathbb{Z}) \times \mathbb{Z}$ and we can choose $S = \{0, 1, \ldots, \alpha_1 - 1\} \times \{0\}$. From

$$
X^{\alpha_1}_1 X_2 = (q_1 \lambda_{12})^{\alpha_1} X_2 X^{\alpha_1}_1 = \lambda_{12}^{\alpha_1} X_2 X^{\alpha_1}_1
$$

and Corollary 4.6 follows that $B_m^{(1)} \simeq T_{\lambda_{12}^{\alpha_1}}$. It can only have finite-dimensional irreducible representations if $\lambda_{12}^{\alpha_1}$ is a root of unity. Assuming this, any such representation is $r$-dimensional, where $r$ is the order of $\lambda_{12}^{\alpha_1}$, and is parametrized by $\mathbb{C}^r \times \mathbb{C}^r \ni (\rho, \mu)$ with basis

$$
M_m = \text{Span}\{ v_j := X_2^j v_0 \mid j = 0, 1, \ldots, r - 1\},
$$

where $X^{\alpha_1}_1 v_0 = \rho v_0$ and relations

$$
X^{\alpha_1}_1 v_j = \lambda_{12}^{\alpha_1 j} \rho v_j,
X_2 v_j = \begin{cases} v_{j+1}, & 0 \leq j < r - 1, \\
\mu v_0, & j = r - 1.
\end{cases}
$$

Therefore by Theorem 5.1,

$$
M = \text{Span}\{ w_{ij} = X_2^i v_j \mid 0 \leq i < \alpha_1, 0 \leq j < r\}.
$$

Using the commutation relations and the formulas in Lemma 6.6 we can write down the action as follows.

$$
X_1 w_{ij} = \begin{cases} w_{i+1,j}, & 0 \leq i < \alpha_1 - 1, \\
\lambda_{12}^{\alpha_1} \rho w_{0,j}, & i = \alpha_1 - 1,
\end{cases}
Y_1 w_{ij} = \begin{cases} (1 - \lambda)(1 - q_1)^{-1} \lambda_{12}^{-\alpha_1 j} \rho^{-1} w_{\alpha_1 - 1,j}, & i = 0, \\
(1 - \lambda q_1^{-i})(1 - q_1)^{-1} w_{i-1,j}, & 0 < i \leq \alpha_1 - 1,
\end{cases}
$$

(6.10)

$$
X_2 w_{ij} = \begin{cases} q_1^{-i} \lambda_{21}^j w_{i,j+1}, & 0 \leq j < r - 1, \\
q_1^{-i} \lambda_{21}^j w_{i,0}, & j = r - 1,
\end{cases}
Y_2 w_{ij} = \begin{cases} \lambda_{2} \mu^{-1} \lambda(1 - q_2)^{-1} w_{i,r-1}, & j = 0, \\
\lambda_{2} \lambda(1 - q_2)^{-1} w_{i,j-1}, & 0 < j \leq r - 1.
\end{cases}
$$

The action can be illustrated in the following way:
6.4. The case $m = n^{(1)}_X$, $\lambda \neq 0$, $\omega$ contains no 1-break and $q_1$ is not a root of unity. By Corollary 6.2, $\lambda \neq q_1^k$ for all $k \in \mathbb{Z}$. Now $Z_m^\lambda = \{0\} \times \mathbb{Z}$ so $G_m = \{0\} \times \mathbb{Z}$. $M_m$ is one-dimensional with basis $v_0$, say, and $X_2 = \rho$ on $M_m$ while $Y_2X_2 = \lambda(1-q_2)^{-1} \neq 0$ on $M_m$. We choose $S = \mathbb{Z} \times \{0\}$. Then a basis for $M$ is

$$C = \{v_j := X_1^j v_0 \mid j \geq 0\} \cup \{v_j := \zeta_j Y_1^{-j} v_0 \mid j < 0\},$$

where we determine $\zeta_j$ by requiring that $X_1 v_j = v_{j+1}$ for all $j$. Explicitly we have for $j < 0$,

$$\zeta_j = \frac{(1-q_1)^{-j}}{(1 - \lambda q_1^{-j})(1 - \lambda q_1^{-j-1}) \ldots (1 - \lambda q_1)}.$$

Using the commutation relations and the formulas in Lemma 6.6 we get the action on $M = \text{Span}\{v_j \mid j \in \mathbb{Z}\}$.

$$(6.11) \quad X_1 v_j = v_{j+1}, \quad X_2 v_j = q_1^{-j} \lambda_1^{-j} \rho v_j,$$

$$Y_1 v_j = \frac{1 - \lambda q_1^{-j+1}}{1 - q_1} v_{j-1}, \quad Y_2 v_j = \lambda_1^j (1 - q_2)^{-1} \rho^{-1} v_j,$$

and a corresponding diagram

6.5. The case $m = n^{(2)}_X$, $\lambda \neq 0$. Here $\gamma_1 = 0$ while $\gamma_2 = \lambda (q_2 - 1)$. By Corollary 6.2, $\omega$ does not contain any breaks. We have $G_m = \mathbb{Z}^2$ and $G_m = Z_m^2 = K_2$.

We will need some lemmas in order to proceed.

Lemma 6.9. For $k, l \in \mathbb{Z}$ we have

$$(6.12) \quad Z_1^k Z_2^l = q_1^{\delta_{l,1}} \lambda_1^l \lambda_2^l Z_1^k Z_2^l,$$

where $\delta_{l,1} = \max\{0, l\}$.

Proof. Relations (2.6)–(2.8) can be rewritten in the more compact form

$$Z_1^k Z_2^l = q_1^{k \delta_{l,1}} \lambda_1^l \lambda_2^l Z_1^k Z_2^l,$$

where $\delta_{l,1}$ is the Kronecker symbol. After repeated application of this, (6.12) follows. \qed

By Lemma 6.6 we have for $k, l \in \mathbb{Z}$,

$$(6.13) \quad \sigma_1^k \sigma_2^l(t_1) = (1 - q_1)^{-1} \text{ mod } m,$$

$$(6.14) \quad \sigma_1^k \sigma_2^l(t_2) = \lambda q_1^k q_2^l \text{ mod } m.$$

Lemma 6.10. Let $k, l \in \mathbb{Z}$ and let $m = \min\{|k|, |l|\}$. Then, as operators on $M_m$, we have

$$(6.15) \quad Z_1^k Z_1^l = \begin{cases} Z_1^{k+l}, & kl \geq 0, \\ (1 - q_1)^{-m} Z_1^{k+l}, & kl < 0, \end{cases}$$

$$(6.16) \quad Z_2^k Z_2^l = \begin{cases} Z_2^{k+l}, & kl \geq 0, \\ \lambda^m q_2^{-2l} (\text{sgn} l) m/2 Z_2^{k+l}, & kl < 0. \end{cases}$$
Proof. Direct calculation using (6.13) and (6.14). For example if $k > 0$ and $l < 0$ we have
\[
Z_2^k Z_2^l = X_2^k Y_2^{-l} = X_2^{k-l} \sigma_2(t_2) Y_2^{-l-1} = \\
= X_2^{k-1} Y_2^{-l-1} \sigma_2(t_2) = X_2^{k-1} Y_2^{-l-1} \lambda q_2^{-l} = \ldots = \\
= \lambda q_2^{-l} \ldots \lambda q_2^{-l-(m-1)} Z_2^{k+l} = \\
= \lambda^{m-l-m-(m-1)/2} Z_2^{k+l}.
\]
\[\square\]

Lemma 6.11. Let $k, l \in \mathbb{Z}$ and let $m = \min\{|k|, |l|\}$. Then, as operators on $M_m$,
\begin{equation}
Z_1^k Z_1^l = Z_1^l Z_1^k,
\end{equation}
and
\begin{equation}
Z_2^k Z_2^l = c(k, l) Z_2^l Z_2^k,
\end{equation}
where
\begin{equation}
c(k, l) = \begin{cases} 1, & kl \geq 0, \\ q_2, & kl < 0. \end{cases}
\end{equation}

Proof. Follows directly from Lemma 6.10. \[\square\]

Lemma 6.12. Let $g = (g_1, g_2) \in \mathbb{Z}^2 = \mathcal{G}_m$ and set $r_g = \varphi_m(a_g a_g)^{-1}$ where $\varphi_m$ is the projection $R \to R/m = K$. Then
\begin{equation}
r_g = (1 - q_1)^{\frac{|g_1|}{2}} (\lambda^{-1} q_2^{-1})^{\frac{|g_2|}{2}}
\end{equation}
and $(a_g)^{-1} = r_g a_g = r_g Z_2^{g_1} Z_1^{-g_1}$ as operators on $M_m$.

Proof. We have
\[
a_g a_g = (Z_1^{g_1} Z_2^{g_2}) Z_1^{g_1} Z_2^{g_2} = Z_2^{-g_2} Z_1^{-g_1} Z_1^{g_1} Z_2^{g_2} = Z_2^{-g_2} Z_1^{g_1} Z_2^{g_2},
\]
by Lemma 6.9. Thus by Lemma 6.10,
\[
\varphi_m(a_g a_g) = (1 - q_1)^{-\frac{|g_1|}{2}} \lambda^{\frac{|g_1|}{2}} q_2^{-\frac{|g_2|}{2}}
\]
which proves the formula. The last statement is immediate. \[\square\]

We consider the three subcases corresponding to the rank of the free abelian group $K_2$.

6.5.1. The case $m = n^{(2)}_\lambda$, $\lambda \neq 0$, rank $K_2 = 0$. $G_m = K_2 = \{0\}$ so $B_m^{(1)} = R$ which is commutative, hence $M_m = C v_0$ for some $v_0$, and $S = \mathbb{Z}^2$. Thus $C = \{a_g v_0 \mid g \in \mathbb{Z}^2\}$ is a basis for $M$ and using Lemma 6.10 and Lemma 6.9 we obtain that the action of $X_i$ is given by

\begin{equation}
X_1 a_g v_0 = \begin{cases} a_{g+c_1} v_0, & g_1 \geq 0, \\ (1 - q_1)^{-\frac{|g_1|}{2}} a_{g+c_1} v_0, & g_1 < 0, \end{cases}
\end{equation}
\begin{equation}
X_2 a_g v_0 = \begin{cases} (q_1 \lambda_{12})^{-g_1} a_{g+c_2} v_0, & g_2 \geq 0, \\ (q_1 \lambda_{12})^{-g_1} \lambda q_2^{-g_2} a_{g+c_2} v_0, & g_2 < 0. \end{cases}
\end{equation}

The action of $Y_i$ on the basis is deduced uniquely from
\begin{equation}
Y_1 X_1 a_g v_0 = (1 - q_1)^{-1} a_g v_0,
\end{equation}
\begin{equation}
Y_2 X_2 a_g v_0 = \lambda q_1^{-g_1} q_2^{-g_2} a_g v_0,
\end{equation}
which hold by (6.13) and (6.14).
6.5.2. The case $m = n^{(2)}_{\lambda}, \lambda \neq 0, \text{rank } K_2 = 1$. Let $(a, b)$ be a basis element. Since $G_m = K_2$ which is of rank one, $D_m^{(1)} \simeq \mathbb{C}[T, T^{-1}]$ by Corollary 4.6 so $M_m$ is one-dimensional. As before we let $M_m = \mathbb{C}v_0$. Then $Z^a_1 Z^b_2 v_0 = \rho v_0$ for some $\rho \in \mathbb{C}^*$.

We assume $a \neq 0$. The case $b \neq 0$ can be treated similarly. By changing basis, we can assume that $a > 0$. Choose $S = \{0, 1, \ldots, a - 1\} \times \mathbb{Z}$. The corresponding basis for $M$ is

$$C = \{w_{ij} := X_1^i Z_2^j v_0 \mid 0 \leq i \leq a - 1, j \in \mathbb{Z}\}.$$ 

We now aim to apply Theorem 5.4. If $0 \leq i < a - 1$ then clearly $X_1 w_{ij} = w_{i+1,j}$.

And

$$X_1 w_{a-1,j} = X_1^a Z_2^j v_0 \in \mathbb{C}Z_2^{j-a} v_0 = \mathbb{C}w_{0,j-a}.$$ 

We want to compute the coefficient of $w_{0,j-a}$. Similarly to the proof of Theorem 5.4 we have, using Lemma 6.12, Lemma 6.9 and (6.16),

$$X_1 w_{a-1,j} = Z_2^j v_0 = (Z^a_1 Z^b_2 r_{(a,b)} Z_1^a Z_2^b) Z_1^a Z_2^b v_0 =$$

$$= r_{(a,b)}(q_1^{a_1} q_2^{b_1}) \lambda_{12}^{a_1} \lambda_2^{a} Z_2^b Z_1^a \rho v_0 =$$

$$= (\lambda^{-1} q_2^{(b-1)/2} \lambda_2^{a(j-b)}) \lambda_2^{a(j-b)} \rho C_0 w_{0,j-b},$$

where

$$C_0 = \begin{cases} 1, & \text{if } (j,b) \in \{1, 2, \ldots, \min(j,b)\} \times \{1, 2, \ldots, \min(j,b)\}/2, \\ b \leq 0, & \text{if } b > 0. \end{cases}$$

Using Lemma 6.9 one easily get the action of $X_2$ on the basis. We conclude that

$$X_1 w_{ij} = \begin{cases} w_{i+1,j}, & 0 \leq i < a - 1, \\ (\lambda^{-1} q_2^{(b-1)/2} \lambda_2^{a(j-b)}) \lambda_2^{a(j-b)} \rho C_0 w_{0,j-b}, & i = a - 1. \end{cases} \tag{6.23}$$

$$X_2 w_{ij} = \begin{cases} q_1^{-1} \lambda_1^{a_i} w_{i,j+1}, & j \geq 0, \\ q_1^{-1} \lambda_1^{a_i} \lambda_2^{a(j-b)} w_{i,j+1}, & j < 0. \end{cases} \tag{6.24}$$

The action of the $Y_i$ is uniquely determined by

$$Y_1 X_1 v_{ij} = (1 - q_1)^{-1} v_{ij},$$

$$Y_2 X_2 v_{ij} = \lambda q_1^{-1} q_2^{-j} v_{ij},$$

which hold by (6.13)–(6.14). See Figure 1 for a visual representation.
6.5.3. The case $m = n^{(2)}$, $\lambda \neq 0$, rank $K_2 = 2$. Let $s_1 = a = (a_1, a_2)$, $s_2 = b = (b_1, b_2)$ be a basis for $G_m = K_2$ over $\mathbb{Z}$. We can assume that $a_1, b_1 \geq 0$ and that $d := \lfloor a_2 / b_2 \rfloor > 0$.

By Corollary 4.6, $B_m^{(1)} \simeq T_\nu$ for some $\nu$ which we will now determine. Using Lemma 6.9 and Lemma 6.11 we have, as operators on $M_m$,

$$Z^a_1 Z^a_2 Z^b_1 Z^b_2 = q_1^{-h_1} \lambda^{1 h_1 a_2} c(a_2, b_2) Z^a_1 Z^b_2 Z^a_1 Z^b_2 =$$

$$= q_1^{-h_1} \lambda^{1 h_1 a_2 - b_2 a_2} c(a_2, b_2) Z^a_1 Z^b_2 Z^a_1 Z^b_2.$$

We conclude that $B_m^{(1)} \simeq T_\nu$ where

$$\nu = \lambda^{d} q_1^{-h_1 a_2} c(a_2, b_2).$$

The function $c$ was defined in (6.19), $d = a_1 b_2 - b_1 a_2$ and $\mathbb{F} := \max\{0, k\}$ for $k \in \mathbb{Z}$.

For $M_m$ to be finite-dimensional it is thus necessary that this $\nu$ is a root of unity. Assume this and let $r$ denote its order. Then $\dim M_m = r$. Let

$$\{v_0, v_1, \ldots, v_{r-1}\}$$

be a basis such that

$$Z^a_1 Z^b_2 v_j = \nu^j \rho v_j,$$

$$Z^b_1 Z^a_2 v_j = \begin{cases} v_{j+1}, & 0 \leq j < r - 1, \\ \mu v_0, & j = r - 1, \end{cases}$$

where $\rho, \mu \in \mathbb{C}^*$.

The next step is to determine a set $S \subseteq \hat{G}_m = \mathbb{Z}^2$ of representatives for the set of cosets $\hat{G}_m / G_m = \mathbb{Z}^2 / K_2$ which makes it possible to write down the action of the algebra later. We proceed as follows.

Recall that $K_2 = \mathbb{Z} \cdot (a_1, a_2) \oplus \mathbb{Z} \cdot (b_1, b_2)$. Let $d_1$ be the smallest positive integer such that $(d_1, 0) \in K_2$. We claim that $d_1 = d / \gcd(a_2, b_2)$. Indeed $d_1$ must be of the form $k a_1 + l b_1$ where $k, l \in \mathbb{Z}$ and $k a_2 + l b_2 = 0$ with $\gcd(k, l) = 1$. For such $k, l$, $k | b_2, l | a_2$ and $b_2 / k = -a_2 / l =: p > 0$.

Then $\gcd(a_2 / p, b_2 / p) = 1$ which implies that $\gcd(a_2, b_2) = p$. Thus $d_1 = k a_1 + l b_1 = (b_2 a_1 - a_2 b_1) / p = d / \gcd(a_2, b_2)$ as claimed.

Next, let $d_2$ denote the smallest positive integer such that some $K_2$-translation of $(0, d_2)$ lies on the $x$-axis, i.e. such that

$$(0, d_2) + K_2 \cap \mathbb{Z} \times \{0\} \neq \emptyset.$$

Such an integer exists because if we write $\gcd(a_2, b_2) = k a_2 + l b_2$, then

$$(0, k a_2 + l b_2) - k (a_1, a_2) - l (b_1, b_2) = (-k a_1 - l b_1, 0).$$

On the other hand if $(0, d_2) + k a_1 + l b_1 \in \mathbb{Z} \times \{0\}$, i.e. if $d_2 = k a_2 + l b_2$, then $\gcd(a_2, b_2) | d_2$. Therefore $d_2 = \gcd(a_2, b_2)$.

We also see that for any point in $\mathbb{Z}^2$ of the form $(x, d_2)$ there is a $g \in K_2$ such that $(x, d_2) + g \in \mathbb{Z} \times \{0\}$.

Suppose now that for some $k, l \in \mathbb{Z}$,

$$k (a_1, a_2) + l (b_1, b_2) \in K_2 \cap \{0, 1, \ldots, d_1 - 1\} \times \{0, 1, \ldots, d_2 - 1\}.$$
is a set of representatives for \( \mathbb{Z}^2 / K_2 \). In particular we get from Corollary 5.3 that \( \dim M \) is finite and
\[
\dim M / \dim M_m = |S| = d_1 d_2 = a_1 b_2 - b_1 a_2.
\]

We fix now integers \( a'_2, b'_2 \) such that
\[
(6.29) \quad d_2 = \text{GCD}(a_2, b_2) = a'_2 a_2 + b'_2 b_2
\]
and such that \(-a'_2 a_1 - b'_2 b_1 \in \{0, 1, \ldots, a_1 - 1\}\). This can be done because for any \( p \in \mathbb{Z}, (a'_2, b'_2) := (a'_2 a_2 / d_2, b'_2 a_2 / d_2) \) also satisfies \( a'_2 a_2 + b'_2 b_2 = d_2 \) but now
\[
-a'_2 a_1 - b'_2 b_1 = -(a'_2 a_2 / d_2) a_1 - (b'_2 a_2 / d_2) b_1 = -a'_2 a_1 - b'_2 b_1 - pd_1.
\]

We set
\[
(6.30) \quad s = -a'_2 a_1 - b'_2 b_1.
\]

Let \( (i, j) \in S \). We have the following reductions in \( \mathbb{Z}^2 \) modulo \( K_2 \).
\[
(1, 0) + (i, j) = \begin{cases} (i + 1, j), & 0 \leq i < d_1 - 1, \\ (0, j), & i = d_1 - 1, \end{cases}
\]
\[
(0, 1) + (i, j) = \begin{cases} (i, j + 1), & 0 \leq j < d_2 - 1, \\ (i + s, 0), & j = d_2 - 1, i + s \leq d_1 - 1, \\ (i + s - d_1 + 1, j + s > d_1 - 1. \end{cases}
\]

From this we can understand how the \( X_i \) act on the support of \( M \), see Figure 2 for an example. By Theorem 5.1 the set
\[
C = \{ w_{ijk} := X^i_1 X^j_2 v_k \mid 0 \leq i < d_1, 0 \leq j < d_2, 0 \leq k < r \} 
\]
is a basis for \( M \) where \( v_k \) is the basis \((6.26)\) for \( M_m \).

If \( 0 \leq i < d_1 - 1 \) we clearly have \( X_1 w_{ijk} = w_{i+1,j,k} \). Suppose \( i = d_1 - 1 \). Then by Lemma 6.9,
\[
X_1 w_{ijk} = X^i_1 X^j_2 v_k = d_{ij}^a \chi_{12}^{i,j} X_2^j X_1^i v_k.
\]

Thus we must express \( X^i_1 \) in terms of \( Z_1^{a_1}, Z_2^{a_2} \) and \( Z_1^{b_1}, Z_2^{b_2} \). Since \((d_1, 0) = b_2 / d_2 a - a_2 / d_2 b \) we have
\[
(6.31) \quad (Z_1^{a_1}, Z_2^{a_2}) yb / dz (Z_1^{b_1}, Z_2^{b_2}) - yb / dz = C_1^{-1} X_1^{a_1},
\]
as operators on \( M_m \) for some constant \( C_1^{-1} \) which we must calculate.

Lemma 6.13. The constant \( C_1 \) defined in \((6.31)\) is given by
\[
(6.32) \quad C_1^{-1} = r_a (q_1^{a_2 / d_2} q_{12}^{b_1 / d_2} \chi_{12}^{b_1 b_2} \chi_{12}^{1 - b_2} / d_2^{1 - b_1}) / \left( q_1^{a_2 / d_2} (q_1^{a_2 / d_2} \chi_{12}^{b_1 b_2} \chi_{12}^{1 - b_2} / d_2^{1 - b_1}) / \left(0, -b_2 a_2 / d_2\right) C_1', \right)
\]
where the $r_g$, $g \in \mathbb{Z}^2$ are given by (6.20),

$$C'_1 = \begin{cases} (1 - q_1)^{-\min\{a_1b_2/d_2, |a_1a_2/d_2|\}}, & a_2b_2 > 0, \\ 1, & a_2b_2 \leq 0, \end{cases}$$

$k = \max\{0, k\}$ for $k \in \mathbb{Z}$ and $d_2 = \text{GCD}(a_2, b_2)$.

**Proof.** If $b_2 > 0$ for example, we have by Lemma 6.9

$$(Z_1^{a_1}Z_2^{a_2})^{b_2/d_2} = q_1^{-a_1d_2\lambda_1^{1-a_1}}(q_1^{-a_1d_2\lambda_1^{1-a_1}})^2 \ldots
$$

$$\ldots (q_1^{-a_1d_2\lambda_1^{1-a_1}})^{b_2/d_2 - 1} Z_1^{a_2d_2/d_2} Z_2^{a_2b_2/d_2} =
$$

$$= (q_1^{-a_1d_2\lambda_1^{1-a_1}})^{b_2/d_2 - 1} Z_1^{a_2d_2/d_2} Z_2^{a_2b_2/d_2}.$$  

When $b_2 < 0$ we get a similar calculation where $r_{a_2}^{-b_2/d_2}$ appears by Lemma 6.12.

$(Z_1^{b_1}Z_2^{b_2})^{-a_2/d_2}$ can analogously be expressed as a multiple of $Z_1^{b_1a_2/d_2} Z_2^{a_2b_2/d_2}$.

We then commute $Z_2^{a_2b_2/d_2}$ and $Z_1^{-a_1a_2/d_2}$ using Lemma 6.9. As a last step we use Lemma 6.10 and obtain two more factors.

We conclude that

$$X_1 w_{ijk} = \begin{cases} w_{i+1,j,k}, & i < d_1 - 1, \\ q_1^{d_1\lambda_1^{1-a_1}} C_1 \nu^{b_2/d_2} \mu^{k_1'}, \mu^{k_2}, & i = d_1 - 1. \end{cases}$$

Here

$$k - a_2/d_2 = r_1^{k_1} + k_1'' \quad \text{with} \quad 0 \leq k_1'' < r.$$  

Next we turn to the description of how $X_2$ acts on the basis $C$. If $0 \leq j < d_2 - 1$ we have $X_2 w_{ijk} = q_1^{-i} \lambda_1^{1-a_1} w_{i+1,j,k}$ by Lemma 6.9. Suppose $j = d_2 - 1$. Then, as in the first step of the proof of Theorem 5.4,

$$(6.34) \quad X_2 w_{ijk} = q_1^{-i} \lambda_1^{1-a_1} X_1^{d_2} v_k = q_1^{-i} \lambda_1^{1-a_1} X_1^{d_2} (X_2^{d_2} r_{(-s,d_2)} Z_2^{-d_2} Z_1^{d_2} Z_2^s) v_k.$$  

By (6.16) and (6.20),

$$(6.35) \quad X_2^{d_2} r_{(-s,d_2)} Z_2^{-d_2} Z_1^s = r_{(-s,d_2)} r_1^{-1} (0, -d_2) Z_1^s =
$$

$$(1 - q_1)^s (\lambda^{1-a_1} q_1^{d_2-1/2})^{d_2} (\lambda^{-1} q_1^{d_2-1/2})^{d_2} Z_1^s =
$$

$$(1 - q_1)^s (\lambda^{1-a_1} q_1^{d_2-1/2})^{d_2} Z_1^s.$$  

We must express $Z_1^{-s} Z_2^{d_2}$ in the generators of the algebra $B_{m}^{(1)}$ in order to calculate its action on $v_k$.

$$(6.36) \quad (Z_1^{a_1} Z_2^{a_2})^{c_2} (Z_1^{b_1} Z_2^{b_2})^{c_2'} = C_2^{-1} Z_1^{-s} Z_2^{d_2},$$

for some $C_2 \in \mathbb{C}^*$ since the degree on both sides are equal by (6.29) and (6.30). Similarly to the proof of Lemma 6.13,

$$(6.37) \quad C_2^{-1} = \frac{q_1^{-a_1d_2\lambda_1^{1-a_1}} (q_1^{-a_1d_2\lambda_1^{1-a_1}})(c_2^1 - 1/2) \cdot r_1 \cdot q_1^{-a_1d_2\lambda_1^{1-a_1}} a_2^{b_2} b_2^{d_2} c_2' C_2',}$$

and

$$C_2' = \begin{cases} 1, & \alpha_2^{b_2} b_2^{d_2} \geq 0, \\ (1 - q_1)^{-\min\{a_1b_2, |a_1a_2|\}}, & \alpha_2^{b_2} b_2^{d_2} < 0, \end{cases}$$

$$C_2'' = \begin{cases} 1, & \alpha_2^{b_2} b_2^{d_2} \geq 0, \\ \lambda^{d_2-b_2}, & \alpha_2^{b_2} b_2^{d_2} < 0, \end{cases}$$

$$C_2'' = \begin{cases} 1, & \alpha_2^{b_2} b_2^{d_2} \geq 0, \\ \lambda^{d_2-b_2}, & \alpha_2^{b_2} b_2^{d_2} < 0, \end{cases}$$
where \( m' = \min\{\|a_2 a'_2\|, \|b_2 b'_2\|\} \). Furthermore, letting
\[
(6.38) \quad b'_2 + k = r k'_2 + k''_2, \quad \text{where } 0 \leq k'_2 < r
\]
we have by (6.27)–(6.28),
\[
(6.39) \quad (Z'^{a}_1 Z'^{a}_2)^{r} (Z'^{b}_1 Z'^{b}_2)^{k'_{2}} v_{k} = \rho^{a} \rho^{a_{2}} \mu^{a_{2}} v_{k''_{2}}.
\]
If \( i + s \leq d_1 - 1 \) we can now write down the action of \( X_2 \) on \( w_{ijk} \) by combining (6.34)–(6.37), (6.39) to get a multiple of \( w_{i+s,0,k''_2} \). However if \( i + s > d_1 - 1 \), we must reduce further because then \( (i + s,0) \notin S \). Let
\[
(6.40) \quad k''_2 - a_2/d_2 = r k'_2 + k''_2, \quad \text{where } 0 \leq k''_2 < r.
\]
Then by the calculations for the action of \( X_1^{d_1} \) on \( M_m \),
\[
X_1^{d_1} v_{k''_2} = X_1^{i+s-d_1} X_1^{d_1} v_{k''_2} = C_1 \mu^{k'_2} \rho^{k'_2 d_2} \rho^{b_2} \rho^{b_2} w_{i+s,0,k''_2}.
\]
Summing up, \( M \) has a basis
\[
\{ w_{ijk} \mid 0 \leq i < d_1, 0 \leq j < d_2, 0 \leq k < r \}
\]
and \( X_1, X_2 \) act on this basis as follows.
\[
(6.41) \quad X_1 w_{ijk} = \begin{cases} \frac{w_{i+1,j,k}}{q_1^{d_1} \lambda_{12}} C_1 \rho^{b_2} \rho^{b_2} \mu^{k'_2} w_{0,j,k'}, & i = d_1 - 1, \\ \frac{w_{i,j+1,k}}{q_1^{d_1} \lambda_{12}} C_1 \rho^{b_2} \rho^{b_2} \mu^{k'_2} w_{i,k',0}, & i < d_1 - 1, \end{cases}
\]
\[
X_2 w_{ijk} = (q_1 \lambda_{12})^{-i}. \]

where \( C_1 \) is given by (6.32), \( C_2 \) by (6.37) and \( \nu \) by (6.25). The parameters \( \rho \) and \( \mu \) come from the action (6.27), (6.28) of \( B^{(1)}_m \) on \( M_m \) and \( k'_2, k''_2 \) are defined in (6.33), (6.38) and (6.40).

The action of the \( Y_i \) is uniquely determined by
\[
(6.42) \quad Y_1 X_1 w_{ijk} = (1 - q_1)^{-1} w_{ijk},
\]
\[
Y_2 X_2 w_{ijk} = \lambda q_1^{-i} q_2^{-j} w_{ijk}.
\]

We remark that the case \( q_1 = q_2 \) corresponds to \( a = (a_1, a_2) = (1, -1) \). Then \( d_2 = 1, d_1 = d = |b_1 + b_2| \) and \( s = 1 \). \( X_1 \) and \( X_2 \) will act on the support in the same direction, cyclically as in Figure 3. The explicit action can be deduced from the above more general case noting that here \( k''_2 = k, k'_2 = 0 \) and
\[
k'_2 = k'_3 = \begin{cases} 0, & k < r - 1, \\ 1, & k = r - 1, \end{cases}
\]
\[
k''_1 = k''_3 = \begin{cases} k, & k < r - 1, \\ 0, & k = r - 1. \end{cases}
\]
6.6. The case $m \notin \{v_μ^{(i)} | μ \in \mathbb{C}, i = 1, 2\}$. This is the generic case. We have $Z_m^2 = Q$ by Corollary 6.8. Our statements here generalize without any problem to the case of arbitrary rank.

Assume first that the $q_i$ are roots of unity of orders $a_k (i = 1, 2)$ and that $ω$ does not contain any 1-breaks or 2-breaks. Then by Corollary 6.2 and Proposition 6.4 we have $G_m = \mathbb{Z}^2$. Thus $G_m = (a_1 \mathbb{Z}) \times (a_2 \mathbb{Z})$. Moreover,

$$X_1^{a_1}X_2^{a_2} = λ_{12}^{a_2}X_2^{a_2}X_1^{a_1}$$

so $B_m^{(1)} \simeq T_{λ_{12}^{a_2}}$ by Corollary 4.6. This algebra has only finite dimensional representations if $λ_{12}^{a_2}$ is a root of unity. Assuming this, let $r$ be the order of $λ_{12}^{a_2}$.

Then there are $ρ, μ \in \mathbb{C}^*$ and $M_m$ has a basis $v_0, v_1, \ldots, v_{r-1}$ such that

$$X_1^{a_1}v_i = λ_{12}^{a_2}ρv_i$$

$$X_2^{a_2}v_i = \begin{cases} v_{i+1} & 0 \leq i < p - 1 \\ μv_0 & i = p - 1 \end{cases}$$

Choose $S = \{0, 1, \ldots, a_1 - 1\} \times \{0, 1, \ldots, a_2 - 1\}$. The corresponding basis for $M$ is $C = \{w_{ijk} := X_1^{i}X_2^{j}v_k | 0 \leq i < a_1, 0 \leq j < a_2, 0 \leq k < r\}$. The following formulas are easily deduced using (2.6)–(2.8).

$$X_1w_{ijk} = \begin{cases} w_{i+1,j,k} & k < a_1 - 1, \\ λ_{12}^{a_1(a_2k+j)}ρw_{0,jk} & k = a_1 - 1. \end{cases}$$

(6.43)

$$X_2w_{ijk} = (q_1λ_{12})^{-i} \begin{cases} w_{i,j+1,k} & l < a_2 - 1, \\ w_{i,0,j+1} & l = a_2 - 1, i < r - 1, \\ μw_{i,00} & l = a_2 - 1, i = r - 1. \end{cases}$$

The action of $Y_1, Y_2$ is determined by

$$Y_1X_1w_{ijk} = q_1^{-i}(α_1 - [i]_{q_1})w_{ijk},$$

$$Y_2X_2w_{ijk} = q_1^{-q_2^2-j}(α_1 - [j]_{q_1}(1 + (q_1 - 1)α_1))w_{ijk}.$$ (6.44)

In all other cases one can show using the same argument that $\dim M_n = 1$ for all $n \in \text{supp}(M)$ and that $M$ can be realized in a vector space with basis $\{w_{ij} \}_{i,j} \subseteq I$, where $I = I_1 \times I_2$ is one of the following sets

$$N_{d_0} \times N_{d_2}, \quad N_{d_0} \times \mathbb{Z}^{d_1}, \quad \mathbb{Z}^{d_0} \times N_{d_2}, \quad \mathbb{Z} \times Z,$$

$$Z^{d_0} \times Z, \quad Z \times Z^{d_1}, \quad Z^{d_0} \times Z^{d_1}, \quad Z^{d_0} \times Z^{d_1},$$

where $N_{d_0} = \{0, 1, \ldots, d_0 - 1\}, Z^{d_1} = \{k \in Z | ±k \geq 0\}$ and $d_i$ is the order of $q_i$ if finite. The action of the generators is given by the following formulas.

$$X_1w_{ij} = \begin{cases} w_{i+1,j} & (i + 1, j) \in I, \\ μλ_{12}^{d_2}w_{0,j} & (i + 1, j) \notin I, I_1 = N_{d_0} \text{ and } α_1 \neq [j]_{q_1}, \\ 0, & \text{otherwise}, \end{cases}$$

(6.45)

$$X_2w_{ij} = (q_1λ_{12})^{-i} \begin{cases} w_{i,j+1} & (i, j + 1) \in I, \\ μw_{i,0} & (i, j + 1) \notin I, I_2 = N_{d_2} \text{ and } α_2 \neq [j]_{q_2}(1 + (q_1 - 1)α_1), \\ 0, & \text{otherwise}, \end{cases}$$
\[ Y_1 w_{ij} = q_i^{i+1}(\alpha_1 - [i - 1]q_1), \]
\[
\begin{cases}
  w_{1-1,j}, & (i - 1, j) \in I, \\
  (\rho a_{12}^{i,j})^{-1} w_{d_1-1,j}, & (i - 1, j) \notin I, I_1 = N_{d_1} \text{ and } \alpha_1 \neq [i - 1]q_1, \\
  0, & \text{otherwise},
\end{cases}
\]
\[ Y_2 w_{ij} = \lambda_2^{-i-j+1}(\alpha_2 - [j - 1]q_2(1 + (q_1 - 1)\alpha_1)). \]
\[
\begin{cases}
  w_{2,i+1}, & (i, j + 1) \in I, \\
  \mu^{-1} w_{i,d_2-1}, & (i, j + 1) \notin I, I_1 = N_{d_2} \\
  0, & \text{and } \alpha_2 \neq [j - 1]q_2(1 + (q_1 - 1)\alpha_1), \text{ otherwise.}
\end{cases}
\]

Thus we have proved the following result.

**Theorem 6.14.** Let \( A \) be a quantized Weyl algebra of rank two with two proper parameters \( q_1, q_2 \in \mathbb{C} \setminus \{0, 1\} \). Then any simple weight \( A \)-module with no proper inner breaks is isomorphic to one of the modules defined by formulas (6.6), (6.7), (6.8), (6.9), (6.10), (6.11), (6.21-6.22), (6.23-6.24), (6.41-6.42), (6.43-6.44) or (6.45-6.46).

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**References**


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